PERGAMON Computers and Mathematics with Applications 40 (2000) 21-35

An International Journal
computers & mathematics with applications

www.elsevier.nl/locate/camwa

Nonorthogonal Wavelet Approximation with Rates of Deterministic Signals

G. A. ANASTASSIOU
Department of Mathematical Sciences, The University of Memphis
Memphis, TN 38152, U.S.A.
anastasg@msci.memphis.edu

S. CAMBANIS*
Department of Statistics, University of North Carolina
Chapel Hill, NC 27599-3260, U.S.A.

Abstract—An n^{th} order asymptotic expansion is produced for the $\mathcal{L}_2$-error in a nonorthogonal (in general) wavelet approximation at resolution 2^{-k} of deterministic signals f. These signals over the whole real line are assumed to have n continuous derivatives of bounded variation. The engaged nonorthogonal (in general) scale function φ fulfills the partition of unity property, and it is of compact support. The asymptotic expansion involves only even terms of products of integrals involving φ with integrals of squares of (the first $\lceil n/2 \rceil - 1$ only) derivatives of f.

Keywords—Mean-error, Wavelet nonorthogonal approximations and expansions, Rates of convergence.

TRIBUTE TO STAMATIS CAMBANIS

The impressive Athens-Greece scene of 1968 in education was one of brain drain to the U.S.A. This was already underway and is highly celebrated. At that time, I was a high school student. I was preparing for my entrance to Athens University (1970), when in the private high quality seminary (J. Mantas) I attended, I found out (from my mentor and great geometer, the late J. Ioannides) about the rising megastar at Princeton University, Stamatis Cambanis. Earlier, Cambanis had gone through the same seminary, where not only had he excelled as a student, but he had authored several books in mathematics and physics which were used later by my generation of students as textbooks. Obviously, to many students, including myself, Cambanis was a shining role model. Then, later in 1979, at the peak of the Greek brain drain exodus to the U.S.A., I again found the name of Cambanis, through his great articles on probabilistic inequalities. Now, Cambanis was a gigantic authority in his many fields of expertise and a full professor in the famous School of Statistics at the University of North Carolina. Later, since we are both Greeks and have common academic interests, we established contact. For example, in my conference on Approximation Theory in Memphis, Tennessee in 1991, Cambanis was one of the main speakers as a great expert on the overlaps of probability and approximation theory. Obviously, I, as the younger mathematician, was very eager to publish work with Cambanis, who invited me to Chapel Hill in February 1993, where we started this article. This collaboration

*Deceased.

0898-1221/00/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0898-1221(00)00137-1

continued only through the Spring of 1993. Unfortunately, Cambanis' bad illness of cancer stopped further collaboration. He died on April 12, 1995 at the age of only 51, in Chapel Hill, North Carolina. Cambanis was not only a great scientist, but he was a very pleasant, easy-going, simple, and friendly man, always smiling, never worrying, and always helpful and encouraging. That is why he had so many friends from all over the world, including myself, who now mourn his early death. His contributions to science are outstanding, well known, and with a lot of implications. Cambanis was an angel who left early; his memory among people who knew him will be eternal. My sorrow is even greater since our collaboration stopped only at the first paper. We, Greeks of diaspora, due to the fact that we are scattered here and there, find it difficult to work together, but when we manage to collaborate, the results usually are outstanding. So that is another reason I mourn Cambanis' loss—for the lost opportunities for future successful collaborations with another Greek. My dearest friend Stamatis, I will remember you forever with the highest esteem.

G. A. Anastassiou

1. INTRODUCTION

Multiresolution signal decomposition and wavelet orthonormal bases of $L_2(\mathbf{R})$ have received more and more attention recently in huge numbers of mathematical, signal, and image processing articles; e.g., see [1–5]. However, the nonorthogonal case is, so far, less discussed.

Here, we only mention that a multiresolution decomposition of $L_2(\mathbf{R})$ is a sequence $\{V_k\}_{k=-\infty}^{+\infty}$ of closed subspaces of $L_2(\mathbf{R})$ such that for all $k, j \in \mathbf{Z}$,

(1) $V_k \subset V_{k+1}$,
(2) $\bigcup_{k=-\infty}^{+\infty} V_k$ is dense in $L_2(\mathbf{R})$,
(3) $\bigcap_{k=-\infty}^{+\infty} V_k = \emptyset$,
(4) $f(t) \in V_k$ iff $f(2t) \in V_{k+1}$, and
(5) $f(t) \in V_k \Leftrightarrow f(t - 2^{-k} \cdot j) \in V_k$.

In the orthogonal case, the approximation of $f \in L_2(\mathbf{R})$ at resolution 2^{-k} is the orthogonal projection $\hat{f}_k$ of f on V_k. This is calculated by the use of a wavelet orthonormal basis $\{\varphi_{k,j}(t) := 2^{k/2} \cdot \varphi(2^k t - j)\}_{j=-\infty}^{+\infty}$ for V_k which is generated by the *scale* function $\varphi \in L_2(\mathbf{R})$. A simple example of orthogonal scale functions of compact support and having the partition of unity property is

$$\varphi_0(x) = \begin{cases} 1, & -\dfrac{1}{2} \le x < \dfrac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

In many applications, such as image compression or edge detection, the orthogonal setting is not enough, since some natural constraints cannot be achieved. For example, it is impossible to make use of finite impulse response linear phase filters. To overcome such difficulties in general, we perform a different kind of $\mathcal{L}_2$-approximation. Namely, here the approximants to $f \in L_2(\mathbf{R})$ at resolution 2^{-k} will be the operators

$$(B_k f)(x) := \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right) \cdot \varphi\left(2^k x - j\right),$$

where φ is a Lebesgue measurable bounded function with compact support having the partition of unity property, and *not* generating (in general) an orthonormal basis $\varphi_{k,j}$ for V_k.

A well-known example of a *nonorthogonal* φ, as described above, is the $m \ge 2^{\text{nd}}$-order cardinal B-spline N_m defined through convolution

$$N_m(x) := (N_{m-1} * N_1)(x) = \int_0^1 N_{m-1}(x - t)\, dt,$$

where N_1 is the characteristic function of the interval $[0,1)$. We obtain that

(i) $N_m(x) = (1/(m-1)!) \cdot \sum_{\tau=0}^{m}(-1)^\tau \cdot \binom{m}{\tau} \cdot (x-\tau)_+^{m-1}$, $m \geq 2$,
(ii) support $N_m = [0,m]$,
(iii) $N_m(x) > 0$, for $0 < x < m$,
(iv) $\sum_{j=-\infty}^{+\infty} N_m(x-j) = 1$, all $x \in \mathbf{R}$, and
(v) N_m is bounded and Lebesgue measurable.

For a greater discussion and properties of N_m, see [2, pp. 85–86]. So, we prove in Theorem 1 that $(B_k f)$ converges to f as $k \to +\infty$ in $\mathcal{L}_2$ norm, under mild appropriate assumptions on f. In fact, we find an asymptotic expansion for $e_k(f) := \|B_k f - f\|_2$, which is the L_2 approximation error at resolution 2^{-k}. This is an n^{th}-order asymptotic expansion having certain degree of smoothness (the existing n derivatives of the deterministic signal f).

The exact rate of convergence and asymptotic constant are determined and their dependence on f and on φ are found. For related work about deterministic signals f in the orthogonal case, see [1; 4, Theorem 3; 5; 6].

To the best of our knowledge, no similar treatment for the *nonorthogonal* deterministic case exists in the literature. We feel that our results will find applications especially in the areas of the image compression and stochastic analysis.

2. WAVELET APPROXIMATION AT RESOLUTION 2^k

Here is our main result.

THEOREM 1. *Let f be such that $f, f', f'', \dots, f^{(n)}$ are functions in $L_1(\mathbf{R})$ and $f^{(n)} \in BV(\mathbf{R}) \cap C(\mathbf{R})$, $n \geq 2$. Let φ be a Lebesgue measurable bounded function with support $\varphi \subseteq [-a,a]$, $0 < a < +\infty$, such that*

$$\sum_{j=-\infty}^{+\infty} \varphi(x-j) = 1, \qquad \forall x \in \mathbf{R}. \tag{1}$$

Denote

$$(B_k f)(x) := \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right) \cdot \varphi\left(2^k x - j\right) \tag{2}$$

and

$$e_k^2(f) := \int_{-\infty}^{+\infty} \left(f(x) - (B_k f)(x)\right)^2 dx, \qquad k \in \mathbf{Z}. \tag{3}$$

Set

$$\beta := \begin{cases} [2a], & \text{if } 2a \neq \text{integer}, \\ 2a-1, & \text{if } 2a \text{ is an integer}, \end{cases} \tag{4}$$

where $[\cdot]$ is the integral part function. Set

$$\ell := \begin{cases} \dfrac{n-1}{2}, & n \text{ is odd}, \\ \dfrac{n-2}{2}, & n \text{ is even}, \end{cases} = \left\lceil \frac{n}{2} \right\rceil - 1, \tag{5}$$

where $\lceil \cdot \rceil$ is the ceiling of the number. Call

$$\rho_\gamma(f) := \frac{(-1)^\gamma}{(2\gamma)!} \int_{-\infty}^{+\infty} \left(f^{(\gamma)}(t)\right)^2 dt, \qquad \gamma = 1, \dots, \ell. \tag{6}$$

Call ($\gamma = 1, \dots, \ell$)

$$\lambda_\gamma(\varphi) := \int_{-\infty}^{+\infty} \varphi(u) \cdot \left\{ -2 \cdot u^{2\gamma} + \sum_{q=1}^{\beta} q^{2\gamma} \cdot (\varphi(u+q) + \varphi(u-q)) \right\} du. \tag{7}$$

Then, for $k \in \mathbb{N}$, *we obtain*

$$e_k^2(f) = \sum_{\gamma=1}^{\ell^*} \frac{\rho_\gamma(f) \cdot \lambda_\gamma(\varphi)}{4^{k\gamma}} + o\left(\frac{1}{2^{k(n-2)}}\right), \tag{8}$$

where

$$\ell^* := \begin{cases} \ell, & \text{if } n \text{ is even,} \\ \ell - 1, & \text{if } n \text{ is odd.} \end{cases}$$

REMARK 1. We observe that $e_k(f) \to 0$ as $k \to +\infty$, i.e., for $n \geq 2$ we get that $B_k(f) \to f$ as $k \to +\infty$ in $\mathcal{L}_2$-norm.

Notice that the odd powers of 2^{-k} are absent in the asymptotic expansion. In particular, when $n = 2$ we get that $e_k(f) = o(1)$, and when $n = 3$ we find that $e_k^2(f) = o(2^{-k})$.

Next, we denote by

$$\lambda_1(\varphi) := \int_{-\infty}^{+\infty} \varphi(u) \left\{ -2 \cdot u^2 + \sum_{q=1}^{\beta} q^2 \cdot (\varphi(u+q) + \varphi(u-q)) \right\} du,$$

and when $n = 4$ we see that

$$e_k^2(f) = \frac{-\lambda_1(\varphi) \int_{-\infty}^{+\infty} (f'(t))^2 \, dt}{2^{2k+1}} + o\left(\frac{1}{2^{2k}}\right).$$

Furthemore, in the case of $n \geq 4$, when $\lambda_1(\varphi) \neq 0$, we obtain

$$2^{2k} e_k^2(f) \to \frac{-\lambda_1(\varphi) \int_{-\infty}^{+\infty} (f'(t))^2 \, dt}{2},$$

as $k \to +\infty$, and this rate cannot be improved by additional differentiability of f greater than first order.

PROOF OF THEOREM 1. Here, we would like to establish for $e_k^2(f)$ the asymptotic expansion (8). This is done in the next four sections.

2.1. Unfolding

We start by unfolding $e_k^2(f)$ as far as possible. Notice that

$$e_k^2(f) = \int_{-\infty}^{+\infty} f^2(x) \, dx - 2 \int_{-\infty}^{\infty} f(x) \cdot (B_k f)(x) \, dx + \int_{-\infty}^{+\infty} (B_k f)^2(x) \, dx, \qquad k \in \mathbf{Z}. \tag{9}$$

First, we would like to prove that $e_k(f) < +\infty$. Since $f, f', f'', \dots, f^{(n)} \in L_1(\mathbf{R})$ and $f^{(n)} \in BV(\mathbf{R})$ ($n \geq 2$), by Remark 1 of [6] we have that $f, f', \dots, f^{(n-1)}$ are all bounded, of bounded variation, and uniformly continuous on the real line and tend to zero as $|x| \to +\infty$. Also, $f^{(n)}$ is bounded. Furthermore, $f^2 \in L_1(\mathbf{R})$ (i.e., $\|f\|_2 < +\infty$), along with $f^2 \in BV(\mathbf{R})$. Thus, from Lemma 1 of [6], we get that

$$\sum_{j=-\infty}^{+\infty} \left| f\left(\frac{j}{2^k}\right) \right|^r < +\infty, \qquad \text{any } k \in \mathbf{Z}, \tag{10}$$

for $r = 1, 2$ (for the case $r = 1$ notice that $|f| \in L_1(\mathbf{R}) \cap BV(\mathbf{R})$).

Next, we need to prove that

$$\xi_1 := \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \left| f\left(\frac{i}{2^k}\right)\right| \cdot \left| f\left(\frac{j}{2^k}\right)\right| \int_{-\infty}^{+\infty} \left|\varphi\left(2^k x - i\right)\right| \cdot \left|\varphi\left(2^k x - j\right)\right| \, dx < +\infty. \tag{11}$$

By a change of variable, we get

$$\xi_1 = \frac{1}{2^k} \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \left| f\left(\frac{i}{2^k}\right)\right| \cdot \left| f\left(\frac{j}{2^k}\right)\right| \cdot \psi(i-j), \tag{12}$$

where

$$\psi(q) := \int_{-\infty}^{+\infty} |\varphi(y)| \cdot |\varphi(y+q)| \, dy \geq 0 \tag{13}$$

and $\psi(q) < +\infty$, $q \in \mathbf{Z}$. Hence, we would like to prove that

$$\xi_2 := \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \left| f\left(\frac{i}{2^k}\right)\right| \cdot \left| f\left(\frac{j}{2^k}\right)\right| \cdot \psi(i-j) < +\infty. \tag{14}$$

But (cf. (50))

$$\begin{aligned}
\xi_2 &= \left(\sum_{i=-\infty}^{+\infty} \left(f\left(\frac{i}{2^k}\right)\right)^2 \right) \cdot \psi(0) + \sum_{q=1}^{\beta} \left\{ \sum_{j=-\infty}^{+\infty} \left| f\left(\frac{j+q}{2^k}\right)\right| \cdot \left| f\left(\frac{j}{2^k}\right)\right| \cdot \psi(q) \right. \\
&\quad \left. + \sum_{i=-\infty}^{+\infty} \left| f\left(\frac{i}{2^k}\right)\right| \cdot \left| f\left(\frac{i+q}{2^k}\right)\right| \cdot \psi(-q) \right\} \\
&= \left(\sum_{i=-\infty}^{+\infty} \left(f\left(\frac{i}{2^k}\right)\right)^2 \right) \cdot \psi(0) \\
&\quad + \sum_{q=1}^{\beta} \left\{ \sum_{i=-\infty}^{+\infty} \left| f\left(\frac{i}{2^k}\right)\right| \cdot \left| f\left(\frac{i+q}{2^k}\right)\right| \cdot (\psi(q) + \psi(-q)) \right\} =: \xi_3.
\end{aligned} \tag{15}$$

That is, we want to prove $\xi_3 < +\infty$ (i.e., $\xi_1 < +\infty$ iff $\xi_3 < +\infty$). Really, from (10), (13), and earlier comments on f, we have that

$$\begin{aligned}
&\sum_{i=-\infty}^{+\infty} \left| f\left(\frac{i}{2^k}\right)\right| \cdot \left| f\left(\frac{i+q}{2^k}\right)\right| \cdot (\psi(q) + \psi(-q)) \\
&\qquad \leq \|f\|_\infty \left(\sum_{i=-\infty}^{+\infty} \left| f\left(\frac{i}{2^k}\right)\right| \right) \cdot (\psi(q) + \psi(-q)) < +\infty.
\end{aligned}$$

Therefore, again from (10), (13), and β being finite, we get that $\xi_3 < +\infty$, equivalently that $\xi_1 < +\infty$. Hence, from (11) we find that

$$\begin{aligned}
&\sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} f\left(\frac{i}{2^k}\right) \cdot f\left(\frac{j}{2^k}\right) \int_{-\infty}^{+\infty} \varphi\left(2^k x - i\right) \cdot \varphi\left(2^k x - j\right) \, dx \\
&\qquad = \int_{-\infty}^{+\infty} (B_k f)^2(x) \, dx = \|B_k f\|_2^2 < +\infty,
\end{aligned} \tag{16}$$

in particular,

$$\|B_k f\|_2 < +\infty. \tag{17}$$

And, by

$$\|f - B_k f\|_2 \leq \|f\|_2 + \|B_k f\|_2 < +\infty,$$

we get that

$$e_k(f) < +\infty, \qquad k \in \mathbf{Z}. \tag{18}$$

Next, we need to establish that

$$\theta_1 := \sum_{j=-\infty}^{+\infty} \left| f\left(\frac{j}{2^k}\right) \right| \int_{-\infty}^{+\infty} |f(x)| \cdot \left| \varphi\left(2^k x - j\right) \right| dx < +\infty. \tag{19}$$

Notice that

$$\begin{aligned} \theta_1 &= \frac{1}{2^k} \sum_{j=-\infty}^{+\infty} \left| f\left(\frac{j}{2^k}\right) \right| \int_{-a}^{a} \left| f\left(\frac{y+j}{2^k}\right) \right| \cdot |\varphi(y)| \, dy \\ &\leq \frac{1}{2^k} \left(\sum_{j=-\infty}^{+\infty} \left| f\left(\frac{j}{2^k}\right) \right| \right) \cdot \|f\|_\infty \int_{-a}^{a} |\varphi(y)| \, dy < +\infty. \end{aligned} \tag{20}$$

The right-hand side of (20) is finite by (10) and f, φ are bounded functions. Hence, $\theta_1 < +\infty$. That is,

$$\int_{-\infty}^{+\infty} f(x) \cdot (B_k f)(x) \, dx = \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right) \int_{-\infty}^{+\infty} f(x) \cdot \varphi\left(2^k x - j\right) \, dx. \tag{21}$$

We have established now that (cf. (9), (16), and (21))

$$\begin{aligned} e_k^2(f) = \int_{-\infty}^{+\infty} f^2(x) \, dx &+ \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} f\left(\frac{i}{2^k}\right) f\left(\frac{j}{2^k}\right) \int_{-\infty}^{+\infty} \varphi\left(2^k x - i\right) \cdot \varphi\left(2^k x - j\right) \, dx \\ &-2 \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right) \int_{-\infty}^{+\infty} f(x) \cdot \varphi\left(2^k x - j\right) \, dx. \end{aligned} \tag{22}$$

2.2. A Basic Asymptotic Expansion

Here, we find the appropriate asymptotic expansion for

$$\begin{aligned} \int_{-\infty}^{+\infty} f \cdot (B_k f) \, dx &= \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right) \int_{-\infty}^{+\infty} f(x) \cdot \varphi\left(2^k x - j\right) \, dx \\ &= \frac{1}{2^k} \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right) \int_{-\infty}^{+\infty} f\left(\frac{y+j}{2^k}\right) \cdot \varphi(y) \, dy \\ &= \frac{1}{2^k} \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right) \int_{-a}^{a} f\left(\frac{y+j}{2^k}\right) \cdot \varphi(y) \, dy. \end{aligned} \tag{23}$$

But (by Taylor's theorem),

$$\begin{aligned} f\left(\frac{y+j}{2^k}\right) &= \sum_{r=0}^{n-1} \frac{f^{(r)}\left(j/2^k\right)}{r!} \left(\frac{y}{2^k}\right)^r + \frac{\left(y/2^k\right)^{n-1}}{(n-1)!} \\ &\times \int_0^1 (1-s)^{n-1} \cdot f^{(n)}\left(\frac{j}{2^k} + \frac{y}{2^k} \cdot s\right) ds, \qquad k \in \mathbf{Z}. \end{aligned} \tag{24}$$

Thus,

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)\cdot(B_kf)(x)\,dx &= \frac{1}{2^k}\sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right)\int_{-a}^{a}\left\{\sum_{r=0}^{n-1}\frac{f^{(r)}\left(j/2^k\right)}{r!}\cdot\frac{y^r}{2^{kr}}+\frac{y^{n-1}}{2^{k(n-1)}\cdot(n-1)!}\right.\\ &\quad\times\left.\int_0^1(1-s)^{n-1}\cdot f^{(n)}\left(\frac{j+y\cdot s}{2^k}\right)ds\right\}\cdot\varphi(y)\,dy\\ &= \frac{1}{2^k}\sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right)\left\{\sum_{r=0}^{n-1}\frac{1}{2^{kr}\cdot r!}\cdot f^{(r)}\left(\frac{j}{2^k}\right)\cdot m_r\right.\\ &\quad\left.+\frac{1}{2^{k(n-1)}\cdot(n-1)!}\cdot\gamma_{kn}^{(j)}(f)\right\}.\end{aligned}\tag{25}$$

Here,

$$m_r := \int_{-a}^{a} y^r\cdot\varphi(y)\,dy,\qquad r=0,\dots,n-1,\tag{26}$$

and

$$\gamma_{kn}^{(j)}(f) := \int_{-a}^{a} y^{n-1}\left(\int_0^1(1-s)^{n-1}\cdot f^{(n)}\left(\frac{j+y\cdot s}{2^k}\right)ds\right)\cdot\varphi(y)\,dy,\tag{27}$$

with $j\in\mathbf{Z}$. Hence,

$$\int_{-\infty}^{+\infty} f(x)\cdot(B_kf)(x)\,dx = \frac{1}{2^k}\left\{\sum_{r=0}^{n-1}\frac{m_r}{2^{kr}\cdot r!}\cdot\mathcal{R}_1+\frac{\mathcal{R}_2}{2^{k(n-1)}\cdot(n-1)!}\right\},\tag{28}$$

where

$$\mathcal{R}_1 := \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right)\cdot f^{(r)}\left(\frac{j}{2^k}\right)\tag{29}$$

and

$$\mathcal{R}_2 := \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right)\cdot\gamma_{kn}^{(j)}(f).\tag{30}$$

Equality (28) is valid given that $\mathcal{R}_1$ and $\mathcal{R}_2$ converge, which we prove next. Notice that for $0\le r\le n-1:\ g:=f\cdot f^{(r)}, g', g'', \dots, g^{(n-r)}\in L_1(\mathbf{R})$ and $g^{(n-r)}\in BV(\mathbf{R})$. From [6, Lemma 2(b)], we obtain

$$\frac{1}{2^k}\sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right)\cdot f^{(r)}\left(\frac{j}{2^k}\right) = \int_{-\infty}^{+\infty} f(x)\cdot f^{(r)}(x)\,dx + o\left(\frac{1}{2^{(n-r)k}}\right),\tag{31}$$

where $0\le r\le n-1$. Again, for [6, Lemma 3] (or from [1, Lemma 2]), we get

$$\int_{-\infty}^{\infty} f\cdot f^{(r)}\,dx = \begin{cases}0, & r \text{ is odd},\\ (-1)^m\int_{-\infty}^{+\infty}\left(f^{(m)}(t)\right)^2dt, & r=2m.\end{cases}\tag{32}$$

Thus,

$$\frac{1}{2^k}\sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right)\cdot f^{(r)}\left(\frac{j}{2^k}\right) = \begin{cases} o\left(\frac{1}{2^{(n-r)k}}\right), & r=\text{odd},\\ (-1)^m\int_{-\infty}^{+\infty}\left(f^{(m)}(t)\right)^2dt + o\left(\frac{1}{2^{(n-2m)k}}\right), & r=2m,\end{cases}\tag{33}$$

where $0\le r\le n-1$. That is, we have proved that $\mathcal{R}_1$ converges.

Next, we would like to estimate

$$\frac{1}{2^k}\cdot\mathcal{R}_2=\frac{1}{2^k}\sum_{j=-\infty}^{+\infty} f\left(\frac{j}{2^k}\right)\left(\int_{-a}^{a} y^{n-1}\left(\int_0^1 (1-s)^{n-1}\cdot f^{(n)}\left(\frac{j+y\cdot s}{2^k}\right)ds\right)\cdot\varphi(y)\,dy\right).$$

From [6, Remark 1], we get that $M:=\|f^{(n)}\|_\infty<+\infty$. And so, we find

$$\left|\int_0^1 (1-s)^{n-1}\cdot f^{(n)}\left(\frac{j+y\cdot s}{2^k}\right)ds\right|\le M\int_0^1(1-s)^{n-1}\,ds=\frac{M}{n}.$$

Therefore,

$$\left|\frac{1}{2^k}\cdot\mathcal{R}_2\right|\le\left(\frac{1}{2^k}\sum_{j=-\infty}^{+\infty}\left|f\left(\frac{j}{2^k}\right)\right|\right)\cdot\frac{M\cdot\theta}{n},$$

where

$$\theta:=\int_{-a}^{a}|y|^{n-1}\cdot|\varphi(y)|\,dy<+\infty. \tag{34}$$

Since $|f|\in L_1(\mathbf{R})\cap BV(\mathbf{R})$, by Lemma 1 of [6], we get that

$$\frac{1}{2^k}\sum_{j=-\infty}^{+\infty}\left|f\left(\frac{j}{2^k}\right)\right|=\int_{-\infty}^{+\infty}|f(t)|\,dt+o(1). \tag{35}$$

Hence,

$$\left|\frac{1}{2^k}\cdot\mathcal{R}_2\right|\le\left(\int_{-\infty}^{+\infty}|f(t)|\,dt+o(1)\right)\cdot\frac{M\cdot\theta}{n}<+\infty.$$

That is, $\mathcal{R}_2$ converges, too.

Set

$$K:=M\cdot\theta<+\infty \tag{36}$$

and

$$L:=\int_{-\infty}^{+\infty}|f(t)|\,dt<+\infty. \tag{37}$$

Then,

$$\begin{aligned}\left|\frac{\mathcal{R}_2}{2^k\cdot 2^{k(n-1)}\cdot(n-1)!}\right|&\le(L+o(1))\cdot\frac{K}{2^{k(n-1)}\cdot n!}\\&=\frac{LK}{2^{k(n-1)}\cdot n!}+\frac{Ko(1)}{2^{k(n-1)}\cdot n!}=o\left(\frac{1}{2^{k(n-2)}}\right)+o\left(\frac{1}{2^{k(n-1)}}\right)\\&=o\left(\frac{1}{2^{k(n-2)}}\right)+o\left(\frac{1}{2^{k(n-2)}}\right)=o\left(\frac{1}{2^{k(n-2)}}\right),\qquad k\in\mathbf{N},\quad n\ge 2.\end{aligned}$$

So, we have proved that

$$\frac{\mathcal{R}_2}{2^k\cdot 2^{k(n-1)}\cdot(n-1)!}=o\left(\frac{1}{2^{k(n-2)}}\right),\qquad k\in\mathbf{N},\quad n\ge 2. \tag{38}$$

Using (31) and (38) into (28), we get

$$\begin{aligned}&\int_{-\infty}^{+\infty} f(x)(B_kf)(x)\,dx\\&=\sum_{r=0}^{n-1}\frac{m_r}{2^{kr}\cdot r!}\left\{\int_{-\infty}^{+\infty} f(x)\cdot f^{(r)}(x)\,dx+o\left(\frac{1}{2^{(n-r)k}}\right)\right\}+o\left(\frac{1}{2^{k(n-2)}}\right),\end{aligned}\tag{39}$$

where $k\in\mathbf{N}$, $n\ge 2$.

Next, we work on (39). In the case of n being odd, then $n-1=2\ell$ is even. So, we obtain (by (32))

$$\int_{-\infty}^{+\infty} f(x)(B_kf)(x)\,dx = \left\{\sum_{\gamma=0}^{\ell} \frac{m_{2\gamma}}{2^{k2\gamma}\cdot(2\gamma)!}\left\{(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt + o\left(\frac{1}{2^{(n-2\gamma)k}}\right)\right\}\right\}$$
$$+\left\{\sum_{\gamma=0}^{\ell-1}\frac{m_{2\gamma+1}}{2^{k(2\gamma+1)}\cdot(2\gamma+1)!}\cdot o\left(\frac{1}{2^{(n-2\gamma-1)k}}\right)\right\}+o\left(\frac{1}{2^{k(n-2)}}\right)$$
$$=\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\cdot\frac{1}{4^{k\gamma}}\cdot(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt$$
$$+\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\cdot\frac{1}{2^{k2\gamma}}\cdot o\left(\frac{1}{2^{(n-2\gamma)k}}\right)$$
$$+\sum_{\gamma=0}^{\ell-1}\frac{m_{2\gamma+1}}{(2\gamma+1)!}\,\frac{1}{2^{k(2\gamma+1)}}\cdot o\left(\frac{1}{2^{(n-2\gamma-1)k}}\right)+o\left(\frac{1}{2^{k(n-2)}}\right)$$
$$=\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\,\frac{1}{4^{k\gamma}}\cdot(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt+\left(\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\right)\cdot o\left(\frac{1}{2^{kn}}\right)$$
$$+\left(\sum_{\gamma=0}^{\ell-1}\frac{m_{2\gamma+1}}{(2\gamma+1)!}\right)\cdot o\left(\frac{1}{2^{kn}}\right)+o\left(\frac{1}{2^{k(n-2)}}\right)$$
$$=\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\,\frac{1}{4^{k\gamma}}(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt+o\left(\frac{1}{2^{kn}}\right)+o\left(\frac{1}{2^{k(n-2)}}\right)$$
$$=\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\,\frac{1}{4^{k\gamma}}(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt+o\left(\frac{1}{2^{k(n-2)}}\right).$$

Thus, when n is odd ($n-1=2\ell$) we have proved that

$$\int_{-\infty}^{+\infty} f(x)(B_kf)(x)\,dx=\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\,\frac{1}{4^{k\gamma}}(-1)^{\gamma}$$
$$\times\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt+o\left(\frac{1}{2^{k(n-2)}}\right),\qquad n\ge 2,\quad k\in\mathbf{N}. \tag{40}$$

We continue work on (39). In the case of n being even, then $n-2=:2\cdot\ell$ is even. So, we obtain (by (32))

$$\int_{-\infty}^{+\infty} f(x)(B_kf)(x)\,dx = \left\{\sum_{\gamma=0}^{\ell} \frac{m_{2\gamma}}{2^{k2\gamma}\cdot(2\gamma)!}\left\{(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt + o\left(\frac{1}{2^{(n-2\gamma)k}}\right)\right\}\right\}$$
$$+\left\{\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma+1}}{2^{k(2\gamma+1)}\cdot(2\gamma+1)!}\cdot o\left(\frac{1}{2^{(n-2\gamma-1)k}}\right)\right\}+o\left(\frac{1}{2^{k(n-2)}}\right)$$
$$=\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\,\frac{1}{4^{k\gamma}}(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt+\left(\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\right)\cdot o\left(\frac{1}{2^{kn}}\right)$$
$$+\left(\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma+1}}{(2\gamma+1)!}\right)\cdot o\left(\frac{1}{2^{kn}}\right)+o\left(\frac{1}{2^{k(n-2)}}\right).$$

Hence, when n is even ($n-2 =: 2\cdot\ell$), we have proved that

$$\int_{-\infty}^{+\infty} f(x)(B_k f)(x)\,dx = \sum_{\gamma=0}^{\ell} \frac{m_{2\gamma}}{(2\gamma)!}\frac{1}{4^{k\gamma}}(-1)^{\gamma} \times \int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt + o\left(\frac{1}{2^{k(n-2)}}\right), \qquad n\ge 2, \quad k\in\mathbf{N}. \tag{41}$$

Letting ℓ as in (5), we finally arrive at the asymptotic expansion for

$$\int_{-\infty}^{+\infty} f(x)(B_k f)(x)\,dx = \sum_{\gamma=0}^{\ell} \frac{m_{2\gamma}}{(2\gamma)!}\frac{1}{4^{k\gamma}}(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt + o\left(\frac{1}{2^{k(n-2)}}\right), \tag{42}$$

where $k\in\mathbf{N}$ and $n\ge 2$. Here,

$$m_{2\gamma} := \int_{-a}^{a} y^{2\gamma}\cdot\varphi(y)\,dy, \qquad \gamma = 0,1,\dots,\ell. \tag{43}$$

It is interesting to notice from (1) that

$$m_0 := \int_{-\infty}^{+\infty}\varphi(y)\,dy = 1. \tag{44}$$

Really, from (1), we obtain that

$$\sum_{j=-\infty}^{+\infty}\varphi(x+j) = 1, \qquad \forall x\in\mathbf{R}. \tag{1*}$$

Furthermore (from (1*)),

$$\int_{-\infty}^{+\infty}\varphi(x)\,dx = \sum_{j=-\infty}^{+\infty}\int_{j}^{j+1}\varphi(x)\,dx = \sum_{j=-\infty}^{+\infty}\int_0^1\varphi(x+j)\,dx = \int_0^1\left(\sum_{j=-\infty}^{+\infty}\varphi(x+j)\right)dx \overset{(1^*)}{=} \int_0^1 1\,dx = 1,$$

proving (44).

2.3. Third Term Asymptotic Expansion Follows

Next, we would like to find a suitable asymptotic expansion for

$$\int_{-\infty}^{+\infty}(B_k f)^2(x)\,dx = \sum_{i=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty} f\left(\frac{i}{2^k}\right) f\left(\frac{j}{2^k}\right)\int_{-\infty}^{+\infty}\varphi\left(2^k x - i\right)\cdot\varphi\left(2^k x - j\right)dx. \tag{45}$$

Equality (45) makes sense due to (16). By a change of variable, we find that

$$\int_{-\infty}^{+\infty}(B_k f)^2(x)\,dx = \frac{1}{2^k}\sum_{i=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty} f\left(\frac{i}{2^k}\right)\cdot f\left(\frac{j}{2^k}\right)\cdot\sigma_{i-j}, \tag{46}$$

where

$$\sigma_{i-j} := \int_{-\infty}^{+\infty}\varphi(u)\cdot\varphi(u+i-j)\,du \tag{47}$$

exists for any $i,j\in\mathbf{Z}$.

Since the support of $\varphi(u)$ is in $[-a,a]$, the support of $\varphi(u+i-j)$ is in $[-a+j-i, a+j-i]$. So, $\varphi(u)$, $\varphi(u+i-j)$ have no common support iff $|j-i| \geq 2a$. That is, $\varphi(u)$, $\varphi(u+i-j)$ have common support iff $|j-i| \leq \beta$ where β is defined by (4). Hence, $\sigma_{i-j} \neq 0$ iff $|j-i| \leq \beta$. Thus,

$$\sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} = \sum_{i=j} + \sum \sum_{i-j=\pm 1} + \cdots + \sum \sum_{i-j=\pm\beta} \tag{48}$$

and

$$\sum \sum_{i-j=\pm q} = \sum_{i=j+q} \sum_{j} + \sum_{i} \sum_{j=i+q}, \tag{49}$$

for all $1 \leq q \leq \beta$.

Finally, we get that

$$\sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} = \sum \sum_{i=j=-\infty}^{+\infty} + \sum_{q=1}^{\beta} \left(\sum_{i=j+q} \sum_{j=-\infty}^{+\infty} + \sum_{i=-\infty}^{+\infty} \sum_{j=i+q} \right). \tag{50}$$

Therefore,

$$\int_{-\infty}^{+\infty} (B_k f)^2(x)\, dx = \left(\frac{1}{2^k} \sum_{i=-\infty}^{+\infty} f^2\left(\frac{i}{2^k}\right) \right) \cdot \sigma_0$$
$$+ \sum_{q=1}^{\beta} \left\{ \frac{1}{2^k} \left(\sum_{j=-\infty}^{+\infty} f\left(\frac{j+q}{2^k}\right) \cdot f\left(\frac{j}{2^k}\right) \right) \cdot \sigma_q + \frac{1}{2^k} \left(\sum_{i=-\infty}^{+\infty} f\left(\frac{i}{2^k}\right) \cdot f\left(\frac{i+q}{2^k}\right) \right) \cdot \sigma_{-q} \right\},$$

where

$$\sigma_0 = \int_{-\infty}^{+\infty} \varphi^2(u)\, du. \tag{51}$$

That is, we obtain

$$\int_{-\infty}^{+\infty} (B_k f)^2(x)\, dx = \left(\frac{1}{2^k} \sum_{i=-\infty}^{+\infty} f^2\left(\frac{i}{2^k}\right) \right) \cdot \sigma_0$$
$$+ \sum_{q=1}^{\beta} \left\{ \frac{1}{2^k} \left(\sum_{i=-\infty}^{+\infty} f\left(\frac{i}{2^k}\right) \cdot f\left(\frac{i+q}{2^k}\right) \right) \cdot (\sigma_q + \sigma_{-q}) \right\}. \tag{52}$$

Notice that $f^2, (f^2)', \ldots, (f^2)^{(n)} \in L_1(\mathbf{R})$, all continuous and $(f^2)^{(n)} \in BV(\mathbf{R})$. Then, from [6, Lemma 2b], we get

$$\frac{1}{2^k} \sum_{i=-\infty}^{+\infty} f^2\left(\frac{i}{2^k}\right) = \int_{-\infty}^{+\infty} f^2(t)\, dt + o\left(\frac{1}{2^{kn}}\right). \tag{53}$$

By Taylor's theorem, we see that

$$f\left(\frac{i+q}{2^k}\right) = \sum_{r=0}^{n-1} \frac{f^{(r)}\left(i/2^k\right)}{r!} \left(\frac{q}{2^k}\right)^r + \frac{\left(q/2^k\right)^{n-1}}{(n-1)!} \int_0^1 (1-s)^{n-1} \cdot f^{(n)}\left(\frac{i+q\cdot s}{2^k}\right) ds. \tag{54}$$

Hence,

$$\frac{1}{2^k} \sum_{i=-\infty}^{+\infty} f\left(\frac{i}{2^k}\right) \cdot f\left(\frac{i+q}{2^k}\right) = \frac{1}{2^k} \sum_{i=-\infty}^{+\infty} \left\{ \left(\sum_{r=0}^{n-1} \frac{q^r}{2^{kr} \cdot r!} \cdot f\left(\frac{i}{2^k}\right) \cdot f^{(r)}\left(\frac{i}{2^k}\right) \right) + \frac{q^{n-1}}{2^{k(n-1)} \cdot (n-1)!} \right.$$
$$\left. \times \left(f\left(\frac{i}{2^k}\right) \int_0^1 (1-s)^{n-1} \cdot f^{(n)}\left(\frac{i+q\cdot s}{2^k}\right) ds \right) \right\} \tag{55}$$
$$= \sum_{r=0}^{n-1} \frac{q^r}{2^{kr} \cdot r!} \cdot E_1 + \frac{q^{n-1}}{2^{k(n-1)} \cdot (n-1)!} \cdot E_2.$$

Here, as before (33),

$$E_1 := \frac{1}{2^k} \sum_{i=-\infty}^{+\infty} f\left(\frac{i}{2^k}\right) \cdot f^{(r)}\left(\frac{i}{2^k}\right)$$
$$= \begin{cases} o\left(\dfrac{1}{2^{(n-r)k}}\right), & r \text{ is odd}, \\ (-1)^m \int_{-\infty}^{+\infty} \left(f^{(m)}(t)\right)^2 dt + \left(\dfrac{1}{2^{(n-2m)k}}\right), & r = 2m, \end{cases} \tag{56}$$

where $0 \le r \le n-1$. That is, E_1 converges. Also,

$$E_2 := \frac{1}{2^k} \sum_{i=-\infty}^{+\infty} f\left(\frac{i}{2^k}\right) \int_0^1 (1-s)^{n-1} \cdot f^{(n)}\left(\frac{i+q\cdot s}{2^k}\right) ds. \tag{57}$$

We still need to prove that E_2 converges so that (55) is valid. Again, $M := \|f^{(n)}\|_\infty < +\infty$. Thus,

$$\left|\int_0^1 (1-s)^{n-1} \cdot f^{(n)}\left(\frac{i+q\cdot s}{2^k}\right) ds\right| \le \frac{M}{n}$$

and

$$|E_2| \le \left(\frac{1}{2^k} \sum_{i=-\infty}^{+\infty} \left|f\left(\frac{i}{2^k}\right)\right|\right) \cdot \frac{M}{n} \qquad \text{(by (35))}$$
$$= \left(\int_{-\infty}^{+\infty} |f(t)|\, dt + o(1)\right) \cdot \frac{M}{n} < +\infty.$$

That is, E_2 converges. Furthermore (by (37) $L < +\infty$), we get

$$\frac{q^{n-1}}{2^{k(n-1)} \cdot (n-1)!} \cdot |E_2| \le \frac{M \cdot q^{n-1}}{n! \cdot 2^{k(n-1)}} \cdot (L + o(1)) = \frac{\tau_1}{2^{k(n-1)}} + \frac{\tau_2}{2^{k(n-1)}} o(1)$$
$$= o\left(\frac{1}{2^{k(n-2)}}\right) + o\left(\frac{1}{2^{k(n-1)}}\right) = o\left(\frac{1}{2^{k(n-2)}}\right),$$
$$\left(\tau_1 := \frac{M \cdot L \cdot q^{n-1}}{n!} < +\infty,\ \tau_2 := \frac{M \cdot q^{n-1}}{n!} < +\infty,\ k \in \mathbf{N},\ n \ge 2\right).$$

That is, we have proved that

$$\frac{q^{n-1}}{2^{k(n-1)} \cdot (n-1)!} \cdot E_2 = o\left(\frac{1}{2^{k(n-2)}}\right), \tag{58}$$

where $k \in \mathbf{N}$, $n \ge 2$. Next, we estimate

$$\sum_{r=0}^{n-1} \frac{q^r}{2^{kr} \cdot r!} \cdot E_1 \overset{(56)}{=} \sum_{(r \text{ odd}=1)}^{n-1} \frac{q^r}{2^{kr} \cdot r!} \cdot o\left(\frac{1}{2^{(n-r)k}}\right)$$
$$+ \sum_{\substack{(r \text{ even}=0 \\ r=2m)}}^{n-1} \frac{q^r}{2^{kr} \cdot r!} \left((-1)^m \int_{-\infty}^{+\infty} \left(f^{(m)}(t)\right)^2 dt + o\left(\frac{1}{2^{(n-2m)k}}\right)\right)$$
$$= \left(\sum_{r \text{ odd}=1}^{n-1} \frac{q^r}{r!}\right) o\left(\frac{1}{2^{kn}}\right) + \sum_{\substack{(r \text{ even}=0 \\ r=2m)}}^{n-1} \frac{q^r}{r!} \frac{1}{2^{kr}} (-1)^m \int_{-\infty}^{+\infty} \left(f^{(m)}(t)\right)^2 dt$$

$$+\left(\sum_{\substack{(r\ \text{even}=0\\ r=2m)}}^{n-1}\frac{q^r}{r!}\right)\cdot o\left(\frac{1}{2^{kn}}\right)$$

$$=\left(\sum_{\substack{(r\ \text{even}=0\\ r=2m)}}^{n-1}\right)\frac{q^r}{r!}\,\frac{1}{2^{kr}}(-1)^m\int_{-\infty}^{+\infty}\left(f^{(m)}(t)\right)^2 dt+o\left(\frac{1}{2^{kn}}\right).$$

Following definition (5), we get

$$\sum_{r=0}^{n-1}\frac{q^r}{2^{kr}\cdot r!}\cdot E_1=\sum_{\gamma=0}^{\ell}\frac{q^{2\gamma}}{(2\gamma)!}\,\frac{1}{4^{k\gamma}}(-1)^\gamma\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt+o\left(\frac{1}{2^{kn}}\right), \tag{59}$$

where $k\in\mathbf{N}$, $n\geq 2$. Thus (see (55), (58), and (59)),

$$\frac{1}{2^k}\sum_{i=-\infty}^{+\infty}f\left(\frac{i}{2^k}\right)\cdot f\left(\frac{i+q}{2^k}\right)=\sum_{\gamma=0}^{\ell}\frac{q^{2\gamma}}{(2\gamma)!}\,\frac{1}{4^{k\gamma}}(-1)^\gamma\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt+o\left(\frac{1}{2^{k(n-2)}}\right), \tag{60}$$
$$k\in\mathbf{N},\quad n\geq 2.$$

Next, from (52), (53), and (60) we find ($n\geq 2$, $k\in\mathbf{N}$)

$$\int_{-\infty}^{+\infty}(B_kf)^2(x)\,dx=\left(\int_{-\infty}^{+\infty}f^2(t)\,dt+o\left(\frac{1}{2^{kn}}\right)\right)\cdot\sigma_0+\sum_{q=1}^{\beta}\left\{\left(\sum_{\gamma=0}^{\ell}\frac{q^{2\gamma}}{(2\gamma)!}\frac{1}{4^{k\gamma}}(-1)^\gamma\right.\right.$$
$$\left.\left.\times\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt+o\left(\frac{1}{2^{k(n-2)}}\right)\right)\cdot(\sigma_q+\sigma_{-q})\right\}$$
$$=\left(\int_{-\infty}^{+\infty}f^2(t)\,dt\right)\cdot\sigma_0+\sum_{q=1}^{\beta}\left\{\left\{\sum_{\gamma=0}^{\ell}\frac{q^{2\gamma}}{(2\gamma)!}\,\frac{1}{4^{k\gamma}}(-1)^\gamma\right.\right.$$
$$\left.\left.\times\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt\right\}\cdot(\sigma_q+\sigma_{-q})\right\}+o\left(\frac{1}{2^{k(n-2)}}\right).$$

Consequently ($n\geq 2,\ k\in\mathbf{N}$),

$$\int_{-\infty}^{+\infty}(B_kf)^2(x)\,dx=\left\{\sum_{\gamma=0}^{\ell}\left(\frac{(-1)^\gamma\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt}{(2\gamma)!\cdot 4^{k\gamma}}\right)\left(\sum_{q=1}^{\beta}q^{2\gamma}\cdot(\sigma_q+\sigma_{-q})\right)\right\}$$
$$+\left(\int_{-\infty}^{+\infty}f^2(t)\,dt\right)\cdot\sigma_0+o\left(\frac{1}{2^{k(n-2)}}\right).$$

And,

$$\int_{-\infty}^{+\infty}(B_kf)^2(x)\,dx=\left\{\sum_{\gamma=1}^{\ell}\left(\frac{(-1)^\gamma\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2 dt}{(2\gamma)!\cdot 4^{k\gamma}}\right)\left(\sum_{q=1}^{\beta}q^{2\gamma}\cdot(\sigma_q+\sigma_{-q})\right)\right\}$$
$$+\left(\int_{-\infty}^{+\infty}(f(t))^2\,dt\right)\left(\sigma_0+\sum_{q=1}^{\beta}(\sigma_q+\sigma_{-q})\right)+o\left(\frac{1}{2^{k(n-2)}}\right),\qquad n\geq 2,\quad k\in\mathbf{N}. \tag{61}$$

Hence,

$$\sigma_0+\sum_{q=1}^{\beta}(\sigma_q+\sigma_{-q})=\int_{-\infty}^{+\infty}\varphi(u)\cdot\varphi(u)\,du$$

$$
\begin{aligned}
&+\sum_{q=1}^{\beta}\left(\int_{-\infty}^{+\infty}\varphi(u)\cdot\varphi(u+q)\,du+\int_{-\infty}^{+\infty}\varphi(u)\cdot\varphi(u-q)\,du\right)\\
&=\int_{-\infty}^{+\infty}\varphi(u)\left\{\varphi(u)+\sum_{q=1}^{\beta}\varphi(u+q)+\sum_{q=1}^{\beta}\varphi(u-q)\right\}du\\
&=\int_{-\infty}^{+\infty}\varphi(u)\left\{\sum_{q=-\beta}^{\beta}\varphi(u-q)\right\}du\\
&=\int_{-\infty}^{+\infty}\varphi(u)\left(\sum_{q=-\infty}^{+\infty}\varphi(u-q)\right)du\\
&\overset{(1)}{=}\int_{-\infty}^{+\infty}\varphi(u)\cdot 1\,du=1,\qquad \text{by (44)},
\end{aligned}
$$

i.e.,

$$
\sigma_0+\sum_{q=1}^{\beta}(\sigma_q+\sigma_{-q})=1. \tag{62}
$$

Therefore ($k\in\mathbf{N}$, $n\geq 2$),

$$
\begin{aligned}
\int_{-\infty}^{+\infty}(B_kf)^2(x)\,dx=&\left\{\sum_{\gamma=1}^{\ell}\left(\frac{(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2dt}{(2\gamma)!\cdot 4^{k\gamma}}\right)\left(\sum_{q=1}^{\beta}q^{2\gamma}\cdot(\sigma_q+\sigma_{-q})\right)\right\}\\
&+\int_{-\infty}^{+\infty}f^2(t)\,dt+o\left(\frac{1}{2^{k(n-2)}}\right).
\end{aligned} \tag{63}
$$

2.4. Wrapping Up

Putting things together: we have that (see (22),(42),(63); $k\in\mathbf{N}$, $n\geq 2$)

$$
\begin{aligned}
e_k^2(f)=&\int_{-\infty}^{+\infty}f^2(t)\,dt+\left\{\sum_{\gamma=1}^{\ell}\left(\frac{(-1)^{\gamma}\cdot\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2dt}{(2\gamma)!\cdot 4^{k\gamma}}\right)\left(\sum_{q=1}^{\beta}q^{2\gamma}\cdot(\sigma_q+\sigma_{-q})\right)\right\}\\
&+\int_{-\infty}^{+\infty}f^2(t)\,dt+o\left(\frac{1}{2^{k(n-2)}}\right)-2\cdot\sum_{\gamma=0}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\frac{1}{4^{k\gamma}}\\
&\times(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2dt-2\cdot o\left(\frac{1}{2^{k(n-2)}}\right)\\
=&\left\{\sum_{\gamma=1}^{\ell}\left(\frac{(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2dt}{(2\gamma)!\cdot 4^{k\gamma}}\right)\left(\sum_{q=1}^{\beta}q^{2\gamma}\cdot(\sigma_q+\sigma_{-q})\right)\right\}\\
&-2\sum_{\gamma=1}^{\ell}\frac{m_{2\gamma}}{(2\gamma)!}\frac{1}{4^{k\gamma}}(-1)^{\gamma}\int_{-\infty}^{+\infty}\left(f^{(\gamma)}(t)\right)^2dt+o\left(\frac{1}{2^{k(n-2)}}\right)\\
\overset{(6)}{=}&\left\{\sum_{\gamma=1}^{\ell}\frac{\rho_{\gamma}(f)}{4^{k\gamma}}\left(\sum_{q=1}^{\beta}q^{2\gamma}\cdot(\sigma_q+\sigma_{-q})\right)\right\}-2\sum_{\gamma=1}^{\ell}\rho_{\gamma}(f)\frac{m_{2\gamma}}{4^{k\gamma}}+o\left(\frac{1}{2^{k(n-2)}}\right).
\end{aligned}
$$

And ($k\in\mathbf{N}$, $n\geq 2$),

$$\begin{aligned} e_k^2(f) &= \sum_{\gamma=1}^{\ell}\sum_{q=1}^{\beta}\left(\frac{\rho_\gamma(f)}{4^{k\gamma}}\right)\cdot q^{2\gamma}\cdot(\sigma_q+\sigma_{-q}) + \sum_{\gamma=1}^{\ell}(-2m_{2\gamma})\frac{\rho_\gamma(f)}{4^{k\gamma}} + o\left(\frac{1}{2^{k(n-2)}}\right) \\ &= \sum_{\gamma=1}^{\ell}\left\{\left(\sum_{q=1}^{\beta}\left(\frac{\rho_\gamma(f)}{4^{k\gamma}}\right)\cdot q^{2\gamma}\cdot(\sigma_q+\sigma_{-q})\right) - 2m_{2\gamma}\cdot\frac{\rho_\gamma(f)}{4^{k\gamma}}\right\} + o\left(\frac{1}{2^{k(n-2)}}\right) \\ &= \sum_{\gamma=1}^{\ell}\left\{\frac{\rho_\gamma(f)}{4^{k\gamma}}\left(\left(\sum_{q=1}^{\beta} q^{2\gamma}\cdot(\sigma_q+\sigma_{-q})\right) - 2m_{2\gamma}\right)\right\} + o\left(\frac{1}{2^{k(n-2)}}\right). \end{aligned} \tag{64}$$

But,

$$\begin{aligned} &\sum_{q=1}^{\beta} q^{2\gamma}\cdot(\sigma_q+\sigma_{-q}) - 2m_{2\gamma} \overset{\{(26),(43),(47)\}}{=} \sum_{q=1}^{\beta} q^{2\gamma} \\ &\quad\times\left(\int_{-\infty}^{+\infty}\varphi(u)\cdot\varphi(u+q)\,du + \int_{-\infty}^{+\infty}\varphi(u)\cdot\varphi(u-q)\,du\right) \\ &\quad - 2\int_{-\infty}^{+\infty} u^{2\gamma}\cdot\varphi(u)\,du = -2\int_{-\infty}^{+\infty} u^{2\gamma}\cdot\varphi(u)\,du \\ &\quad + \sum_{q=1}^{\beta} q^{2\gamma}\left(\int_{-\infty}^{+\infty}\varphi(u)(\varphi(u+q)+\varphi(u-q))\,du\right) \\ &= -2\int_{-\infty}^{+\infty} u^{2\gamma}\cdot\varphi(u)\,du + \int_{-\infty}^{+\infty}\varphi(u)\left(\sum_{q=1}^{\beta} q^{2\gamma}\cdot(\varphi(u+q)+\varphi(u-q))\right)du \\ &= \int_{-\infty}^{+\infty}\varphi(u)\left\{-2u^{2\gamma} + \sum_{q=1}^{\beta} q^{2\gamma}\cdot(\varphi(u+q)+\varphi(u-q))\right\}du \overset{(7)}{=} \lambda_\gamma(\varphi) < +\infty, \end{aligned}$$

for all $\gamma = 1,\dots,\ell$. That is,

$$\sum_{q=1}^{\beta} q^{2\gamma}\cdot(\sigma_q+\sigma_{-q}) - 2m_{2\gamma} = \lambda_\gamma(\varphi), \tag{65}$$

for all $\gamma = 1,\dots,\ell$. Also, from (64) we get that ($k \in \mathbf{N}$, $n \geq 2$)

$$e_k^2(f) = \sum_{\gamma=1}^{\ell}\left\{\frac{\rho_\gamma(f)}{4^{k\gamma}}\left[\left(\sum_{q=1}^{\beta} q^{2\gamma}\cdot(\sigma_q+\sigma_{-q})\right) - 2m_{2\gamma}\right]\right\} + o\left(\frac{1}{2^{k(n-2)}}\right). \tag{66}$$

Now it is clear that (65) and (66) imply (8). ∎

REMARK 2. In Theorem 1 and its proof, no orthogonality condition in any form was assumed for φ! The set of these φ is very rich indeed.

REFERENCES

1. S. Cambanis and E. Masry, Wavelet approximation of deterministic and random signals: Convergence properties and rates, *IEEE Transactions of Information Theory* **40**, 1013–1029 (1994).
2. C.K. Chui, *An Introduction to Wavelets*, Academic Press, New York, (1992).
3. I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, (1992).
4. S.G. Mallat, Multiresolution approximations and wavelet orthonormal bases of $L_2(\mathbf{R})$, *Trans. Amer. Math. Soc.* **315**, 69–87 (1989).
5. Y. Meyer, *Ondelettes et Opérateurs I, Ondelettes*, Hermann, Paris, (1990).
6. S. Cambanis and E. Masry, Wavelet approximation of deterministic and random signals: Convergence properties and rates, Technical Report No. 352, Center for Stochastic Processes, University of North Carolina (November 1991).