LOCALIZED ENERGY ESTIMATES FOR WAVE EQUATIONS ON (1 + 4)-DIMENSIONAL MYERS-PERRY SPACE-TIMES

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ABSTRACT. Localized energy estimates for the wave equation have been increasingly used to prove various other dispersive estimates. This article focuses on proving such localized energy estimates on (1+4)-dimensional Myers-Perry black hole backgrounds with small angular momenta. The Myers-Perry space-times are generalizations of higher dimensional Kerr backgrounds where additional planes of rotation are availabile while still maintaining axial symmetry. Once it is determined that all trapped geodesics have constant r, the method developed by Tataru and the fourth author, which perturbs off of the Schwarzschild case by using a pseudodifferential multiplier, can be adapted.

1. INTRODUCTION

This article focuses on proving localized energy estimates for wave equations on the family of $(1 + 4)$-dimensional Myers-Perry spacetimes with small angular momenta. The latter are higher dimensional generalizations of the Kerr family of solutions to the Einstein vacuum equations. They are axisymmetric, though with the additional dimension, an extra nonzero angular momentum may be permitted, which further complicates the behavior of the null geodesics. On, e.g., the Schwarzschild and Kerr space-times, such localized energy estimates have been essential to proving other types of dispersive estimates such as Strichartz estimates [45], [69] and pointwise decay estimates [52], [67], [20, 22], [43, 44].

We first describe the localized energy estimates on the Minkowski space-time. To begin, we define

$$\|u\|_{LE_M} = \sup_{j \in \mathbb{Z}} 2^{-\frac{j}{2}} \|u\|_{L^2_{t,x}(\mathbb{R} \times \{|x| \in [2^{j-1}, 2^j]\})}$$

and

$$\|u\|_{LE_M^1} = \|u'\|_{LE_M} + \||x|^{-3/2}u\|_{L^2_{t,x}},$$

where $u' = (\partial_t u, \nabla_x u)$ is the space-time gradient. To measure an inhomogeneous term, we shall use a dual norm

$$\|f\|_{LE_M^*} = \sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \|f\|_{L^2_{t,x}(\mathbb{R} \times \{|x| \in [2^{k-1}, 2^k]\})}.$$

Then for $\Box = -\partial_t^2 + \Delta$, the localized energy estimate for the wave equation states that

$$(1.1) \qquad \|u'\|_{L^\infty_t L^2_x} + \|u\|_{LE_M^1} \lesssim \|u'(0, \cdot)\|_{L^2} + \|\Box u\|_{LE_M^* + L^1_t L^2_x}$$

provided $n \geq 4$. A similar estimate holds when $n = 3$ but the second term in the definition of the LE_M^1 norm must be replaced by, e.g., $\||x|^{-1}u\|_{LE_M}$. Estimates of this type first appeared in [53]. From this original estimate, it is clear that the l_j^∞ norm in the definition of $\| \cdot \|_{LE_M^1}$ can be replaced by square summability, which is stronger, on the angular derivative term and, when $n \geq 4$, on the lower order term. Generalizations of the original estimate have appeared in, e.g., [65], [38], [61], [37], [46], [14] [34], [64], [48, 49], [50, 51] and have ultimately led to (1.1). These estimates have been applied to problems in scattering theory, used to prove long time existence for nonlinear wave equations, and used to prove other dispersive estimates by handling the behavior of the solution in a compact region. This class of estimate is known to be fairly robust and similar estimates are known for small, possibly time dependent, long-range perturbations of the Minkowski metric [1], [48, 49], [50, 51] and for time independent, nontrapping, asymptotically flat perturbations [12], [13], [63].

While proofs that rely on Huygens' principle in odd spatial dimensions (see, e.g., [37]), Plancherel's theorem (see, e.g., [61], [46]), and resolvent estimates (see, e.g., [12], [13]) exist, the most robust argument and the one that is most applicable to the current setting relies, in essence, on a positive commutator argument. See, e.g., [64], [48]. For a choice of function $f(r)$, one multiplies the equation $\Box u$ by $f(r)\partial_r u + \frac{n-1}{2}\frac{f(r)}{r}u$ and integrates by parts. The original estimate of [53], then, corresponds to the choice $f(r) \equiv 1$.

Related procedures have been carried out on the $(1+3)$-dimensional Schwarzschild space-times, beginning with [40]. The Schwarzschild space-times are the class of spherically symmetric, static solutions to Einstein's equations and represent the simplest black hole solutions. When the analysis is carried out here, the phenomenon of trapping is encountered. The Schwarzschild space-time contains null geodesics that stay in a compact region for all times. Specifically trapping occurs on the event horizon $r = 2M$ and on the photon sphere $r = 3M$.

Trapping is a known barrier to localized energy estimates. See, e.g., [57] and [25] (the latter regards the related local smoothing estimate for the Schrödinger equation). And, thus, estimates on Schwarzschild are expected to contain losses when compared to (1.1); recently this was shown rigurously in [59]. By modifying the choice of $f(r)$ and, in particular, requiring vanishing on the trapped sets, localized energy estimates (with losses) were obtained for the wave equation on Schwarzschild space-times in [7, 8, 10, 9], [19, 20], [45].

While trapping occurs on the event horizon $r = 2M$, it was discovered [45] that, by quantifying the red-shift effect in the style of [19], this loss in the localized energy estimates can be negated. On the other hand, the loss at the photon sphere typically arises as a quadratic vanishing of the coefficient on the angular derivatives and the time derivative, though with some, e.g., microlocal analysis this can be improved to merely a logarithmic loss. See [45] and the preceding works [9, 10].

The Kerr family of solutions to Einstein's equations correspond to axially symmetric, rotating black holes. For small angular momentums, the Kerr metrics are small short-range perturbations of the Schwarzschild metric. The structure of the trapping, however, is much more complicated. While still of codimension two in phase space, the location of the trapping may no longer be described merely in physical space. As such, it is provably

impossible [2] to obtain such a localized energy estimate using a first order differential multiplier.

Despite this, three approaches have been developed for proving such on Kerr backgrounds with small angular momenta. In [3], [21, 23, 24], and [68] respectively, the authors proved certain versions of localized energy estimates using somewhat different approaches: [3] uses the existence of a nontrivial Killing tensor (due to [16]), [21, 23, 24] rely on frequency decomposition, and [68] is based on using a pseudodifferential multiplier. The three approaches are intimately related [15]. The current study, however, relies most heavily on the ideas from [68].

Though we shall currently focus only on the small angular momenta regime, we mention some recent related works [24], [4, 5, 6] that permit large momenta. We also mention here a few places [45], [69], [67], [52], [42], [20, 22], [43, 44] where example applications of these estimates on black hole backgrounds can be found. In particular, in these works, the localized energy estimates are used to prove other measures of dispersion. They are used to obtain information about solutions in a compact region where the geometry is most difficult and are combined with known Minkowski estimates near infinity where the metric may be viewed as a small perturbation of the Minkowski metric. Such arguments are akin to those appearing much earlier in, e.g., [36] and to those used in, e.g., [61], [13], [37], [46, 47], [48, 49], [51] to prove exterior domain analogs of well-known boundaryless estimates.

Higher dimensional black hole space-times have been derived, based partially in interest coming from string theory. Our motivation for studying this problem comes from the connections to the problem of black hole stability, which is described e.g. in [21]. It is well-known that classes of quasilinear wave equations have global small data solutions in $(1 + n)$-dimensions for $n \geq 4$, but in order to guarantee such for $n = 3$, one must take advantage of some nonlinear structure. See, e.g., [62]. As such, one might consider such stability problems in higher dimensions first where the nonlinear structure may be less essential. The reader can compare, e.g., the proof of the stability of Minkowski space [18] with that in higher dimensions [17].

While the extra dimensions permit black holes with different topologies (see e.g. [26, 27], [31], [32], [33], [35], [39], [58], and the references therein), we shall study only those which are most closely related to the standard Kerr solutions. In particular, a higher dimensional analog of the Schwarzschild space times was derived in [66], and the analogs of Kerr are from [54]. In higher dimensions, as there are more directions in which one may rotate and yet maintain axial symmetry, a broader family of solutions, called Myers-Perry space-times, may be examined.

Localized energy estimates were proved independently on the (hyperspherical) $(1+n)$-dimensional Schwarzschild space-times of [66] in [41] and [60]. Rather than examining generic dimension, we examine specifically $(1+4)$-dimensions. In particular, as in [68], our proof will rely heavily on the integrability of the geodesic flow. This is known for Myers-Perry black holes with two distinct angular momenta [56], [70], but, to our knowledge, not generically. In $(1 + 4)$-dimensions, however, this covers the entire family. Our main theorem, Theorem 4.1, states that for small angular momenta there is a localized energy estimate for the wave equation on such $(1 + 4)$-dimensional Myers-Perry solutions.

This paper is structured as follows. The next section contains a review of the localized energy estimates on $(1 + 4)$-dimensional Schwarzschild space-times. In what follows, we shall perturb off of this result. We modify the existing estimate of [41] by incorporating the redshift effect and by smoothing out the multiplier near the photon sphere, which will simplify the microlocal analysis that comes later. In the third section, we present the Myers-Perry metric and its most relevant properties. We, there, analyze the trapped geodesics and, in particular, prove that they each lie on surfaces of constant r. In the fourth section, we define our local energy spaces, and in the final section, we prove the main estimate, Theorem 4.1.

2. Localized energy on $(1 + 4)$-dimensional Schwarzschild

Here we shall rely on the approach of [45] and [41]. Estimates of a similar form were also proved in [60]. Akin to the relationship between [45] and [68], the estimate of [41] plays a key role as we assume that $a, b \ll 1$ and argue perturbatively.

In the sequel, we shall be employing pseudodifferential multipliers. So we first seek to modify the multiplier of [41], which is only C^2 at the photon sphere, to make it smooth in a neighborhood of the photon sphere. Moreover, as in [45], we shall include the arguments of [19] that take advantage of the red-shift effect and allow us to prove an estimate that does not degenerate at the event horizon. This was previously done in [60], and we include it here for completeness.

We assume a basic familiarity with [41] and shall not reproduce every calculation here.

The metric for the $(1+n)$-dimensional hyperspherical Schwarzschild space-time, which was discovered in [66], is

$$ds^2 = -\Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)dt^2 + \Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)^{-1} dr^2 + r^2 d\omega^2,$$

where $r = r_s$ represents the event horizon and $d\omega^2$ is the line element on the $(n - 1)$-dimensional unit sphere. Here $d = n - 3$, which allows us to quickly compare to the $n = 3$ case. We define $r_{ps} = \Big(\frac{d+3}{2}\Big)^{\frac{1}{d+1}} r_s$ as $r = r_{ps}$ is the location of the photon sphere. $K = \partial_t$ is the Killing vector field that is timelike in the domain of outer communication. It extends into the interior of the black hole, becoming null on the event horizon and spacelike in the interior.

As in the more typical $(1+3)$-dimensional case, the singularity at $r = r_s$ is a coordinate singularity. Setting

$$r^* = \int_{r_{ps}}^{r} \Big(1 - \Big(\frac{r_s}{s}\Big)^{d+1}\Big) ds$$

and $v = t + r^*$, the metric becomes

$$ds^2 = -\Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)dv^2 + 2\, dv\, dr + r^2 d\omega^2.$$

While these coordinates are nonsingular at the event horizon, surfaces of constant v are null and not convenient for posing Cauchy problems for the wave equation. To this end, we introduce, as in [45],

$$\tilde{v} = v - \mu(r), \quad \mu \in C^\infty$$

subject to

- $\mu(r) \geq r^*$ for $r > r_s$ and $\mu(r) = r^*$ for $r > (r_s + r_{ps})/2$,
- $\mu'(r) > 0$ and $2 - \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)\mu'(r) > 0$.

The metric, in the $(\tilde{v}, r, \omega)$ coordinates, then becomes

$$(2.1) \quad ds^2 = -\left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right) d\tilde{v}^2 + 2\left(1 - \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)\mu'(r)\right) d\tilde{v}\, dr$$
$$+ \left(2\mu'(r) - \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)(\mu'(r))^2\right) dr^2 + r^2\, d\omega^2.$$

The first condition on μ shows that these coordinates coincide with the Schwarzschild coordinates away from the event horizon. The second condition guarantees that $\tilde{v} =$constant slices are space-like.

For a choice $0 < r_e < r_s$, we seek to solve the wave equation

$$(2.2) \qquad\qquad\qquad \Box_S u = f,$$

where $\Box_S = \nabla^\alpha \partial_\alpha$ when the background metric is given by (2.1), in

$$\mathcal{M}_R = \{\tilde{v} \geq 0, r \geq r_e\}$$

with initial data on

$$\Sigma_R^- = \mathcal{M}_R \cap \{\tilde{v} = 0\},$$

which is space-like. We use

$$\Sigma_R^+ = \mathcal{M}_R \cap \{r = r_e\}$$

to denote the lateral boundary.

We shall work with a nondegenerate energy, rather than the conserved energy associated to the Killing vector field ∂_t, as was used in [41]. This will help us, as in [45], to cleanly utilize the red-shift effect. We define the energy on an arbitrary $\tilde{v} = \tilde{v}_0$ slice to be

$$E[u](\tilde{v}_0) = \int_{\mathcal{M}_R \cap \{\tilde{v}=\tilde{v}_0\}} \left((\partial_r u)^2 + (\partial_{\tilde{v}} u)^2 + |\overline{\nabla} u|^2\right) r^{d+2}\, dr\, d\omega.$$

The initial energy $E[u](\Sigma_R^-)$ and the outgoing energy $E[u](\Sigma_R^+)$ are defined as

$$E[u](\Sigma_R^-) = E[u](0), \quad E[u](\Sigma_R^+) = \int_{\Sigma_R^+} \left((\partial_r u)^2 + (\partial_{\tilde{v}} u)^2 + |\overline{\nabla} u|^2\right) r_e^{d+2}\, d\tilde{v}\, d\omega.$$

We modify the localized energy spaces to reflect the quadratic loss at the photon sphere. To do so, we set

$$\|u\|_{LE_S} = \left\|\left(\frac{r - r_{ps}}{r}\right)u\right\|_{LE_M},$$

and

$$\|u\|_{LE_S^1} = \|\partial_r u\|_{LE_M} + \|\partial_{\tilde{v}} u\|_{LE_S} + \|\overline{\nabla} u\|_{LE_S} + \|r^{-3/2} u\|_{L^2(\mathcal{M}_R)}.$$

Note that the loss due to trapping only occurs on the $\partial_{\tilde{v}}$ and $\overline{\nabla}$ components. We analogously set

$$\|f\|_{LE_S^*} = \left\|\left(\frac{r - r_{ps}}{r}\right)^{-1} f\right\|_{LE_M^*}.$$

Then, we have the following theorem, which is largely from [41] and [60].

Theorem 2.1. *For u solving (2.2), we have*

$$E[u](\Sigma_R^+) + \sup_{\tilde{v}} E[u](\tilde{v}) + \|u\|_{LE_S^1}^2 \lesssim E[u](\Sigma_R^-) + \|f\|_{LE_S^*}^2.$$

For any Lorentzian metric g, the energy-momentum tensor for the wave equation is given by

$$Q_{\alpha\beta}[u] = \partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta} \partial^\gamma u \partial_\gamma u,$$

satisfying $\nabla^\alpha Q_{\alpha\beta}[u] = \partial_\beta u \Box_g u$. For a C^1 vector field X, a scalar field q, and a 1-form m, we set

$$P_\alpha[u, X, q, m] = Q_{\alpha\beta}[u] X^\beta + q u \partial_\alpha u - \frac{1}{2}(\partial_\alpha q) u^2 + \frac{1}{2} m_\alpha u^2.$$

It, thus, follows that

(2.3) $$\nabla^\alpha P_\alpha[u, X, q, m] = \Box_g u \Big(X u + q u \Big) + Q[u, X, q, m],$$

where

$$Q[u, X, q, m] = Q_{\alpha\beta}[u] \pi^{\alpha\beta} + q \partial^\alpha u \partial_\alpha u + m_\alpha u \partial^\alpha u + \frac{1}{2}(\nabla^\alpha m_\alpha - \nabla^\alpha \partial_\alpha q) u^2.$$

Here

$$\pi_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha)$$

denotes the deformation tensor of X.

The theorem will follow from a proper choice of X, q, and m. Indeed, we will construct such satisfying the following lemma.

Lemma 2.2. *There exist X, q, and m that are smooth, spherically symmetric, and K-invariant in $r \geq r_s$. Moreover, X is bounded in $(\tilde{v}, r, \omega)$ coordinates and $X(dr)(r_s) < 0$; $|q(r)| \lesssim r^{-1}$ and $|q'(r)| \lesssim r^{-2}$; m has compact support in r and $\langle m, dr \rangle(r_s) > 0$; and*

$$Q[u, X, q, m] \gtrsim r^{-(d+3)}(\partial_r u)^2 + \Big(\frac{r - r_{ps}}{r} \Big)^2 (r^{-(d+3)}(\partial_{\tilde{v}} u)^2 + r^{-1} |\not\nabla u|^2) + r^{-3} u^2.$$

Proof. We begin by working in the (t, r, ω) coordinates and set

$$X = X_1 + \delta X_2, \quad \delta \ll 1$$

with

$$X_1 = f(r)\Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big) \partial_r, \quad X_2 = -b(r)\Big(\partial_r - \Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)^{-1} \partial_t\Big).$$

For smooth choices of f and b, these are smooth on $\mathcal{M}_R$ in the (v, r, ω) coordinates. Indeed,

$$X_1 = f(r)\Big(\Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big) \partial_r + \partial_v\Big), \quad X_2 = -b(r) \partial_r.$$

The choice of X_1 is inspired by the preceding work [41], though to ease calculations in the sequel, we shall mollify the original multiplier near the photon sphere. The vector field X_2 is inspired by [19]. See [60] for the corresponding argument in generic dimension. This component allows us to take advantage of the red shift effect.

To begin, for $g(r) = \frac{r^{d+2}-r_{ps}^{d+2}}{r^{d+2}}$ and $h(r) = \ln\left(\frac{r^{d+1}-r_s^{d+1}}{\frac{d+1}{2}r_s^{d+1}}\right)$, we set

$$f_1(r) = g(r) + \frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}a(h(r)),$$

where

$$a(x) = \begin{cases} x, & x \leq 0, \\ x - \frac{2}{3\alpha^2}x^3 + \frac{1}{5\alpha^4}x^5, & 0 \leq x \leq \alpha, \\ \frac{8\alpha}{15}, & x \geq \alpha. \end{cases}$$

Here $\alpha = 5-$. The function a serves to smooth out the logarithm near infinity to insure that f_1 is everywhere increasing. For later purposes, we note that a''' is everywhere non-positive. With the further choice of

$$q_1[f_1] = \frac{1}{2}\left[\left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)r^{-(d+2)}\partial_r(r^{d+2}f_1(r))\right],$$

it was shown in [41] that

$$Q[u, X_1[f_1], q_1, 0] \gtrsim \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)r^{-(d+3)}(\partial_r u)^2 + \frac{1}{r}\left(\frac{r-r_{ps}}{r}\right)^2|\nabla u|^2$$
$$+ r^{-3}\left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)u^2$$

where $X_1[f_1] = f_1\left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)\partial_r$.

We would like to replace f_1 by a similar multiplier F that is smooth at the photon sphere. In order for $Q[u, X_1[f_1], q_1, 0]$ to still satisfy the inequality above, it is sufficient to pick F to be increasing, bounded, and $l(F) > 0^1$ where

$$l(F) = -\frac{1}{4}r^{-(d+2)}\partial_r\left[\left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)r^{d+2}\partial_r\left\{\left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)r^{-(d+2)}\partial_r(F(r)r^{d+2})\right\}\right].$$

Let

$$F(r) = f_1(r) + \frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}\chi(r)\left((\psi_N * a)(h(r)) - Q_2(r) - a(h(r))\right)$$
$$= g(r) + \frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}\left[(1 - \chi)(r)a(h(r)) + \chi(r)\left((\psi_N * a)(h(r)) - Q_2(r)\right)\right].$$

Here ψ_N is a standard mollifier, which is an approximation of the identity for large N. See, e.g., [28]. Q_2 is a second order polynomial (with very small coefficients) that is selected so that all derivatives up to order 2 of $(\psi_n * a)(h(r)) - Q_2(r)$ coincide with those of $a(h(r))$ at $r = r_{ps}$. The function $\chi(r)$ is a smooth cutoff that is supported in a small neighborhood of r_{ps} and is the identity on a smaller neighborhood of r_{ps}. Since $a \in C^2$ on the support of χ, given any $\varepsilon > 0$, we can choose N sufficiently large so that

$$|\partial^\alpha(F(r) - f_1(r))| < \varepsilon, \quad 0 \leq \alpha \leq 2.$$

We proceed to show that F preserves the desired properties of f_1 on the support of χ if N is chosen sufficiently large. By construction, we have that $F(r_{ps}) = f_1(r_{ps}) = 0$. By

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[1]$l(f_1)$ is the coefficient of u^2 in $Q[u, X_1[f_1], q_1, 0]$

choosing N sufficiently large so that

$$|\partial(F(r) - f_1(r))| < \frac{1}{2} \inf_{r \in \mathrm{supp}\, \chi} f_1'(r),$$

it follows easily that $F'(r) > 0$.

It finally remains to show that $l(F) > 0$. We write

(2.4)

$$l(F) = l(f_1) + \left[l(F - f_1) + \frac{1}{4} \frac{d+2}{d+3} \frac{r_{ps} r_s^{d+1}}{r^{d+2}} \chi(r) \left(1 - \left(\frac{r_s}{r} \right)^{d+1} \right)^2 (\psi_N * a)'''(h(r))(h'(r))^3 \right]$$

$$- \frac{1}{4} \frac{d+2}{d+3} \frac{r_{ps} r_s^{d+1}}{r^{d+2}} \chi(r) \left(1 - \left(\frac{r_s}{r} \right)^{d+1} \right)^2 (\psi_N * a''')(h(r))(h'(r))^3.$$

As

$$\left[l(F - f_1) + \frac{1}{4} \frac{d+2}{d+3} \frac{r_{ps} r_s^{d+1}}{r^{d+2}} \chi(r) \left(1 - \left(\frac{r_s}{r} \right)^{d+1} \right)^2 (\psi_N * a)'''(h(r))(h'(r))^3 \right]$$

only contains derivatives up to order 2 of the mollified a, it can be made uniformly small, and an argument similar to that employed to show that $F'(r) > 0$ can be used to show $l(F) > 0$ provided that

$$- \frac{1}{4} \frac{d+2}{d+3} \frac{r_{ps} r_s^{d+1}}{r^{d+2}} \chi(r) \left(1 - \left(\frac{r_s}{r} \right)^{d+1} \right)^2 (\psi_N * a''')(h(r))(h'(r))^3 > 0,$$

which follows easily from the non-positivity of a''' on the relevant region.

This gives that

$$Q[u, X_1[F], q_1[F], 0] \gtrsim \left(1 - \left(\frac{r_s}{r} \right)^{d+1} \right) r^{-(d+3)} (\partial_r u)^2 + \frac{1}{r} \left(\frac{r - r_{ps}}{r} \right)^2 |\nabla u|^2$$

$$+ r^{-3} \left(1 - \left(\frac{r_s}{r} \right)^{d+1} \right) u^2$$

for a smooth choice of F.

This choice of F, however, does not give a bounded multiplier at the event horizon. To rectify this, we set ρ to be a smooth, increasing function satisfying

$$\rho(R) = \begin{cases} R, & R \geq -1, \\ -2, & R \leq -3, \end{cases}$$

and let $\rho_\varepsilon(R) = \varepsilon^{-1} \rho(\varepsilon R)$. Finally, we set

$$f(r) = \frac{1}{r^{d+2}} \rho_\varepsilon(r^{d+2} F(r))$$

in the definition of X_1. Recalculating, with the abbreviated notation $X_1[f] = X_1$ and $q_1[f] = q_1$, we have

$$Q[u, X_1, q_1, 0] \geq C \left(1 - \left(\frac{r_s}{r} \right)^{d+1} \right)^2 r^{-(d+3)} (\partial_r u)^2 + C \frac{1}{r} \left(\frac{r - r_{ps}}{r} \right)^2 |\nabla u|^2 + l(f).$$

As before, $l(f) \gtrsim r^{-3} \left(1 - \left(\frac{r_s}{r} \right)^{d+1} \right)$ except when $\varepsilon r^{d+2} F(r) < -1$. We record that

$$l(f) = \rho'(\varepsilon r^{d+2} F(r)) l(F) - O(\varepsilon) \rho''(\varepsilon r^{d+2} F(r)) - O\left(\varepsilon^2 \left(1 - \left(\frac{r_s}{r} \right)^{d+1} \right)^{-1} \right) \rho'''(\varepsilon r^{d+2} F(r)).$$

For the multiplier X_2, we calculate

$$Q\left[u, X_2, -\frac{d+2}{2}\frac{b(r)}{r}, 0\right] = -\frac{1}{2}b'(r)\left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)\left(\partial_r u - \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^{-1}\partial_t u\right)^2$$

$$+ \frac{(d+1)r_s^{d+1}}{2}\frac{b(r)}{r^{d+2}}\left(\partial_r u - \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^{-1}\partial_t u\right)^2$$

$$+ \left[\frac{1}{2}b'(r) - \frac{b(r)}{r}\right]|\slashed{\nabla}u|^2 + \frac{d+2}{4}\nabla^\alpha\partial_\alpha\left(\frac{b(r)}{r}\right)u^2.$$

Inspired by this, we choose $b(r)$ to be smooth, positive and decreasing on $[r_s, (r_s+3r_{ps})/4)$, and supported in $[r_s, (r_s + 3r_{ps})/4]$. For later purposes, we set

$$m_t = \frac{(d+1)r_s^{d+1}}{r^{d+2}}b(r)\gamma, \quad m_r = \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^{-1}\frac{(d+1)r_s^{d+1}}{r^{d+2}}b(r)\gamma, \quad m_\omega = 0.$$

Then,

$$Q\left[u, X_2, -\frac{d+2}{2}\frac{b(r)}{r}, m\right] \geq \frac{(d+1)r_s^{d+1}}{2}\frac{b(r)}{r^{d+2}}\left(\partial_r u - \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^{-1}\partial_t u + \gamma u\right)^2$$

$$+ \left[\frac{1}{2}b'(r) - \frac{b(r)}{r}\right]|\slashed{\nabla}u|^2 - \frac{(d+1)r_s^{d+1}}{2}\frac{b(r)}{r^{d+2}}\gamma^2 u^2$$

$$+ \frac{d+2}{4}\nabla^\alpha\partial_\alpha\left(\frac{b(r)}{r}\right)u^2 + (d+1)\frac{r_s^{d+1}}{r^{d+2}}\partial_r(b(r)\gamma)u^2$$

Combining, we obtain

$$(2.5) \quad Q\left[u, X_1 + \delta X_2, q_1 - \delta\frac{d+2}{2}\frac{b(r)}{r}, \delta m\right] \geq C\left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^2 r^{-(d+3)}(\partial_r u)^2$$

$$+ C\frac{1}{r}\left(\frac{r - r_{ps}}{r}\right)^2|\slashed{\nabla}u|^2 + C\delta b(r)\left(\partial_r u - \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^{-1}\partial_t u + \gamma u\right)^2 + n(r)u^2$$

provided $\delta > 0$ is sufficiently small. Here

$$n(r) = -\delta\frac{(d+1)r_s^{d+1}}{2}\frac{b(r)}{r^{d+2}}\gamma^2 + \delta\frac{d+2}{4}\nabla^\alpha\partial_\alpha\left(\frac{b(r)}{r}\right) + \delta(d+1)\frac{r_s^{d+1}}{r^{d+2}}\partial_r(b(r)\gamma) + l(f).$$

Before proceeding to bound n from below, we address the fact that the right side of (2.5) fails to control ∂_t away from the event horizon. This is easily remedied by setting

$$q_2(r) = \chi_{r>(r_s+r_{ps})/2}r^{-(d+3)}\left(\frac{r - r_{ps}}{r}\right)^2,$$

and recomputing

$$Q\left[u, X_1 + \delta X_2, q_1 - \delta\frac{d+2}{2}\frac{b(r)}{r} + \delta_1 q_2, \delta m\right].$$

Provided that $\delta_1 \ll \delta$ and bootstrapping the many negligible terms, we obtain

$$Q\left[u, X_1 + \delta X_2, q_1 - \frac{d+2}{2}\frac{b(r)}{r} + \delta_1 q_2, m\right] \geq C\left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^2 r^{-(d+3)}(\partial_r u)^2$$

$$+ C\frac{1}{r}\left(\frac{r - r_{ps}}{r}\right)^2|\slashed{\nabla}u|^2 + C\delta b(r)\left(\partial_r u - \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^{-1}\partial_t u + \gamma u\right)^2$$

$$+ \delta_1 q_2\left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^{-1}(\partial_t u)^2 + n(r)u^2.$$

As we may easily observe that

$$X(dr)(r_s) = -\delta b(r_s) < 0, \quad \langle m, dr \rangle(r_s) = \frac{d+1}{r_s} b(r_s)\gamma(r_s) > 0,$$

provided that $\gamma(r_s) > 0$, it remains to choose γ so that n is positive where $\varepsilon r^{d+2} f(r) < -1$. We suppose that supp $\gamma \subset \{r < r_{ps}\}$, $0 \leq \gamma \leq 1$, and $\gamma' > -1$. For $r > r_{ps}$, the lower bound follows from that for $l(f)$ by the support properties of $b(r)$ and γ. For $r < r_{ps}$, we write

$$n = l(f) + \delta \frac{(d+1)r_s^{d+1}}{r^{d+2}} b(r)\gamma'(r) - O(\delta).$$

The lower bound on γ', then, implies that $n \geq l(f) - O(\delta)$, which is positive where $\varepsilon r^{d+2} f(r) \geq -1$ for sufficiently small δ.

Finally, in the region $\varepsilon r^{d+2} f(r) < -1$,

$$n \geq \delta \frac{(d+1)r_s^{d+1}}{r^{d+2}} b(r)\gamma'(r) - O(\delta) - O(\varepsilon)\rho''(\varepsilon r^{d+2} F(r))$$

$$- O\left(\varepsilon^2 \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^{-1}\right)\rho'''(\varepsilon r^{d+2} F(r)).$$

While γ' may be taken to be positive, the requirement that $0 \leq \gamma \leq 1$ places restrictions on how large it may be. On average, γ' can be at most $O(e^{c/\varepsilon})$ on this interval. But since the interval of integration $\varepsilon r^{d+2} f(r) < 1$ is of length $e^{-c/\varepsilon}$,

$$\int_{\varepsilon r^{d+2} f(r) < -1} \delta + \varepsilon |\rho''(\varepsilon r^{d+2} f(r))| + \varepsilon^2 \left(1 - \left(\frac{r_s}{r}\right)^{d+1}\right)^{-1} |\rho'''(\varepsilon r^{d+2} f(r))| \, dr \lesssim \varepsilon + e^{-c/\varepsilon} \ll \delta,$$

provided that ε is sufficiently small. And this completes the proof of the lemma.   □

Proof of Theorem 2.1. With the previous lemma in hand, we only tersely describe the remainder of the proof as the details follow from obvious modifications of the argument in [45]. We allow X, q, and m to be as in the lemma. From the divergence relation (2.3) and the fact that K is a Killing vector field, we have

$$\nabla^\alpha P_\alpha[u, X + CK, q, m] = (\Box_g u)((X + CK)u + qu) + Q[u, X, q, m]$$

for a large constant C. Integrating over $0 < \tilde{v} < \tilde{v}_0$, and $r > r_e$, we obtain

$$\int \left(\Box_g u((X+CK)u+qu)+Q[u, X, q, m]\right) dVol = \int \langle d\tilde{v}, P[u, X+CK, q, m]\rangle r^{d+2} \, dr \, d\omega \Big|_{\tilde{v}=0}^{\tilde{v}=\tilde{v}_0}$$

$$- \int_{r=r_e} \langle dr, P[u, X + CK, q, m]\rangle r_e^{d+2} \, d\tilde{v} d\omega.$$

For C large enough and r_e near enough to r_s, we claim

(2.6) $$E[u](\tilde{v}_1) \approx - \int_{\tilde{v}=\tilde{v}_1} \langle d\tilde{v}, P[u, X + CK, q, m]\rangle \, r^{d+2} \, dr \, d\omega, \quad \tilde{v}_1 \geq 0$$

and

(2.7) $$\langle dr, P[u, X + CK, q, m]\rangle \gtrsim (\partial_r u)^2 + (\partial_{\tilde{v}} u)^2 + |\nabla u|^2 + u^2, \quad r = r_e.$$

Upon proving these, a localized energy estimate, though with weight $r^{-(d+3)/2}$ rather than $r^{-1/2-}$ at infinity on the ∂_r and $\partial_{\tilde{v}}$ terms, follows immediately from the lemma and the Schwarz inequality.

To prove (2.6), we note that, for $r > r_s$,

$$\langle d\tilde{v}, P[u, X + CK]\rangle = (X(d\tilde{v}) + C)\langle d\tilde{v}, P[u, \partial_{\tilde{v}}]\rangle + X(dr)\langle d\tilde{v}, P[u, \partial_r]\rangle$$

$$\approx C\Big[(\partial_{\tilde{v}}u)^2 + \Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)(\partial_r u)^2 + |\nabla u|^2\Big] + (\partial_r u)^2.$$

Indeed, due to the boundedness of X, $X(d\tilde{v}) + C \approx C$ for C large enough. The condition $X(dr)(r_s) < 0$ is used to obtain the nondegenerate ∂_r contribution. And by continuity, the same expression extends to $r > r_e$ for r_e sufficiently close to r_s. To complete the proof of (2.6), we note that the decay of q, the compact support of m, and the Schwarz inequality reduce controlling the lower order terms to proving a straightforward Hardy-type inequality such as

$$\int_{r_e}^{\infty} r^{-2} u^2 r^{d+2}\, dr \lesssim C^{-1/2} \int_{r_e}^{\infty} \Big[C\Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big) + 1\Big](\partial_r u)^2\, r^{d+2}\, dr.$$

For (2.7), we similarly compute

$$\langle dr, P[u, X + CK, 0, m]\rangle \gtrsim C(\partial_{\tilde{v}}u)^2 + |\nabla u|^2 - \Big(1 - \frac{r_s^{d+1}}{r_e^{d+1}}\Big)(\partial_r u)^2 + u^2,$$

when $0 < r_s - r_e \ll 1$. Here we have used the lateral boundary condition on m as provided by the lemma. As

$$qu\langle dr, du\rangle - \frac{1}{2}u^2\langle dr, dq\rangle \ll C(\partial_{\tilde{v}}u)^2 + u^2 + \Big(1 - \frac{r_s^{d+1}}{r_e^{d+1}}\Big)(\partial_r u)^2,$$

provided that C is large enough and $r_s - r_e$ is sufficiently small, these terms can be bootstrapped into the above. We can now translate these bounds to any $0 < r_e < r_s$ from local thoery. This completes the proof of the theorem with the weaker weights at infinity.

In order to get the sharp weights at infinity, we can rely on the methods of [48] for small perturbations of Minkowski space. See the sharper statement of the results that follow from that proof in [42, Lemma 3.3]. Indeed, for some constant $R \gg r_{ps}$ and a smooth β so that $\beta(r) \equiv 0$ for $r < R$ and $\beta(r) \equiv 1$ for $r > 2R$, then an application of the results of [48] complete the proof provided that

$$\|[\Box_g, \beta]u\|_{LE_M^*},$$

can be bounded. But as $[\Box_g, \beta]$ is supported in a compact region, it follows immediately that the above is controlled using the previously established estimate with weaker weights at infinity. $\square$

3. The Metric

Here we introduce the class of $(1+4)$-dimensional Myers-Perry black holes. These space-times are higher dimensional analogs of the well-known Kerr solutions of Einstein's equations. With the additional dimension, an extra angular momentum parameter is introduced.

The line element for the $(1+4)$-dimensional Myers-Perry solution is given by

$$(3.1) \quad ds^2 = -dt^2 + \frac{r_s^2}{\rho^2}\Big[dt + a\sin^2\theta d\phi + b\cos^2\theta d\psi\Big]^2 + \frac{\rho^2}{4\Delta}dx^2$$
$$+ \rho^2 d\theta^2 + (x+a^2)\sin^2\theta d\phi^2 + (x+b^2)\cos^2\theta d\psi^2.$$

Here r_s is a parameter, depending on the mass of the black hole, which corresponds to the Schwarzschild radius. Moreover,

$$\rho^2 = x + a^2\cos^2\theta + b^2\sin^2\theta,$$
$$\Delta = (x+a^2)(x+b^2) - r_s^2 x.$$

The parameters a, b are angular momentum parameters. We are using the notational conventions that we learned from [30]. In particular, $x = r^2$, where r is the commonly used "radial" variable.

The event horizons are given by the solutions to $\Delta = 0$. In particular, they are $x = x_\pm$,

$$x_\pm = \frac{1}{2}\Big(r_s^2 - a^2 - b^2 \pm \sqrt{(r_s^2 - a^2 - b^2)^2 - 4a^2b^2}\Big).$$

We shall primarily be concerned with the domain of outer communication $x > x_+$.

For future reference, we note that

$$\sqrt{-g} = \frac{1}{2}\sin\theta\cos\theta\rho^2$$

and that the inverse metric is given by

$$g^{tt} = \frac{1}{\rho^2}\Big[(a^2-b^2)\sin^2\theta - \frac{(x+a^2)[\Delta + r_s^2(x+b^2)]}{\Delta}\Big],$$

$$g^{t\phi} = \frac{ar_s^2(x+b^2)}{\rho^2\Delta}, \quad g^{t\psi} = \frac{br_s^2(x+a^2)}{\rho^2\Delta}$$

$$g^{\phi\phi} = \frac{1}{\rho^2}\Big[\frac{1}{\sin^2\theta} - \frac{(a^2-b^2)(x+b^2) + b^2 r_s^2}{\Delta}\Big]$$

$$g^{\psi\psi} = \frac{1}{\rho^2}\Big[\frac{1}{\cos^2\theta} + \frac{(a^2-b^2)(x+a^2) - a^2 r_s^2}{\Delta}\Big]$$

$$g^{\phi\psi} = -\frac{abr_s^2}{\rho^2\Delta}, \quad g^{xx} = 4\frac{\Delta}{\rho^2}, \quad g^{\theta\theta} = \frac{1}{\rho^2}.$$

As expected, the Myers-Perry black holes serve as higher dimensional analogs to the axisymmetric Kerr black holes. The structure of the singularities, horizons, and ergoregions display similarities to those of the more familiar Kerr space-times. Further similarities include superradiance, the construction of maximal analytic extensions, and the existence of hidden symmetries. See [54], [55], [29], and the references therein. Some properties do differ, such as the existence of ultra-spinning black holes [54], [55].

In what follows, we focus our comparisons on the behavior of null geodesics. In particular, we show that each trapped geodesics has a constant r. Like the Carter tensor [16] for Kerr, the Myers-Perry space-times contain hidden symmetries [29] in all dimensions, but as is mentioned therein this is insufficient to guarantee the separability of the Hamilton-Jacobi equations in generic dimensions. The Hamilton-Jacobi equations can be separated in the special case that there are, at most, two distinct rotations parameters.

See [56] and [70]. Specifically for the $(1+4)$-dimensional solutions, the hidden symmetries and separability appear in [30], and it is this exposition that closely informs what immediately follows.

Let $E = -\partial_t$, $\Phi = \partial_\phi$, $\Psi = \partial_\psi$, and K denote the constants of motion. Then we have, again with notations from [30],

$$(3.2) \qquad \rho^4 \dot{\theta}^2 = \Theta,$$

$$(3.3) \qquad \rho^4 \dot{x}^2 = 4\mathcal{X},$$

$$(3.4) \qquad \rho^2 \dot{t} = E\rho^2 + \frac{r_s^2(x+a^2)(x+b^2)}{\Delta}\mathcal{E},$$

$$(3.5) \qquad \rho^2 \dot{\phi} = \frac{\Phi}{\sin^2\theta} - \frac{ar_s^2(x+b^2)}{\Delta}\mathcal{E} - \frac{(a^2-b^2)\Phi}{x+a^2},$$

$$(3.6) \qquad \rho^2 \dot{\psi} = \frac{\Psi}{\cos^2\theta} - \frac{br_s^2(x+a^2)}{\Delta}\mathcal{E} + \frac{(a^2-b^2)\Psi}{x+b^2}.$$

Here we have abstracted

$$(3.7) \qquad \mathcal{E} = E + \frac{a\Phi}{x+a^2} + \frac{b\Psi}{x+b^2}$$

$$(3.8) \qquad \Theta = E^2(a^2\cos^2\theta + b^2\sin^2\theta) - \frac{1}{\sin^2\theta}\Phi^2 - \frac{1}{\cos^2\theta}\Psi^2 + K$$

$$(3.9) \qquad \mathcal{X} = \Delta\left[E^2 x + (a^2-b^2)\left(\frac{\Phi^2}{x+a^2} - \frac{\Psi^2}{x+b^2}\right) - K\right] + r_s^2(x+a^2)(x+b^2)\mathcal{E}^2.$$

By writing $(a^2 - b^2) = (x+a^2) - (x+b^2)$, we see that

$$-r_s^2 x(a^2-b^2)\left(\frac{\Phi^2}{x+a^2} - \frac{\Psi^2}{x+b^2}\right) + r_s^2(x+a^2)(x+b^2)\left(\frac{a^2\Phi^2}{(x+a^2)^2} + \frac{b^2\Psi^2}{(x+b^2)^2}\right)$$
$$= r_s^2(b^2\Phi^2 + a^2\Psi^2).$$

Making this substitution in the definition of $\mathcal{X}$, we obtain

$$\mathcal{X} = \Delta(E^2 x - K) + (a^2-b^2)\left(\Phi^2(x+b^2) - \Psi^2(x+a^2)\right)$$
$$+ r_s^2\left(E^2(x+a^2)(x+b^2) + 2aE\Phi(x+b^2) + 2bE\Psi(x+a^2) + (b\Phi+a\Psi)^2\right).$$

We have reduced our analysis to understanding this cubic polynomial as the zeros of $\mathcal{X}$ correspond to turning points of (3.3).

At least one of the parameters E, K, Φ, Ψ should be nonzero. As (3.2) disallows $E = K = 0$, the above representation for $\mathcal{X}$ is always nondegenerate.

We first examine the case that $E = 0$. Rewriting (3.8)

$$\Theta = \left[K + (\Phi - Ea)^2 - \Phi^2 + (\Psi - Eb)^2 - \Psi^2\right] - \frac{(\Phi - Ea\sin^2\theta)^2}{\sin^2\theta} - \frac{(\Psi - Eb\cos^2\theta)^2}{\cos^2\theta},$$

it follows from (3.2) that

$$K + (\Phi - Ea)^2 - \Phi^2 + (\Psi - Eb)^2 - \Psi^2 \geq 0.$$

Thus, in particular, when $E = 0$, we must have $K > 0$. In this case, $\mathcal{X}$ is a quadratic equation with leading coefficient $-K$, and we note that $\mathcal{X}(x_+) \geq 0$ since $\Delta(x_+) = 0$.

Thus, since $K > 0$, there is a single zero with multiplicity 1 outside of the event horizon. $\mathcal{X}$ must change sign at this point from positive to negative, which creates a single right turning point for (3.3) and no trapped geodesic can exist.

We next examine the case $E \neq 0$. Here $\mathcal{X}$ is a cubic polynomial with a positive leading coefficient. As it is obvious from the original formulation (3.9) of $\mathcal{X}$ that $\mathcal{X}(x_+) > 0$, it follows that there is always one root of $\mathcal{X}$ to the left of x_+. Three cases then must be examined: (1) $\mathcal{X}$ has no roots greater than x_+, (2) it has two distinct roots in $x > x_+$, or (3) it has a since double root in this region. If there are no roots, then $\mathcal{X} > 0$ in this region and all null geodesics escape to infinity. If there are two distinct roots x_1, x_2 so that $x_+ < x_1 < x_2$, then $\mathcal{X}$ changes sign from positive to negative and negative to positive respectively at x_1 and x_2. Thus x_1 is a right turning point and x_2 is a left turning point, which does not support a trapped geodesic. A double root x_0 of $\mathcal{X}$ corresponds to a steady state and implies that all other geodesics converge to x_0 in one direction and either $x = 0$ or ∞ in the other direction, which are thus not trapped.

The above analysis shows that along any trapped null geodesic we have constant x, which corresponds to a double root of $\mathcal{X}$.

4. Localized energy spaces

We now return to the $(1 + 4)$-dimensional Myers-Perry space-time with metric given by (3.1), and we seek to define localized energy norms that reflect the more complicated trapping on this background and to state our main theorem.

We let τ, Ξ, Φ, Ψ, and Θ denote the Fourier variables for t, x, ϕ, ψ, and θ. We are abusing notation as we early used some of these to denote constants of motion. From this point forward, however, the notation will be reserved solely for the Fourier variables.

We set

$$p(x, \theta, \tau, \Xi, \Phi, \Psi, \Theta) = g^{tt}\tau^2 + 2g^{t\phi}\tau\Phi + 2g^{t\psi}\tau\Psi + g^{\phi\phi}\Phi^2 + g^{\psi\psi}\Psi^2 + 2g^{\phi\psi}\Phi\Psi + g^{xx}\Xi^2 + g^{\theta\theta}\Theta^2$$

to be the Hamiltonian, which vanishes on any null geodesic. The Hamilton flow indicates

$$(4.1) \qquad \dot{x} = \frac{\partial p}{\partial \Xi} = \frac{8\Delta}{\rho^2}\Xi$$

(4.2)
$$\dot{\Xi} = -\frac{\partial p}{\partial x} = -\left(g^{tt}_{,x}\tau^2 + 2g^{t\phi}_{,x}\tau\Phi + 2g^{t\psi}_{,x}\tau\Psi + g^{\phi\phi}_{,x}\Phi^2 + g^{\psi\psi}_{,x}\Psi^2 + 2g^{\phi\psi}_{,x}\Phi\Psi + g^{xx}_{,x}\Xi^2 + g^{\theta\theta}_{,x}\Theta^2 \right).$$

Equivalent to the latter, we have

$$\rho^2\dot{\Xi} = -\frac{\partial}{\partial x}(\rho^2 p) + \frac{\partial}{\partial x}(\rho^2)p$$

$$= R_{a,b}(x, \tau, \Phi, \Psi)\Delta^{-2} + \frac{\partial}{\partial x}(\rho^2)p - 4(2x + a^2 + b^2 - r_s^2)\Xi^2.$$

To be explicit, we have

$$R_{a,b} = \Big(x^2(x+b^2)(x-2r_s^2+b^2)+a^4(x+b^2)^2+a^2(b^2(4x^2-2r_s^2x+r_s^4)+2x^2(x-r_s^2)+2b^4x) \Big) \tau^2$$
$$+ 2ar_s^2(x+b^2(2x-r_s^2)+b^4)\tau\Phi + 2br_s^2(x^2+a^2(2x-r_s^2)+a^4)\tau\Psi$$
$$+ (b^2(x^2+2(r_s^2+b^2)x+(r^4-b^4))-a^2(x^2-2b^2(x-r^2)+b^4))\Phi^2$$
$$+ (a^6-a^4(b^2+2r_s^2-2x)-b^2x^2+a^2(r_s^4-2r_s^2x+x(-2b^2+x)))\Psi^2$$
$$- 2abr_s^2(2x-r_s^2+a^2+b^2)\Phi\Psi.$$

Along any null geodesic, we have that $p = 0$. Moreover, as we proved in Section 3, trapped null geodesics have constant x, and thus, by (4.1), $\Xi \equiv 0$. Hence, corresponding to any trapped null geodesic, there must be a root $r_{a,b}(\tau, \Phi, \Psi)$ solving

$$R_{a,b}(r_{a,b}^2, \tau, \Phi, \Psi) = 0.$$

Note that $\tau^{-2}R_{a,b}$ is a fourth order polynomial in x and $\tau^{-2}R_{0,0} = x^3(x-2r_s^2)$ has a simple root at $x = r_{ps}^2$. Moreover, near $x = r_{ps}^2$, there is a constant C_{mp} so that $|\Phi|, |\Psi| \le C_{mp}|\tau|$ since $p = 0$. Hence $\tau^{-2}R_{a,b}$ is a small perturbation of $\tau^{-2}R_{0,0}$ for small enough a, b, and has a simple root near $r = r_{ps}$.

It will be more convenient to work with the r coordinate, $x = r^2$, and its corresponding Fourier variable $\xi = 2r\Xi$ instead of x and Ξ. For future reference, we record that

$$(4.3) \qquad \partial_r(\rho^2 p) = -2rR_{a,b}\Delta^{-2} + \partial_r(\frac{\Delta}{r^2})\xi^2$$

$$(4.4) \qquad \partial_\xi(\rho^2 p) = 2\frac{\Delta}{r^2}\xi.$$

We are now ready to define the local energy spaces, in a similar fashion to [68]. Naively, one would like to replace the $r - r_{ps}$ factor in [41] by a quantization of $r - r_{a,b}(\tau, \Phi, \Psi)$. The problem with this approach is that any such quantization depends nontrivially on τ, which makes uniform energy bounds on $\tilde{v}$-slices difficult to prove. Instead, near $r = r_{ps}$ we can write

$$p = g^{tt}(\tau - \tau_1(r,\theta,\xi,\Psi,\Phi,\Theta))(\tau - \tau_2(r,\theta,\xi,\Psi,\Phi,\Theta))$$

for some real smooth 1-homogeneous symbols τ_1 and τ_2. We define

$$c_i(r,\theta,\xi,\Psi,\Phi,\Theta) = \chi_{\ge 1}(r - r_{a,b}(\tau_i,\Phi,\Psi))$$

where $\chi_{\ge 1}$ is a smooth symbol which equals 1 for frequencies $\gg 1$ and 0 for frequencies $\lesssim 1$. The role of the cutoff is to transform the homogeneous symbol into a classical one. Note that since low frequencies are already controlled without degeneracy near the trapped set, this makes no difference in our estimates.

We use the symbols c_i to define microlocally weighted function spaces in a neighborhood $V \times \mathbb{S}^3$ of $\{r = r_{ps}\} \times \mathbb{S}^3$. We set

$$\|u\|_{L^2_{c_i}}^2 = \|c_i^w(x,D)u\|_{L^2}^2 + \|u\|_{H^{-1}}^2.$$

For the inhomogeneous term, we define the dual norm

$$\|g\|_{c_i L^2}^2 = \inf_{c_i(x,D)g_1+g_2=g} (\|g_1\|_{L^2}^2 + \|g_2\|_{H^1}^2).$$

For the remainder of the paper, we shall be using the r variable rather than the x variable from Section 3. In another abuse of notation, x is now used to denote $x = r\omega$ analogous to standard spherical coordinates.

Pick $\chi(r)$ to be a cutoff function supported in V and equal to 1 near $r = r_{ps}$. We set our local energy norm to be

(4.5)
$$\|u\|_{LE^1_{mp}} = \|\chi(D_t - \tau_2(x,D))\chi u\|_{L^2_{c_1}} + \|\chi(D_t - \tau_1(x,D))\chi u\|_{L^2_{c_2}} + \|(1-\chi^2)\partial_t u\|_{LE_M}$$
$$+ \|(1-\chi^2)\overline{\nabla} u\|_{LE_M} + \|\partial_r u\|_{LE_M} + \|r^{-3/2} u\|_{L^2}.$$

For the inhomogeneous term, we set the dual norm to be

(4.6)
$$\|f\|_{LE^*_{mp}} = \|(1-\chi)f\|_{LE^*_M} + \|\chi f\|_{c_1 L^2 + c_2 L^2}$$

Letting $\square_{mp}$ denote the d'Alembertian in the Myers-Perry metric, the main result in our paper is the following:

Theorem 4.1. *For $|a|, |b|$ sufficiently small and for some $r_- < r_e < r_+$, let u solve $\square_{mp} u = f$ in $\mathcal{M}_R$. Then*

(4.7)
$$\|u\|^2_{LE^1_{mp}} + \sup_{\tilde{v}} E[u](\tilde{v}) + E[u](\Sigma^+_R) \lesssim E[u](\Sigma^-_R) + \|f\|^2_{LE^*_{mp}}.$$

in the sense that the left hand side is finite and the inequality holds whenever the right hand side is finite.

5. Proof of Theorem 4.1

We now prove the localized energy estimate from the previous section assuming $\max(|a|, |b|) \leq \epsilon_0 \ll 1$. The arguments are fundamentally the same as the ones in [68], and some of the results will be cited without proof.

The main idea is to use the multiplier method. Unfortunately, due to the complicated nature of the trapping, no differential operator provides us with a positive local energy norm. Instead we rely on the smoothed out (near the photonsphere) vector field X to control the LE_{mp} norm away from the trapped set, to which we add a pseudodifferential correction near the trapped set. Since we would also like to establish uniform energy bounds, we will pick the correction to be a differential operator of first order in t, which will allow us to integrate by parts with respect to time.

If one integrates (2.3) on the domain
$$D = \{0 < \tilde{v} < \tilde{v}_0, \ r > r_e\},$$
local energy estimates are established as long as the boundary terms satisfy

(5.1) $$BDR[u] \leq c_1 E[u](\Sigma^-_R) - c_2(E[u](\tilde{t}_0) + E[u](\Sigma^+_R)), \qquad c_1, c_2 > 0$$
and $Q[u, X, q, m] \geq 0$.

From here one, we will use the sub(super)scripts S and mp to denote the Schwarzschild and the Myers-Perry metrics respectively. By choosing X, q and m like in Section 3, we know that

(5.2) $$Q^S[u, X, q, m] \gtrsim r^{-4}(\partial_r u)^2 + \left(1 - \frac{r_{ps}}{r}\right)^2 (r^{-4}(\partial_{\tilde{v}} u)^2 + r^{-1}|\overline{\nabla} u|^2) + r^{-3} u^2.$$

and the boundary terms satisfy (5.1)

The same computation can be performed for the Kerr metric. Precisely, with ∂ standing for ∂_t and ∂_x, $x = r\omega$,

$$(5.3) \qquad |\partial^\alpha[(g_{mp})_{ij} - (g_S)_{ij}]| \leq c_\alpha \frac{\epsilon_0}{r^{2+|\alpha|}}, \qquad |\partial^\alpha[(g_{mp})^{ij} - (g_S)^{ij}]| \leq c_\alpha \frac{\epsilon_0}{r^{2+|\alpha|}}$$

and thus one can easily check that

$$(5.4) \qquad |P_\alpha^S[u, X, q, m] - P_\alpha^{mp}[u, X, q, m]| \lesssim \frac{\epsilon_0}{r^2}|\partial u|^2,$$

respectively

$$(5.5) \qquad |Q^S[u, X, q, m] - Q^{mp}[u, X, q, m]| \lesssim \epsilon_0 \left(\frac{1}{r^2}|\partial u|^2 + \frac{1}{r^4}|u|^2 \right).$$

Thus (5.1) still holds; however, we can only say that

$$(5.6) \quad Q^{mp}[u, X, q, m] \gtrsim r^{-4}|\partial_r u|^2 + \left[\left(1 - \frac{r_{ps}}{r} \right)^2 - C\epsilon_0 \right] (r^{-4}|\partial_{\tilde{v}} u|^2 + r^{-1}|\nabla\!\!\!\!/\, u|^2) + r^{-3}u^2,$$

and the right hand side is no longer positive definite near $r = r_{ps}$.

In order to correct this, we will add a pseudodifferential correction to X and q. In order to quantize the symbols, we will use a Weyl calculus with respect to the metric $dV_{mp} = r\rho^2 dr d\omega$. We thus slightly abuse the notation and redefine the Weyl quantization as

$$s^w := \frac{r}{\rho} \, s^w \, \frac{\rho}{r}$$

so that real symbols get quantized to self-adjoint operators with respect to $L^2(dV_{mp})$.

Let

$$S = is_1^w + s_0^w \partial_t, \qquad E = e_0^w + \frac{1}{i} e_{-1}^w \partial_t$$

where $s_1 \in S^1$, $s_0, e_0 \in S^0$ and $e_{-1} \in S^{-1}$ are real symbols with kernels supported close to $r = r_{ps}$. One can now compute

$$(5.7) \qquad \Re \int_D \square_{mp} u \cdot (S + E)u \, dV_{mp} = \int_D Qu \cdot u \, dV_{mp} + BDR^{mp}[u, S, E]$$

where

$$(5.8) \qquad Q = \frac{1}{2}([\square_{mp}, S] + \square_{mp} E + E\square_{mp}) = q_2^w + 2q_1^w D_t + q_0^w D_t^2 + q_{-1}^w D_t^3$$

with $q_j \in S^j$.

Since the Weyl quantization is only with respect to the spatial variables, we define the matrix-valued pseudodifferential operator

$$\tilde{Q} = \begin{pmatrix} q_2^w & q_1^w \\ q_1^w & q_0^w \end{pmatrix}.$$

Let

$$(5.9) \qquad IQ^{mp}[u, S, E] = \int_D q_2^w u \cdot \overline{u} + 2\Re q_1^w u \cdot \overline{D_t u} + q_0^w D_t u \overline{D_t u} \, dV_{mp}$$

and note that

$$IQ^{mp}[u, S, E] = \int_0^{\tilde{v}_0} \langle \tilde{Q}\vec{u}, \vec{u} \rangle d\tilde{v}$$

where $\vec{u} = (u, D_t u)$.

We will pick S and E so that $q_{-1}^w = 0$; in this case, after integrating by parts in time, (5.7) becomes

$$(5.10) \qquad \Re \int_D \Box_{mp} u \cdot (S + E)u \, dV_{mp} = IQ^{mp}[u, S, E] + BDR^{mp}[u, S, E].$$

The exact form of the boundary terms does not matter, since we will pick the symbols $s_i \in \epsilon_0 S^i$ to be small when $\epsilon_0 \ll 1$. In particular, since there are no boundary terms at $r = r_e$, this means that

$$|BDR^{mp}[u, S, E]| \lesssim \epsilon_0 (E[u](0) + E[u](\tilde{v}_0)).$$

The local energy estimates now follow if we can pick S and E so that

$$(5.11) \qquad \int_D Q^{mp}[u, X, q, m] \, dV_{mp} + IQ^{mp}[u, S, E] \gtrsim \|u\|_{LE_{mp}}^2.$$

Near $r = r_{ps}$, the principal symbol of the quadratic form on the left in (5.11) is (see (5.8))

$$\frac{1}{2i}\{p, X + s\} + p(q + e)$$

where for convenience we denote

$$s = s_1 + \tau s_0, \qquad e = e_0 + \tau e_{-1}.$$

In order for (5.11) to hold, the above symbol must dominate the principal part of the LE_{mp} norm, i.e.

$$\frac{1}{2i}\{p, X + s\} + p(q + e) \gtrsim c_2^2(\tau - \tau_1)^2 + c_1^2(\tau - \tau_2)^2 + \xi^2.$$

Unfortunately this is not enough. Indeed, since $\tilde{Q}$ is a matrix valued operator of second order, the Fefferman-Phong inequality does not hold. What we do instead is write the left hand side as a sum of squares dominating the right hand side. We have:

Lemma 5.1. *Let ϵ_0 be sufficiently small. Then there exist smooth homogeneous symbols $s \in \epsilon_0(S_{hom}^1 + \tau S_{hom}^0)$, $e \in \epsilon_0(S_{hom}^0 + \tau S_{hom}^{-1})$ so that for r close to r_{ps} we have*

$$(5.12) \qquad \rho^2 \left(\frac{1}{2i}\{p, X + s\} + p(q + e) \right) = \sum_{j=1}^{11} \mu_j^2 \gtrsim c_2^2(\tau - \tau_1)^2 + c_1^2(\tau - \tau_2)^2 + \xi^2$$

for some $\mu_j \in S_{hom}^1 + \tau S_{hom}^0$ that depend smoothly on a, b and in addition satisfy the conditions:

   i) *μ_i are differential operators for $a = b = 0$, $i = \overline{1, 9}$*

   ii) *μ_{10} and μ_{11} are small, in the sense that*

$$(\mu_{10}, \mu_{11}) \in \sqrt{\epsilon_0}(S_{hom}^1 + \tau S_{hom}^0).$$

Proof. We will start by reformulating the results of Section 2 in the setting of pseudo-differential operators. We know that near $r = r_{ps}$ the symbol of X is $if(r)(r - r_{ps})\xi$ for some smooth function $f > 0$. Then near $r = r_{ps}$, we can write

$$Q^S[u, X, q, m] = q^{S, \alpha\beta} \partial_\alpha u \partial_\beta u + q^{S, 0} u^2$$

where the principal symbols are

$$q^S = q^{S,\alpha\beta}\eta_\alpha\eta_\beta = \frac{1}{2i}\{p_S, X\} + qp_S, \qquad q^{S,0} = -\frac{1}{2}\nabla^\alpha\partial_\alpha q.$$

The symbol of $\Box_S$ is

$$p_S = -\left(1 - \frac{r_s^2}{r^2}\right)^{-1}\tau^2 + \left(1 - \frac{r_s^2}{r^2}\right)\xi^2 + \frac{1}{r^2}\lambda^2$$

where λ stands for the spherical Fourier variable.

We can now compute

(5.13)
$$\begin{aligned}
r^2 q^S &= \frac{1}{2i}\{r^2 p_S, X\} + (q - r^{-1}f(r)(r - r_{ps}))(r^2 p_S) \\
&= \alpha_S^2(r)\tau^2 + \beta_S^2(r)\xi^2 + \tilde{q}(r)(r^2 p_S)
\end{aligned}$$

where, near $r = r_{ps}$,

$$\alpha_S^2(r) = \frac{r^3(r + \sqrt{2}r_s)f(r)(r - r_{ps})^2}{(r^2 - r_s^2)^2},$$

$$\beta_S^2(r) = (r^2 - r_s^2)f + (r - r_{ps})(f'(r^2 - r_s^2) - rf),$$

respectively

$$\tilde{q}(r) = q - r^{-1}f(r)(r - r_{ps}).$$

The facts that $f > 0$ and $\left[\left(1 - \frac{r_s^2}{r^2}\right)^{-1}(r - r_{ps})f(r)\right]' > 0$ are used to write the first two coefficients as squares.

Due to the results of Section 2, we know that

(5.14)
$$q^S \gtrsim \xi^2 + (r - r_{ps})^2(\tau^2 + \lambda^2), \qquad q^{S,0} > 0$$

near $r = r_{ps}$, which implies that $\tilde{q}$ is a multiple of $(r - r_{ps})^2$ and moreover we can write

$$-g^{tt}r^2\tilde{q} = \nu(r)\alpha_S^2(r), \qquad 0 < \nu < 1.$$

The symbol λ^2 of the spherical Laplacian can also be written as sums of squares of differential symbols,

$$\lambda^2 = \sum_{i=1}^{6}\lambda_i^2$$

where in Euclidean coordinates we can write

$$\{\lambda_i\} = \{x_k\eta_j - x_j\eta_k, 1 \leq k < j \leq 4\}.$$

This leads to the sum of squares

(5.15) $$r^2 q^S = (1 - \nu(r))\alpha_S^2(r)\tau^2 + \beta_S^2(r)\xi^2 + \nu(r)\alpha_S^2(r)r^{-2}\left(\sum_{i=1}^{6}\lambda_i^2 + (r^2 - r_s^2)\xi^2\right).$$

The natural counterpart for the Myers-Perry spacetime is the symbol

$$\tilde{s} = if(r)(r - r_{a,b}(\tau, \Phi, \Psi))\xi.$$

This is a homogeneous symbol that coincides with X in the Schwarzschild case $a = b = 0$, and it is well defined for r near r_{ps} and $|\Phi|, |\Psi| < C_{mp}|\tau|$. In particular it is well defined in a neighborhood of the characteristic set $p = 0$, which is all we need.

We can now compute the Poisson bracket on the characteristic set $\{p = 0\}$:

$$\frac{1}{i}\{\rho^2 p, \tilde{s}\} = -(\rho^2 p)_r f(r)(r - r_{a,b}(\tau, \Phi, \Psi)) + \xi(\rho^2 p)_\xi \partial_r\left(f(r)(r - r_{a,b}(\tau, \Phi, \Psi))\right)$$

$$= 2rf(r)R_{a,b}\Delta^{-2}(r - r_{a,b}(\tau, \Phi, \Psi))$$

$$+ \left[2\frac{\Delta}{r^2}\partial_r\left(f(r)(r - r_{a,b}(\tau, \Phi, \Psi))\right) - (\partial_r\frac{\Delta}{r^2})f(r)(r - r_{a,b}(\tau, \Phi, \Psi))\right]\xi^2.$$

Since $r_{a,b}(\tau, \Phi, \Psi)$ is the unique zero of $R_{a,b}$ near $r = r_{ps}$ and is close to r_{ps}, it follows that we can write

$$(5.16) \quad \frac{1}{2i}\{\rho^2 p, \tilde{s}\} = \alpha^2(r, \tau, \Phi, \Psi)\tau^2(r - r_{a,b}(\tau, \Phi, \Psi))^2 + \beta^2(r, \tau, \Phi, \Psi)\xi^2 \qquad \text{on } \{p = 0\}$$

where $\alpha, \beta \in S_{hom}^0$ are positive symbols.

Unfortunately $\tilde{s}$ is not a polynomial in τ. To remedy that we first note that

$$\tilde{s} - if(r)(r - r_{ps})\xi \in \epsilon_0 S_{hom}^1.$$

Hence by (the simplest form of) the Malgrange preparation theorem we can write

$$\frac{1}{i}\tilde{s} = (r - r_{ps})f(r)\xi + s_1(r, \xi, \theta, \Theta, \Phi, \Psi) + s_0(r, \xi, \theta, \Theta, \Phi, \Psi)\tau) + \gamma(\tau, r, \xi, \theta, \Theta, \Phi, \Psi)p$$

with $s_1 \in \epsilon_0 S_{hom}^1$, $s_0 \in \epsilon_0 S_{hom}^0$ and $\gamma \in \epsilon_0 S_{hom}^{-1}$. Then we define the desired symbol s by

$$s = i(s_1 + s_0\tau).$$

We also define

$$\alpha_i = \frac{2|\tau_i|}{\tau_1 - \tau_2}\alpha(r, \tau_i, \Phi, \Psi)(r - r_{a,b}(\tau_i, \Phi, \Psi)) \in S_{hom}^0, \qquad \beta_i = \beta(r, \tau_i, \Phi, \Psi),$$

and let C be a large constant. Then we can find $e \in \epsilon_0(S_{hom}^0 + \tau S_{hom}^{-1})$ so that (5.12) holds for

$$\mu_i^2 = \frac{\lambda_i^2}{r^2(\lambda^2 + (r^2 - r_s^2)\xi^2}\frac{\nu}{4}(\alpha_1(\tau - \tau_2) - \alpha_2(\tau - \tau_1))^2, \qquad i = \overline{1, 6},$$

$$\mu_7^2 = \frac{(r^2 - r_s^2)\xi^2}{r^2(\lambda^2 + (r^2 - r_s^2)\xi^2}\frac{\nu}{4}(\alpha_1(\tau - \tau_2) - \alpha_2(\tau - \tau_1))^2$$

$$\mu_8^2 = \frac{1 - \nu}{4}(\alpha_1(\tau - \tau_2) + \alpha_2(\tau - \tau_1))^2, \qquad \mu_9^2 = \frac{1}{2}(\beta_1^2 + \beta_2^2 - C\epsilon_0)\xi^2$$

$$\mu_{10}^2 = \frac{(C\epsilon_0 - \beta_2^2 + \beta_1^2)(\tau - \tau_2)^2}{2(\tau_1 - \tau_2)^2}\xi^2, \qquad \mu_{11}^2 = \frac{(C\epsilon_0 - \beta_1^2 + \beta_2^2)(\tau - \tau_1)^2}{2(\tau_1 - \tau_2)^2}\xi^2$$

The proof is identical to the one in Lemma 4.3 of [68], so we shall skip it.

$\square$

To finish the proof, we will pick S and E so that (5.11) holds. Since s and e are homogeneous, we need to truncate away the low frequencies, so we redefine

$$s := \chi_{>1}s, \qquad e := \chi_{>1}e$$

where χ is a smooth symbol equal to 1 for frequencies greater than 1 and 0 for frequencies $\ll 1$. We also need to truncate near the trapped set. Let χ be a smooth cutoff function

supported near r_{ps} which equals 1 in a neighborhood of r_{ps}, chosen so that we have a smooth partition of unity in r,

$$1 = \chi^2(r) + \chi_o^2(r).$$

We now define the operators

$$S = \chi s^w \chi, \qquad E = \chi e^w \chi - e_{aux}^w D_t$$

where the operator e_{aux}^w is chosen so that the D_t^3 term in (5.8) vanishes:

$$g^{tt} e_{aux}^w + e_{aux}^w g^{tt} = q_{-1}^w$$

The proof that (5.11) holds for small enough ϵ_0 is identical to the one in [68].

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