Available online at www.sciencedirect.com

SCIENCE DIRECT®

Appl. Comput. Harmon. Anal. 19 (2005) 49–60

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

# Wavelet-based simulation of fractional Brownian motion revisited

Vladas Pipiras

*Department of Statistics and Operations Research, Smith Building, CB #3260, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599, USA*

Received 22 April 2003; revised 1 March 2004; accepted 24 January 2005

Available online 14 March 2005

Communicated by Richard Gundy

**Abstract**

We reexamine the wavelet-based simulation procedure for fractional Brownian motion proposed by Abry and Sellan. We clarify in what sense the wavelet-based simulation procedure works, shed light on the structure of associated fractional low- and high-pass filters, and consequently suggest some modifications to the simulation algorithm.


*MSC:* 60G18; 41A58; 60F15

*Keywords:* Fractional Brownian motion; Wavelets; Low- and high-pass filters; Zero moments; FARIMA time series; Simulation

## 1. Introduction

The goal of this paper is to reexamine and clarify the wavelet-based simulation procedure for fractional Brownian motion proposed by Abry and Sellan [1] and also summarized in Abry et al. [2]. *Fractional Brownian motion* (fBm, in short) is a stochastic process $\{B_H(t)\}_{t\in\mathbb{R}}$, $H \in (0,1)$, having an integral representation

$$B_H(t) = k_H \int_{\mathbb{R}} \left((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\right) \mathrm{d}B(u), \quad t \in \mathbb{R}, \tag{1.1}$$

*E-mail address:* pipiras@email.unc.edu.

1063-5203/$ – see front matter 
doi:10.1016/j.acha.2005.01.002

where $\{B(u)\}_{u\in\mathbb{R}}$ is a standard Brownian motion, $x_+ = \max\{0, x\}$ and $k_H$ is a normalizing constant. The choice of

$$k_H = \left(\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}\right)^{1/2} \tag{1.2}$$

leads to $EB_H(1)^2 = 1$ or standard fBm, while setting

$$k_H = \big(\Gamma(H+1/2)\big)^{-1} \tag{1.3}$$

allows to write fBm as fractional integral of the Gaussian white noise $\mathrm{d}B(u)/\mathrm{d}u$. Here, $\Gamma(\cdot)$ is the usual gamma function. FBm has *stationary increments* and is *self-similar with exponent $H$*, that is, for any $c > 0$, processes $B_H(ct)$ and $c^H B_H(t)$ have the same finite-dimensional distributions. Since it is the only (up to a constant) Gaussian process with these two characteristics, fBm has been extensively studied in theory and also widely used in applications where its increments serve as a paradigm for long-range dependent, fractal or $1/f$-noise discrete-time series. The facts stated above and more information on fBm can be found in Section 7 of Samorodnitsky and Taqqu [15], Embrechts and Maejima [9], and Doukhan et al. [8]. See also numerous references therein.

Sellan [16], Meyer et al. [12] have recently established an almost surely and uniformly on compact intervals convergent expansion of fBm in wavelets which decorrelates the high-frequencies, namely,

$$B_H(t) = \sum_{k=-\infty}^{\infty} \Phi_H(t-k)S_k^{(H)} + \sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{-jH}\Psi_H\big(2^j t - k\big)\varepsilon_{j,k} - b_0, \tag{1.4}$$

where $\Phi_H$ and $\Psi_H$ are a suitably chosen biorthogonal scaling function and a wavelet, respectively, $S_k^{(H)}$, $k \in \mathbb{Z}$, is a partial sum process of a FARIMA$(0, H-1/2, 0)$ sequence with independent Gaussian innovations $\mathcal{N}(0,1)$, $\varepsilon_{j,k}$, $j \geqslant 0$, $k \in \mathbb{Z}$, are independent Gaussian $\mathcal{N}(0,1)$ random variables and $b_0$ is a random variable such that $B_H(0) = 0$. Some of the details behind the decomposition (1.4) as well as related terminology can be also found in Section 2 below. In particular, the functions $\Phi_H$ and $\Psi_H$ are defined through a related orthogonal scaling function $\phi$ and a wavelet $\psi$ associated with a multiresolution analysis (MRA, in short). Meyer et al. [12] have used the Lemarié–Meyer MRA functions $\phi$ and $\psi$ because of their appealing smoothness properties. Other wavelet bases, for example, the Daubechies MRA, are possible as well (see Remark 10 in [12, p. 488]). FARIMA sequences which appear in (1.4) through their partial sums are celebrated discrete-time linear series (see, for example, [5]). The sequence $S_k^{(H)}$ is often referred to as a (nonstationary) FARIMA$(0, H+1/2, 0)$ rather than as a partial sum process of a (stationary) FARIMA$(0, H-1/2, 0)$ sequence.

Decorrelation of the high frequencies of (1.4) which refers to independence (decorrelation) of the Gaussian coefficients $\varepsilon_{j,k}$, allows for a fast simulation of fBm by using pyramidal Mallat-type algorithm (fast wavelet transform). Practical implementation of the decomposition (1.4) to simulate fBm was proposed by Abry and Sellan [1]. Let

$$S_k^{(H)}(l) = 2^{l(H+1)} \int_{\mathbb{R}} \big(B_H(t) + b_0\big) g\big(2^l t - k\big)\,\mathrm{d}t, \tag{1.5}$$

be the conveniently normalized approximation coefficients in the wavelet expansion of fBm at the scale $2^{-l}$, where the function $g:\mathbb{R}\mapsto\mathbb{R}$ is biorthogonal to the scaling function $\Phi_H$ appearing in (1.4). The algorithm involves defining low- and high-pass filters, denoted hereafter by $u^{(s)}$ and $v^{(s)}$ with

$$s=H+\frac{1}{2}, \tag{1.6}$$

respectively, and can be represented as

$$S_{\cdot}^{(H)}(l)=u^{(s)}*\bigl(\uparrow_2 S_{\cdot}^{(H)}(l-1)\bigr)+v^{(s)}*(\uparrow_2\varepsilon_{l-1,\cdot}), \tag{1.7}$$

where $*$ stands for a convolution and the standard operator $(\uparrow_2 x)$ inserts zeros between the elements of a sequence $x$. As shown by Abry and Sellan [1], the fractional filters $u^{(s)}$ and $v^{(s)}$ satisfy the relations

$$u^{(s)}=f^{(s)}*u,\qquad v^{(s)}=g^{(s)}*v, \tag{1.8}$$

where the filters $f^{(s)}=\{f_n^{(s)}\}$ and $g^{(s)}=\{g_n^{(s)}\}$ are defined through the $z$-transformations as

$$f^{(s)}(z)=\bigl(1+z^{-1}\bigr)^s=\sum_{n=-\infty}^{\infty}f_n^{(s)}z^{-n}, \tag{1.9}$$

$$g^{(s)}(z)=\bigl(1-z^{-1}\bigr)^{-s}=\sum_{n=-\infty}^{\infty}g_n^{(s)}z^{-n}, \tag{1.10}$$

respectively, and $u$ and $v$ are the low- and high-pass filters associated with the initial MRA corresponding to the scaling function $\phi$ and the wavelet $\psi$. Observe that the filters $f^{(s)}$ and $g^{(s)}$ differ from those in [1] by a multiplicative constant. This happens (see Remark 3 at the end of Section 2) because we consider conveniently normalized approximation coefficients (1.5). In practice, since the filters $u^{(s)}$ and $v^{(s)}$ are infinite, and $g_n^{(s)}$ may diverge as $n\to\infty$, Abry and Sellan [1] suggested to set

$$u^{(s)}=f^{(1)}*tf^{(d)}*u,\qquad v^{(s)}=g^{(1)}*tg^{(d)}*v, \tag{1.11}$$

where

$$d=H-\frac{1}{2}, \tag{1.12}$$

and $tf^{(d)}$ and $tg^{(d)}$ stand for $f^{(d)}$ and $g^{(d)}$ truncated at some a priori chosen cutoff level. The idea then is to generate a FARIMA$(0,H+1/2,0)$ sequence of finite length and use the scheme (1.7) with truncated filters (1.11) to generate a much longer process $S_k^{(H)}(l)$ at the desired approximation level $l$. The suitably normalized sequence $S_k^{(H)}(l)$ is taken for the approximation of fBm at the scale $2^{-l}$.

In this work, we aim to shed more light on the wavelet-based method to simulate fBm. In Section 2, we clarify in what sense the approximation coefficients $S_k^{(H)}(l)$ are suitable approximations for fractional Brownian motion. We also revisit the Mallat-type scheme (1.7) from the perspective of time series analysis. In Section 3, we reexamine the fractional low- and high-pass filters $u^{(s)}$ and $v^{(s)}$. In particular, we bring into consideration the number of vanishing moments of the underlying orthogonal MRA and also investigate the decay of the resulting filters $u^{(s)}$ and $v^{(s)}$. Modifications to the Abry and Sellan algorithm are discussed in Section 4. The interested reader may also want to see the accompanying paper Pipiras [14] where we explore the usefulness of the wavelet-based simulation of fBm, compare it to other simulation methods and provide further guidelines for the use of the wavelet-based simulation.

## 2. Another look at the approximation coefficients

In this section, we make a few observations concerning the approximation coefficients $S_k^{(H)}(l)$ in (1.5). The next result shows that these coefficients can be used as approximations of fBm. By the framework of Meyer et al. [12] below, we mean that the scaling function $\psi$ and the wavelet $\phi$ entering into (1.4) through $\Phi_H$ and $\Psi_H$ correspond to the Lemarié–Meyer MRA considered by Meyer et al. [12].

**Proposition 2.1.** *Under the framework of Meyer et al.* [12], *we have for* $\varepsilon$, $0 < \varepsilon < H$,

$$\sup_{t\in[0,1]} \left|2^{-lH} S_{[2^l t]}^{(H)}(l) - \big(B_H(t) + b_0\big)\right| \leqslant C 2^{-l(H-\varepsilon)} \tag{2.1}$$

*almost surely, where a random variable $C$ depends on $H$, $\varepsilon$ and the scaling function $\phi$, and* $[x]$ *stands for an integer part function of $x \in \mathbb{R}$.*

**Proof.** The function $\Phi_H$ is defined in [12] through the Fourier transformation

$$\hat{\Phi}_H(x) = \left(\frac{1-e^{-ix}}{ix}\right)^{H+1/2} \hat{\phi}(x),$$

where $\phi$ is the scaling function corresponding to the Lemarié–Meyer MRA. (By convention, the Fourier transform $\hat{\phi} \in L^2(\mathbb{R})$ of a function $\phi \in L^2(\mathbb{R})$ is defined by $\hat{\phi}(x) = \int_{\mathbb{R}} e^{-itx}\phi(t)\,\mathrm{d}t$.) The function $g$, biorthogonal to $\Phi_H$, is then defined by

$$\hat{g}(x) = \left(\frac{1-e^{ix}}{-ix}\right)^{-(H+1/2)} \hat{\phi}(x)$$

and, under the framework of Meyer et al. [12], is infinitely many times differentiable with its derivatives decaying faster than any polynomial. Observe from the definition of $g$ that $\|g\|_{L^1} = \int_{\mathbb{R}} g(t)\,\mathrm{d}t = \hat{g}(0) = \hat{\phi}(0) = 1$ where the last equality follows, for example, from (3.1) in [12].

Relation (1.5) and $\|g\|_{L^1} = 1$ imply that, for $t \in \mathbb{R}$,

$$\left|2^{-lH} S_k^{(H)}(l) - \big(B_H(t) + b_0\big)\right| \leqslant \int_{\mathbb{R}} \left|B_H(s) - B_H(t)\right| 2^l \left|g\big(2^l s - k\big)\right| \mathrm{d}s. \tag{2.2}$$

It is then enough to argue that, for arbitrarily small $\varepsilon > 0$, there is a random variable $C$ such that

$$\left|B_H(s) - B_H(t)\right| \leqslant C|s-t|^{H-\varepsilon} + C|s-t|^{H+\varepsilon} \tag{2.3}$$

for all $s \in \mathbb{R}$ and $t \in [0,1]$. Indeed, by substituting (2.3) into (2.2) and making a change of variables, we would obtain that

$$\begin{aligned}\left|2^{-lH} S_{[2^l t]}^{(H)}(l) - B_H(t)\right| &\leqslant C 2^{-l(H-\varepsilon)} \int_{\mathbb{R}} \left|w - \big(2^l t - \big[2^l t\big]\big)\right|^{H\pm\varepsilon} \left|g(w)\right| \mathrm{d}w \\ &\leqslant C' 2^{-l(H-\varepsilon)} \int_{\mathbb{R}} |w|^{H\pm\varepsilon} \left|g(w)\right| \mathrm{d}w = C'' 2^{-l(H-\varepsilon)},\end{aligned}$$

since $0 \leqslant 2^l t - [2^l t] < 1$ and where $|x|^{H\pm\varepsilon} = |x|^{H+\varepsilon} + |x|^{H-\varepsilon}$ for $x \in \mathbb{R}$.

An application of the Kolmogorov's criterion (see, for example, [10, p. 53]) yields that fBm has sample paths which are Hölder continuous of the order $H-\varepsilon$ for any $\varepsilon \in (0, H)$ (see also Theorem 4.1.1 in [9]). This result allows to bound the left-hand side of (2.3) by $C|s-t|^{H-\varepsilon}$ when $s$ and $t$ belong to $[0,1]$. On the other hand, by the law of the iterated logarithm for fBm (see, for example, [13, Theorem 1.1]), we have $|B_H(s)| \leqslant C|s|^H(\ln\ln|s|)^{1/2}$ as $s \to \infty$. This result allows to bound the left-hand side of (2.3) by $C|s-t|^{H+\varepsilon}$ when $s \in \mathbb{R}\setminus[0,1]$ and $t\in[0,1]$. □

The random variable $b_0$ can be eliminated in (2.1) by assuming that an approximating FARIMA sequence starts at 0. Hence, set

$$\tilde{S}_k^{(H)}(l) = S_k^{(H)}(l) - S_0^{(H)}(l), \quad k, l \in \mathbb{Z}. \tag{2.4}$$

**Corollary 2.1.** *Under the assumptions of Proposition* 2.1, *we have for* $\varepsilon$, $0<\varepsilon<H_1$,

$$\sup_{t\in[0,1]} \left|2^{-lH}\tilde{S}_{[2^l t]}^{(H)}(l) - B_H(t)\right| \leqslant C2^{-l(H-\varepsilon)} \tag{2.5}$$

*almost surely, where $C$ is a random variable.*

**Proof.** The bound (2.5) follows from (2.1) and the fact that, since $B_H(0)=0$,

$$\left|2^{-lH}S_0^{(H)}(l) - b_0\right| \leqslant C2^{-l(H-\varepsilon)},$$

which is a consequence of (2.1). □

In practice, fBm is therefore approximated by a normalized sequence $\tilde{S}_k^{(H)}(l)$. It becomes clear from the proof above that Proposition 2.1 and Corollary 2.1 are true in other situations as well, as long as (1.5) holds and the function $g$ has a sufficient decay at infinity. The interval $t\in[0,1]$ in (2.1) and (2.5) can be replaced by other compact intervals. This replacement, however, affects the random variable $C$ in the corresponding bounds.

**Remarks.** (1) The approximating process $2^{-lH}\tilde{S}_{[2^l t]}^{(H)}(l)$, $t\in[0,1]$, in (2.4) has jumps at the dyadic points $t=k2^{-l}$, $k=0,\dots,2^l$. We can also define a continuous and easy to implement approximation to fBm by linearly interpolating the values of the previous approximation at the dyadic points, namely, as

$$\hat{S}_t^{(H)}(l) = \tilde{S}_{[2^l t]}^{(H)}(l) + \left(2^l t - \left[2^l t\right]\right)\left(\tilde{S}_{[2^l t]+1}^{(H)}(l) - \tilde{S}_{[2^l t]}^{(H)}(l)\right), \quad t\in[0,1]. \tag{2.6}$$

By using the fact that $0 \leqslant 2^l t - [2^l t] < 1$, the relation (2.5) and the Hölder continuity of fBm of the order $H-\varepsilon$ with any $\varepsilon\in(0,H)$, we deduce that

$$\sup_{t\in[0,1]} \left|2^{-lH}\hat{S}_t^{(H)}(l) - B_H(t)\right| \leqslant C2^{-l(H-\varepsilon)}, \tag{2.7}$$

where $C$ is a random variable.

(2) Results analogous to Proposition 2.1, Corollary 2.1 and the remark above in a deterministic situation are well known. See, for example, [6, pp. 202–206]. Observe also that the approximations of fBm in (2.7) and (2.5) are more accurate for $H$ closer to 1. This is natural because the paths of fBm get smoother as $H$ increases.

Focus now on the pyramidal Mallat-type scheme (1.7). Since (1.4) is a wavelet decomposition of fBm (see, in particular, (1.5)), we know that the right-hand side of (1.7) defines a FARIMA$(0, H + 1/2, 0)$ sequence. We shall conclude this section by providing an alternative proof of this result. The proof sheds light on the structure of the relation (1.7) and should be of independent interest as well. More generally, consider a FARIMA$(0, s, 0)$, $s \in \mathbb{R}$, sequence $X = \{X_n\}$ and a sequence $\varepsilon = \{\varepsilon_n\}$ of independent $\mathcal{N}(0, 1)$ random variables, so-called Gaussian white noise. When $s < 1/2$, FARIMA$(0, s, 0)$ sequence $X = \{X_n\}_{n\in\mathbb{Z}}$ is stationary and can be defined through the $z$-transformation as

$$X(z) = \left(1 - z^{-1}\right)^{-s}\varepsilon(z) \tag{2.8}$$

or, element-wise, as

$$X_n = \left(g^{(s)} * \varepsilon\right)_n = \sum_{k=0}^{\infty} g_k^{(s)}\varepsilon_{n-k}, \tag{2.9}$$

where $\varepsilon$ is a Gaussian white noise. One can extend the definition to the case $s \geqslant 1/2$ by setting

$$X(z) = \left(1 - z^{-1}\right)^{-s}\varepsilon(z) = \left(1 - z^{-1}\right)^{-[s+1/2]}\left(1 - z^{-1}\right)^{-(s-[s+1/2])}\varepsilon(z), \tag{2.10}$$

where the $z$-transformation $(1 - z^{-1})^{-1}$ corresponds to a partial sum operation (and hence the process (2.10) is not stationary). Suppose that the sequences $X$ and $\varepsilon$ are independent. We will show that the process

$$Y = u^{(s)} * (\uparrow_2 X) + v^{(s)} * (\uparrow_2 \varepsilon), \tag{2.11}$$

is indeed another FARIMA$(0, s, 0)$ sequence with independent $\mathcal{N}(0, 1)$ innovations. Turning to the $z$-transformations, the relation (2.11) is equivalent to

$$Y(z) = u^{(s)}(z)X\left(z^2\right) + v^{(s)}(z)\varepsilon\left(z^2\right). \tag{2.12}$$

Since $X$ is a FARIMA$(0, s, 0)$ sequence, we have from (2.8) and (2.10) that $X(z) = (1 - z^{-1})^{-s}\xi(z)$, where $\xi = \{\xi_n\}$ is a Gaussian white noise sequence, independent of the series $\varepsilon$. Then, by using (1.9) and (1.10), and the identity $1 - z^{-2} = (1 + z^{-1})(1 - z^{-1})$, we obtain that

$$\begin{aligned} Y(z) &= \left(1 + z^{-1}\right)^{s}\left(1 - z^{-2}\right)^{-s}u(z)\xi\left(z^2\right) + \left(1 - z^{-1}\right)^{-s}v(z)\varepsilon\left(z^2\right) \\ &= \left(1 - z^{-1}\right)^{-s}\left(u(z)\xi\left(z^2\right) + v(z)\varepsilon\left(z^2\right)\right) =: \left(1 - z^{-1}\right)^{-s}\eta(z). \end{aligned} \tag{2.13}$$

By Lemma 2.1 below, the sequence $\eta = \{\eta_n\}$ is a Gaussian white noise and hence the sequence $Y$ is indeed a FARIMA$(0, s, 0)$ sequence by definition.

**Lemma 2.1.** *Let $\xi = \{\xi_n\}$ and $\varepsilon = \{\varepsilon_n\}$ be two independent Gaussian white noise sequences. Let also $u$ and $v$ be, respectively, the low- and high-pass filters associated with an orthonormal MRA. Then, the sequence $\eta = \{\eta_n\}$ defined in the $z$-notation by*

$$\eta(z) = u(z)\xi\left(z^2\right) + v(z)\varepsilon\left(z^2\right), \tag{2.14}$$

*is a Gaussian white noise sequence as well.*

**Proof.** Since (2.14) is a reconstruction scheme, it is enough to show that, given a Gaussian white noise $\eta$, the sequences $\xi = \downarrow_2(u^\vee * \eta)$ and $\varepsilon = \downarrow_2(v^\vee * \eta)$, where $(x^\vee)_n = x_{-n}$, are two independent

Gaussian white noise sequences as well. By using the identity $\downarrow_2(u^\vee * u) = \delta_0$ for the low-pass filter $u$ corresponding to an orthogonal MRA, the covariance function $r^\xi = \{r_n^\xi\}$ of $\xi$ can be expressed as $r^\xi = \downarrow_2(u^\vee * u) = \delta_0$, which shows that $\xi$ is a Gaussian white noise. The same holds for the sequence $\varepsilon$. To show that $\xi$ and $\varepsilon$ are independent, it is enough to prove that $\downarrow_2(u^\vee * v) = 0$. This identity holds for the low- and high-pass filters $u$ and $v$ corresponding to an orthogonal MRA. □

**Remarks.** (1) The Mallat-type synthesis relation (2.13) is easy to generalize for other Gaussian stationary linear sequences. Suppose that $X = \{X_n\}$ is a Gaussian linear process given by $X_n = \sum_{k=-\infty}^{\infty} b_k \xi_{n-k}$ or

$$X(z) = b(z)\xi(z) \tag{2.15}$$

in the $z$-notation, where $b(z)$ is the $z$-transform of $\{b_n\}$ and $\xi = \{\xi_n\}$ consists of independent $\mathcal{N}(0,1)$ random variables. Then, arguing as in (2.12) and (2.13) above, the process

$$Y(z) = u^b(z)X\big(z^2\big) + v^b(z)\varepsilon\big(z^2\big) \tag{2.16}$$

with

$$u^b(z) = \frac{b(z)}{b(z^2)}u(z), \qquad v^b(z) = b(z)v(z), \tag{2.17}$$

is a Gaussian linear sequence having the same linear representation as the process $X$ (and, in particular, having the same probability law as the process $X$).

(2) For example, if $X$ is an AR(1) process represented by (2.15) with $b(z) = (1 - a_1 z^{-1})^{-1}$ and $|a_1| < 1$ (see [5]), then

$$\frac{b(z)}{b(z^2)} = \big(1 - a_1 z^{-2}\big)\sum_{k=0}^{\infty} a_1^k z^{-k} = 1 + a_1 z + \sum_{k=2}^{\infty}\big(a_1^k - a_1^{k-1}\big)z^{-k}.$$

If, for example, the filters $u$ and $v$ correspond to the Haar MRA, then $u^b = \{u_n^b\}$ and $v^b = \{v_n^b\}$ are given by

$$u_n^b = 2^{1/2}\begin{cases} a_1^n - a_1^{n-2}, & \text{if } n \geqslant 3,\\ a_1^2, & \text{if } n = 2,\\ 1 + a_1, & \text{if } n = 1,\\ 1, & \text{if } n = 0, \end{cases} \qquad v_n^b = \big(-2^{1/2}\big)\begin{cases} a_1^n - 2a_1^{n-1} + a_1^{n-2}, & \text{if } n \geqslant 3,\\ a_1^2 - 2a_1, & \text{if } n = 2,\\ 1 - a_1, & \text{if } n = 1,\\ 1, & \text{if } n = 0. \end{cases}$$

One appealing feature of the fractional filters $u^{(s)}$ and $v^{(s)}$ in (1.8) is that they can be conveniently expressed by using (1.9) and (1.10).

(3) If one uses the approximation coefficients

$$s_k^{(H)}(l) = 2^{-ls} S_k^{(H)}(l) = \int_{\mathbb{R}} \big(B_H(t) + b_0\big)2^{l/2} g\big(2^l t - k\big)\,\mathrm{d}t$$

of the wavelet expansion of fBm at the scale $2^{-l}$ (with $s = H + 1/2$), then the scheme (1.7) can be expressed as

$$s_\cdot^{(H)}(l) = \big(2^{-s}u^{(s)}\big) * \big(\uparrow_2 s_\cdot^{(H)}(l-1)\big) + \big(2^{-s}v^{(s)}\big) * \big(\uparrow_2 2^{-(l-1)s}\varepsilon_{l-1,\cdot}\big). \tag{2.18}$$

Observe that the scheme (2.18) is that considered by Abry and Sellan [1] but also note that $2^{-js}$ should be replaced by $2^{js}$ in (3) of their paper and $G^{(s)}(z)$ should be defined as $2^{-s}(1 - z^{-1})^{-s}$ in (10) (see also pp. 81–82 in Abry et al. [2]). We preferred to work with the coefficients $S_k^{(H)}(l)$ rather than $s_k^{(H)}(l)$ for simplicity of the formulas and also since $S_k^{(H)}(l)$ is a FARIMA sequence of the same variance for all $l$.

## 3. Fractional low- and high-pass filters reexamined

In this section, we shed light on the fractional low- and high-pass filters $u^{(s)}$ and $v^{(s)}$ given by (1.8), (1.9), and (1.10). To do so, suppose that an orthogonal MRA associated with the original filters $u$ and $v$ has $N$ zero moments. Then, under mild assumptions on the functions $\phi$ and $\psi$,

$$u(z) = \left(1 + z^{-1}\right)^N u_0(z), \qquad v(z) = \left(1 - z^{-1}\right)^N v_0(z), \tag{3.1}$$

for some filters $u_0$ and $v_0$ (see, for example, [6] or [11]). By using these representations, we deduce the following elementary result.

**Proposition 3.1.** *Suppose that an orthogonal MRA associated with the low- and high-pass filters $u$ and $v$ has $N$ vanishing moments and, consequently, under mild assumptions, that the relations* (3.1) *hold. Then, the fractional low- and high-pass filters $u^{(s)}$ and $v^{(s)}$ of* (1.8), (1.9) *and* (1.10) *can be represented as*

$$u^{(s)}(z) = \left(1 + z^{-1}\right)^{N+s} u_0(z) = f^{(N+s)}(z) u_0(z), \tag{3.2}$$

$$v^{(s)}(z) = \left(1 - z^{-1}\right)^{N-s} v_0(z) = g^{(s-N)}(z) v_0(z), \tag{3.3}$$

*where $u_0$ and $v_0$ are the filters defined through the relation* (3.1).

The advantage of representing $u^{(s)}$ and $v^{(s)}$ by (3.2) and (3.3), respectively, rather than by $u^{(s)}(z) = f^{(s)}(z)u(z) = (1 + z^{-1}) f^{(d)}(z)u(z)$ and $v^{(s)}(z) = g^{(s)}(z)v(z) = (1 - z^{-1})^{-1} g^{(d)}(z)v(z)$ as in (1.11) used by Abry and Sellan [1], is that the filters $f^{(N+s)}$ and $g^{(s-N)}$ decay much faster than the corresponding filters $f^{(d)}$ and $g^{(d)}$ above when $N$ is taken large. Indeed by using the asymptotic relation

$$(-1)^n f_n^{(-a)} = g_n^{(a)} = \prod_{j=1}^{n} \frac{a + j - 1}{j} = \frac{\Gamma(n + a)}{\Gamma(n + 1)\Gamma(a)} \sim \frac{n^{a-1}}{\Gamma(a)},$$

as $n \to +\infty$ (with $a \in \mathbb{R}$ not an integer), we obtain that

$$f_j^{(N+s)} \sim (-1)^j \frac{j^{-1-N-s}}{\Gamma(-N - s)}, \qquad g_j^{(s-N)} \sim \frac{j^{-1-N+s}}{\Gamma(-N + s)}, \tag{3.4}$$

as $j \to +\infty$. Compare (3.4) to analogous relation with $s$ replaced by $d$ and $N = 1$ corresponding to the filters $f^{(d)}$ and $g^{(d)}$. The decay of the filters $f^{(\cdot)}$ and $g^{(\cdot)}$ is important when truncating them in practice at some a priori chosen cutoff level $\varepsilon$. The representations (3.2) and (3.3) allow to show that the fractional low- and high-pass filters $u^{(s)}$ and $v^{(s)}$ can often be characterized by a fast decay as well.

**Proposition 3.2.** *Under the assumptions of Proposition* 3.1, *suppose that $N + s > 0$, $N - s > 0$ and the filters $u_0 = \{u_{0,n}\}$ and $v_0 = \{v_{0,n}\}$ satisfy $|u_{0,n}| \leqslant C|n|^{-p_u}$ and $|v_{0,n}| \leqslant C|n|^{-p_v}$ for $p_u, p_v > 1$. Then, for $n \in \mathbb{Z} \setminus \{0\}$,*

$$\left|u_n^{(s)}\right| \leqslant C_u |n|^{-((N+s+1)\wedge p_u)}, \qquad \left|v_n^{(s)}\right| \leqslant C_v |n|^{-((N-s+1)\wedge p_v)}, \tag{3.5}$$

*where $a \wedge b = \min\{a, b\}$ and the constants $C_u$ and $C_v$ above do not depend on $n$. In particular, when $u$ and $v$* (*equivalently, $u_0$ and $v_0$*) *have finite length, the exponents $p_u$ and $p_v$ in* (3.5) *can be removed.*

**Proof.** Consider, for example, the fractional low-pass filter $u^{(s)}$. By using (3.2), (3.4) and the assumptions of the proposition, we obtain that, for $n \geqslant 2$ and generic constants $C$, $C'$,

$$\begin{aligned}
\big|u_n^{(s)}\big| &\leqslant \sum_{j=-\infty}^{\infty} \big|f_{n-j}^{(N+s)}\big||u_{0,j}| \leqslant C \sum_{j=-\infty}^{\infty} \big(1+|n-j|\big)^{-1-N-s}\big(1+|j|\big)^{-p_u} \\
&\leqslant C \int_{\mathbb{R}} \big(1+|u-n|\big)^{-1-N-s}\big(1+|u|\big)^{-p_u}\,\mathrm{d}u \leqslant Cn^{-1-N-s} \int_{-\infty}^{1} \big(1+|u|\big)^{-p_u}\,\mathrm{d}u \\
&\quad + C \int_{1}^{n-1} |u-n|^{-1-N-s}|u|^{-p_u}\,\mathrm{d}u + Cn^{-p_u} \int_{n-1}^{\infty} \big(1+|u-n|\big)^{-1-N-s}\,\mathrm{d}u \\
&\leqslant Cn^{-(N+s+1)\wedge p_u} + Cn^{-N-s-p_u} \int_{1/n}^{1-1/n} (1-w)^{-1-N-s} w^{-p_u}\,\mathrm{d}w \leqslant C'n^{-(N+s+1)\wedge p_u}.
\end{aligned}$$

The case $n \leqslant 2$ and then that of the high-pass filter $v^{(s)}$ can be considered in a similar way. □

In Table 1, we give an idea on the length of truncated fractional filters $u^{(s)}$ and $v^{(s)}$ for a chosen cutoff level $\varepsilon$, the Daubechies MRA's with the number of vanishing moments $N$ and the choice of $s = 1.25$. More precisely, the length of a truncated filter $u^{(s)}$, for example, is computed as follows. Observe that

$$\big|u_n^{(s)}\big| \leqslant \sum_{k=k_0}^{k_1} |u_{0,k}|\big|f_{n-k}^{(N+s)}\big|, \tag{3.6}$$

where $\{u_{0,k},\ k = k_0, \dots, k_1\}$ is the finite filter associated with $u$ through (3.1). We choose the length of a truncated filter $u^{(s)}$ as the smallest $n_0 - k_0 + 1$ with $n_0 \geqslant k_1$ for which the right-hand side of (3.6) is bounded by $\varepsilon$ for any $n \geqslant n_0$. (Note that it is enough to find the first such $n$ because the right-hand side decreases monotonically as $n \geqslant k_1$ increases.) For the filter $u_0 = \{u_{0,k}\}$ in (3.6), we chose the properly normalized values given in Table 6.2 of Daubechies [6, p. 196]. Observe from Table 1 that the lengths of truncated filters become significantly smaller for large $N$ at small values of $\varepsilon$. This observation is relevant in practice because, when $N$ is larger, we can choose truncated filters of significantly smaller length.

Table 1
Lengths of truncated filters $u^{(s)}$ and $v^{(s)}$ at a cutoff $\varepsilon$ with $s = 1.25$ and the Daubechies MRA with $N$ vanishing moments

| Filters | Cutoff $\varepsilon$ | $N = 1$ | $N = 3$ | $N = 6$ | $N = 10$ |
|---|---|---|---|---|---|
| $u^{(s)}$ | $10^{-4}$ | 17 | 17 | 21 | 28 |
| | $10^{-7}$ | 123 | 48 | 36 | 38 |
| | $10^{-10}$ | 1009 | 164 | 70 | 56 |
| | $10^{-15}$ | 34,769 | 1425 | 257 | 121 |
| $v^{(s)}$ | $10^{-4}$ | $\approx 4 \times 10^4$ | 38 | 23 | 28 |
| | $10^{-7}$ | $\approx 4 \times 10^8$ | 422 | 57 | 42 |
| | $10^{-10}$ | $\approx 4 \times 10^{12}$ | 5162 | 171 | 72 |
| | $10^{-15}$ | $\approx 4 \times 10^{19}$ | $\approx 3 \times 10^5$ | 1220 | 206 |

## 4. Modifications of the simulation algorithm

Recall from Sections 1 and 2 that the basic idea behind the wavelet-based synthesis of fBm is to generate an initial FARIMA$(0, H+1/2, 0)$ sequence, apply to it the recursive Mallat-type scheme (1.7) and finally take the resulting FARIMA$(0, H+1/2, 0)$ sequence as an approximation to fBm. Since fractional low- and high-pass filters are truncated in practice, the length of a FARIMA sequence essentially doubles after each application of the scheme (1.7). Practical implementation of this simulation procedure was proposed by Abry and Sellan [1]. We suggest to make the following important modifications to their algorithm.

*Modifications*:

1. Fractional low- and high-pass filters $u^{(s)}$ and $v^{(s)}$ with $s = H + 1/2$ which enter into (1.7), should be computed by using the relations (3.2) and (3.3), and truncated at some cutoff level by using the arguments around (3.6).
2. An initial FARIMA$(0, H+1/2, 0)$ sequence can be taken of a different length.
3. We propose to simulate an initial FARIMA$(0, H+1/2, 0)$ sequence exactly.

We now explain each of these modifications in greater detail.

In regard to the *first* modification above, Abry and Sellan [1] computed the fractional filters through (1.11) which involves truncating the sequences $f^{(d)}$ and $g^{(d)}$. Since these sequences decay very slowly, fractional filters chosen by Abry and Sellan [1] are unnecessary too long. Moreover, in contrast to (3.6), there is no control over the size of their elements.

Concerning the *second* modification above, let $J$ enter into the time scale $2^{-J}$ at which the resulting FARIMA$(0, H+1/2, 0)$ sequence is taken as an approximation to fBm. Abry and Sellan [1] take the same $J$ for the number of times that the Mallat-type scheme (1.7) is recursively applied to an initial FARIMA$(0, H+1/2, 0)$ sequence. This means in particular that, if the number of desired fBm observations $K$ is large and $J$ is relatively small, then one needs to generate an initial FARIMA sequence of a large approximate length $K2^{-J}$. The constraint that (1.7) is used exactly $J$ times, is not necessary. In fact, as argued by Pipiras [14], it does not really matter (in the sense specified in that paper) what the length of an initial FARIMA sequence is. If this is so, then one natural choice for the length of an initial FARIMA sequence is the smallest number which make the use of (1.7) possible when accounting for boarder effect. An appeal of this choice is that the simulation of fBm then becomes truly wavelet-based. For example, there is no need to generate a very long initial FARIMA sequence by using other simulation methods.

If $r$ is the maximum length of the fractional low- and high-pass filters truncated at some cutoff level, then the smallest possible length of an initial FARIMA$(0, H+1/2, 0)$ sequence is

$$k_0 = r + 1. \tag{4.1}$$

Observe that after applying the scheme (1.7) to a FARIMA sequence of length (4.1), we would obtain $(2k_0 - 1 - r) + 1 = r + 2$ number of points of a new FARIMA sequence which are unaffected by boarder effect. Here, $2k_0 - 1$ is the number of points after the operation $\uparrow_2$ and $(-r)$ takes into account the boarder effect. By repeating this argument, the number of points of a resulting FARIMA sequence after recursively applying (1.7) $m$ times, is $r + 2^m$. In practice, we choose $m$ so that $r + 2^m$ is larger than the number of desired points of fBm.

Our *third* modification concerns the initial FARIMA$(0, H + 1/2, 0)$ sequence. Abry and Sellan [1] generate it directly by using definition through

$$X_n = \left(g^{(1)} * tg^{(d)} * \xi\right)_n,$$

where $tg^{(d)}$ is the filter $g^{(d)}$ truncated at a prescribed cutoff level and $\xi$ is a discrete Gaussian white noise. Since the elements of the filter $g^{(d)}$ decay extremely slowly, the truncated filter is very long, its computation and that of the convolution with $\xi$ are time consuming and, moreover, the sequence $X_n$ is not exactly a FARIMA$(0, H + 1/2, 0)$ sequence. We suggest to generate the initial FARIMA$(0, H + 1/2, 0)$ sequence exactly.

At least two exact simulation methods are available in the statistical literature: the Durbin–Levinson algorithm and the circulant matrix embedding method. These methods are nicely summarized by Bardet et al. [3] but see also Brockwell and Davis [5] for the Durbin–Levinson algorithm and Dietrich and Newsam [7] for the circulant matrix embedding method. The Durbin–Levinson algorithm is exact but time consuming when generating long time series. We can nevertheless often use it because, in view of (4.1) and Table 1, the length of an initial FARIMA$(0, H + 1/2, 0)$ series may be taken quite small. The circulant matrix embedding method, on the other hand, is considered exact and not time consuming.

The modified wavelet-based algorithm to simulate fBm, written in MATLAB, is available from the author upon request. The use and usefulness of the wavelet-based simulation of fBm are further discussed in [14].

**Remark.** Generating a FARIMA$(0, H + 1/2, 0)$ sequence is at the core of the wavelet-based algorithm to simulate fBm. Starting with an initial FARIMA$(0, H + 1/2, 0)$ sequence, we recursively apply to it the scheme (1.7) to obtain each time an approximately twice longer FARIMA$(0, H + 1/2, 0)$ sequence. Would it be possible to start with a fBm sequence and then apply to it an analogous (1.7)-type scheme to obtain a twice longer fBm sequence? This question is important because, in contrast to using FARIMA sequences, we would in principle obtain not an approximation to fBm but a fBm sequence itself. Meyer et al. [12] showed in Section 8 that generating fBm in this way is indeed possible in theory. To implement their procedure in practice, we would need to compute associated fractional low- and high-pass filters which enter into the Mallat-type scheme (1.7). This task, however, is numerically much more difficult than in the case of FARIMA sequences.

To understand this, consider a fractional Gaussian noise (fGn, in short) sequence $\{Z_H(k)\}_{k\in\mathbb{Z}}$ defined as increments of fBm $\{B_H(k)\}_{k\in\mathbb{Z}}$ at integer times. These two sequences are equivalent in the sense that starting with one of them, we can find the other one either by taking a partial sum or by taking the increments. The difference is that fGn is a stationary sequence and hence slightly easier to manipulate. By Proposition 2.1 in [4], fGn can be represented as

$$Z_H(k) = \sum_{j=-\infty}^{\infty} b_j \varepsilon_{k-j}, \quad k \in \mathbb{Z},$$

where $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ are independent $\mathcal{N}(0, 1)$ random variables and the sequence $\{b_j\}_{j\in\mathbb{Z}}$ is defined through its discrete Fourier transform as

$$\left|\hat{b}(x)\right|^2 = EB_H^2(1)\sin(\pi H)\Gamma(2H+1)(1-\cos x)\sum_{n=-\infty}^{\infty}|2\pi n + x|^{-2H-1}, \quad x \in (-\pi, \pi).$$

Then, by using Remark 2 at the end of Section 2, the fractional low- and high-pass filters associated with fGn can be expressed as

$$u^b(z) = \frac{b(z)}{b(z^2)} u(z), \qquad v^b(z) = b(z)v(z),$$

where $b(z) = \sum_{j=-\infty}^{\infty} b_j z^{-j}$. In contrast to the case of FARIMA$(0, \lambda, 0)$ sequences, there is no easy way or formula to compute the elements $b_j$. Moreover, the filters $u^b(z)$ and $v^b(z)$ are not easy to determine or to study either.

## Acknowledgment

The author thanks an anonymous referee for many useful comments and suggestions.

## References

[1] P. Abry, F. Sellan, The wavelet-based synthesis for the fractional Brownian motion proposed by F. Sellan and Y. Meyer: Remarks and fast implementation, Appl. Comput. Harm. Anal. 3 (4) (1996) 377–383.
[2] P. Abry, P. Flandrin, M.S. Taqqu, D. Veitch, Wavelets for the analysis, estimation and synthesis of scaling data, in: K. Park, W. Willinger (Eds.), Self-Similar Network Traffic and Performance Evaluation, Wiley, New York, 2000, pp. 39–88.
[3] J.-M. Bardet, G. Lang, G. Oppenheim, A. Philippe, M.S. Taqqu, Generators of long-range dependent processes: A survey, in: P. Doukhan, G. Oppenheim, M.S. Taqqu (Eds.), Long-Range Dependence: Theory and Applications, Birkhäuser, Boston, 2003, pp. 579–623.
[4] J. Beran, Statistics for Long-Memory Processes, Chapman & Hall, New York, 1994.
[5] P.J. Brockwell, R.A. Davis, Time Series: Theory and Methods, second ed., Springer, New York, 1991.
[6] I. Daubechies, Ten Lectures on Wavelets, CBMS–NSF Ser., vol. 61, SIAM, Philadelphia, 1992.
[7] C.R. Dietrich, G.N. Newsam, Fast and exact simulation of stationary Gaussian processes through circulant embedding of the covariance matrix, SIAM J. Sci. Comput. 18 (4) (1997) 1088–1107.
[8] P. Doukhan, G. Oppenheim, M.S. Taqqu (Eds.), Long-Range Dependence: Theory and Applications, Birkhäuser, Boston, 2003.
[9] P. Embrechts, M. Maejima, Selfsimilar Processes, Academic Press, 2002.
[10] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, second ed., Springer, New York, 1997.
[11] S. Mallat, A Wavelet Tour of Signal Processing, Academic Press, Boston, 1998.
[12] Y. Meyer, F. Sellan, M.S. Taqqu, Wavelets, generalized white noise and fractional integration: The synthesis of fractional Brownian motion, J. Fourier Anal. Appl. 5 (5) (1999) 465–494.
[13] S. Orey, Growth rate of certain Gaussian processes, in: Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, CA, 1970/1971), vol. II: Probability Theory, Univ. California Press, Berkeley, CA, 1972, pp. 443–451.
[14] V. Pipiras, On the usefulness of wavelet-based simulation of fractional Brownian motion, Preprint, 2003.
[15] G. Samorodnitsky, M.S. Taqqu, Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance, Chapman & Hall, New York/London, 1994.
[16] F. Sellan, Synthèse de mouvements browniens fractionnaires à l'aide de la transformation par ondelettes, C. R. Acad. Sci. Paris Sér. I 321 (1995) 351–358.