

# Nonabelian Discrete Symmetries, Fermion Mass Textures and Large Neutrino Mixing

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## Abstract

Nonabelian discrete groups are an attractive tool to describe fermion masses and mixings. They have nonsinglet representations which seem particularly suitable for distinguishing the lighter generations from the heavier ones. Also, they do not suffer from the extra constraints a continuous group must obey, *e.g.* limits on extra particles. Some of the simplest groups are the nonabelian discrete subgroups of  $SO(3)$  and  $SU(2)$ , the so called dihedral groups  $D_n$  and dicyclic groups  $Q_{2n}$ , which both have only singlet and doublet representations. After studying which vacuum expectation value (VEV) directions of representations of dihedral and dicyclic groups preserve which subgroups, we construct a simple model based on the group  $Q_6 \times Q_6$ . The model reproduces the masses and mixings of all quarks and leptons, including neutrinos. It has a large mixing angle in the  $\mu - \tau$  neutrino sector, in accordance with the recent SuperKamiokande results, while keeping a small quark mixing in the bottom - charm sector. The reason is similar to the one found in the literature based on the  $SU(5)$  group: the large *left* handed mixing angle in the lepton sector corresponds to the large unphysical *right* handed in the down quark sector. The large mixing is also responsible for the different hierarchies of the two heaviest families in the up and down sector, and can be summarized as the order of magnitude relation:  $\frac{m_s}{m_b} \sim \tan(\theta_{\mu\tau}) \sqrt{\frac{m_c}{m_t}}$ .

## I. INTRODUCTION

The question of the origin of fermion masses and mixings is one of the most pressing questions in the Standard Model. Namely, fermion masses and mixings are merely input parameters and in order to get a handle on so many arbitrary parameters one has to go beyond the Standard Model. One of the most promising ways to go there is to utilize flavor symmetries so that one understands the mass couplings as parameters of flavor symmetry breaking. The major hope of such approach is to minimize the number of input parameters and therefore have verifiable predictions. Such approaches have been pursued for many years with the quark masses and mixings and sometimes with charged lepton masses as well.

On the other hand, during the last couple of years we have seen a steady increase in quantity and improvement in quality of neutrino data, and moreover the first compelling evidence that neutrinos do have a mass. The SuperKamiokande atmospheric data [1] strongly indicate that there is a large nonvanishing mixing of the muon neutrino. The same experiment and several others [2] also indicate presence of neutrino mixing of the electron neutrinos coming from the Sun, and there is further [3], though not nearly as strong [4], evidence of other neutrino mixing phenomena.

The simplest explanation of such data is that neutrinos have a nonvanishing mass, and similarly to quarks, mix with other neutrinos due to the misalignment of flavor and mass eigenstates. The smallness of the overall scale of neutrino mass compared to other observed fermions, can be understood by the see-saw mechanism [5]. On the other hand, the neutrino mixing is generally explained quantitatively with different neutrino mass textures and many such examples exist in the literature (for recent reviews see [6,7]).

It is thus an important question how to address all fermion masses and mixings, including neutrinos, in a viable and simple flavor theory. One of the interesting questions that arises is how to explain the large mixing in the neutrino sector between the second and third generation<sup>1</sup>, when the corresponding mixing angle in the quark sector is very small. However, the key is to observe that the observed quark mixing angles pertain to the left handed sector. It might be that their right handed mixing angles are large but they are unphysical and hence unobservable. However, in a larger, possibly unified theory, the right handed quark sector might be related to the left handed lepton sector. Such is the case in SU(5) for example, where the **5** representation contains a *right* handed quark and a *left* handed lepton. An interesting set of textures exploring this left-right relation in SU(5) is found in [8], and still other may be found in [9]. A particularly interesting aspect of this approach is that the large mixing can come solely from the neutrino Dirac mass, regardless of the details of the right handed Majorana mass matrix. Namely, if in the basis where the charged lepton mass matrix is diagonal, the neutrino Dirac matrix is of the form (with left handed lepton doublets multiplying from the left)

$$m^\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

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<sup>1</sup>We do not consider the possibility of more than three neutrino species.

where  $\sigma$  is a number of order unity, and zeroes represent the much smaller neglected entries, the resulting light neutrino mass matrix

$$m_{light}^\nu = m^\nu \mathbf{M}_N^{-1} m^{\nu T} = 1/M_{33} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma^2 & \sigma \\ 0 & \sigma & 1 \end{pmatrix} \quad (2)$$

has only one nonvanishing eigenvalue, and a large mixing angle. Note that the result does not depend on details of the right handed Majorana mass matrix  $M$ , so long as  $M_{33} \neq 0$ .

Such models must in general use, additional global symmetries in order to get the desired texture. The question then arises if one could use only *discrete* symmetries [10] and build such textures. Such symmetries may be the only remnant of the grand unified or stringy origin of an underlying theory. Especially suitable are non-abelian discrete symmetries, which in general prove to be more restrictive and thus more predictive. Another advantage of discrete symmetries is that one can explain their origin as remnants of a broken gauge symmetry, or more generally, of a string theory. The permutation symmetry group  $S_3$  has been often used for building neutrino mass matrices [11]. Other models based on nonabelian discrete groups have, surprisingly, not been studied much in connection with quark mass textures [12–15] and even less so for neutrino mass textures: the quaternionic group  $Q$  and dicyclic  $Q_6$  has been studied in [16] in connection with the neutrino magnetic moment, and  $\Delta(75)$ , a discrete subgroup of  $SU(3)$  in [17].

In this paper we study flavor theories with nonabelian discrete subgroups of a gauged  $SU(2)$  in order to obtain suitable textures for both quark and lepton mass matrices. Such groups have other motivations as well: cosmological applications, i.e. such as Alice strings [18]; also the  $SU(2)$  origin might be interesting alternative starting point for grand unification.

In Section 2 we review and compare two sets of discrete groups: the dihedral groups  $D_m$  which are subgroups of  $SO(3)$ , and dicyclic groups  $Q_{2n}$  which are subgroups of  $SU(2)$ . As we will see, just as the  $SU(2)$  is the spinorial generalization of  $SO(3)$ , so are the  $Q_{2n}$  spinorial generalizations of  $D_{2n}$ , and we argue why these groups should be taken seriously in model building.

There are two reasons why  $D_m$  and  $Q_{2n}$  groups are particularly interesting for building flavor models. One is the presence of only singlet and doublet representations. This enables one to accommodate the three generations minimally in such representations, and at the same time somehow distinguish the heavy third family from the other two [13]. The second reason is that these groups, even for small  $m$  or  $n$ , have a rich structure of subgroups, making it possible to use the scales of symmetry breakings as the origin of fermion mass and mixing hierarchies.

In Section 3 we enumerate the possible symmetry breaking directions of the vacuum expectation values (VEVs). In Section 4 we present a model based on  $Q_6 \times Q_6$  for which in Section 5 we demonstrate desirable quark mass and mixing hierarchies and an acceptable large neutrino mixing, in a similar way as in [8]. Finally, Section 6 is the conclusion.

## II. NONABELIAN DISCRETE SUBGROUPS OF SO(3) AND SU(2)

Dihedral groups  $D_m$  are the groups of rotations of a regular  $n$ -agon in three dimensions (regarded as a two-faced entity - "dihedral") and are discrete nonabelian subgroups of SO(3). They can be defined as

$$D_m = \{a, b | a^m = e, b^2 = e, aba = b\}. \quad (3)$$

They are of order  $2m$ . Simplest examples are  $D_2$  which has four elements and is isomorphic to  $K = Z_2 \times Z_2$  (the Klein group),  $D_3$  has six elements and is isomorphic to the permutation group  $S_3$ ,  $D_4$  has eight elements and represents the symmetries of a square, etc.

Subgroups of  $D_m$  include a  $Z_m$  (generated by  $a$ ) and  $m$   $Z_2$ 's (each one generated by  $ba^l$ ,  $l = 0, 1, \dots, m - 1$ ) and further subgroups depending on whether  $m$  is prime or not. For example,  $D_{2n}$  has in addition two subgroups  $D_n$  (one generated by  $a^2$  and  $b$ , and the other generated by  $a^2$  and  $ba$ ) and  $n$  Klein ( $=Z_2 \times Z_2$ ) groups (each generated by  $a^n$  and one of the  $ba^l$ ,  $l = 0, \dots, n - 1$ ), and possible further subgroups if  $n$  is not prime.

Particularly useful matrix representation for the groups  $D_{2n}$  is given by

$$a = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (4)$$

where  $\theta = 2\pi/m$ .

Dicyclic groups  $Q_{2n}$  are spinorial generalizations of  $D_{2n}$  and are subgroups of SU(2). They can be defined as

$$Q_{2n} = \{a, b | a^{2n} = e, b^2 = a^n, aba = b\}. \quad (5)$$

and are of order  $4n$ . The spinorial generalization may be seen here from the property  $b^4 = e$  (compared to  $b^2 = e$  in  $D_m$ ). Smallest dicyclic groups include  $Q_2$  which has four elements and is isomorphic to  $Z_4$ ,  $Q_4$  with eight elements and isomorphic to the quaternionic group, etc.

Subgroups of  $Q_{2n}$  are  $Z_{2n}$  (generated by  $a$ ) and  $n$   $Z_4$ 's (each generated by  $a^n$  and one of  $ba^l$ ,  $l = 0, \dots, n - 1$ ) and more subgroups depending on whether  $n$  is prime or not.

For  $Q_{2n}$  the matrix representation (4) generalizes to

$$a = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (6)$$

where  $\theta = 4\pi/2n$ .

Representations of both  $D_{2n}$  and  $Q_{2n}$  have four singlets and  $n - 1$  doublets, while representations of  $D_{2n+1}$  have 2 singlets and  $n$  doublets. As we wish to compare the dihedral and dicyclic groups of the same order, we will not consider  $D_{2n+1}$  further. Let us denote the four singlets  $1, 1', 1'', 1'''$  and the doublets as  $2_i$ ,  $1 = 1, \dots, n - 1$ . The multiplication table for the representations of both  $D_m$  and  $Q_{2n}$  can be found in [13,19] The behavior of the odd-numbered doublets (*i.e.*  $2_{2i+1}$ ) is spinorial, and in the special case when  $n$  is odd,  $1''$  and  $1'''$  are spinorial too.

It is useful to exhibit the decomposition of the  $SO(3)$  and  $SU(2)$  representations under the  $D_{2n}$  and  $Q_{2n}$ . We start by identifying the  $\mathbf{2}$  of  $SU(2)$  with the first spinorial doublet  $2_1$  of  $Q_{2n}$ , and then find the decomposition of other  $SU(2)$  irreps from the decomposition of products of  $\mathbf{2}$ 's. For  $SO(3)$  it is necessary to identify  $\mathbf{3}$  with  $1' + 2_1$ . Notice that the  $Q_{2n}$  matrix representation in (6) is the 2-dimensional representation  $2_1$  of  $Q_{2n}$ , while 3-dimensional representation (4) of  $D_{2n}$  contains representations  $1' + 2_1$  of  $D_{2n}$ . The decomposition of the lowest  $SO(3)$  and  $SU(2)$  representations under  $D_{2n}$  and  $Q_{2n}$  are as follows

$SO(3) \rightarrow$	$D_{2n}$	$SU(2) \rightarrow$	$Q_{2n}$
$\mathbf{3} \rightarrow$	$1' + 2_1$	$\mathbf{2} \rightarrow$	$2_1$
		$\mathbf{3} \rightarrow$	$1' + 2_2$
$\mathbf{5} \rightarrow$	$1 + 2_1 + 2_2$	$\mathbf{4} \rightarrow$	$2_1 + 2_3$
$\mathbf{7} \rightarrow$	$1' + 2_1 + 2_2 + 2_3$	$\mathbf{5} \rightarrow$	$1 + 2_2 + 2_4$
$\mathbf{9} \rightarrow$	$1 + 2_1 + 2_2 + 2_3 + 2_4$	$\mathbf{6} \rightarrow$	$2_1 + 2_3 + 2_5$
		$\mathbf{7} \rightarrow$	$1 + 2_2 + 2_4 + 2_6$
		$\mathbf{8} \rightarrow$	$2_1 + 2_3 + 2_5 + 2_7$
		$\mathbf{9} \rightarrow$	$1 + 2_2 + 2_4 + 2_6 + 2_8$

(7)

etc., for sufficiently high  $n$ . Of course, for a given  $D_{2n}$  (or  $Q_{2n}$ ), there are only  $n - 1$  doublets, and for the doublets with indices higher than  $n - 1$  one has to use the identifications  $2_n \equiv 1'' + 1'''$ ,  $2_i \equiv 2_{2n-i}$  and  $2_0 \equiv 1 + 1'$ . The odd-dimensional  $SU(2)$  representations cannot contain spinorial doublets of  $Q_{2n}$ , since they are vectorial. On the other hand, in  $D_{2n}$  there is no notion of spinors and all doublets appear in the representations of  $SO(3)$  which are all odd-dimensional. Since the highest doublet representation is  $2_{n-1}$  (and first appears in  $\mathbf{2n} - \mathbf{2}$  of  $SU(2)$ ) the decomposition along the  $T_3$  direction is actually  $\text{mod}(n)$ , which correspond to the subgroup  $Z_{2n}$  for  $Q_{2n}$  (because of the half-integer isospins in spinorial representations). Similarly, the decomposition under  $T_3$  of  $SO(3)$  also shows the decomposition of irreps under  $D_{2n}$  as elements of the  $Z_{2n}$  subgroup (notice that here the highest doublet  $2_{n-1}$  first appears in  $\mathbf{4n} - \mathbf{4}$  of  $SO(3)$ ).

We see that from model building viewpoint both groups have their advantages.  $Q_{2n}$  groups offer more choice in putting generations in complete  $SU(2)$  multiplets<sup>2</sup> without going to higher representations, where one has to worry about the constraints on extra matter or anomaly cancellations. Namely, in order to cancel anomalies choosing to put the generations of fermions in a complete  $SU(2)$  representations will satisfy the cancellation conditions linear in  $SU(2)$ . Other anomaly cancellation conditions do not have to be necessarily satisfied in a low-energy effective theory because of the in principle unknown contributions from heavy fermions [20]. Thus, we can put the three generations into  $1 + 1 + 1$  or  $1 + 2$  or a  $3$  of  $SU(2)$ , while in  $SO(3)$  we use only  $1 + 1 + 1$  or  $3$ . On the other hand  $D_{2n}$  groups have more subgroups, and therefore more choice in symmetry breaking. It will then depend on the underlying theory to decide whether to pick a  $SU(2)$  or  $SO(3)$  group.

In the next two sections we first list the symmetry breaking directions, and then we build a model based on  $Q_6 \times Q_6$ .

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<sup>2</sup>This flavor  $SU(2)$  group is not to be confused with the usual electroweak  $SU(2)_W$ .

### III. SYMMETRY BREAKING

An interesting thing to note in (7) is that  $\mathbf{5}$  of  $SO(3)$  (or  $SU(2)$ ) contains a singlet under any  $D_{2n}$  or  $Q_{2n}$ . Indeed, the VEV of the singlet in  $\mathbf{5}$  written as a symmetric traceless  $3 \times 3$  matrix

$$\langle \mathbf{5} \rangle = v \text{diag}(1, 1, -2) \quad (8)$$

does leave invariant a larger group. This group is generated by the matrix representation of  $D_{2n}$  in (4), with any  $\theta$  from 0 to  $2\pi$ . This invariance group, loosely speaking, consists of a  $SO(2)$ , generated by the matrices  $a$  in Eq. (4) and a  $Z_2$ , generated by  $b$  in Eq. (4). These do not commute. More precisely, in the case of  $SO(3)$  the subgroup is  $O(2)$  with two connected components: one consists of all rotations around the z-axis and is connected to the identity of  $SO(3)$ , and the other one has  $180^\circ$  rotation around an axis in the x-y plane. This group is generated by elements from (4) for any  $\theta$  between 0 and  $2\pi$ . In the case of  $SU(2)$  this generalizes to the so-called  $Pin(2)$  group (generated by elements from (6) for any  $\theta$  between 0 and  $4\pi$ .) which has interesting applications for Alice strings in cosmology [18].

Next, notice that for a given  $n$ , the first  $SU(2)$  representation which contains a singlet of  $Q_{2n}$  but not of  $Q_{2n+2}, Q_{2n+4}, \dots$ , is  $\mathbf{2n+1}$ , since the highest doublet in it,  $2_{2n}$ , is identified by  $2_{2n} \equiv 2_0 \equiv 1 + 1'$ , and similarly  $\mathbf{4n+2}$  of  $SO(3)$  for  $D_{2n}$ . Thus

$$SO(3) \xrightarrow{\langle \mathbf{4n+2} \rangle} D_{2n}, \quad SU(2) \xrightarrow{\langle \mathbf{2n+1} \rangle} Q_{2n}. \quad (9)$$

We next identify the symmetry breaking directions within  $D_{2n}$  and  $Q_{2n}$ .

#### Symmetry breaking in $D_{2n}$

The  $1'$  is part of the triplet in  $SO(3)$ . The triplet is

$$\mathbf{3} = \begin{pmatrix} 2_1 \\ 1' \end{pmatrix} \quad (10)$$

and as can be seen from (4), a VEV of  $1'$  preserves all transformations involving  $a = R_{12}(2\pi/2n)$  (but not the ones involving  $b$ ) and therefore preserves  $Z_{2n}$ . On the other hand  $2_1$  preserves one of the  $2n$   $Z_2$  subgroups generated by  $ba^l$  ( $l = 0, 1, \dots, 2n-1$ ) depending on the VEV direction. In particular  $\langle 2_1 \rangle = (v, \frac{\cos(2\pi l/2n)-1}{\sin(2\pi l/2n)}v)^T$  preserves the  $Z_2$  generated by  $ba^l$  ( $(ba^l)^2 = 1$  for any  $l = 0, 1, \dots, 2n-1$ ). Since this  $Z_2$  is not invariant under the  $1'$ , simultaneous VEVs of  $1'$  and  $2_1$  will break  $D_{2n}$  completely.

The VEV of the next doublet  $2_2$  conserves one of the  $n$  Klein ( $=Z_2 \times Z_2$ ) subgroups generated by  $a^n$  and  $ba^l$  since  $(a^n)^2 = 1$  and  $(ba^l)^2 = 1$ ). This can be seen from the decomposition of  $\mathbf{5} = 1 + 2_1 + 2_2$  under  $D_{2n}$

$$\mathbf{5} = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix} + \begin{pmatrix} 0 & 0 & v_1 \\ 0 & 0 & v_2 \\ v_1 & v_2 & 0 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 & 0 \\ w_2 & -w_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (11)$$

The VEVs of the singlet and the doublet  $2_1$  were discussed above. The last term is the  $2_2$  of  $D_{2n}$ . Its VEV is invariant under a Klein group generated by  $a^n = \text{diag}(-1, -1, 1)$  and

$ba^l$  ( $l = 1, 2, \dots, n-1$ ), provided  $w_2 = \frac{\cos(4\pi l/2n)-1}{\sin(4\pi l/2n)}w_1$ . When there is no relation between  $w_1$  and  $w_2$ , the VEV of  $2_2$  simply leaves invariant a  $Z_2$  subgroup generated by  $a^n$ .

So far we have shown which symmetry breaking directions are possible with  $1'$ , and the two doublets  $2_1$  and  $2_2$ . In the same manner one can analyze other doublets  $2_i$ , with  $i = 3, 4, \dots, n-1$ , however as we will concentrate in the next section on  $Q_6$  and  $D_6$  which only have  $2_1$  and  $2_2$ , we do not study further doublet VEVs here.

The only thing that remains is to show which symmetries remain unbroken when  $1''$  or  $1'''$  get a VEV. We have seen before that in multiplication tables for representations  $1'' + 1'''$  behave as the (fictitious) doublet  $2_n$  of  $D_{2n}$  or  $Q_{2n}$ . Also, as we have seen before,  $2_n$  is in the  $2\mathbf{n} + \mathbf{1}$  representation of  $SO(3)$  and corresponds to  $T_3 = \pm n$ . This tells us that they preserve the symmetry  $Z_n$  (generated by  $a^2$ ) of  $D_{2n}$ . In addition, when the VEVs of both  $1''$  and  $1'''$  are equal, in  $D_{2n}$  they will leave invariant also  $b$ , so that the total symmetry left invariant by VEVs of  $1'' + 1'''$  in  $D_{2n}$  is  $D_n$ .

We can summarize the symmetry breaking chains as follows

$$\begin{aligned}
D_{2n} & \xrightarrow{\langle 1' \rangle} & Z_{2n} = \{a\} , \\
D_{2n} & \xrightarrow{\langle 2_1 \rangle = \begin{pmatrix} v \\ \frac{c_l-1}{s_l}v \end{pmatrix}} & Z_2 = \{ba^l\} , & l = 0, 1, \dots, 2n-1 , \\
D_{2n} & \xrightarrow{\langle 2_2 \rangle = \begin{pmatrix} v \\ \frac{c_{2l}-1}{s_{2l}}v \end{pmatrix}} & K = Z_2 \times Z_2 = \{a^n, ba^l\} , & l = 0, 1, \dots, n-1 \\
D_{2n} & \xrightarrow{\langle 1'' \rangle = \langle 1''' \rangle} & D_n = \{a^2, b\} & ; \langle 1'' \rangle \neq \langle 1''' \rangle \longrightarrow Z_n = \{a^2\}
\end{aligned} \tag{12}$$

where  $c_l = \cos 2\pi l/2n$  and  $s_l = \sin 2\pi l/2n$ .

### Symmetry breaking in $Q_{2n}$

In  $SU(2)$ , the triplet can be represented as  $2 \times 2$  traceless matrix  $\Delta$  that transforms as  $\Delta \rightarrow U\Delta U^\dagger$ . The components of this triplet are

$$1' = \begin{pmatrix} f_1 & 0 \\ 0 & -f_1 \end{pmatrix} , \quad 2_2 = \begin{pmatrix} 0 & f_2 \\ g_2 & 0 \end{pmatrix} \tag{13}$$

so that  $\langle 1' \rangle = \text{diag}(v, -v)$  is invariant under  $a$  from (6) ( $\langle 1' \rangle \rightarrow a \langle 1' \rangle a^\dagger$ ), but not  $b$ , and therefore also preserves  $Z_{2n}$ . This is also easily understood, since  $1'$  has a vanishing third component of isospin, and the multiplication tables allow for isospin multiplications up to  $\text{mod}(n)$ . On the other hand, the vectorial doublet  $2_2$  of  $Q_{2n}$  does preserve one of the  $n$   $Z_4$  symmetries of  $Q_{2n}$  generated by  $ba^l$ ,  $l = 0, 1, \dots, n-1$  (since  $(ba^l)^4 = (a^n)^2 = 1$ ), and this is achieved when the VEV points in the direction

$$\langle 2_2 \rangle = \begin{pmatrix} 0 & v \\ -ve^{4\pi il/2n} & 0 \end{pmatrix} \tag{14}$$

The  $2_1$  spinorial doublet of  $Q_{2n}$  does not preserve any subgroup of  $Q_{2n}$  so its non-zero VEV breaks the group to nothing. However, one has to stress that spinorial couplings itself often preserve further accidental discrete symmetries, and we will make use of those in the next section.

Similarly to  $D_{2n}$ ,  $1''$  and  $1'''$  break  $Q_{2n}$  to  $Z_n$  generated by  $a^2$ , but cannot be left invariant under  $b$ . To summarize,

$$\begin{aligned}
Q_{2n} &\xrightarrow{\langle 1' \rangle} && Z_{2n} = \langle a \rangle , \\
Q_{2n} &\xrightarrow{\langle 2_1 \rangle} && \textit{nothing} , \\
Q_{2n} &\xrightarrow{\langle 2_2 \rangle = \begin{pmatrix} v \\ -ve^{i4\pi l/2n} \end{pmatrix}} && Z_4 = \langle a^n, ba^l \rangle , l = 0, 1, \dots, n-1 , \\
Q_{2n} &\xrightarrow{\langle 1'' \rangle, \langle 1''' \rangle} && Z_n = \langle a^2 \rangle .
\end{aligned} \tag{15}$$

#### IV. THE MODEL

Several flavor models based on groups  $Q_{2n}$  and  $D_{2n}$  exist. Flavor theories based on  $Q_6$  and higher  $Q_{2n}$  groups were studied in [13,14]. The quaternionic group and  $Q_6$  were also used to study neutrino magnetic moments [16].

In building a model, we must be guided by simplicity, but at the same time by predictivity as well. Building models on a single low order flavor group, such as  $Q_6$  or  $D_6$  will inevitably require some fine tuning to explain all patterns of fermion masses and mixings, simply because the number of parameters is too small. On the other hand, using a too large group leads to many free parameters which defeats the purpose. A model which lies somewhere between these two extremes based on  $D_6 \times D_6$  was presented recently in [15]. It is one of the purposes of this paper to present an alternative model based on group  $Q_6 \times Q_6$ . This model utilizes the advantage of the  $Q_{2n}$  groups over  $D_{2n}$ 's: there are more possibilities for dividing the three generations among the irreps so that the anomaly cancellation requirements are trivially satisfied. We will also show that the smaller number of symmetry breaking possibilities is enough to build a viable model.

Let us first show why a single  $Q_6$  is not sufficient, unless a fine tuning is imposed. Assuming there are no extra generations, we have three possibilities for the assignment of fermions:  $1+1+1$ ,  $1+2_1$  and  $1'+2_2$ . The left handed quark doublets  $Q_L^i$  cannot be assigned to  $1+1+1$  since then, if there is no fine tuning, the diagonalization of mass matrices will require large left handed mixings and therefore lead to large nondiagonal entries in the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Suppose next that  $Q_L^i$  are in  $1+2_1$ . Then the right handed up quarks  $u_R^i$  cannot be in  $1+2_1$  since then all the up quark masses will be of the same order because the product  $2_1 \times 2_1$  contains a singlet. If  $u_R^i$  is in  $1'+2_2$  then masses of the up and charm quark transform as one of the representations  $1''+1''' + 2_1$ . However, this would predict that the up and charm masses are equal or at least of the same order. Namely the VEVs of  $1''$  or  $1'''$  couple with equal strength to both up and charm,



and similarly for the VEV of a  $2_2$  (unless one fine tunes one of the components to have zero VEV). Thus we can assign the up quarks only to  $1 + 1 + 1$ . However, the down quarks then cannot be in  $1 + 1 + 1$  since that would predict similar ratios of masses in both up and down sectors in the absence of fine tuning. Down quarks cannot be assigned to  $1' + 2_2$  either because of the same reason as the up quarks cannot, as explained above. Finally, assignment of down quarks to  $1 + 2_1$  would give all the three down quarks the mass of the same order. A similar discussion can be given for the last case when the  $Q_L^i$  are assigned to  $1' + 2_2$ .

Therefore, we conclude that in the absence of fine tuning, a single  $Q_6$  group is not enough to explain the quark masses and mixings, let alone the lepton sector. We are therefore led to a bigger group, possibly a product of  $Q_6$  with another group. As a simple example we look at the direct product of two  $Q_6$  groups.

The flavor model is based on the group  $Q_6 \times Q_6$  where the Standard Model fermions (and right handed neutrinos) are assigned into the following representations:  $Q_L^i = (1 + 2_1, 1)$ ,  $u_R^i = (1, 1 + 2_1)$ ,  $d_R^i = (1, 1' + 2_2)$ ,  $L_L^i = (1' + 2_2, 1)$ ,  $e_R^i = (1, 1 + 2_1)$  and  $\nu_R^i = (1, 1' + 2_2)$ .

There are no new fermions added to the SM field content, except for the right handed neutrinos  $\nu^c$ , which we assume get a large mass. We notice that, if the  $Q_6 \times Q_6$  originated from  $SU(2) \times SU(2)$ , the above assignment satisfies Witten's requirement [21] of even number of doublets in each  $SU(2)$  (corresponding to number of  $2_1$ s in each  $Q_6$ ).

We assume the global symmetry  $Q_6 \times Q_6$  (which is a subgroup of a gauged  $SU(2) \times SU(2)$  symmetry broken at a high scale by a pair of **7**'s) itself gets broken at some very high scale by a set of scalar fields, *flavons*, possibly through the Froggatt-Nielsen mechanism [22]. The Standard Model Higgs is neutral under the  $Q_6 \times Q_6$ . We assume that all of the flavons with "minimal"  $Q_6$  numbers exist:  $(1', 1), (1, 1'), (2_2, 1), (1, 2_2), (2_1, 1)$  and  $(1, 2_1)$ , but no combined representations such as  $(1', 1')$ .

We assume the following pattern of symmetry breaking

$$Q_6 \times Q_6 \xrightarrow{\langle 1' \rangle} Z_6 \times Z_6 \xrightarrow{\langle 2_2 \rangle} Z_2 \times Z_2 \xrightarrow{\langle 2_1 \rangle} \text{nothing} \quad (16)$$

The important thing to notice is that although no remnant of the  $Q_6$  is left unbroken after  $2_1$  gets a VEV, the first generation fermions will still be massless at the tree level. This is because the Yukawa couplings coming from the product of two spinorial representations carry an additional  $Z_2$  symmetry, so that only some linear combinations of fermion fields get a mass. This symmetry is of course broken in other sectors of the theory, so that the first generation masses will be generated through loop corrections. Thus the first generation masses will in general be suppressed by some small loop factor, and the precise predictions here will depend crucially on the flavon sector. In the next Section we concentrate on the masses and mixings of the heaviest two generations while we only make general comments about the first generation and leave more detailed analysis for a future publication.

## V. FERMION MASSES AND MIXINGS IN THE MODEL

We assume that the flavon VEVs satisfy the following hierarchy:

$$\langle 1' \rangle \equiv v' \approx \langle 2_2 \rangle \equiv (v'\sigma, -v'\sigma) \gg \langle 2_1 \rangle \equiv (\epsilon_1 M, \epsilon_2 M) \quad (17)$$

with  $\sigma \sim 1, \epsilon_i \ll v'/M \ll 1$ .

We will assume that the VEVs in both  $Q_6$ s are comparable for corresponding fields (*i.e.*  $(2_2, 1) \approx (1, 2_2)$ ). This assumption is natural if one assumes a “left-right” symmetric theory ( $Q_6 \times Q_6$  or the primordial  $SU(2) \times SU(2)$ )<sup>3</sup>.

In the limit  $\epsilon_i = 0$  we get

$$m^u \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad m^d \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k^d \sigma & -k^d \sigma & 1 \end{pmatrix}, \quad m^e \sim \begin{pmatrix} 0 & 0 & h^e \sigma \\ 0 & 0 & -h^e \sigma \\ 0 & 0 & 1 \end{pmatrix},$$

$$m^\nu \sim \begin{pmatrix} g^\nu \sigma^2 & -g^\nu \sigma^2 & h^\nu \sigma \\ -g^\nu \sigma^2 & g^\nu \sigma^2 & -h^\nu \sigma \\ k^\nu \sigma & -k^\nu \sigma & 1 \end{pmatrix}, \quad M_{\mathbf{N}} \sim \begin{pmatrix} 0 & g_N & h_N \lambda \\ g_N & 0 & h_N \lambda \\ h_N \lambda & h_N \lambda & 1 \end{pmatrix} \quad (18)$$

where  $\lambda \equiv \sigma v'/M \ll 1$ , and  $g, h, k$  are some numbers of order one. In the quark sector at this level only the top and bottom get their mass, while the CKM matrix is equal to the unit matrix. Notice the large mixing of the second and third generations in the down quark and charged lepton sectors, which is similar to SU(5) scenarios [8,24].

The ratio of the bottom to top mass is given by  $m_b/m_t = (\frac{v'}{M})(\frac{v}{\bar{v}})$ , where  $v$  and  $\bar{v}$  are the VEVs of the Standard Model Higgs doublets that couple to down and up sector respectively. Since we do not know the ratio of the two electroweak Higgs VEVs, we can conclude only that  $\frac{v'}{M} = 1 \sim 1/60$ . Similarly in the charged lepton sector only the tau lepton gets its mass at this level, and it is of the order of the b quark mass.

The last two matrices are the Dirac neutrino and the right handed neutrino mass matrices and we need to find the mass matrix of the light neutrinos. The inverse of the Majorana mass matrix can be computed exactly

$$M_{\mathbf{N}}^{-1} \sim \begin{pmatrix} \left(\frac{h_N \lambda}{g_N}\right)^2 & \frac{1}{fg_N} + \left(\frac{h_N \lambda}{g_N}\right)^2 & -\frac{h_N \lambda}{g_N} \\ \frac{1}{fg_N} + \left(\frac{h_N \lambda}{g_N}\right)^2 & \left(\frac{h_N \lambda}{g_N}\right)^2 & -\frac{h_N \lambda}{g_N} \\ -\frac{h_N \lambda}{g_N} & -\frac{h_N \lambda}{g_N} & 1 \end{pmatrix} \quad (19)$$

where  $f \equiv 1/(1 - 2(h_N \lambda)^2/g_N^2)$  and an overall factor of  $f$  has been absorbed in the overall mass scale. Then we find that the mass matrix for the light neutrino masses reduces to the following form

$$m_{light}^\nu = -m^\nu M_{\mathbf{N}}^{-1} m^{\nu T} \sim \begin{pmatrix} g\sigma^2 & -g\sigma^2 & h\sigma \\ -g\sigma^2 & g\sigma^2 & -h\sigma \\ h\sigma & -h\sigma & 1 + k\sigma^2 \end{pmatrix} \quad (20)$$

where as before  $g, h, k$  are some numbers of order one. Notice that the orders of magnitude in the light neutrino matrix do not depend on the details of the Majorana mass matrix, a feature similar to those of abelian flavor theories with positive charges [25].

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<sup>3</sup>In the usual left-right symmetric theory the parameters were chosen so that the left-right symmetry is broken maximally [23]. Here we assume that the parameters are such that the VEVs do not break the symmetry between the sectors, so that their VEVs are equal in both sectors.

A 45° rotation of the first two lefthanded leptons brings us to the following basis for the charged leptons and the light neutrinos

$$m^e \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2}h^e\sigma \\ 0 & 0 & 1 \end{pmatrix}, \quad m_{light}^\nu \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2g\sigma^2 & -\sqrt{2}h\sigma \\ 0 & -\sqrt{2}h\sigma & 1+k\sigma^2 \end{pmatrix} \quad (21)$$

leading to two neutrino masses of the same order and one zero mass neutrino. These matrices are next diagonalized leading to the maximal mixing in the  $\nu_\mu$ - $\nu_\tau$  sector

$$\tan \theta_{\mu\tau} \sim \sigma. \quad (22)$$

This is consistent with the results of SuperKamiokande.

Second generation fermion masses and mixing angles will be generated by factors of order  $\epsilon$ . When the VEV of  $2_1$  is turned on (i.e.  $\epsilon_i \neq 0$ ), it produces the following order of magnitude entries for the mass matrices:

$$\begin{aligned} m^u &\sim \begin{pmatrix} g^u \epsilon_2^2 & -g^u \epsilon_1 \epsilon_2 & h^u \epsilon_2 \\ -g^u \epsilon_1 \epsilon_2 & g^u \epsilon_1^2 & -h^u \epsilon_1 \\ k^u \epsilon_2 & -k^u \epsilon_1 & 1 \end{pmatrix}, \\ m^d &\sim \begin{pmatrix} g^d \sigma \epsilon_2 & -g^d \sigma \epsilon_2 & h^d \epsilon_2 \\ -g^d \sigma \epsilon_1 & g^d \sigma \epsilon_1 & -h^d \epsilon_1 \\ k^d \sigma & -k^d \sigma & 1 \end{pmatrix}, \\ m^e &\sim \begin{pmatrix} g^e \sigma \epsilon_2 & -g^e \sigma \epsilon_1 & h^e \sigma \\ -g^e \sigma \epsilon_2 & g^e \sigma \epsilon_1 & -h^e \sigma \\ k^e \epsilon_2 & -k^e \epsilon_1 & 1 \end{pmatrix} \end{aligned} \quad (23)$$

where  $g^i, h^i, k^i$  are some unknown couplings of order 1.

It is important to remember that all three matrices have still one zero eigenvalue because of the accidental symmetry of the Yukawa couplings. However, this symmetry is broken in other sectors of the theory, and so there will be small loop corrections to the Yukawa matrices eventually generating the lightest mass eigenvalues. We will comment on this in a minute.

The masses and mixings are to the leading order

$$\begin{aligned} \frac{m_\mu}{m_\tau} &\approx (g^e - h^e k^e) \sigma \sqrt{\epsilon_1^2 + \epsilon_2^2} \sim \sigma \epsilon, \\ \frac{m_s}{m_b} &\approx (g^d - h^d k^d) \sigma \sqrt{\epsilon_1^2 + \epsilon_2^2} \sim \sigma \epsilon, \\ \frac{m_c}{m_t} &\approx (g^u - h^u k^u) (\epsilon_1^2 + \epsilon_2^2) \sim \sigma \epsilon^2, \\ \theta_{\mu\tau} &\sim O(1), \quad \theta_{cb} \sim \epsilon \end{aligned} \quad (24)$$

More quantitative relations are obtained if one makes further assumptions that the order one constants  $g^i, h^i, k^i$  perhaps vary by a factor of two or so. Also, the VEVs in the two different  $Q_6$ s do not have to be equal in both sectors. For example, if the VEVs in  $\langle (2_1, 1) \rangle$  and  $\langle (1, 2_1) \rangle$  differ by a factor of 3 we get relations

$$\begin{aligned} \frac{m_\mu}{m_\tau} &\sim \sigma \epsilon, \quad \frac{m_s}{m_b} \sim \frac{\sigma \epsilon}{3}, \quad \frac{m_c}{m_t} \sim \frac{\epsilon^2}{3} \\ \theta_{\mu\tau} &\sim O(1), \quad \theta_{cb} \sim \epsilon/3 \approx \sqrt{\frac{m_c}{3m_t}} \end{aligned} \quad (25)$$

The relation between the masses of charged leptons and down quarks is the usual Georgi-Jarlskog relation. However, notice the additional factor of  $\sqrt{3}$  in the  $\theta_{cb}$  expression which makes it in better agreement with data. However, one has to stress again the uncertainty with factors of order one at this level of predictivity.

Let us remark on the size of parameters  $\epsilon \sim < 2_1 > / M$ , which characterizes the size of the VEVs of the spinorial doublets  $2_1$ , and  $v'/M$ , which characterizes the size of the VEVs of the singlets  $1'$  and doublets  $2_2$ . Relations (24) ( or (25) ) tell us that  $\epsilon$  is of order  $1/20$  or so, using our assumption  $\sigma \sim O(1)$ . This then tells us that  $v'/M$  is somewhere between  $1$  and  $1/20$ , and does not have to be small. However, since we want to control the size of the first generation masses (see next paragraph), we do not want this number to be too close to  $1$ .

Finally, let us comment on the first generation masses and mixings. Such terms can be generated at the tree level only by higher dimensional terms and will be suppressed. Also, the first generation masses and mixings are not strictly zero at the loop level either since the flavor symmetries of  $Q_6$  and its subgroups are completely broken by the VEVs of the  $2_1$  doublets. This will generate masses at the loop level, which will however be suppressed by the loop factors. This will be for example notable in the down and charged lepton sector where the VEVs of the components of the  $2_2$  doublet will not be in general exactly equal to the negative of each other. Thus, we expect all first generation fermion masses, including light neutrinos, to be smaller compared to second and third generation masses. We also expect suppressed mixing angles involving first generation fermions, including neutrinos, which seems to favor the small angle MSW solution. We leave more precise statements concerning the first generation for a future publication.

## VI. CONCLUSIONS

Nonabelian discrete groups are an attractive tool to describe fermion masses and mixings. They have nonsinglet representations which seem particularly suitable for distinguishing the lighter generations from the heavier ones. Also, they do not suffer from the extra constraints a continuous group must obey, *e.g.* limits on extra particles (gauge or pseudogoldstone, depending on the continuous group being local or global). One of the simplest groups are the nonabelian discrete subgroups of  $SO(3)$  and  $SU(2)$ , the so called dihedral groups  $D_n$  and dicyclic groups  $Q_{2n}$ , which both have only singlet and doublet representations. Such groups have also a rich structure of subgroups which makes it possible to use the hierarchies in the symmetry breaking chain as the origin of hierarchies in fermion masses and mixings. Equations (12) and (15) summarize which VEVs break dihedral and dicyclic groups to which subgroups.

As an example, we constructed a simple model based on the group  $Q_6 \times Q_6$ . The model reproduces the masses and mixings of all quarks and leptons, including neutrinos. It has a large mixing angle in the  $\mu - \tau$  neutrino sector, while keeping a small quark mixing in the bottom - charm sector. The reason is similar to the one found in the literature based on the  $SU(5)$  group [8]: the large *left* handed mixing angle in the lepton sector corresponds to a the large unphysical *right* handed in the down quark sector. The large mixing is also responsible

for the different hierarchies of the two heaviest families in the up and down sector, and can be summarized as the order of magnitude relation

$$\frac{m_s}{m_b} \sim \tan(\theta_{\mu\tau}) \sqrt{\frac{m_c}{m_t}}. \quad (26)$$

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**Note Added** Another paper using nonabelian discrete symmetry to describe quark and lepton masses appeared recently, using the double tetrahedral group [26].

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