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Analysis of Quark Mixing Using Binary Tetrahedral Flavor Symmetry

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Abstract

Using the binary tetrahedral group $T^{'}$, the three angles and phase of the quark CKM mixing matrix are pursued by symmetry-breaking which involves $T^{'}$-doublet VEVs and the Chen-Mahanthappa CP-violation mechanism. The NMRT$^{'}$M, Next-to-Minimal-Renormalizable -T$^{'}$-Model is described, and its one parameter comparison to experimental data is explored.

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1 Introduction

To go beyond the standard model based on $G = SU(3) \times SU(2) \times U(1)$ generally has the aim of relating some of the many parameters therein. Well-known possibilities include grand unification $G \in G_{GUT}$, otherwise extending the gauge group $G \in G'$, supersymmetry, technicolor, and finally horizontal or flavor symmetry G_F, a global group commuting with G.

In the present paper we study further the use of G_F, in particular the choice $G_F = T^{'}$, the binary tetrahedral group. This group can combine the advantages of its central quotient $T \equiv A_4$ for leptons with the incorporation of three quark families in a (2+1) pattern with the third much heavier family treated asymmetrically.

We shall employ Higgs scalars which are all electroweak doublets. An alternative approach would be to use electroweak singlets, so-called *flavons*, but that would necessitate non-renormalizable or irrelevant operators which we eschew.

In recent work, two of the present authors, together with Kephart [1], presented a simplified model based on $T^{'}$ flavor symmetry. The principal simplification was that the CKM mixing angles [#4] involving the third quark family were taken to vanish $\Theta_{23} = \Theta_{13} = 0$.

In terms of the scalar field content, all scalar fields are taken to be doublets under electroweak $SU(2)$ with vacuum values which underly the symmetry breaking. Great simplification was originally achieved by the device of restricting scalar fields to irreducible representations of $T^{'}$ which are singlets and triplets only, without any $T^{'}$ doublets. There was a good reason for this because the admission of $T^{'}$-doublet scalars enormously complicates the symmetry breaking. This enabled the isolation of the Cabibbo angle Θ_{12} and to a very reasonable prediction thereof, namely [1] $\tan 2\Theta_{12} = (\sqrt{2})/3$.

Within the same simplified model, in a subsequent paper [2], the departure of Θ_{12} from this $T^{'}$ prediction was used to make predictions for the departure of the neutrino PMNS angles θ_{ij} from their tribimaximal values [3]. Also in that model [4], we suggested a smoking-gun $T^{'}$ prediction for leptonic decay of the standard model Higgs scalar. Other related works are [5–15].

In the present article, we examine the addition of $T^{'}$-doublet scalars. As anticipated in [1], this allows more possibilities of $T^{'}$ symmetry breaking

[#4] Note that here upper case Θ_{ij} refer to quarks (CKM) and lower case θ_{ij} will refer to neutrinos (PMNS).

and permits non-zero values for Θ_{23}, Θ_{13} and δ_{KM}. We present an explicit $(T^{'} \times Z_2)$ model and investigate for all the CKM angles.

To understand the incorporation of $T^{'}$-doublet scalars and to make the present article self-contained, it is necessary to review the previous simplified model employed in [1, 2, 4] in which $T^{'}$-doublet scalars were deliberately excluded in order to isolate the Cabibbo angle Θ_{12}. We here adopt the global symmetry $(T^{'} \times Z_2)$.

Note that we focus on a renormalizable model with few if any free parameters and focus on the mixing matrix rather than on masses as the former is more likely to have a geometrical interpretation while without adding many extra parameters the masses are unfortunately not naturally predicted. This is especially true for the lighter quarks; for the t quark the flavor group assignments allow it much heavier mass.

2 The Previous Simplified Model, MRT$^{'}$M

By $MRT^{'}M$, we mean Minimal Renormalizable $T^{'}$ Model. Actually the global symmetry, to restrict the Yukawa couplings is $(T^{'} \times Z_2)$.

Left-handed quark doublets $(t,b)_L, (c,d)_L, (u,d)_L$ are assigned under $(T^{'} \times Z_2)$ as

$$\begin{array}{lll} \begin{pmatrix} t \\ b \end{pmatrix}_L & \mathcal{Q}_L & (\mathbf{1_1}, +1) \\ \left.\begin{array}{l} \begin{pmatrix} c \\ s \end{pmatrix}_L \\ \begin{pmatrix} u \\ d \end{pmatrix}_L \end{array}\right\} & Q_L & (\mathbf{2_1}, +1), \end{array} \tag{1}$$

and the six right-handed quarks as

$$\begin{array}{lll} t_R & & (\mathbf{1_1}, +1) \\ b_R & & (\mathbf{1_2}, -1) \\ \left.\begin{array}{l} c_R \\ u_R \end{array}\right\} & \mathcal{C}_R & (\mathbf{2_3}, -1) \\ \left.\begin{array}{l} s_R \\ d_R \end{array}\right\} & \mathcal{S}_R & (\mathbf{2_2}, +1). \end{array} \tag{2}$$

The leptons are assigned as

$$
\left.\begin{array}{l}
\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \\
\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \\
\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L
\end{array}\right\} L_L(3,+1) \quad
\begin{array}{ll}
\tau_R^- \; (1_1,-1) & N_R^{(1)} \; (1_1,+1) \\
\mu_R^- \; (1_2,-1) & N_R^{(2)} \; (1_2,+1) \\
e_R^- \; (1_3,-1) & N_R^{(3)} \; (1_3,+1),
\end{array}
\tag{3}
$$

Next we turn to the symmetry breaking and the necessary scalar sector with its own potential [#5] and Yukawa coupling to the fermions, leptons and quarks.

The scalar fields in the previous simplified model were namely the two $T^{'}$ triplets and two $T^{'}$ singlets

$$
H_3(3,+1); \; H_3^{'}(3,-1); \; H_{1_1}(1_1,+1); \; H_{1_3}(1_3,-1) \tag{4}
$$

which leads to CKM angles $\Theta_{23} = \Theta_{13} = 0$. That model was used to derive a formula for the Cabibbo angle [1], to predict corrections [2] to the tribimaximal values [3] of PMNS neutrino angles, and to make a prediction for Higgs boson decay [4].

The Yukawa couplings for the $T^{'}$-triplet and $T^{'}$-singlet scalars were as follows:

$$
\begin{aligned}
\mathcal{L}_Y = {} & \frac{1}{2} M_1 N_R^{(1)} N_R^{(1)} + M_{23} N_R^{(2)} N_R^{(3)} \\
& + \Big\{ Y_1 \left(L_L N_R^{(1)} H_3 \right) + Y_2 \left(L_L N_R^{(2)} H_3 \right) + Y_3 \left(L_L N_R^{(3)} H_3 \right) \\
& + Y_\tau \left(L_L \tau_R H_3' \right) + Y_\mu \left(L_L \mu_R H_3' \right) + Y_e \left(L_L e_R H_3' \right) \Big\} \\
& + Y_t (\{\mathcal{Q}_L\}_{\mathbf{1_1}} \{t_R\}_{\mathbf{1_1}} H_{\mathbf{1_1}}) \\
& + Y_b (\{\mathcal{Q}_L\}_{\mathbf{1_1}} \{b_R\}_{\mathbf{1_2}} H_{\mathbf{1_3}}) \\
& + Y_{\mathcal{C}} (\{Q_L\}_{\mathbf{2_1}} \{\mathcal{C}_R\}_{\mathbf{2_3}} H_{\mathbf{3}}^{'}) \\
& + Y_{\mathcal{S}} (\{Q_L\}_{\mathbf{2_1}} \{\mathcal{S}_R\}_{\mathbf{2_2}} H_{\mathbf{3}}) \\
& + \text{h.c.}.
\end{aligned}
\tag{5}
$$

[#5] The scalar potential will not be examined explicitly. We assume that it has enough parameters to accommodate the required VEVs in a finite neighborhood of parameter values.

3 Choice of the Present Model, NMRT$^{'}$M

By $NMRT^{'}M$ we mean Next-to Minimal Renormalizable $T^{'}$ Model.

We introduce one $T^{'}$ doublet scalar in an explicit model. Non-vanishing Θ_{23} and Θ_{13} will be induced by symmetry breaking due to the addition the $T^{'}$ doublet scalar.

The possible choices under $(T^{'} \times Z_2)$ for the new scalar field are:

$$\mathbf{A} \quad H_{2_1}(2_1, +1) \tag{6}$$

$$\mathbf{B} \quad H^{'}_{2_3}(2_3, -1) \tag{7}$$

$$\mathbf{C} \quad H^{'}_{2_2}(2_2, -1) \tag{8}$$

$$\mathbf{D} \quad H_{2_3}(2_3, +1) \tag{9}$$

The fields in Eqs.(6,7,8,9) allow respectively Yukawa couplings:

$$\mathbf{A} \quad Y_{Qt} Q_L t_R H_{2_1} + h.c. \tag{10}$$

$$\mathbf{B} \quad Y_{Qb} Q_L b_R H^{'}_{2_3} + h.c. \tag{11}$$

$$\mathbf{C} \quad Y_{\mathcal{QC}} \mathcal{Q}_L \mathcal{C}_R H^{'}_{2_2} + h.c. \tag{12}$$

$$\mathbf{D} \quad Y_{\mathcal{QS}} \mathcal{Q}_L \mathcal{S}_R H_{2_3} + h.c. \tag{13}$$

This leads potentially to different extensions of the $MRT^{'}M$. For simplicity we keep only one additional term, **D**, inspired by the Chen-Mahanthappa mechanism [16] for CP violation. We shall keep $Y_{\mathcal{QS}}$ real and CP violation will arise from the imaginary part of $T^{'}$ Clebsch-Gordan coefficients.

The vacuum expectation value (VEV) for H_{2_3} is taken with the alignment

$$< H_{2_3} >= V_{2_3}(1, 1) \tag{14}$$

while as in [1] the other VEVs include

$$< H_3 >= V(1, -2, 1) \tag{15}$$

4 Predictions of NMRT$'$M (D)

From the Yukawa term **D** and the vacuum alignment we can derive for the down-quark mass matrix

$$D = \begin{pmatrix} M_b & \frac{1}{\sqrt{2}} Y_{\mathcal{QS}} V_{2_3} & \frac{1}{\sqrt{2}} Y_{\mathcal{QS}} V_{2_3} \\ 0 & \frac{1}{\sqrt{3}} Y_{\mathcal{S}} V & -2\sqrt{\frac{2}{3}} \omega Y_{\mathcal{S}} V \\ 0 & \sqrt{\frac{2}{3}} Y_{\mathcal{S}} V & \frac{1}{\sqrt{3}} \omega Y_{\mathcal{S}} V \end{pmatrix} \tag{16}$$

where $M_b = Y_b V_{1_3}$ and $\omega = e^{i\pi/3}$.

The hermitian squared mass matrix $\mathcal{D} \equiv D D^\dagger$ for the charge $(-1/3)$ quarks is then

$$\mathcal{D} = \begin{pmatrix} M_b'^2 & \frac{1}{\sqrt{6}} Y_{\mathcal{S}} Y_{\mathcal{QS}} V V_{2_3} (1 - 2\sqrt{2}\omega^2) & \frac{1}{\sqrt{6}} Y_{\mathcal{S}} Y_{\mathcal{QS}} V V_{2_3} (\omega^2 + \sqrt{2}) \\ \frac{1}{\sqrt{6}} Y_{\mathcal{S}} Y_{\mathcal{QS}} V V_{2_3} (1 - 2\sqrt{2}\omega^{-2}) & 3 (Y_{\mathcal{S}} V)^2 & -\frac{\sqrt{2}}{3} (Y_{\mathcal{S}} V)^2 \\ \frac{1}{\sqrt{6}} Y_{\mathcal{S}} Y_{\mathcal{QS}} V V_{2_3} (\omega^{-2} + \sqrt{2}) & -\frac{\sqrt{2}}{3} (Y_{\mathcal{S}} V)^2 & (Y_{\mathcal{S}} V)^2 \end{pmatrix} \tag{17}$$

where $M_b'^2 = M_b^2 + (Y_{\mathcal{QS}} V_{2_3})^2$.

Note that in this model the mass matrix for the charge $+2/3$ quarks is diagonal [#6] so the CKM mixing matrix arises purely from diagonalization of $\mathcal{D}$ in Eq.(17). The presence of the complex T' Clebsch-Gordan in Eq. (17) permits a Chen-Mahanthappa origin [16] for the KM CP violating phase.

In Eq.(17) the 2×2 sub-matrix for the first two families coincides with the result discussed earlier [1] and hence the successful Cabibbo angle formula $\tan 2\Theta_{12} = (\sqrt{3})/2$ is preserved as follows.

The relevant 2×2 submatrix of $\mathcal{D}$ is proportional to

$$\mathcal{D}_{2\times 2} = \begin{pmatrix} 3 & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & 1 \end{pmatrix} \tag{18}$$

whose diagonalization leads to the Cabibbo angle formula

$$\tan 2\Theta_{12} = \sqrt{3}/2. \tag{19}$$

[#6] This uses the approximation that the electron mass is $m_e = 0$; *c.f.* ref. [1].

For m_b^2 the experimental value is $17.6 GeV^2$ [17] although the CKM angles and phase do not depend on this overall normalization.

Actually our results depend only on assuming that the ratio $(Y_{\mathcal{QS}} V_{2_3} / Y_{\mathcal{S}} V)$ is much smaller than one.

Defining

$$\mathcal{D}^{'} = 3\mathcal{D}/(Y_{\mathcal{S}} V)^2 \tag{20}$$

we find

$$\mathcal{D}^{'} = \begin{pmatrix} \mathcal{D}^{'}_{11} & Ae^{i\psi_1} & A\eta e^{i\psi_2} \\ Ae^{-i\psi_1} & 9 & -\sqrt{2} \\ A\eta e^{-i\psi_2} & -\sqrt{2} & 3 \end{pmatrix} \tag{21}$$

in which we denoted

$$\mathcal{D}^{'}_{11} = 3M_b^{'2}/(Y_{\mathcal{S}} V)^2 \tag{22}$$

$$A = \left(\sqrt{\frac{3}{2}}\right) \left(\frac{Y_{\mathcal{QS}} V_{2_3}}{Y_{\mathcal{S}} V}\right) |1 - 2\sqrt{2}\omega^2| \tag{23}$$

$$\eta = \left| \frac{\omega^2 + \sqrt{2}}{1 - 2\sqrt{2}\omega^2} \right| = 0.33615... \tag{24}$$

$$\tan\psi_1 = \frac{-\sqrt{6}}{1 + \sqrt{2}} = -1.01461... \tag{25}$$

$$\tan\psi_2 = \frac{\sqrt{3}}{2\sqrt{2} - 1} = 0.94729... \tag{26}$$

To arrive at predictions for the other CKM mixing elements other than the Cabibbo angle (*i.e.* $\Theta_{13}, \Theta_{23}, \delta_{KM}$) one needs only to diagonalize the matrix $\mathcal{D}^{'}$ in Eq.(21) by

$$\mathcal{D}^{'}_{diagonal} = V^{\dagger}_{CKM} \mathcal{D}^{'} V_{CKM} \tag{27}$$

We write the mixing matrix as

$$V_{CKM} = \begin{pmatrix} 1 & V_{ts} & V_{td} \\ V_{cb} & \cos\Theta_{12} & \sin\Theta_{12} \\ V_{ub} & -\sin\Theta_{12} & \cos\Theta_{12} \end{pmatrix} \tag{28}$$

and substituting Eq.(28) into Eq.(27) and using Eq.(21) leads to

$$\begin{pmatrix} V_{cb} \\ V_{ub} \end{pmatrix} = \frac{1}{\hat{\mathcal{D}}'_{11}} \begin{pmatrix} \mathcal{D}'_{11} - 3 & -\sqrt{2} \\ -\sqrt{2} & \mathcal{D}'_{11} - 9 \end{pmatrix} \begin{pmatrix} Ae^{-i\psi_1} \\ Ae^{-i\psi_2} \end{pmatrix} \tag{29}$$

where $\hat{\mathcal{D}}'_{11} = (\mathcal{D}'_{11} - 6 - \sqrt{11})(\mathcal{D}'_{11} - 6 + \sqrt{11})$.

while from unitarity it follows that

$$\begin{pmatrix} V_{ts} \\ V_{td} \end{pmatrix} = - \begin{pmatrix} \cos\Theta_{12} & -\sin\Theta_{12} \\ \sin\Theta_{12} & \cos\Theta_{12} \end{pmatrix} \begin{pmatrix} V^*_{cb} \\ V^*_{ub} \end{pmatrix} \tag{30}$$

The strategy now is to calculate the CP-violating Kobayashi-Maskawa phase given by

$$\delta_{KM} = \gamma = \arg\left(-\frac{V_{ud}V^*_{ub}}{V_{cd}V^*_{cb}} \right) \tag{31}$$

and using Eqs.(28,29) we arrive at the formula in terms of $\mathcal{D}_{11}$

$$\delta_{KM} = \gamma_{T'} = \arg\left[\frac{-\sqrt{2} + (\mathcal{D}'_{11} - 9)\eta e^{-i(\psi_1 - \psi_2)}}{(\mathcal{D}'_{11} - 3) - \sqrt{2}\eta e^{-i(\psi_1 - \psi_2)}} \right] = \arg[\Gamma(\mathcal{D}'_{11})] \tag{32}$$

where Γ, a function of $\mathcal{D}'_{11}$, is defined for later use.

In Fig. 1, we show a plot of $\gamma_{T'}$ versus $\mathcal{D}'_{11}$ using Eq.(32) and taking the range of experimentally-allowed $\gamma \equiv \delta_{KM}$ from the global fit [18] prompts us to use a value $\mathcal{D}'_{11} = 19 \pm 2$ in the subsequent analysis.

From the preceding equations (28,29) we find a formula for

$$|V_{ub}/V_{cb}| = |\tan\Theta_{13}\sin\Theta_{23}| \tag{33}$$

using unitarity, Eq.(30), from the form for the ratios of CKM matrix elements

$$|V_{td}/V_{ts}| = \left| \frac{\sin\Theta_{12} + \Gamma(\mathcal{D}'_{11})\cos\Theta_{12}}{\cos\Theta_{12} - \Gamma(\mathcal{D}'_{11})\sin\Theta_{12}} \right| \tag{34}$$

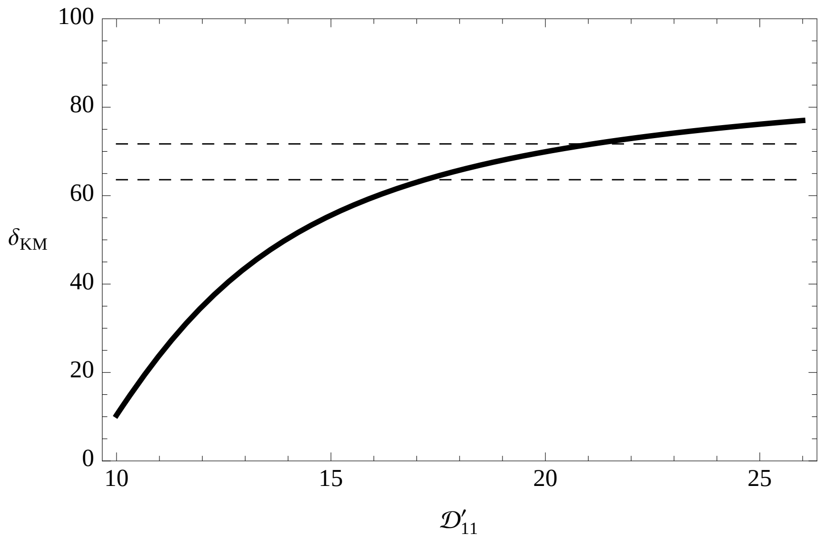

Figure 1: The vertical axis is the value of $\delta_{KM} \equiv \gamma_{T'}$ in degrees and the horizontal axis is the value of $\mathcal{D}'_{11}$ defined in the text. The dashed horizontal lines give the 1σ range for δ_{KM} allowed by the global fit of [18].

Fig. 2 shows a plot of $|V_{td}/V_{ts}|$ as a function of $\mathcal{D}'_{11}$. It requires a value of $\mathcal{D}'_{11}$ of approximately 16 which is sufficiently close to that in Fig. 1.

For the value of $|V_{ub}/V_{cb}|$ there is approximately a factor two between the prediction (higher) and the best value from [18].

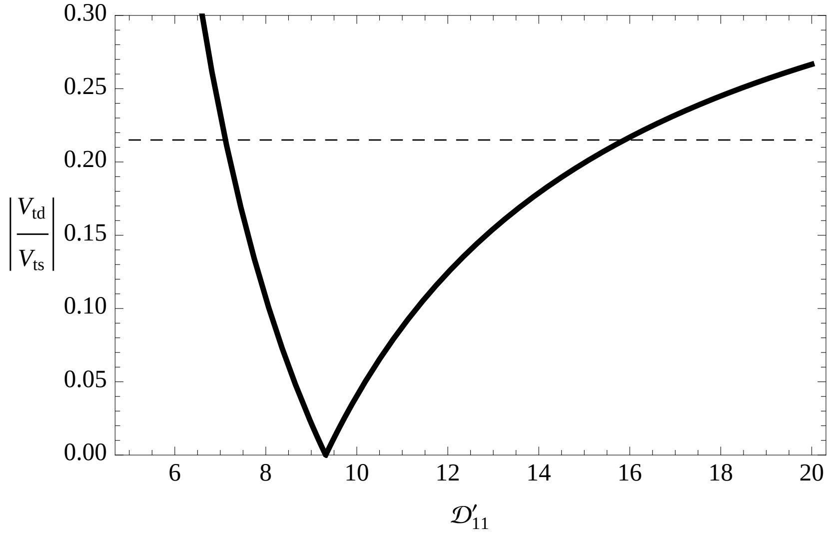

Figure 2: The vertical axis is the value of $|V_{td}/V_{ts}|$ and the horizontal axis is the value of $\mathcal{D}'_{11}$ defined in the text. The dashed horizontal lines give the value with small error allowed by the global fit of [18].

5 Discussion

Note that once the off-diagonal third-family elements in Eq.(17) are taken as much smaller than the elements involved in the Cabibbo angle, the two KM angles and the CP phase are predicted by the present NMRT$'$M in general agreement so this vindicates the hope expressed in [1].

With regard to alternative NMRT$'$M models discussed earlier the possibilities **A** and **C** modify the charge-2/3 mass matrix where we take flavor and mass eigenstates coincident. The final possibility **C** does modify the charge (-1/3) mass matrix but does not permit CP violation to arise from the Chen-Mahanthappa mechanism as in the **D** model we have analysed both here and in [19].

With respect to the article [19] which was letter length, the present article presents more technical detail and figures to clarify the results and predictions merely stated in [19] without explanation.

In summary, we have reported results of studying mixing angles by exploying the binary tetrahedral group (T') as a global discrete flavor symmetry commuting with the local gauge symmetry $SU(3) \times SU(2) \times U(1)$ of the standard model of particle phenomenology. The results are encouraging to pursue this direction of study.

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