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A relation between approaches to integrability in superconformal Yang-Mills theory

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Abstract: We make contact between the infinite-dimensional non-local symmetry of the type-IIB superstring on $AdS_5 \times S^5$ and a non-abelian infinite-dimensional symmetry algebra for the weakly coupled superconformal gauge theory. We explain why the planar limit of the one-loop dilatation operator is the hamiltonian of a spin chain, and show that it commutes with the $g^2N = 0$ limit of the non-abelian charges.

Keywords: 1/N Expansion, Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence, Global Symmetries.

Contents

1. Introduction

It has long been conjectured [1] that there might be integrable structures in four-dimensional quantum gauge theory, analogous to the known integrable structures in two-dimensional sigma models and possibly extending what is known for self-dual gauge fields in four dimensions (see for example [2]–[5]). Any such result is bound to be related to the planar or large-N limit of gauge theories [6], for a simple reason. There is no chance that quantum SU(N) gauge theory would turn out to be integrable for any given N, because the phenomena it describes (such as nuclear physics for $N = 3$ when quarks are included) are far too complicated. But the phenomena are believed to simplify for $N \to \infty$ (for example, confining theories become free in this limit), making integrability conceivable.

A possible clue of this has appeared some time ago in studies of high energy scattering in gauge theories [7, 8]. Lately, two different developments have pointed to integrable structures in the large-N limit of four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM). One development, which was stimulated by the BMN model of the plane wave limit of $\mathrm{AdS}_5 \times S^5$ [9], has involved the study of the dilatation operator in perturbation theory. Initially in some special cases [10] and later in greater generality [11, 12] following additional work [13, 14, 15], it has been argued that the one-loop anomalous dimension operator can be interpreted as one of the commuting hamiltonians of an integrable spin chain. This result depends on the one-loop anomalous dimensions of twist two operators, which were also computed in [16, 17]. That the one-loop anomalous dimension operator in non-supersymmetric gauge theory is integrable asymptotically for large angular momentum at least for a very large class of operators had been discovered earlier in a parallel line of development [18]–[21] .

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The other development, for which the bosonic theory gave a clue [22], is that the classical Green-Schwarz superstring action for $\mathrm{AdS}_5 \times S^5$, constructed in [23], has turned out [24] to possess a hierarchy of non-local symmetries, presumably implying that the world-sheet theory is a two-dimensional integrable system, analogous to many other such systems that are known. (See [26, 27] for construction of nonlocal conserved charges in sigma models, and [28] for an extensive introduction to a variety of types of integrable two-dimensional model.) Experience with other two-dimensional systems indicates that the non-local nature of these symmetries gives them the potential to constrain the perturbative string spectrum (i.e. the gauge theory spectrum at $N = \infty$) without usefully constraining the exact string amplitudes summed over genus (corresponding to an exact solution of gauge theory at all N — too much to ask for, as noted above). One does, however, hope that these symmetries are relevant to gauge theory in the large-N limit for all values of g^2N, not just at $g^2N = \infty$ where the classical analysis in [24] applies. A step in this direction has been obtained by showing [29] that analogous non-local symmetries hold in the Berkovits description [30] of $\mathrm{AdS}_5 \times S^5$.

The present paper aims at a modest step toward relating these two developments. Starting with the non-local symmetries of [24], we will try to deduce *why* the one-loop anomalous dimension operator is the hamiltonian of an integrable spin chain. We begin in section 2 by guessing how the non-local symmetries should act on a chain of Yang-Mills partons at $g^2N = 0$. The symmetries generate a non-abelian algebra that has been called the Yangian, and there is a natural (and standard) way for the Yangian to be realized in a chain of partons or spins.

We conjecture that this is the $g^2N = 0$ limit of how the Yangian operators act in Yang-Mills theory. We then argue, also in section 2, that the one-loop anomalous dimension operator must commute with the $g^2N = 0$ limit of the Yangian. Then in the rest of the paper, we verify the commutativity explicitly using formulas and properties developed in [11, 12]. Generally, for $1+1$-dimensional systems, the local operators that commute with the Yangian are the integrable hamiltonians, so this commutativity means that the one-loop dilatation operator is the hamiltonian of an integrable spin chain. See for example [33].

The argument showing that the one-loop anomalous dimension operator commutes with the $g^2N = 0$ limit of the Yangian is special to one loop. Beyond one loop, we do not expect the dilatation operator to commute with the Yangian. The general structure is that, like all the generators of the superconformal group $\mathrm{PSU}(2,2|4)$, the exact dilatation operator is one of the generators of the Yangian. Many of the generators of the Yangian, including the dilatation operator, receive perturbative corrections beyond one loop. For example, in higher orders, there are perturbative corrections to the dilatation operator that do not conserve the number of partons (i.e. the length L of the spin chain), so in general this system cannot be viewed as a conventional spin chain with the partons as spins. Certainly, then, in general the Yangian generators receive corrections.

For some special sets of states, such as sets considered in [9, 10, 12], the quantum numbers are such as to prevent creation and annihilation of partons, and it is plausible (and has been proposed) that in such such sets of states, the exact dilatation operator is the hamiltonian of an integrable spin chain. It may be that the higher generators Q^A

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of the Yangian have no corrections when restricted to such sectors; this might lead to an interpretation of the exact dilatation operator as an integrable hamiltonian in such a sector. Of the appendices in this paper, only appendix C develops material that is actually used in the main text.

2. Non-local generators

We begin by recalling how non-local symmetries arise in two-dimensional sigma models [26]. One considers a model with a group G of symmetries; the Lie algebra of G has generators T_A obeying $[T_A, T_B] = f_{AB}^C T_C$. The action of G is generated by a current $j^{\mu\,A}$ that is conserved, $\partial_\mu j^{\mu\,A} = 0$. Nonlocal charges arise if in addition the Lie algebra valued current $j_\mu = \sum_A j_\mu^A T_A$ can be interpreted as a flat connection,

$$\partial_\mu j_\nu - \partial_\nu j_\mu + [j_\mu, j_\nu] = 0\,. \tag{2.1}$$

(Indices of j_μ are raised and lowered using the Lorentz metric in two dimensions.) The conservation of j_μ leads in the usual fashion to the existence of conserved charges that generate the action of G:

$$J^A = \int_{-\infty}^{\infty} dx\, j^{0\,A}(x,t)\,. \tag{2.2}$$

In addition, a short computation using (2.1) reveals that

$$Q^A = f_{BC}^A \int_{-\infty}^{\infty} dx \int_x^{\infty} dy\, j^{0\,B}(x,t)\, j^{0\,C}(y,t) - 2\int_{-\infty}^{\infty} dx\, j_1^A(x,t) \tag{2.3}$$

is also conserved. (For the moment we take the spatial direction to be noncompact, although in string theory it is more relevant to compactify on a circle with periodic boundary conditions. When one does compactify, Q^A cannot be defined, as the restriction to $x < y$ does not make sense.) Under repeated commutators, the Q^A generates an infinite-dimensional symmetry algebra that has been called the Yangian. The Yangian has a basis $\mathcal{J}_n^A$ where $\mathcal{J}_0^A = J^A$, $\mathcal{J}_1^A = Q^A$, and $\mathcal{J}_n^A$ is an n-local operator that arises in the $(n-1)$-form commutator of Q's. Since we will work in this paper mainly with the generators J^A and Q^A, we have given them those special names.

The detailed algebraic structure of the Yangian is rather complicated and will not be needed in the present paper. We pause, however, to briefly explain how the Yangian is related to the much simpler partial Kac-Moody algebra that can also be defined in such systems [2, 31, 32]. (This is also explained briefly in [33]–[36].) The Yangian generates by Poisson brackets some transformations of the fields Φ that we write schematically $\delta\Phi = \sum_{n,A} \epsilon_{n,A} \mathcal{J}_n^A(\Phi)$, where $\epsilon_{n,A}$ are infinitesimal parameters. These transformations are symmetries of the classical equations of motion, since the Yangian generators commute with the hamiltonian. The partial Kac-Moody algebra is generated by infinitesimal transformations $\delta\Phi = \sum_{n,A} \tilde{\epsilon}_{n,A} \delta_n^A(\Phi)$, where $\tilde{\epsilon}_{n,A}$ are another set of parameters and the objects δ_n^A are certain infinitesimal symmetries of the equations. Usually, one considers transformations with field-independent coefficients ϵ or $\tilde{\epsilon}$. However, the equivalence relation generated by the symmetries is obtained by letting ϵ and $\tilde{\epsilon}$ be arbitrary, so it does not matter if they

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are field-dependent. In the problem at hand, the Yangian and partial Kac-Moody symmetries are different, but become equivalent if one lets ϵ (or $\tilde{\epsilon}$) be field-dependent. This is shown explicitly in appendix A. Thus the Yangian and partial Kac-Moody algebras are different but generate the same equivalence relation. The relation between them is somewhat similar to the relation between commutative and non-commutative Yang-Mills gauge transformations as given by the Seiberg-Witten map [37].

There are also discrete spin systems, that is systems in which the dynamical variables live on a one-dimensional lattice rather than on the real line, that have the same Yangian symmetry. The lattice definition of J^A is clear. We assume that the spins at each site i have G symmetry, and let J_i^A be the symmetry operators at the i^{th} site. The total charge generator for the whole system is then

$$J^A = \sum_i J_i^A \,. \tag{2.4}$$

What about Q^A? At least the bilocal part of (2.3) has an obvious discretization:

$$Q^A = f^A_{BC} \sum_{i<j} J_i^B J_j^C \,. \tag{2.5}$$

This turns out to be the right formula, in many of the most commonly studied lattice integrable systems. Note that one has made no attempt to discretize the second term in (2.3). This proves to be unnecessary.[1] The bare generators (2.4) and (2.5) satisfy:

$$\begin{aligned}
[J^A, J^B] &= f_{ABC} J^C \\
[J^A, Q^B] &= f_{ABC} Q^C \,.
\end{aligned} \tag{2.6}$$

An analog of (2.5) in gauge theory at $g^2N = 0$ is described in appendix B.

One generator of $\mathrm{PSU}(2,2|4)$, called the dilatation generator D, is of special importance. In the radial quantization of four-dimensional superconformal Yang Mills theory on $\mathbb{R} \times S^3$, D is the hamiltonian.[2] Using conformal invariance to identify $\mathbb{R} \times S^3$ with $\mathbb{R}^4$, the states of the quantum theory on $\mathbb{R} \times S^3$ are in one-to-one correspondence with local operators $\mathcal{O}(x)$, where the correspondence is $|\mathcal{O}\rangle \sim \lim_{x\to 0} \mathcal{O}(x)|0\rangle$. In the large-$N$ limit of the gauge theory, we focus on single-trace local operators. Such an operator is the trace of a product of letters where a letter is as follows. A letter is one of the elementary fields $\phi^I = \phi^{I\mathcal{A}}(x)\mathcal{T}^{\mathcal{A}}$, $\psi^a_\alpha = \psi^{a\mathcal{A}}_\alpha(x)\mathcal{T}^{\mathcal{A}}$, $F_{\mu\nu} = F^{\mathcal{A}}_{\mu\nu}(x)\mathcal{T}^{\mathcal{A}}$, or the (symmetrized) n^{th} derivative of one of those, for some $n > 0$. (The indices $1 \le I \le 6$, $1 \le a \le 4$ label the vector and spinor SU(4) R-symmetry representations.) A single-trace operator $\mathcal{O}(x)$ is said to be of length L if it is the trace of a product of L letters. In the correspondence between operators and states, the letters form a basis for the one-particle states of the free $\mathcal{N} = 4$ vector multiplet on $\mathbb{R} \times S^3$.

[1] In many simple models, it is also impossible for elementary reasons. One would expect a discretization of $\int dx\, j_1^A$ to be of the form $\sum_i j_i^A$ where j_i^A acts on the i^{th} site and transforms in the adjoint representation. Frequently, there is no such operator except J_i^A. But taking j_i^A to be a multiple of J_i^A would simply add to Q^A a multiple of J^A; this is a change of basis that does not affect the structure of the algebra.

[2] D is conjugate in $\mathrm{PSU}(2,2|4)$ to $\frac{1}{2}(P_0 + K_0)$, where P_μ, K_μ are the translations and special conformal transformations. For radial quantization on a hyperboloid, see [25] .

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We really want to consider a gauge-invariant state that is a single trace $\mathcal{O}(x) = \mathrm{Tr}\,\Phi_1(x)\Phi_2(x)\ldots\Phi_L(x)$ of a possibly very large number of fields Φ_i, each of which is one of the letters considered above. As in many papers cited in the introduction, we think of the choice of a given $\mathcal{O}$ as representing in free field theory a state of a chain of L "spins" (which we also call "partons"). Our "spins," therefore, are simply one-particle states in the $\mathcal{N} = 4$ vector multiplet quantized on S^3. In identifying the possible operators $\mathcal{O}$ with the states of a spin chain, one ignores the cyclic symmetry of the trace. One studies all possible states of the spin chain, even though in the application to gauge theory one only wants the (gauge invariant) cyclically symmetric states.

Our basic assumption in this paper is that in $\mathcal{N} = 4$ super Yang-Mills theory at $g^2N = 0$, with J_i^A understood as the PSU$(2,2|4)$ generators of the i^{th} parton, (2.5) is the correct formula for the Yangian generators Q^A. Our assumption, in other words, is that the bilocal symmetry deduced from [24] goes over to (2.5) for $g^2N \to 0$. Of course, in any case (2.4) is the appropriate free field formula for the J^A, so we do not need to state any hypothesis for these generators. And no further assumption is needed for the higher charges in the Yangian; they are generated by repeated commutators of the Q^A. So our hypothesis about Q^A completely determines the form of the Yangian in the free-field limit.

Now we consider what happens when g^2N is not quite zero. Some generators of the Yangian do not receive quantum corrections. For example, the spatial translation symmetries and the Lorentz generators are uncorrected, because the theory can be regularized in a way that preserves them. But the dilatation operator D – the generator of scale transformations — certainly is corrected. The corrections to the eigenvalues of D are called anomalous dimensions.

We assume, in view of [24], that the $\mathcal{N} = 4$ Yang-Mills theory in the planar limit does have Yangian symmetry for all g^2N. If so, the corrections modify the form of the generators, but preserve the commutation relations. One of the commutation relations says that Q^A transforms in the adjoint representation of the global group PSU$(2,2|4)$ generated by J^A: $[J^A, Q^B] = f^{ABC}Q^C$. We will write J^A and Q^A for the charges at $g^2N = 0$, and δJ^A and δQ^A for the corrections to them of order g^2N. We write $\tilde{J}^A$ and $\tilde{Q}^A$ for the exact generators (which depend on g^2N), so $\tilde{J}^A = J^A + (g^2N)\delta J^A + \mathcal{O}((g^2N)^2)$, and likewise for $\tilde{Q}^A$. To preserve the commutation relations, we have

$$[\delta J^A, Q^B] + [J^A, \delta Q^B] = f^{ABC}\delta Q^C\,. \tag{2.7}$$

We are now going to make an argument for the Yangian that parallels one used in [11] for the PSU$(2,2|4)$ generators. We consider the special case of this relation in which A is chosen so that J^A is the dilatation operator D. We also pick a basis Q^B of the Q's to diagonalize the action of D, so the PSU$(2,2|4)$ algebra reads in part $[D, Q^B] = \lambda^B Q^B$, where λ^B is the bare conformal dimension of Q^B. Then (2.7) gives us

$$[\delta D, Q^B] + [D, \delta Q^B] = \lambda^B \delta Q^B\,. \tag{2.8}$$

However, in perturbation theory, operators only mix with other operators of the same classical dimension. So just as $[D, Q^B] = \lambda^B Q^B$, we have $[D, \delta Q^B] = \lambda^B \delta Q^B$. Combining this with (2.7), we have therefore

$$[\delta D, Q^B] = 0\,. \tag{2.9}$$

Precisely the same argument was used in [11] to show that $[\delta D, J^A] = 0$; this was a step in determining δD. Combining this with (2.9), we see that δD must commute with the $g^2 N = 0$ limit of the whole Yangian.

The structure of perturbation theory implies in addition that the operator δD is a sum of operators local along the chain; this fact has been exploited in [9] and many subsequent papers. (In fact, δD, as described explicitly in [11], is a sum of operators that act on nearest neighbor pairs.) The operators of this type that commute with the Yangian — where here we mean the Yangian representation most commonly studied in lattice integrable models, which for us is the one generated at $g^2N = 0$ by J^A and Q^A — are called the hamiltonians of the integrable spin chain. Thus, from our assumption about the free-field limit of the Yangian, we have been able, starting with the nonlocal symmetries found in [24], to deduce the basic conclusion of [12], found earlier in a special case in [10], that δD is a hamiltonian of an integrable spin chain.

In the remainder of this paper, we will verify this picture by proving directly, using formulas from [11, 12], that it is true that δD commutes with the Yangian. Since its commutativity with J^A was already used in [11] to compute δD, we only need to verify that $[\delta D, Q^A] = 0$.

From what we have said, it is clear that the appearance of a hamiltonian that commutes with the Yangian depends on expanding to first order near $g^2N = 0$. In the exact theory, at a nonzero value of g^2N, one would simply say that the exact dilatation operator $\mathcal{D}$, which of course depends on g^2N, is one of the generators of the Yangian. It is not the case in the exact theory that one has a Yangian algebra and also a dilatation operator that commutes with it.

In string theory, or Yang-Mills theory, one really wants to compactify the string (or the spin chain) on a circle with periodic boundary conditions, since the string is closed. This makes it impossible to define the Yangian, because the restriction of the integration region in (2.3) to $x < y$ does not make sense. The global PSU$(2,2|4)$ generators J^A still make sense, of course, and so do some globally defined operators — traces of holonomies, which one might think of as Casimir operators of the Yangian. These Casimir operators, which commute with PSU$(2,2|4)$, perhaps can be used as an aid in computing the spectrum of $\mathcal{N} = 4$ super Yang-Mills theory in the planar limit. Some of these Casimir operators are odd under charge conjugation (which is the symmetry that reverses the order of the spin chain), so the fact that they commute with PSU$(2,2|4)$ would lead to degeneracies among states of opposite charge conjugation properties, as found and exploited in [14].

3. Commutation of Q^A with the planar one-loop hamiltonian

Now we will prove that (2.9)

$$[\delta D, Q^A] = 0 \tag{3.1}$$

is true, using the properties of δD for the super Yang Mills theory. In this section, to simplify notation, and in view of its interpretation as the hamiltonian of an integrable spin chain, we refer to δD as H. Actually, we will show that the commutator $[H, Q^A]$ is the lattice version of a total derivative, in the following sense. The general form of H for a chain of length L is that it is a sum of operators each of which only acts on nearest neighbors,

$$H = \sum_{i=1}^{L-1} H_{i,i+1} \,. \tag{3.2}$$

A lattice version of a total derivative is an expression such as

$$q^A = \sum_{i=1}^{L-1} \left(J_i^A - J_{i+1}^A\right) = J_1^A - J_{\mathrm{L}}^A \,, \tag{3.3}$$

which is a sum of difference operators along the chain and only acts at the ends of the chain.

We will show, using the specific form of H determined in [11], that

$$[H, Q^A] = q^A \,, \tag{3.4}$$

where q^A is such a total derivative. For an infinite chain (which would correspond more closely to the $1+1$-dimensional field theory studied in [24]) and assuming no spontaneous symmetry breaking so that surface terms at infinity can be dropped, the total derivative term in (3.4) vanishes, and in that sense $[H, Q^A] = 0$ for an infinite chain. For a finite chain with periodic boundary conditions, the situation is similar though more subtle. (3.4) together with the fact that $[H, J^A] = 0$ implies that the commutator of H with any generator of the Yangian is the integral of a lattice total derivative. Therefore, for a finite chain with periodic boundary conditions, where a total derivative will sum to zero, the commutator of H with a Casimir operator of the Yangian (which is well-defined with periodic boundary conditions) vanishes.

Let V_F be the space of one-particle states in free $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R} \times S^3$. (Thus the set of letters as defined above is a vector space basis for V_F.) In the spin chain that is relevant to planar Yang-Mills theory at $g^2 N = 0$, V_F is the space of states of a single spin. H_{12} acts on a two-spin system, which as a representation of $\mathrm{PSU}(2,2|4)$ is simply $V_F \otimes V_F$. The decomposition of $V_F \otimes V_F$ in irreducible representations of $\mathrm{PSU}(2,2|4)$ is surprisingly simple and plays an important role in [11]. The decomposition, which we heuristically explain in appendix C, is

$$V_F \otimes V_F = \bigoplus_{j=0}^{\infty} V_j \,, \tag{3.5}$$

where, apart from some exceptions at small j, V_j can be characterized as a representation whose superconformal primary (or highest weight vector) is an R-singlet of angular momentum $j-2$. ($\bigoplus_{j=0}^{\infty} V_j$ merely designates a direct sum of modules V_j, and *not* any sum on lattice sites.) We also will need to know how the $\mathrm{PSU}(2,2|4)$ quadratic Casimir operator $J^2 = \sum_A J^A J^A$ (described more precisely in appendix C) acts on $V_F \otimes V_F$. With

J_1^A and J_2^A denoting the PSU(2, 2|4) generators of the first and second spins, respectively, the quadratic Casimir operator of the two-spin system is

$$J_{12}^2 = \sum_A (J_1^A + J_2^A)(J_1^A + J_2^A)\,. \tag{3.6}$$

Astonishingly, just as if the group were SU(2) instead of PSU(2, 2|4), this operator has eigenvalue $j(j+1)$ when acting on V_j:

$$J_{12}^2 V_j = j(j+1)V_j\,, \quad j = 0, 1, 2, \ldots\,. \tag{3.7}$$

This fact is used in [11] and below; we give a brief explanation of it in appendix C.

According to [11], H is a sum of two-body operators as in (3.2), where the basic two-body operator is

$$H_{12} = \sum_{j=0}^{\infty} 2h(j) P_{12,j}\,. \tag{3.8}$$

Here $P_{12,j}$ is the operator that projects the two-body Hilbert space $V_F \otimes V_F$ onto V_j, and $h(j)$ are the harmonic numbers $h(j) = \sum_{n=1}^{j} \frac{1}{n}$ for $j \in \mathbb{Z}_+$ (one defines $h(0) = 0$). According to our hypothesis (2.5), the bilocal Yangian generator Q^A is also a (non-local) sum of two-body operators, $Q_A = \sum_{i<j} Q_{ij}^A$, where the basic two-body operator is

$$Q_{ij}^A = \sum_{B,C} f_{BC}^A J_i^B J_j^C\,. \tag{3.9}$$

We observe the identity

$$Q_{ij}^A = \frac{1}{4}\,[\, J_{ij}^2, q_{ij}^A\,]\,, \tag{3.10}$$

where J_{ij}^2 is the quadratic Casimir operator of the two-particle system, and q_{ij}^A is the difference operator

$$q_{ij}^A = J_i^A - J_j^A\,. \tag{3.11}$$

We will first prove (3.4) for a system of two spins. For this purpose, we use (3.10) to write

$$[H_{12}, Q_{12}^A] = \frac{1}{4}[H_{12}, [J_{12}^2, q_{12}^A]]\,. \tag{3.12}$$

Then acting on a two-particle state $|\lambda(j)\rangle$ that is contained in V_j (and so has eigenvalues of H_{12} and J_{12}^2 given above), we have

$$[H_{12}, Q_{12}^A]|\lambda(j)\rangle = \frac{1}{4}(H_{12}J_{12}^2 - j(j+1)H_{12} - 2h(j)J_{12}^2 + 2h(j)j(j+1))\; q_{12}^A|\lambda(j)\rangle\,. \tag{3.13}$$

We will show in section 3.1 that the action of q_{12}^A on a state in V_j can be written as a linear combination of states in V_{j-1} and V_{j+1}, i.e. for any $|\lambda(j)\rangle \in V_j$, we have

$$q_{12}^A|\lambda(j)\rangle = |\chi^A(j-1)\rangle + |\rho^A(j+1)\rangle\,, \tag{3.14}$$

where $|\chi^A(j-1)\rangle \in V_{j-1}$ and $|\rho^A(j+1)\rangle \in V_{j+1}$. Given this fact, from (3.13) we have

$$\begin{aligned}[H_{12}, Q_{12}^A]|\lambda(j)\rangle &= j(h(j) - h(j-1))|\chi^A(j-1)\rangle +, (j+1)(h(j+1) - h(j))|\rho^A(j+1)\rangle \\ &= q_{12}^A|\lambda(j)\rangle\,. \end{aligned} \tag{3.15}$$

We used the fact that $h(j) - h(j-1) = 1/j$.

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Now let us consider a chain of more than two spins. H is a sum of nearest neighbor terms $H_{i,i+1}$, while Q^A is a bilocal sum of two-body operators $Q^A_{j,k}$ with $j<k$. We have

$$0=\left[H_{i,i+1}\,,\sum_{j<k,(j,k)\neq(i,i+1)}Q^A_{jk}\right]. \tag{3.16}$$

In fact, terms in the sum in which neither j nor k equals i or $i+1$ are trivially zero. On the other hand, terms with (say) $k>i+1$ and $j=i,i+1$ add up to $f^A_{BC}[H_{i,i+1},(J^B_i+J^B_{i+1})J^C_k]$, which vanishes because $J^B_i+J^B_{i+1}$ is the PSU$(2,2|4)$ generator of the two-spin system and so commutes with $H_{i,i+1}$, as does J^C_k for $k>i+1$.

So the commutator $[H,Q^A]$ collapses to

$$[H,Q^A]=\sum_{i=1}^{L-1}[H_{i,i+1},Q^A_{i,i+1}]=\sum_{i=1}^{L-1}q^A_{i,i+1}=q^A\,, \tag{3.17}$$

where $q^A_{i,i+1}=J^A_i-J^A_{i+1}$ is the difference operator of the two-spin system, and $q^A=J^A_1-J^A_L$. We have established our claim that $[H,Q^A]$ is the lattice version of a total derivative.

3.1 Decomposition of q^AV_j

In this subsection, we consider a system of two spins, with $q^A=q^A_{12}=J^A_1-J^A_2$. We wish to show that q^AV_j is contained in $V_{j+1}\oplus V_{j-1}$. We do this by proving two facts:

1. q^AV_j is contained in the direct sum of V_k with $k-j$ odd.
2. q^AV_j is contained in the direct sum of V_k with $|k-j|\leq 1$.

Clearly the two facts together imply what we want.

Fact (1) follows directly by considering the operator σ that exchanges the two spins, that is the two copies of V_F in $V_F\otimes V_F$. (If the two spins are both fermionic, one exchanges them with a minus sign.) The operator q^A is odd under σ. As explained in appendix C, σ has eigenvalue $(-1)^j$ on V_j. Fact (1) is a direct consequence of these two assertions.

To prove fact (2), we first note that to prove that $q^AV_j\subset\oplus_{k\in T}V_k$, where T is any set of allowed values of k (such as the set $|j-k|\leq 1$) it suffices to show that if $|\psi(j)\rangle$ is a primary state in V_j, then

$$q^A|\psi(j)\rangle\in\oplus_{k\in T}V_k\,. \tag{3.18}$$

Indeed any state in V_j is a linear combination of states $L_1L_2\ldots L_s|\psi(j)\rangle$ where the L's are "raising" operators in PSU$(2,2|4)$ (and are, in fact, either momenta P_μ or global supersymmetries Q^a_α). To study $q^AL_1L_2\ldots L_s|\psi(j)\rangle$, we try to commute q^A to the right so that we can use (3.18). In the process we meet commutators $[q^A,L_i]$, but these are linear combinations of the q^B's, so, assuming (3.18) has been proved for all choices of A, it can still be used. The net effect is that (3.18) implies that $q^AL_1L_2\ldots L_s|\psi(j)\rangle$ is always contained in $\oplus_{k\in T}V_k$; in other words, with T being the set $|k-j|\leq 1$, (3.18) implies fact (2).

Since the q^A transform the same way as J^A, they have the same dimensions. The dimensions of the q^A therefore range from 1 to -1. The value 1 is achieved only for the components of q^A that transform like the momentum operators P^μ.

Let us first prove that $q^A|\psi(j)\rangle \subset \oplus_{k\leq j+1}V_k$. For $j \geq 2$, this follows simply by dimension-counting. In this range, the primary $|\psi(j)\rangle$ has dimension j; it is of course the state of lowest dimension in V_j. The operator q^A has at most dimension 1, so $q^A|\psi(j)\rangle$ has dimension at most $j+1$ (and this value is only achieved if A is such that q^A transforms as one of the momentum operators). Hence $q^A|\psi(j)\rangle \in \oplus_{k\leq j+1}V_k$. To reach the same conclusion for $j = 0, 1$ takes just a little more care. For these values of j, $|\psi(j)\rangle$ has dimension 2. So $q^A|\psi(j)\rangle$ has dimension at most 3, and must be contained in $\oplus_{k\leq 3}V_k$. Given this, fact (1) above implies further that $q^A|\psi(1)\rangle \in V_0 \oplus V_2$, which is what we wanted to prove for $j = 1$. For $j = 0$, fact (1) implies that $q^A|\psi(0)\rangle \in V_1 \oplus V_3$; we wish to prove that in fact $q^A|\psi(0)\rangle \in V_1$. This follows from the $\mathrm{SU}(4)_R$ symmetry. The only state in V_3 of dimension no greater than three is the primary state $|\psi(3)\rangle$, which is $\mathrm{SU}(4)_R$-invariant; but no linear combination of the states $q^A|\psi(0)\rangle$ has this property.

Finally, we must prove the opposite inequality $q^A|\psi(j)\rangle \in \oplus_{k\geq j-1}V_j$. This is equivalent to saying that $\langle\chi|q^A|\psi(j)\rangle = 0$ if $|\chi\rangle \in V_k$ with $k < j-1$. We will use the fact that $\langle\chi|q^A|\psi(j)\rangle$ is the complex conjugate of $\langle\psi(j)|(q^A)^\dagger|\chi\rangle$. (The adjoint is taken in radial quantization, so for example the adjoint of the momentum P_μ is the special conformal generator K_μ.) But $(q^A)^\dagger$ is a linear combination of the q^A's, so we can use the result of the previous paragraph to assert that for $|\chi\rangle \in V_k$, $|(q^A)^\dagger|\chi\rangle$ is a sum of states in V_m with $m \leq k+1$; so this state is orthogonal to $|\psi(j)\rangle$ if $j > k+1$ or in other words if $k < j-1$. This completes the proof of fact (2).

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A. Field-dependent transformations — the Kac-Moody loop algebra and the yangian

Here we supply some details of the relation between the partial Kac-Moody algebra that can be defined in various two-dimensional integrable models and the Yangian. For definiteness, we consider the case of the principal chiral model, which is a two-dimensional model in which the field g takes values in a Lie group G. In these models, along with the Yangian, it is possible to find [2] non-local transformations that obey the algebra of a partial Kac-Moody algebra, by which we mean simply the algebra $[T_{A\,n}, T_{B\,m}] = f_{AB}^C T_{C\,m+n}$, $m, n \geq 0$. We write δ_n^A for the symmetry transformations corresponding to the $T_{A\,n}$. δ_0^A is the standard global symmetry generator, and coincides with the transformation generated by the generator J^A of the Yangian. Just as the Yangian is generated by J^A and the first non-trivial generator Q^A, and partial Kac-Moody algebra is generated by δ_0^A and δ_1^A.

In this model, the infinitesimal transformation generated via Poisson brackets by Q^A, the first non-trivial generator of the Yangian, can be written in terms of the Kac-Moody generators as

$$\{Q^A, g(x,t)\} = -\delta_1^A g(x,t) + \frac{1}{2} f_{ABC} J^B\, \delta_0^C(g(x,t))\,. \tag{A.1}$$

Thus, they do not coincide, but they differ by a term involving the Kac-Moody generator δ_0^A times J^B (by which we mean simply the function of g and its derivatives which is the Yangian generator J^B). Thus, Q^A does not generate the same symmetry as δ_1^A. However, they differ by a field-dependent multiple of δ_0^A. Since the symmetry transformation generated by δ_0^A is also a generator of the partial Kac-Moody algebra (or of the Yangian), it follows that if we want to know if two fields $g(x,t)$ can be related to each other by a symmetry, it does not matter if the symmetry group we use is generated by the Kac-Moody algebra or the Yangian. Any transformation generated by J^A and Q^A, with some coefficients, can be generated by δ_0^A and δ_1^A, with some other coefficients. (The transformation from one set of coefficients to the other depends on $g(x,t)$ because J^B, which appears in (A.1), has such a dependence, so this transformation does not preserve the Kac-Moody or Yangian commutation relations.)

B. Noether currents

In order to make contact with conventional Noether current symmetry analysis, we give the expression for the non-local charge (2.5) in terms of the elementary fields of the super Yang Mills lagrangian

$$\mathcal{L} = \frac{1}{g_{YM}^2}\,\mathrm{Tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^I D^\mu\phi^I - \frac{1}{2}[\phi^I,\phi^J][\phi^I,\phi^J] + \text{fermions}\right). \tag{B.1}$$

For simplicity, we will only consider $A \in so(2,4)$.

In the classical theory, the symmetry currents for the conformal group are given in terms of the improved energy-momentum tensor by

$$j^{A\mu}(x) = \kappa_\nu^A \theta^{\mu\nu}(x)\,, \tag{B.2}$$

where κ_μ^A are the conformal Killing vectors, and

$$\theta^{\mu\nu} = 2\,\mathrm{Tr}\,F^{\mu\rho}F^\nu_\rho + 2\,\mathrm{Tr}\,D^\mu\phi^I D^\nu\phi^I - g^{\mu\nu}\mathcal{L} - \frac{1}{3}\,\mathrm{Tr}(D^\mu D^\nu - g^{\mu\nu}D_\rho D^\rho)\phi^I\phi^I + \text{fermions}\,. \tag{B.3}$$

The currents (B.2) are conserved at any g^2N using the classical interacting equations of motion.

If we set $g^2N = 0$, we note that the *untraced* matrix

$$\begin{aligned}(\theta^{\mu\nu})_i^{\ j} = {} & F^{\mu\rho}F^\nu_\rho + F^{\nu\rho}F^\mu_\rho + \partial^\mu\phi^I\partial^\nu\phi^I + \partial^\nu\phi^I\partial^\mu\phi^I - g^{\mu\nu}\left(\frac{1}{2}F_{\rho\sigma}F^{\rho\sigma} + \partial_\mu\phi^I\partial^\mu\phi^I\right) - \\ & -\frac{1}{3}(\partial^\mu\partial^\nu - g^{\mu\nu}\partial_\rho\partial^\rho)\phi^I\phi^I + \text{fermions}\,, \end{aligned} \tag{B.4}$$

is also conserved, as is $\kappa^A_\nu(\theta^{\mu\nu})_i^{\ j}$. Here i, j are the matrix labels of the gauge group generators $(T^{\mathcal{A}})_i^{\ j}$. It follows that we can construct non-local conserved charges by

$$Q_0^{AB\dots} = \int_M \kappa^A_\nu(\theta^{0\nu})_i^{\ j} \int_M \kappa^B_\rho(\theta^{0\rho})_j^{\ k} \dots, \tag{B.5}$$

where M is an initial value surface in spacetime. In free field theory, this acts on a chain of partons rather as (2.5) does, but we have no idea how to extend the definition to $g^2N \neq 0$.

C. Primary states in the two-particle system

To determine the one-loop dilatation operator in [11], it is necessary to know the decomposition of the two-particle system in free $\mathcal{N} = 4$ super Yang-Mills theory in irreducible representations of PSU$(2, 2|4)$. This decomposition is surprisingly simple. The irreducible representations are conveniently denoted as V_j, $j = 0, 1, 2, \dots$, where the quantum numbers of a superconformal primary $|\psi(j)\rangle \in V_j$ can be conveniently described as follows.

First of all, the representations V_0 and V_1 are exceptional, as they are degenerate representations of PSU$(2, 2|4)$. So we describe them separately first. We let ϕ^I, $I = 1, \dots, 6$ be the elementary scalars of $\mathcal{N} = 4$ super Yang-Mills theory. They transform in the vector representation of the R symmetry group SU$(4) \cong$ SO(6). Superconformal primaries in V_0 and V_1 are the following bilinears in ϕ:

$$\begin{aligned} V_0 &: |\psi(0)\rangle \sim \phi^I\phi^J + \phi^J\phi^I - \frac{1}{3}\delta^{IJ}\phi^K\phi^K \\ V_1 &: |\psi(1)\rangle \sim \phi^I\phi^J - \phi^J\phi^I\,. \end{aligned} \tag{C.1}$$

In case the formula for $|\psi(1)\rangle$ looks strange, note that we need not worry about bose statistics here, because the fields carry U(N) gauge indices that are being suppressed. Thus, the primaries in V_0 and V_1 have dimension two and non-trivial R-symmetry quantum numbers. By contrast, the primaries $|\psi(j)\rangle$ in V_j, $j \geq 2$, are singlets of the R symmetry, and have dimension j. They can be obtained by combining two partons in an SU(4) singlet state with relative angular momentum $j - 2$:

$$V_j : |\psi(j)\rangle \sim \sum_{I=1}^{6}\sum_{k=0}^{j-2} c_k^{(j-2)}\partial^k\phi^I\partial^{j-2-k}\phi^I + \cdots\,. \tag{C.2}$$

Here (see for example eq. 6 in [38]) $c_k^{(j-2)} = (-1)^k/k!^2(j-k-2)!^2$. These precise coefficients ensure that V_j is a conformal primary. To make a superconformal primary, one must add additional terms, also of angular momentum $j - 2$, that are bilinear in fermions or gauge fields instead of scalars. We have indicated these terms by the ellipses in (C.2).

We can give as follows a heuristic explanation of why this is the classification of the V_j. An operator $\mathcal{O}$ acting on the vacuum in free field theory creates a two particle state. Consider the case that particle A is traveling in the $+z$ direction and particle B in the $-z$ direction. There are a total of $16 \times 16 = 256$ helicity states of this type, with 16 for each particle. Of the 16 global supercharges (we only consider supercharges that commute with translations as we have diagonalized the momentum) of the $\mathcal{N}=4$ theory, half annihilate particles moving in the $+z$ direction and a complementary half annihilate particles moving in the $-z$ direction. So altogether each supercharge acts nontrivially in the two particle system and it takes $2^{16/2}=256$ states to represent the 16 supercharges. So in short the global supercharges act irreducibly on these 256 states, and to classify $N=4$ multiplets, the relevant variable is the angular momentum, which is the variable $j-2$ in (C.2). However, this argument breaks down for small j when the supersymmetries fail to act in a nondegenerate fashion, and this is how the exceptional representations V_0 and V_1 appear.

In section 3.1, an important role is played by the behavior of V_j under the operator σ that exchanges the two spins. From the above description of the V_j, it is clear that $\sigma V_j=(-1)^j V_j$. For $j=0,1$, this reflects the fact that the primary in V_0 is symmetric in I,J while that in V_1 is antisymmetric. For $j\geq 2$, it reflects the fact that a state of two scalars with relative angular momentum $j-2$ transforms as $(-1)^{j-2}$ under exchange of the two particles; more explicitly, one can note that $c_k^{(j-2)}=(-1)^j c_{j-2-k}^{(j-2)}$.

As exploited in [11], the quadratic Casimir operator J^2 of PSU(2,2|4) has the amazing property

$$J^2 V_j = j(j+1)V_j\,, \tag{C.3}$$

rather as if the group were SU(2) instead of PSU(2,2|4), and despite the exceptional nature of V_0 and V_1. Since we need this relation in section 3, we sketch a proof. The quadratic Casimir for PSU(2,2|4) is given explicitly by

$$J^2=\frac{1}{2}D^2+\frac{1}{2}L^{\gamma}_{\ \delta}L^{\delta}_{\ \gamma}+\frac{1}{2}\dot{L}^{\dot\gamma}_{\ \dot\delta}\dot{L}^{\dot\delta}_{\ \dot\gamma}-\frac{1}{2}R^c_d R^d_c-\frac{1}{2}[Q^c_\gamma,S^\gamma_c]-\frac{1}{2}[\dot{Q}^c_{\dot\gamma},\dot{S}^{\dot\gamma}_c]-\frac{1}{2}\{P_{\gamma\dot\delta},K^{\gamma\dot\delta}\}\,. \tag{C.4}$$

Here $1\leq\alpha,\dot\alpha\leq 2, 1\leq a\leq 4$. When acting on the superconformal primaries $|\psi(j)\rangle$, the Casimir J^2 can be rewritten as

$$\begin{aligned} J^2|\psi(j)\rangle = \Bigg(&\frac{1}{2}D^2+\frac{1}{2}L^{\gamma}_{\ \delta}L^{\delta}_{\ \gamma}+\frac{1}{2}\dot{L}^{\dot\gamma}_{\ \dot\delta}\dot{L}^{\dot\delta}_{\ \dot\gamma}-\frac{1}{2}R^c_d R^d_c \\ &+\frac{1}{2}\{S^\gamma_c,Q^c_\gamma\}+\frac{1}{2}\{\dot{S}^{\dot\gamma}_c,\dot{Q}_{\dot\gamma c}\}-\frac{1}{2}[K^{\gamma\dot\delta},P_{\gamma\dot\delta}]\Bigg)|\psi(j)\rangle\,. \end{aligned} \tag{C.5}$$

We have used the fact that the primary is annihilated by S and K. (C.5) can be easily evaluated using the commutation relations as

$$J^2|\psi(j)\rangle=\left(\frac{1}{2}D^2+s_1(s_1+1)+s_2(s_2+1)-\frac{1}{2}R^c_dR^d_c+2D\right)|\psi(j)\rangle\,. \tag{C.6}$$

Here s_1 and s_2 are the spin quantum numbers of an operator; a general operator transforms under the four-dimensional rotation group SO(4) $\cong$ SU(2) $\times$ SU(2) with spins (s_1,s_2). For $j\geq 2$, $|\psi(j)\rangle$is annihilated by the $R^c{}_d$ as it is an SU(4)$_R$ singlet, and has dimension j and $s_1=s_2=(j-2)/2$. So we get

$$J^2|\psi(j)\rangle=\left(\frac{1}{2}j^2+\left(\frac{j}{2}-1\right)\frac{j}{2}2+2j\right)|\psi(j)\rangle=j(j+1)|\psi(j)\rangle\,. \tag{C.7}$$

For the cases $j=0$ and $j=1$, the conformal dimension is two, and the spins s_1 and s_2 are zero. The SU(4)$_R$ contribution is $-\frac{1}{2}R^c_dR^d_c=-6$ for the traceless symmetric tensor (the $j=0$ case) giving $J^2|\psi(0)\rangle=0$, and $-\frac{1}{2}R^c_dR^d_c=-4$ for the adjoint representation (the $j=1$ case) giving $J^2|\psi(1)\rangle=2$.

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References

[1] A.M. Polyakov, *Interaction of goldstone particles in two-dimensions. applications to ferromagnets and massive Yang-Mills fields*, *Phys. Lett.* **B 59** (1975) 79; *Hidden symmetry of the two-dimensional chiral fields*, *Phys. Lett.* **B 72** (1977) 224; *String representations and hidden symmetries for gauge fields*, *Phys. Lett.* **B 247** (1979) 82; *Gauge fields as rings of glue*, *Nucl. Phys.* **B 164** (1980) 1971.

[2] L. Dolan, *Kac-moody algebra is hidden symmetry of chiral models*, *Phys. Rev. Lett.* **47** (1981) 1371; *Kac-Moody symmetry group of real self-dual Yang-Mills,Phys. Lett.* **B 113** (1982) 378; *Kac-moody algebras and exact solvability in hadronic physics*, *Phys. Rept.* **109** (1984) 1.

[3] L.-L. Chau, M.-l. Ge and Y.-s. Wu, *The Kac-Moody algebra in the selfdual Yang-Mills equation*, *Phys. Rev.* **D 25** (1982) 1086.

[4] K. Ueno and Y. Nakamura, *Transformation theory for anti(self)dual equations and the Riemann-Hilbert problem*, *Phys. Lett.* **B 109** (1982) 273.

[5] A.D. Popov and C.R. Preitschopf, *Conformal symmetries of the self-dual Yang-Mills equations*, *Phys. Lett.* **B 374** (1996) 71 [hep-th/9512130].

[6] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[7] L.N. Lipatov, *High-energy asymptotics of multicolor QCD and exactly solvable lattice models*, *Sov. Phys. JETP Lett.* **59** (1994) 596 [hep-th/9311037].

[8] L.D. Faddeev and G.P. Korchemsky, *High-energy QCD as a completely integrable model*, *Phys. Lett.* **B 342** (1995) 311 [hep-th/9404173].

[9] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[10] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **03** (2003) 013 [hep-th/0212208].

[11] N. Beisert, *The complete one-loop dilatation operator of $N = 4$ super Yang-Mills theory*, hep-th/0307015.

[12] N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042].

[13] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *J. High Energy Phys.* **09** (2003) 010 [hep-th/0306139].

[14] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 664** (2003) 131 [hep-th/0303060].

[15] A.V. Belitsky, A.S. Gorsky and G.P. Korchemsky, *Gauge/string duality for QCD conformal operators*, *Nucl. Phys.* **B 667** (2003) 3 [hep-th/0304028].

[16] A.V. Kotikov, L.N. Lipatov and V.N. Velizhanin, *Anomalous dimensions of wilson operators in $N = 4$ SYM theory*, *Phys. Lett.* **B 557** (2003) 114 [hep-ph/0301021].

[17] F.A. Dolan and H. Osborn, *Superconformal symmetry, correlation functions and the operator product expansion*, *Nucl. Phys.* **B 629** (2002) 3 [hep-th/0112251].

[18] V.M. Braun, S.E. Derkachov and A.N. Manashov, *Integrability of three-particle evolution equations in QCD*, *Phys. Rev. Lett.* **81** (1998) 2020 [hep-ph/9805225].

[19] A.V. Belitsky, *Fine structure of spectrum of twist-three operators in QCD*, *Phys. Lett.* **B 453** (1999) 59 [hep-ph/9902361]; *Integrability and WKB solution of twist-three evolution equations*, *Nucl. Phys.* **B 558** (1999) 259 [hep-ph/9903512]; *Renormalization of twist-three operators and integrable lattice models*, *Nucl. Phys.* **B 574** (2000) 407 [hep-ph/9907420].

[20] V.M. Braun, S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, *Baryon distribution amplitudes in QCD*, *Nucl. Phys.* **B 553** (1999) 355 [hep-ph/9902375].

[21] S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, *Evolution equations for quark gluon distributions in multi-color QCD and open spin chains*, *Nucl. Phys.* **B 566** (2000) 203 [hep-ph/9909539].

[22] G. Mandal, N.V. Suryanarayana and S.R. Wadia, *Aspects of semiclassical strings in AdS_5*, *Phys. Lett.* **B 543** (2002) 81 [hep-th/0206103].

[23] R.R. Metsaev and A.A. Tseytlin, *Type-IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028].

[24] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, hep-th/0305116.

[25] S. Fubini, A.J. Hanson and R. Jackiw, *New approach to field theory*, *Phys. Rev.* **D 7** (1973) 1732.

[26] M. Lüscher and K. Pohlmeyer, *Scattering of massless lumps and nonlocal charges in the two-dimensional classical nonlinear sigma model*, *Nucl. Phys.* **B 137** (1978) 46.

[27] M. Lüscher, *Quantum nonlocal charges and absence of particle production in the two-dimensional nonlinear sigma model*, *Nucl. Phys.* **B 135** (1978) 1.

[28] L.A. Takhtadjan and L. Fadde'ev, *Hamiltonian methods in the theory of solitons*, Springer Series in Soviet Mathematics, Springer Verlag, 1987;
How algebraic Bethe ansatz works for integrable model, hep-th/9605187.

[29] B.C. Vallilo, *Flat currents in the classical $AdS_5 \times S^5$ pure spinor superstring*, hep-th/0307018.

[30] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, *J. High Energy Phys.* **04** (2000) 018 [hep-th/0001035].

[31] J.H. Schwarz, *Classical symmetries of some two-dimensional models coupled to gravity*, *Nucl. Phys.* **B 454** (1995) 427 [hep-th/9506076].

[32] P. . Goddard and D. I. Olive, *Kac-Moody and virasoro algebras: a reprint volume for physicists*, *Adv. Ser. Math. Phys.* **3** (1988) 1.

[33] D. Bernard, *An introduction to yangian symmetries*, *Int. J. Mod. Phys.* **B 7** (1993) 3517 [hep-th/9211133].

[34] N.J. MacKay, *On the classical origins of yangian symmetry in integrable field theory*, *Phys. Lett.* **B 281** (1992) 90.

[35] D. Bernard and A. Leclair, *Quantum group symmetries and nonlocal currents in 2-D QFT*, *Commun. Math. Phys.* **142** (1991) 99.

[36] D. Bernard, *Hidden yangian in 2D massive current algebras*, *Commun. Math. Phys.* **137** (1991) 191.

JHEP10(2003)017

[37] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[38] A. Mikhailov, *Notes on higher spin symmetries*, hep-th/0201019.