arXiv:1208.5971v4 [hep-th] 9 Sep 2013

Partition Functions for Maxwell Theory on the Five-torus and for the Fivebrane on $\mathbf{S^1} \times \mathbf{T^5}$

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Abstract

We compute the partition function of five-dimensional abelian gauge theory on a five-torus T^5 with a general flat metric using the Dirac method of quantizing with constraints. We compare this with the partition function of a single fivebrane compactified on S^1 times T^5, which is obtained from the six-torus calculation of Dolan and Nappi [arXiv:hep-th/9806016]. The radius R_1 of the circle S^1 is set to the dimensionful gauge coupling constant $g^2_{5YM} = 4\pi^2 R_1$. We find the two partition functions are equal only in the limit where R_1 is small relative to T^5, a limit which removes the Kaluza-Klein modes from the 6d sum. This suggests the 6d $N = (2, 0)$ tensor theory on a circle is an ultraviolet completion of the 5d gauge theory, rather than an exact quantum equivalence.

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1 Introduction

A quantum equivalence between the six-dimensional $N = (2, 0)$ theory of multiple fivebranes compactfied on a circle S^1 and five-dimensional maximally supersymmetric Yang Mills has been conjectured by Douglas and Lambert *et al.* in [1, 2]. In this paper we will study an abelian version of the conjecture where the common five-manifold is a five-torus T^5 with a general flat metric, and find an equivalence only in the weak coupling limit.

The physical degrees of freedom of a single fivebrane are described by an $N = (2, 0)$ tensor supermultiplet which includes a chiral two-form field potential, so even a single fivebrane has no fully covariant action. In order to investigate its quantum theory we were thus led in [3] to compute the partition function instead, which we carried out on the six-torus T^6. We will use this calculation to investigate the partition function of the self-dual three-form field strength restricted to $S^1 \times T^5$ and compare it with the partition function of the five-dimensional Maxwell theory on a twisted five-torus quantized via Dirac constraints in radiation gauge.

Because both the theory and the manifold are so simple, we do not use localization techniques fruitful for non-abelian theories and their partition functions on spheres [4]-[9].

The five-dimensional Maxwell partition function on T^5 is defined[1] as in string theory [13],

$$\begin{aligned} Z^{5d,Maxwell} &\equiv tre^{-2\pi H^{5d}+i2\pi\gamma^i P_i^{5d}} = Z^{5d}_{\text{zero modes}} \cdot Z^{5d}_{\text{osc}}, \\ H^{5d} &= \frac{R_6}{g^2_{5YM}} \int_0^{2\pi} d\theta^2 d\theta^3 d\theta^4 d\theta^5 \sqrt{g}\, \big(\frac{1}{2R_6^2} g^{ii'}\, F_{6i}F_{6i'} + \frac{1}{4} g^{ii'} g^{jj'} F_{ij}F_{i'j'}\big), \\ P_i^{5d} &= \frac{1}{g^2_{5YM} R_6} \int_0^{2\pi} d\theta^2 d\theta^3 d\theta^4 d\theta^5 \sqrt{g}\, g^{jj'}\, F_{6j'}F_{ij}, \end{aligned} \tag{1.1}$$

in terms of the gauge field strength $F_{\tilde{m}\tilde{n}}(\theta^2, \theta^3, \theta^4, \theta^5, \theta^6)$, and constant metric g^{ij}, R_6, γ^i. The partition function of the abelian chiral two-form on a space circle times the five-torus is

$$\begin{aligned} Z^{6d,chiral} &= tr\, e^{-2\pi R_6 \mathcal{H}+i2\pi\gamma^i \mathcal{P}_i} = Z^{6d}_{\text{zero modes}} \cdot Z^{6d}_{\text{osc}}, \\ \mathcal{H} &= \frac{1}{12}\int_0^{2\pi} d\theta^1 \dots d\theta^5 \sqrt{G_5} {G_5}^{ll'} {G_5}^{mm'} {G_5}^{nn'} H_{lmn}(\vec{\theta}, \theta^6)\, H_{l'm'n'}(\vec{\theta}, \theta^6), \\ \mathcal{P}_i &= -\frac{1}{24}\int_0^{2\pi} d\theta^1 ... d\theta^5 \epsilon^{rsumn} H_{umn}(\vec{\theta}, \theta^6)\, H_{irs}(\vec{\theta}, \theta^6) \end{aligned} \tag{1.2}$$

where θ^1 is the direction of the circle S^1. The time direction θ^6 we will use for quantization is common to both theories, and the angles between the circle and the five-torus denoted by α, β^i in [3] have been set to zero. The final results are given in (4.15), (4.16).

We use (1.1,1.2) to compute both the zero mode and oscillator contributions, and find an exact equivalence between the zero mode contributions,

$$Z^{6d}_{\text{zero modes}} = Z^{5d}_{\text{zero modes}}. \tag{1.3}$$

[1]Related work is [10] which appeared after an earlier version of this paper. See also [11],[12].

Not surprisingly, we find the oscillator traces differ by the absence in $Z^{5d}_{\rm osc}$ of the Kaluza-Klein modes generated in $Z^{6d}_{\rm osc}$ from compactification on the circle S^1.

The Kaluza-Klein modes have been associated with instantons in the five-dimensional non-abelian gauge theory in [1, 2, 14, 15], with additional comments given for the abelian limit. It would be interesting to find a systematic way to incorporate these modes in a generalized five-dimensional partition function along the lines of a character, in order to match the partition functions exactly, but we have not done that here. Rather our explicit expressions show an equivalence between the oscillator traces of the two theories only in the limit where the compactification radius R_1 of the circle is small compared to the five-torus T^5.

Other approaches to $N = (2,0)$ theories formulate fields for non-abelian chiral two-forms [16]-[20] which would be useful if the non-abelian six-dimensional theory has a classical description and if the quantum theory can be described in terms of fields. On the other hand the partition functions on various manifolds [21]-[26] can demonstrate aspects of the six-dimensional finite quantum conformal theory presumed responsible for features of four-dimensional gauge theory [27].

In section 2, the contribution of the zero modes to the partition function for the chiral theory on a circle times a five-torus is computed as a sum over the ten integer eigenvalues, and its relation to that of the gauge theory is shown via a fiber bundle approach. In section 3, the abelian gauge theory is quantized on a five-torus using Dirac constraints, and the Hamiltonian and momenta are computed in terms of the oscillator modes. In section 4, we construct the oscillator trace contribution to the partition function for the gauge theory and compare it with that of the chiral two-form. Section 5 contains discussion and conclusions. Appendix A presents details of the Dirac quantization and Appendix B verifies the Hamilton equations of motion. Appendix C regularizes the vacuum energy. Appendix D proves the $SL(5,\mathcal{Z})$ invariance of both partition functions.

2 Zero Modes

The $N = (2,0)$ 6d world volume theory of the fivebrane contains five scalars, two four-spinors and a chiral two-form B_{MN}, which has a self-dual three-form field strength $H_{LMN} = \partial_L B_{MN} + \partial_M B_{NL} + \partial_N B_{LM}$ with $1 \le L, M, N \le 6$,

$$H_{LMN}(\vec{\theta}, \theta^6) = \frac{1}{6\sqrt{-G}} G_{LL'} G_{MM'} G_{NN'} \epsilon^{L'M'N'RST} H_{RST}(\vec{\theta}, \theta^6). \tag{2.1}$$

(2.1) gives $H_{LMN}(\vec{\theta}, \theta^6) = \frac{i}{6\sqrt{|G|}} G_{LL'} G_{MM'} G_{NN'} \epsilon^{L'M'N'RST} H_{RST}(\vec{\theta}, \theta^6)$ for a Euclidean signature metric. In the absence of a covariant Lagrangian, the partition function of the chiral field is defined via a trace over the Hamiltonian [3] as is familiar from string calculations. We display this expression in (1.2) where the metric has been restricted to describe the line element for $S^1 \times T^5$,

$$ds^2 = {R_1}^2 (d\theta^1)^2 + {R_6}^2 (d\theta^6)^2 + \sum_{i,j=2\ldots5} g_{ij} (d\theta^i - \gamma^i d\theta^6)(d\theta^j - \gamma^j d\theta^6) \tag{2.2}$$

with $0 \le \theta^I \le 2\pi$, $1 \le I \le 6$. The parameters R_1 and R_6 are the radii for directions 1 and 6, g_{ij} is a 4d metric, and γ^j are the angles between between 6 and j. So from (2.2),

$$G_{ij} = g_{ij}\,;\; G_{11} = {R_1}^2; \quad G_{i1} = 0\,; \quad G_{66} = {R_6}^2 + g_{ij}\gamma^i\gamma^j; \quad G_{i6} = -g_{ij}\gamma^j\,; \quad G_{16} = 0; \tag{2.3}$$

and the inverse metric is

$$G^{ij} = g^{ij} + \frac{\gamma^i\gamma^j}{R_6^2}; \quad G^{11} = \frac{1}{R_1^2}; \quad G^{1i} = 0; \quad G^{66} = \frac{1}{R_6^2}; \quad G^{i6} = \frac{\gamma^i}{R_6^2}; \quad G^{16} = 0. \tag{2.4}$$

We want to keep the time direction θ^6 common to both theories, so in the 5d expressions (1.1) the indices are on $2 \le \tilde{m}, \tilde{n} \le 6$; whereas the Hamiltonian and momenta in (1.2) sum on $1 \le m, n \le 5$. The common space index is labeled $2 \le i, j \le 5$. To this end, for the metric G_{MN} in (2.3) we introduce the 5-dimensional inverse (in directions 1,2,3,4,5)

$$G_5{}^{ij} = g^{ij}; \qquad G_5{}^{i1} = 0; \qquad G_5{}^{11} = \frac{1}{R_1^2}; \tag{2.5}$$

and the 5-dimensional inverse (in directions 2,3,4,5,6) for the five-torus T^5,

$$\widetilde{G}_5^{ij} = g^{ij} + \frac{\gamma^i\gamma^j}{R_6^2}\,; \qquad \widetilde{G}_5^{i6} = \frac{\gamma^i}{R_6^2}\,; \qquad \widetilde{G}_5^{66} = \frac{1}{R_6^2}. \tag{2.6}$$

The determinants of the metrics are related simply by $\sqrt{G} = R_6\sqrt{G_5} = R_1\sqrt{\widetilde{G}_5} = R_6R_1\sqrt{g}\,$, and $\epsilon_{23456} \equiv \widetilde{G}_5\,\epsilon^{23456} = \widetilde{G}_5$, with corresponding epsilon tensors related by G, G_5, g.

To compute $Z^{6d}_{\text{zero modes}}$ we neglect the integrations in (1.2) and get

$$\begin{aligned}
-2\pi R_6\mathcal{H} &= -\frac{\pi}{6}R_6R_1\sqrt{g}g^{ii'}g^{jj'}g^{kk'}H_{ijk}H_{i'j'k'} - \frac{\pi}{4}\frac{R_6}{R_1}\sqrt{g}(g^{jj'}g^{kk'} - g^{jk'}g^{kj'})H_{1jk}H_{1j'k'},\\
i2\pi\gamma^i\mathcal{P}_i &= -\frac{i\pi}{2}\gamma^i\epsilon^{jkj'k'}H_{1jk}H_{ij'k'} = \frac{i\pi}{3}\gamma^i\epsilon^{jj'kk'}H_{j'kk'}H_{1ij},
\end{aligned} \tag{2.7}$$

where the zero modes of the four fields H_{ijk} are labeled by the integers $n_7, \ldots, n_{10}$. The six fields H_{1jk} have zero mode eigenvalues $H_{123} = n_1$, $H_{124} = n_2$, $H_{125} = n_3$, $H_{134} = n_4$, $H_{135} = n_5$, $H_{145} = n_6$, and the trace on the zero mode operators in (1.2) is

$$\begin{aligned}
Z^{6d}_{\text{zero modes}} = &\sum_{n_1,\ldots,n_6} \exp\{-\frac{\pi}{4}\frac{R_6}{R_1}\sqrt{g}(g^{jj'}g^{kk'} - g^{jk'}g^{kj'})H_{1jk}H_{1j'k'}\}\\
&\cdot\sum_{n_7,\ldots,n_{10}} \exp\{-\frac{\pi}{6}R_6R_1\sqrt{g}g^{ii'}g^{jj'}g^{kk'}H_{ijk}H_{i'j'k'} - \frac{i\pi}{2}\gamma^i\epsilon^{jkj'k'}H_{1jk}H_{ij'k'}\}.
\end{aligned} \tag{2.8}$$

The same sum is obtained from the 5d Maxwell theory (1.1) where the gauge coupling is identified[2] with the radius of the circle $g_{5YM}^2 = 4\pi^2 R_1$, as follows. The zero modes of the gauge theory are eigenvalues of operator-valued fields that satisfy Maxwell equations with no sources. Even classically these solutions have constant F_{ij} which lead to non-zero flux through closed two-surfaces that are not a boundary of a three-dimensional submanifold in T^5. Working in $A_6 = 0$ gauge, if we consider the $U(1)$ gauge field A_i at any time θ^6 as a connection on a principal U(1) bundle with base manifold T^4, then the curvature $F_{ij} = \partial_i A_j - \partial_j A_i$ for $2 \leq i, j \leq 5$ must have integer flux [28, 29], in the sense that

$$n_I = \frac{1}{2\pi}\int_{\Sigma_2^I} F \equiv \frac{1}{2\pi}\int_{\Sigma_2^I} \frac{1}{2} F_{ij}\, d\theta^i \wedge d\theta^j, \qquad n_I \in \mathcal{Z}, \text{ for each } 1 \leq I \leq 6. \tag{2.9}$$

In T^4, the six representative two-cycles Σ_2^I are each a 2-torus constructed by the six ways of combining the four S^1 of T^4 two at a time, given by the cohomology class, $\dim H_2(T^4) = 6$. Relabeling n_I as $n_{i,j}$ and Σ_2^I as $\Sigma_2^{i,j}$, $2 \leq i < j \leq 5$, we have $\int_{\Sigma_2^{g,h}} d\theta^i \wedge d\theta^j = (2\pi)^2(\delta_g^i \delta_h^j - \delta_h^i \delta_g^j)$. So (2.9) is

$$F_{ij} = \frac{n_{i,j}}{2\pi}, \qquad n_{i,j} \in \mathcal{Z} \text{ for } i < j. \tag{2.10}$$

Furthermore we show how the zero mode eigenvalues of F_{6i} are found[3] from those of the conjugate momentum Π^i. In section 3 we derive the form of H^{5d} and P_i^{5d} given in (1.1) from a canonical quantization using a Lorentzian signature metric. In (3.9) the conjugate momentum is defined as

$$\Pi^i = \frac{\sqrt{g}}{4\pi^2 R_1 R_6} g^{ii'} F_{6i'}. \tag{2.11}$$

From the commutation relations (3.12) we can compute its commutator with the holonomy $\int_{\Sigma_1^k} A \equiv \int_{\Sigma_1^k} A_i(\vec{\theta}, \theta^6) d\theta^i$ where Σ_1^k are the four representative one-cycle circles in T^4,

$$\left[\int_{\Sigma_1^k} A_i(\vec{\theta}, \theta^6) d\theta^i, \int \frac{d^4\theta'}{2\pi} \Pi^j(\vec{\theta}', \theta^6)\right] = \frac{i}{2\pi}\int_{\Sigma_1^k} d\theta^j = i\,\delta_k^j. \tag{2.12}$$

Hence an eigenstate ψ of the the zero mode operator $\frac{1}{2\pi}\int d^4\theta' \Pi^k(\vec{\theta}', \theta^6)$ with eigenvalue λ is

$$\psi = e^{i\lambda \int_{\Sigma_1^k} A}\,|0\rangle, \qquad \left(\frac{1}{2\pi}\int d^4\theta' \Pi^k(\vec{\theta}', \theta^6)\right) e^{i\lambda \int_{\Sigma_1^k} A}\,|0\rangle = \lambda\; e^{i\lambda \int_{\Sigma_1^k} A}\,|0\rangle.$$

Since the holonomy is defined mod 2π, thus allowing A to vary by gauges when crossing neighborhoods, but ensuring $e^{i\int_{\Sigma_1^k} A}$ to be a single valued element of the structure group $U(1)$, then the states

$$e^{i\lambda \int_{\Sigma_1^k} A}\,|0\rangle \qquad \text{and} \qquad e^{i\lambda\left(2\pi + \int_{\Sigma_1^k} A\right)}\,|0\rangle \tag{2.13}$$

[2]See for example arXiv:1012.2882, p5 [2].
[3]This point of view is discussed in [30]. See also [10].

must be equivalent, so the eigenvalue λ of the operator $\frac{1}{2\pi}\int d^4\theta' \Pi^k(\vec{\theta}', \theta^6)$ must have integer values $n^{(k)}$,

$$\Pi^k(\vec{\theta}', \theta^6) = \frac{n^{(k)}}{(2\pi)^3}, \qquad n^{(k)} \in \mathcal{Z}^4. \tag{2.14}$$

In this normalization of the zero mode eigenvalues for the gauge theory, we are taking the $d\theta^i$ space integrations into account. So (1.1) gives

$$\begin{aligned} &-2\pi H^{5d} + i2\pi\gamma^i P_i^{5d} \\ &= \Big(-\frac{\pi\sqrt{g}}{R_1 R_6} g^{ii'} F_{6i} F_{6i'} - \frac{\pi R_6}{2R_1}\sqrt{g} g^{ii'} g^{jj'} F_{ij} F_{i'j'} + 2\pi i \gamma^i \frac{\sqrt{g}}{R_1 R_6} g^{jj'} F_{6j'} F_{ij} \Big) (2\pi)^2. \end{aligned} \tag{2.15}$$

We can use the identity

$$-\frac{1}{4}\epsilon^{jkj'k'} H_{1jk} H_{ij'k'} = \frac{1}{6}\epsilon^{jj'kk'} H_{j'kk'} H_{1ij},$$

to rewrite the last term in (2.8) as

$$-\frac{i\pi}{2}\gamma^i \epsilon^{jkj'k'} H_{1jk} H_{ij'k'} = \frac{i\pi}{3}\gamma^i \epsilon^{jj'kk'} H_{j'kk'} H_{1ij},$$

which is equal to the last term in (2.15) if we identify

$$\frac{1}{6}\epsilon^{jj'kk'} H_{j'kk'} = \frac{2\pi\sqrt{g}}{R_1 R_6} g^{jj'} F_{6j'}, \qquad H_{1ij} = 2\pi F_{ij}. \tag{2.16}$$

Then, from (2.16) we have that the first term in (2.15) becomes[4]

$$-\frac{4\pi^3\sqrt{g}}{R_1 R_6} g^{ii'} F_{6i} F_{6i'} = -\frac{\pi}{6}\sqrt{g} R_1 R_6 \; g^{j'g'} g^{gh} g^{g'h'} H_{j'kk'} H_{g'hh'}.$$

Thus with the identifications in (2.16), the 5d Maxwell expression in (2.15) is equal to the

[4]

$g_{jg}\epsilon^{jj'kk'}\epsilon^{gg'hh'} = g(g^{j'g'}g^{kh}g^{k'h'} - g^{j'g'}g^{k'h}g^{kh'} - g^{k'g'}g^{kh}g^{j'h'} + g^{k'g'}g^{j'h}g^{kh'} - g^{kg'}g^{j'h}g^{k'h'} + g^{kg'}g^{k'h}g^{j'h'})$,
$\epsilon^{2345} = 1$ and $\epsilon_{2345} = g\epsilon^{2345} = g$.

6d chiral exponent in (2.8),

$$-2\pi H^{5d}+i2\pi\gamma^i P_i^{5d}=\Big(-\frac{\pi\sqrt{g}}{R_1R_6}g^{ii'}\,F_{6i}F_{6i'}-\frac{\pi R_6\sqrt{g}}{2R_1}g^{ii'}g^{jj'}F_{ij}F_{i'j'}+\frac{i2\pi\sqrt{g}}{R_1R_6}\gamma^i g^{jj'}\,F_{6j'}F_{ij}\Big)(2\pi)^2$$
$$=-t\mathcal{H}+i2\pi\gamma^i\mathcal{P}_i=-\frac{\pi}{6}R_6R_1\sqrt{g}g^{ii'}g^{jj'}g^{kk'}H_{ijk}H_{i'j'k'}-\frac{\pi}{4}\frac{R_6}{R_1}\sqrt{g}(g^{jj'}g^{kk'}-g^{jk'}g^{j'k})H_{1jk}H_{1j'k'}$$
$$-\frac{i\pi}{2}\gamma^i\epsilon^{jkj'k'}H_{1jk}H_{ij'k'}.$$

We now discuss the sum over integers in (2.8). From (2.16), if H_{1jk} are integers, then $2\pi\, F_{ij}$ are integers. If H_{ijk} are integers, then $\frac{1}{6}\epsilon^{jj'kk'}H_{j'kk'}$ are also integers. This implies, again from (2.16), that $\frac{2\pi\sqrt{g}}{R_1R_6}g^{jj'}F_{6j'}$ should be integers, which we justify in (2.10) and (2.14) with (2.11). Thus the Maxwell zero mode trace can be written as

$$Z^{5d}_{\text{zero modes}}=\sum_{n_1\ldots n_6}\exp\{-2\pi^3\frac{R_6\sqrt{g}}{R_1}g^{ii'}g^{jj'}F_{ij}F_{i'j'}\}$$
$$\cdot\sum_{n^7\ldots n^{10}}\exp\{-\frac{4\pi^3\sqrt{g}}{R_1R_6}g^{ii'}F_{6i}F_{6i'}+\frac{i(2\pi)^3\sqrt{g}}{R_1R_6}\gamma^i g^{jj'}\,F_{6j'}F_{ij}\}\quad(2.17)$$

where the integer eigenvalues are $n_1=2\pi F_{23}, n_2=2\pi F_{24}, n_3=2\pi F_{25}, n_4=2\pi F_{34}$,
$n_5=2\pi F_{35}, n_6=2\pi F_{45}$; $(n^7,n^8,n^9,n^{10})\equiv(n^{(2)},n^{(3)},n^{(4)},n^{(5)})$,
for $n^{(k)}\equiv\frac{2\pi\sqrt{g}}{R_1R_6}g^{ki'}F_{6i'}\in\mathcal{Z}^4$. So we have proved the relation (1.3)

$$Z^{6d}_{\text{zero modes}}=Z^{5d}_{\text{zero modes}}\quad(2.18)$$

and the explicit expression is given by (2.8) or (2.17).

3 Dirac Quantization of Maxwell Theory on a Five-torus

To evaluate the oscillator contribution to the partition function in (1.1), we will first quantize the abelian gauge theory on the five-torus with a general metric. The equation of motion is $\partial^{\tilde m}F_{\tilde m\tilde n}=0$. For $F_{\tilde m\tilde n}=\partial_{\tilde m}A_{\tilde n}-\partial_{\tilde n}A_{\tilde m}$, a solution is given by a solution to

$$\partial^{\tilde n}\partial_{\tilde n}A_{\tilde m}=0,\qquad\partial^{\tilde m}A_{\tilde m}=0.\quad(3.1)$$

These have a plane wave solution $A_{\tilde m}(\vec\theta,\theta^6)=f_{\tilde m}(k)e^{ik\cdot\theta}+(f_{\tilde m}(k)e^{ik\cdot\theta})^*$ when

$$\widetilde{G}_L^{\tilde m\tilde n}k_{\tilde m}k_{\tilde n}=0,\qquad k^{\tilde m}f_{\tilde m}=0.\quad(3.2)$$

In order for the operator formalism (1.1) to reproduce a path integral quantization with spacetime metric (2.6), we must canonically quantize H^{5d} and P_i^{5d} via a metric that has zero angles with the time direction, *i.e.* $\gamma^i=0$, and insert γ^i in the partition function merely as the coefficient of P_i^{5d} [13]. Furthermore a Lorentzian signature metric is needed for quantum mechanics, so we modify the metric on the five-torus (2.3), (2.6) to be

$$\widetilde{G}_{L\,ij} = g_{ij}\,;\ \widetilde{G}_{L\,66} = -{R_6}^2;\ \widetilde{G}_{L\,i6} = 0\,;\quad \widetilde{G}_L^{ij} = g^{ij};\ \widetilde{G}_L^{66} = -\frac{1}{R_6^2};\ \widetilde{G}_L^{i6} = 0,\quad \widetilde{G}_L = \det \widetilde{G}_{L\,\tilde{m}\tilde{n}}. \tag{3.3}$$

Solving for k_6 from (3.2) we find

$$k_6 = \pm \frac{\sqrt{-\widetilde{G}_L^{66}}}{\widetilde{G}_L^{66}}\,|k|, \tag{3.4}$$

where $2 \leq i,j \leq 5$, and $|k| \equiv \sqrt{g^{ij}k_ik_j}$. Use the gauge invariance $f_{\tilde{m}} \to f'_{\tilde{m}} = f_{\tilde{m}} + k_{\tilde{m}}\lambda$ to fix $f'_6 = 0$, which is the gauge choice

$$A_6 = 0.$$

This reduces the number of components of $A_{\tilde{m}}$ from 5 to 4. To satisfy (3.2), we can use the $\partial^{\tilde{m}} F_{\tilde{m}6} = -\partial_6\partial^i A_i = 0$ component of the equation of motion to eliminate f_5 in terms of the three f_2, f_3, f_4,

$$f_5 = -\frac{1}{p^5}(p^2 f_2 + p^3 f_3 + p^4 f_4),$$

leaving just three independent polarization vectors corresponding to the physical degrees of freedom of the 5d one-form with Spin(3) content 3. From the Lorentzian Lagrangian

$$\mathcal{L} = -\frac{1}{4}\frac{\sqrt{-\widetilde{G}_L}}{g_{5YM}^2}\widetilde{G}_L^{\tilde{m}\tilde{m}'}\widetilde{G}_L^{\tilde{n}\tilde{n}'}F_{\tilde{m}\tilde{n}}F_{\tilde{m}'\tilde{n}'} = \frac{R_6\sqrt{g}}{4\pi^2 R_1}\Big(-\frac{1}{4}g^{ii'}g^{jj'}F_{ij}F_{i'j'} - \frac{1}{2}\widetilde{G}_L^{66}g^{jj'}F_{6j}F_{6j'}\Big), \tag{3.5}$$

the energy-momentum tensor

$$\mathcal{T}^{m}_{\ \ n} = \frac{\delta\mathcal{L}}{\delta\partial_m A_p}\partial_n A_p - \delta^m_{\ \ n}\,\mathcal{L} \tag{3.6}$$

leads to the Hamiltonian and momenta operators

$$H_c \equiv \int d^4\theta\, \mathcal{T}^{6}_{\ \ 6} = \int d^4\theta\,\Big(\frac{R_6\sqrt{g}}{4\pi^2R_1}\big(-\frac{1}{2}\widetilde{G}_L^{66}g^{ii'}\,F_{6i}F_{6i'} + \frac{1}{4}g^{ii'}g^{jj'}F_{ij}F_{i'j'} - F^{6i}\partial_i A_6\big) + \Pi^6\partial_6 A_6\Big), \tag{3.7}$$

$$P_i \equiv \int d^4\theta\, \mathcal{T}^{6}_{\ \ i} = \int d^4\theta\Big(\frac{R_6\sqrt{g}}{4\pi^2R_1}\big(-\widetilde{G}_L^{66}g^{jj'}\,F_{6j'}F_{ij} - F^{6j}\partial_j A_i\big) + \Pi^6\partial_i A_6\Big), \tag{3.8}$$

where the conjugate momentum is

$$\Pi^i = \frac{\delta\mathcal{L}}{\delta\partial_6 A_i} = -\frac{R_6\sqrt{g}}{4\pi^2R_1}F^{6i} = -\frac{R_6\sqrt{g}}{4\pi^2R_1}\widetilde{G}_L^{66}g^{ii'}F_{6i'},\qquad \Pi^6 = \frac{\delta\mathcal{L}}{\delta\partial_6 A_6} = 0. \tag{3.9}$$

We quantize the Maxwell field on the five-torus with the metric (3.3) in radiation gauge using Dirac constraints [31, 32]. The theory has a primary constraint $\Pi^6(\vec{\theta},\theta^6)\approx 0$. We can express the Hamiltonian (3.7) in terms of the conjugate momentum as

$$H_{can} = \int d^4\theta\Big(-\frac{2\pi^2 R_1}{R_6\sqrt{g}\widetilde{G}_L^{66}}\, g_{ii'}\Pi^i\,\Pi^{i'} + \frac{R_6\sqrt{g}}{16\pi^2 R_1} g^{ii'}g^{jj'}F_{ij}F_{i'j'} - \partial_i\Pi^i\, A_6\Big), \tag{3.10}$$

where the last term has been integrated by parts. The primary Hamiltonian is defined by

$$H_p = \int d^4\theta\Big(-\frac{2\pi^2 R_1}{R_6\sqrt{g}\widetilde{G}_L^{66}}\, g_{ii'}\Pi^i\,\Pi^{i'} + \frac{R_6\sqrt{g}}{16\pi^2 R_1} g^{ii'}g^{jj'}F_{ij}F_{i'j'} - \partial_i\Pi^i\, A_6 + \lambda_1\Pi^6\Big), \tag{3.11}$$

with λ_1 as a Lagrange multiplier. In Appendix A, we use the Dirac method of quantizing with constraints for the radiation gauge conditions $A_6\approx 0$, $\partial^i A_i\approx 0$, and find the equal time commutation relations (A.13), (A.14):

$$\begin{aligned}
&[\Pi^j(\vec{\theta},\theta^6), A_i(\vec{\theta}',\theta^6)] = -i\Big(\delta_i^j - g^{jj'}(\partial_i\frac{1}{g^{kk'}\partial_k\partial_{k'}}\partial_{j'})\Big)\,\delta^4(\theta-\theta'),\\
&[A_i(\vec{\theta},\theta^6), A_j(\vec{\theta}',\theta^6)] = 0, \qquad [\Pi^i(\vec{\theta},\theta^6),\Pi^j(\vec{\theta}',\theta^6)] = 0.
\end{aligned} \tag{3.12}$$

Appendix B shows the Hamilitonian (3.11) to give the correct equations of motion.

In $A_6 = 0$ gauge, the free quantum vector field on the torus is expanded as

$$A_i(\vec{\theta},\theta^6) = \text{ zero modes } + \sum_{\vec{k}\neq 0,\vec{k}\in\mathcal{Z}_4} (f_i^\kappa a_{\vec{k}}^\kappa e^{ik\cdot\theta} + f_i^{\kappa *} a_{\vec{k}}^{\kappa\dagger} e^{-ik\cdot\theta}),$$

where $1\le\kappa\le 3$, $2\le i\le 5$ and k_6 defined in (3.4). The sum is on the dual lattice $\vec{k} = k_i\in\mathcal{Z}_4\neq\vec{0}$. Having computed the zero mode contribution in (2.17), here we consider[5]

$$A_i(\vec{\theta},\theta^6) = \sum_{\vec{k}\neq 0}(a_{\vec{k}\,i}e^{ik\cdot\theta} + a^\dagger_{\vec{k}\,i}e^{-ik\cdot\theta}), \tag{3.13}$$

with polarizations absorbed in

$$a_{\vec{k}\,i} = f_i^\kappa a_{\vec{k}}^\kappa. \tag{3.14}$$

From (3.12) the commutator in terms of the oscillators is

$$\int\frac{d^4\theta d^4\theta'}{(2\pi)^8} e^{-ik_i\theta^i}e^{-ik'_i\theta'^i}[A_i(\vec{\theta},0), A_j(\vec{\theta}',0)] = [(a_{\vec{k}\,i}+a^\dagger_{-\vec{k}\,i}),(a_{\vec{k}'\,j}+a^\dagger_{-\vec{k}'\,j})] = 0. \tag{3.15}$$

The conjugate momentum $\Pi^j(\vec{\theta},\theta^6)$ in (3.9) is expressed in terms of $a_{\vec{k}\,i}, a^\dagger_{-\vec{k}\,i}$ by

$$\Pi^j(\vec{\theta},\theta^6) = -i\frac{R_6\sqrt{g}}{4\pi^2 R_1}\widetilde{G}_L^{66} g^{jj'}\sum_{\vec{k}} k_6\,(a_{\vec{k}\,j'}e^{ik\cdot\theta} - a^\dagger_{\vec{k}\,j'}e^{-ik\cdot\theta}). \tag{3.16}$$

[5]In this mode expansion, we shall pick the plus sign in (3.4) for the root k_6 which solves (3.2).

Then taking the Fourier transform of $\Pi^j(\vec{\theta},\theta^6)$ at $\theta^6=0$, we have

$$\int \frac{d^4\theta}{(2\pi)^4} e^{-ik_i\theta^i}\Pi^j(\vec{\theta},0) = -i\frac{R_6\sqrt{g}}{4\pi^2 R_1}\widetilde{G}_L^{66} g^{jj'} k_6\,(a_{\vec{k}\,j'} - a^\dagger_{-\vec{k}\,j'}). \tag{3.17}$$

From (3.17) and the commutators (3.12) and (3.15), we find

$$\begin{aligned}&\int \frac{d^4\theta d^4\theta'}{(2\pi)^8} e^{-ik_i\theta^i} e^{-ik'_i\theta'^i}[\Pi^j(\vec{\theta},0), A_i(\vec{\theta}',0)]\\ &= -i(\delta_i^j - \frac{g^{jj'}k_i k_{j'}}{g^{kk'}k_k k_{k'}})\delta_{\vec{k},-\vec{k}'}\,\frac{1}{(2\pi)^4} = -i\frac{R_6\sqrt{g}}{4\pi^2 R_1}\widetilde{G}_L^{66} g^{jj'} k_6\,[(a_{\vec{k}\,j'} - a^\dagger_{-\vec{k}\,j'}),(a_{\vec{k}'\,i} + a^\dagger_{-\vec{k}'\,i})].\end{aligned} \tag{3.18}$$

To reach the oscillator commutator (3.24), we define

$$A_{\vec{k}\,i} \equiv a_{\vec{k}\,i} + a^\dagger_{-\vec{k}\,i} = A^\dagger_{-\vec{k}\,i}, \qquad E_{\vec{k}\,i} \equiv a_{\vec{k}\,i} - a^\dagger_{-\vec{k}\,i} = -E^\dagger_{-\vec{k}\,i}, \tag{3.19}$$

$$a_{\vec{k}\,i} = \frac{1}{2}(A_{\vec{k}\,i} + E_{\vec{k}\,i}), \qquad a^\dagger_{\vec{k}\,i} = \frac{1}{2}(A^\dagger_{\vec{k}\,i} + E^\dagger_{\vec{k}\,i}) = \frac{1}{2}(A_{-\vec{k}\,i} - E_{-\vec{k}\,i}). \tag{3.20}$$

Now inverting (3.18) we have

$$[E_{\vec{k}\,j}, A_{\vec{k}'\,i}] = \frac{R_1}{R_6\sqrt{g}\widetilde{G}_L^{66} k_6}\frac{1}{(2\pi)^2}\big(g_{ji} - \frac{k_j k_i}{g^{kk'}k_k k_{k'}}\big)\delta_{\vec{k},-\vec{k}'}, \tag{3.21}$$

and from (3.17) and the relations (3.12) and (3.15),

$$[A_{\vec{k}\,i}, A_{\vec{k}'\,j}] = 0, \qquad [E_{\vec{k}\,i}, E_{\vec{k}'\,j}] = 0. \tag{3.22}$$

Using (3.20),

$$[a_{\vec{k}\,i}, a^\dagger_{\vec{k}'\,j}] = \frac{1}{4}\Big([A_{\vec{k}\,i}, A_{-\vec{k}'\,j}] - [E_{\vec{k}\,i}, E_{-\vec{k}'\,j}] - [A_{\vec{k}\,i}, E_{-\vec{k}'\,j}] + [E_{\vec{k}\,i}, A_{-\vec{k}'\,j}]\Big), \tag{3.23}$$

together with (3.21), (3.22) we find the oscillator commutation relations

$$\begin{aligned}&[a_{\vec{k}\,i}, a^\dagger_{\vec{k}'\,j}] = \frac{R_1}{R_6\sqrt{g}\widetilde{G}_L^{66} k_6}\frac{1}{2(2\pi)^2}\big(g_{ij} - \frac{k_i k_j}{g^{kk'}k_k k_{k'}}\big)\delta_{\vec{k},\vec{k}'},\\ &[a_{\vec{k}\,i}, a_{\vec{k}'\,j}] = 0, \qquad [a^\dagger_{\vec{k}\,i}, a^\dagger_{\vec{k}'\,j}] = 0.\end{aligned} \tag{3.24}$$

In the gauge $\partial^i A_i(\vec{\theta},\theta^6) = 0$, then $k^i a_{\vec{k}\,i} = g^{ij}k_j a_{\vec{k}\,i} = 0,\ k^i a^\dagger_{\vec{k}\,i} = g^{ij}k_j a^\dagger_{\vec{k}\,i} = 0$ as in (3.2), and these are consistent with the commutator (3.24). We will use this commutator to proceed with the evaluation of the Hamiltonian and momenta in (3.7,3.8). In $A_6 = 0$ gauge,

$$H_c = \int d^4\theta\,\frac{R_6\sqrt{g}}{4\pi^2 R_1}\Big(-\frac{1}{2}\widetilde{G}_L^{66} g^{ii'}\,\partial_6 A_i \partial_6 A_{i'} + \frac{1}{4} g^{ii'} g^{jj'} F_{ij}F_{i'j'}\Big), \tag{3.25}$$

which is the Hamiltonian H^{5d} in (1.1). In (3.8) after integrating by parts, we also set the second constraint described in Appendix A $\partial_i \Pi^i = 0$, to find

$$P_i = \frac{1}{4\pi^2 R_1 R_6} \int_0^{2\pi} d\theta^2 d\theta^3 d\theta^4 d\theta^5 \sqrt{g}\; g^{jj'}\; F_{6j'} F_{ij}, \tag{3.26}$$

which is the momenta P_i^{5d} in (1.1).

From (3.25), in terms of the normal mode expansion (3.13),

$$\begin{aligned} H_c = (2\pi)^2 \frac{R_6\sqrt{g}}{R_1} \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} \big(\frac{1}{2}\widetilde{G}_L^{66} g^{ii'} k_6 k_6 + \frac{1}{2}(g^{ii'}g^{jj'} - g^{ij'}g^{ji'})k_j k_{j'}\big)(a_{\vec{k}\,i} a_{-\vec{k}\,i'} e^{2ik_6\theta^6} + a^\dagger_{\vec{k}\,i} a^\dagger_{-\vec{k}\,i'} e^{-2ik_6\theta^6}) \\ + (2\pi)^2 \frac{R_6\sqrt{g}}{R_1} \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} \big(-\frac{1}{2}\widetilde{G}_L^{66} g^{ii'} k_6 k_6 + \frac{1}{2}(g^{ii'}g^{jj'} - g^{ij'}g^{ji'})k_j k_{j'}\big)(a_{\vec{k}\,i} a^\dagger_{\vec{k}\,i'} + a^\dagger_{\vec{k}\,i} a_{\vec{k}\,i'}), \end{aligned} \tag{3.27}$$

with the delta function

$$\int \frac{d^4\theta}{(2\pi)^4} e^{i(k_i - k'_i)\theta^i} = \delta_{\vec{k},\vec{k}'}, \tag{3.28}$$

and where k_6 is given in (3.4). From the on-shell and transverse conditions (3.2), $\widetilde{G}_L^{66} k_6 k_6 + |k|^2 = 0$, and $k^i a_{\vec{k}\,i} = k^i a^\dagger_{\vec{k}\,i} = 0$, so the time-dependence of H_c on θ^6 cancels and

$$H_c = (2\pi)^2 \frac{R_6\sqrt{g}}{R_1} \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} g^{ii'} |k|^2\; (a_{\vec{k}\,i} a^\dagger_{\vec{k}\,i'} + a^\dagger_{\vec{k}\,i} a_{\vec{k}\,i'}). \tag{3.29}$$

Similarly the momenta from (3.26) become

$$P_i = -\frac{R_6\sqrt{g}}{R_1} \widetilde{G}_L^{66} g^{jj'} (2\pi)^2 \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} k_6 k_i\; \big(a_{\vec{k}\,j'} a^\dagger_{\vec{k}\,j} + a^\dagger_{\vec{k}\,j'} a_{\vec{k}\,j}\big). \tag{3.30}$$

Then

$$\begin{aligned} -H_c + i\gamma^i P_i &= \mp\sqrt{-\widetilde{G}_L^{66}} \frac{R_6\sqrt{g}}{R_1}\; (2\pi)^2 \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} |k|\; \big(\pm \frac{|k|}{\sqrt{-\widetilde{G}_L^{66}}} + i\gamma^i k_i\big) g^{jj'} \big(a_{\vec{k}\,j} a^\dagger_{\vec{k}\,j'} + a^\dagger_{\vec{k}\,j} a_{\vec{k}\,j'}\big) \\ &= \mp i\sqrt{-\widetilde{G}_L^{66}} \frac{R_6\sqrt{g}}{R_1}\; (2\pi)^2 \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} |k|\; \big(\pm i \frac{\sqrt{-\widetilde{G}_L^{66}}}{\widetilde{G}_L^{66}}\; |k| + \gamma^i k_i\big) g^{jj'} \big(a_{\vec{k}\,j} a^\dagger_{\vec{k}\,j'} + a^\dagger_{\vec{k}\,j} a_{\vec{k}\,j'}\big). \end{aligned} \tag{3.31}$$

Since we are using a Lorentzian signature metric at this point, $-\widetilde{G}_L^{66} > 0$. Then rewriting in terms of a real Euclidean radius R_6, and making the upper sign choice in (3.4), we have

$$-H_c + i\gamma^i P_i = -i\frac{1}{R_6}\,\frac{R_6\sqrt{g}}{R_1}\,(2\pi)^2 \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} |k|\big(-iR_6|k| + \gamma^i k_i\big) g^{jj'}\big(a_{\vec{k}\,j}a^\dagger_{\vec{k}\,j'} + a^\dagger_{\vec{k}\,j}a_{\vec{k}\,j'}\big). \tag{3.32}$$

Inserting the polarizations as $a_{\vec{k}\,i} = f_i^\kappa a_{\vec{k}}^\kappa$ and $a^\dagger_{\vec{k}\,i} = f_i^{\lambda *} a_{\vec{k}}^{\lambda\dagger}$ from (3.14) in the commutator (3.24) gives

$$[a_{\vec{k}\,i}, a^\dagger_{\vec{k}'\,j}] = \frac{R_1}{R_6\sqrt{g}}\frac{R_6}{|k|}\frac{1}{2(2\pi)^2}\Big(g_{ij} - \frac{k_i k_j}{|k|^2}\Big)\delta_{\vec{k},\vec{k}'} = f_i^\kappa f_j^{\lambda *}[a_{\vec{k}}^\kappa, a_{\vec{k}}^{\lambda\dagger}], \tag{3.33}$$

where we choose the normalization

$$[a_{\vec{k}}^\kappa, a_{\vec{k}'}^{\lambda\dagger}] = \delta^{\kappa\lambda}\delta_{\vec{k},\vec{k}'}. \tag{3.34}$$

Then the polarization vectors satisfy

$$f_i^\kappa f_j^{\lambda *}\delta^{\kappa\lambda} = \frac{R_1}{\sqrt{g}\,|k|}\frac{1}{2(2\pi)^2}\Big(g_{ij} - \frac{k_i k_j}{|k|^2}\Big), \qquad g^{jj'} f_j^\kappa f_{j'}^{\lambda *}\delta^{\kappa\lambda} = \frac{R_1}{\sqrt{g}\,|k|}\frac{1}{2(2\pi)^2}\cdot 3,$$
$$g^{jj'} f_j^\kappa f_{j'}^{\lambda *} = \delta^{\kappa\lambda}\frac{R_1}{\sqrt{g}\,|k|}\frac{1}{2(2\pi)^2}.$$

So the exponent in (1.1) is given by

$$\begin{aligned} -H_c + i\gamma^i P_i &= -i\frac{1}{R_6}\,\frac{R_6\sqrt{g}}{R_1}(2\pi)^2 \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} |k|\big(-iR_6|k| + \gamma^i k_i\big) g^{jj'}\big(2a^\dagger_{\vec{k}\,j}a_{\vec{k}\,j'} + [a_{\vec{k}\,j}, a^\dagger_{\vec{k}\,j'}]\big) \\ &= -i\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} \big(\gamma^i k_i - iR_6|k|\big)\, a_{\vec{k}}^{\kappa\dagger} a_{\vec{k}}^\kappa \quad - \frac{i}{2}\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} \big(-iR_6|k|\big)\,\delta^{\kappa\kappa}. \end{aligned} \tag{3.35}$$

Then the partition function is

$$Z^{5d,Maxwell} \equiv tr\ \exp\{2\pi(-H_c + i\gamma^i P_i)\} = Z^{5d}_{\text{zero modes}}\, Z^{5d}_{\text{osc}}, \tag{3.36}$$

where from (3.35),

$$Z^{5d}_{\text{osc}} = tr\ e^{-2\pi i\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}}\big(\gamma^i k_i - iR_6|k|\big)\, a_{\vec{k}}^{\kappa\dagger}a_{\vec{k}}^\kappa\ -\pi R_6\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}}|k|\,\delta^{\kappa\kappa}}. \tag{3.37}$$

4 Comparison of Oscillator Traces Z^{5d}_{osc} and Z^{6d}_{osc}

In order to compare the partition functions of the two theories, we first review the calculation for the 6d chiral field from [3] setting the angles between the circle and five-torus $\alpha, \beta^i = 0$.

The oscillator trace is evaluated by rewriting (1.2) as

$$-2\pi R_6\mathcal{H} + i2\pi\gamma^i\mathcal{P}_i = \frac{i\pi}{12}\int_0^{2\pi} d^5\theta H_{lrs}\epsilon^{lrsmn}H_{6mn} = \frac{i\pi}{2}\int_0^{2\pi} d^5\theta\sqrt{-G}H^{6mn}H_{6mn}$$
$$= -i\pi\int_0^{2\pi} d^5\theta(\Pi^{mn}H_{6mn} + H_{6mn}\Pi^{mn}) \tag{4.1}$$

where the definitions $H^{6mn} = \frac{1}{6\sqrt{-G}}\epsilon^{mnlrs}H_{lrs}$ and $H_{6mn} = \frac{1}{6\sqrt{-G}G^{66}}\epsilon_{mnlrs}H^{lrs}$ follow from the self-dual equation of motion (2.1). $\Pi^{mn}(\vec{\theta},\theta^6)$, the field conjugate to $B_{mn}(\vec{\theta},\theta^6)$ is defined from the Lagrangian for a general (non-self-dual) two-form $I_6 = \int d^6\theta(-\frac{\sqrt{-G}}{24})H_{LMN}H^{LMN}$, so $\Pi^{mn} = \frac{\delta I_6}{\delta\partial_6 B_{mn}} = -\frac{\sqrt{-G}}{4}H^{6mn}$. The commutation relations of the two-form and its conjugate field $\Pi^{mn}(\vec{\theta},\theta^6)$ are

$$\begin{aligned}[\Pi^{rs}(\vec{\theta},\theta^6), B_{mn}(\vec{\theta}',\theta^6)] &= -\, i\delta^5(\vec{\theta}-\vec{\theta}')(\delta^r_m\delta^s_n - \delta^r_n\delta^s_m),\\ [\Pi^{rs}(\vec{\theta},\theta^6), \Pi^{mn}(\vec{\theta}',\theta^6)] &= [B_{rs}(\vec{\theta},\theta^6), B_{mn}(\vec{\theta}',\theta^6)] = 0.\end{aligned}$$

From the Bianchi identity $\partial_{[L}H_{MNP]} = 0$ and the fact that (2.1) implies $\partial^L H_{LMN} = 0$, then a solution to (2.1) is given by a solution to the homogeneous equations $\partial^L\partial_L B_{MN} = 0$, $\partial^L B_{LN} = 0$. These have a plane wave solution

$$B_{MN}(\vec{\theta},\theta^6) = f_{MN}(p)e^{ip\cdot\theta} + (f_{MN}(p)e^{ip\cdot\theta})^*; \qquad G^{LN}p_Lp_N = 0\,; \qquad p^L f_{LN} = 0; \tag{4.2}$$

and quantum tensor field expansion

$$B_{mn}(\vec{\theta},\theta^6) = \text{zero modes} \quad + \sum_{\vec{p}=p_l\in\mathcal{Z}^5\neq\vec{0}} (f^\kappa_{mn}b^\kappa_{\vec{p}}e^{ip\cdot\theta} + f^{\kappa*}_{mn}b^{\kappa\dagger}_{\vec{p}}e^{-ip^*\cdot\theta}) \tag{4.3}$$

for the three physical polarizations of the 6d chiral two-form [3], $1 \le \kappa \le 3$. Because oscillators with different polarizations commute, each polarization can be treated separately and the result then cubed. Without the zero mode term,

$$B_{mn}(\vec{\theta},\theta^6) = \sum_{\vec{p}\neq 0}(b_{\vec{p}mn}\, e^{ip\cdot\theta} + b^\dagger_{\vec{p}mn}e^{-ip^*\cdot\theta})\,, \tag{4.4}$$

for $b_{\vec{p}mn} = f^1_{mn}b^1_{\vec{p}}$ for example, with a similar expansion for $\Pi^{mn}(\vec{\theta},\theta^6)$ in terms of $c^{6mn\dagger}_{\vec{p}}$. From (4.2) the momentum p_6 is

$$p_6 = -\gamma^i p_i - iR_6\sqrt{g^{ij}p_ip_j + \frac{p_1^2}{R_1^2}}. \tag{4.5}$$

For the gauge choice $B_{6n}=0$, the exponent (4.1) becomes

$$
\begin{aligned}
& -i\pi(2\pi)^5 \sum_{\vec{p}=p_l\in\mathcal{Z}^5\neq 0} ip_6(\mathcal{C}^{6mn\dagger}_{\vec{p}} B_{\vec{p}mn} + B_{\vec{p}mn}\mathcal{C}^{6mn\dagger}_{\vec{p}}) \\
&= -2i\pi \sum_{\vec{p}\neq 0} p_6 \mathcal{C}^{\kappa\dagger}_{\vec{p}} B^{\lambda}_{\vec{p}} f^{\kappa mn}(p) f^{\lambda}_{mn}(p) - i\pi \sum_{\vec{p}\neq 0} p_6 f^{\kappa mn}(p) f^{\kappa}_{mn}(p) \\
&= -2i\pi \sum_{\vec{p}\neq 0} p_6 \mathcal{C}^{\kappa\dagger}_{\vec{p}} B^{\kappa}_{\vec{p}} - i\pi \sum_{\vec{p}\neq 0} p_6 \delta^{\kappa\kappa},
\end{aligned}
\tag{4.6}
$$

with $B_{\vec{p}mn} \equiv b_{\vec{p}mn} + b^{\dagger}_{-\vec{p}mn}$, $\mathcal{C}^{6mn\dagger}_{\vec{p}} \equiv c^{6mn}_{-\vec{p}} + c^{6mn\dagger}_{\vec{p}}$. The polarization tensors have been restored where $1\leq\kappa,\lambda\leq 3$ and the oscillators $B^{\kappa}_{\vec{p}}, \mathcal{C}^{\lambda\dagger}_{\vec{p}}$ satisfy the commutation relation

$$
[B^{\kappa}_{\vec{p}}, \mathcal{C}^{\lambda\dagger}_{\vec{p}}] = \delta^{\kappa\lambda}\ \delta_{\vec{p},\vec{p}'}. \tag{4.7}
$$

So restricting the manifold to a circle times a five-torus in [3] we have

$$
\begin{aligned}
& -2\pi R_6 \mathcal{H} + i2\pi\gamma^i P_i \\
&= -2i\pi \sum_{\vec{p}\in\mathcal{Z}^5\neq 0} \Big(-\gamma^i p_i - iR_6 \sqrt{g^{ij}p_ip_j + \frac{p_1^2}{R_1^2}} \Big) \mathcal{C}^{\kappa\dagger}_{\vec{p}} B^{\kappa}_{\vec{p}} - \pi R_6 \sum_{\vec{p}\in\mathcal{Z}^5} \sqrt{g^{ij}p_ip_j + \frac{p_1^2}{R_1^2}}\ \delta^{\kappa\kappa}
\end{aligned}
\tag{4.8}
$$

The oscillator trace (1.2) is

$$
\begin{aligned}
Z^{6d}_{\text{osc}} &= tr\, e^{-t\mathcal{H}+i2\pi\gamma^i P_i} = tr\, e^{-2i\pi\sum_{\vec{p}\neq 0} p_6 \mathcal{C}^{\kappa\dagger}_{\vec{p}} B^{\kappa}_{\vec{p}} - \pi R_6 \sum_{\vec{p}} \sqrt{g^{ij}p_ip_j + \frac{p_1^2}{R_1^2}}\ \delta^{\kappa\kappa}}, \\
Z^{6d,chiral} &= Z^{6d}_{\text{zero modes}} \cdot \big(e^{-\pi R_6 \sum_{\vec{n}\in\mathcal{Z}^5} \sqrt{g^{ij}n_in_j + \frac{n_1^2}{R_1^2}}} \prod_{\vec{n}\in\mathcal{Z}^5\neq 0} \frac{1}{1 - e^{-2\pi R_6\sqrt{g^{ij}n_in_j + \frac{n_1^2}{R_1^2}} + i2\pi\gamma^i n_i}} \big)^3.
\end{aligned}
\tag{4.9}
$$

Regularizing the vacuum energy as in [3], the chiral field partition function (1.2) becomes

$$
Z^{6d,chiral} = Z^{6d}_{\text{zero modes}} \cdot \Big(e^{R_6\pi^{-3}\sum_{\vec{n}\neq\vec{0}} \frac{\sqrt{G_5}}{(g_{ij}n^in^j + R_1^2(n^1)^2)^3}} \prod_{\vec{n}\in\mathcal{Z}^5\neq\vec{0}} \frac{1}{1 - e^{-2\pi R_6\sqrt{g^{ij}n_in_j + \frac{(n_1)^2}{R_1^2}} + i2\pi\gamma^i n_i}} \Big)^3, \tag{4.10}
$$

where $Z^{6d}_{\text{zero modes}}$ is given in (2.8). Lastly we compute the 5d Maxwell partition function (1.1) from (3.37),

$$
Z^{5d,Maxwell} = Z^{5d}_{\text{zero modes}} \cdot\ tr\ e^{-2i\pi\sum_{\vec{k}\neq\vec{0}}(\gamma^i k_i - iR_6\sqrt{g^{ij}k_ik_j})\, a^{\kappa\dagger}_{\vec{k}}\, a^{\kappa}_{\vec{k}} - \pi\sum_{\vec{k}\neq\vec{0}}(R_6\sqrt{g^{ij}k_ik_j})\,\delta^{\kappa\kappa}}, \tag{4.11}
$$

where $\vec{k} = k_i = n_i \in \mathcal{Z}^4$ on the torus. From the standard Fock space argument

$$tr\ \omega^{\sum_p pa_p^\dagger a_p} = \prod_p \sum_{k=0}^{\infty} \langle k|\omega^{pa_p^\dagger a_p}|k\rangle = \prod_p \frac{1}{1-\omega^p},$$

we perform the trace on the oscillators,

$$Z_{\rm osc}^{5d} = \Big(e^{-\pi R_6 \sum_{\vec n\in\mathcal{Z}^4}\sqrt{g^{ij}n_in_j}} \prod_{\vec n\in\mathcal{Z}^4\neq\vec 0} \frac{1}{1-e^{-i2\pi(\gamma^i n_i - iR_6\sqrt{g^{ij}n_in_j})}}\Big)^3, \tag{4.12}$$

$$Z^{5d,Maxwell} = Z_{\text{zero modes}}^{5d}\cdot\Big(e^{-\pi R_6 \sum_{\vec n\in\mathcal{Z}^4}\sqrt{g^{ij}n_in_j}} \prod_{\vec n\in\mathcal{Z}^4\neq\vec 0} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}-2\pi i\gamma^i n_i}}\Big)^3, \tag{4.13}$$

where $Z_{\text{zero modes}}^{5d}$ is given in (2.17). (4.13) and (4.9) are each manifestly $SL(4,\mathcal{Z})$ invariant due to the underlying $SO(4)$ invariance we have labeled as $i=2,3,4,5$. We use the $SL(4,\mathcal{Z})$ invariant regularization of the vacuum energy reviewed in Appendix C to obtain

$$Z^{5d,Maxwell} = Z_{\text{zero modes}}^{5d}\cdot\Big(e^{\frac{3}{8}R_6\pi^{-2}\sum_{\vec n\neq 0}\frac{\sqrt{g}}{(g_{ij}n^in^j)^{\frac{5}{2}}}} \prod_{\vec n\in\mathcal{Z}^4\neq\vec 0} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}-2\pi i\gamma^i n_i}}\Big)^3, \tag{4.14}$$

where the sum is on the original lattice $\vec n = n^i \in \mathcal{Z}^4 \neq \vec 0$, and the product is on the dual lattice $\vec n = n_i \in \mathcal{Z}^4 \neq \vec 0$. In Appendix D we prove that the product of the zero mode contribution and the oscillator contribution in (4.14) is $SL(5,\mathcal{Z})$ invariant. In (D.32) we give an equivalent expression,

$$Z^{5d,Maxwell} = Z_{\text{zero modes}}^{5d}\cdot\Big(e^{\frac{\pi R_6}{6R_2}} \prod_{n\neq 0} \frac{1}{1-e^{-2\pi\frac{R_6}{R_2}|n|+2\pi i\gamma^2 n}}\Big)^3$$
$$\cdot\Big(\prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} e^{-2\pi R_6 <H>_{p_\perp}} \prod_{n_2\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}+2\pi i\gamma^i n_i}}\Big)^3, \tag{4.15}$$

with $< H >_{p_\perp}$ defined in (C.13). In Appendix D we also prove the $SL(5,\mathcal{Z})$ invariance of the 6d chiral partition function (4.10), using the equivalent form (D.45),

$$Z^{6d,chiral} = Z_{\text{zero modes}}^{6d}\cdot(e^{\frac{\pi R_6}{6R_2}} \prod_{n\in\mathcal{Z}\neq 0} \frac{1}{1-e^{-2\pi\frac{R_6}{R_2}|n|+2\pi i\,\gamma^2\,n\,)}})^3$$
$$\cdot(\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6<H>_{p_\perp}^{6d}} \prod_{n_2\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j+\frac{n_1^2}{R_1^2}}+i2\pi\gamma^i n_i}})^3 \tag{4.16}$$

with $< H >_{p_\perp}^{6d}$ in (D.44). Thus the partition functions of the two theories are both $SL(5,\mathcal{Z})$ invariant, but they are not equal.

The comparison of the 6d chiral theory on $S^1\times T^5$ and the abelian gauge theory on T^5 shows the exponent of the oscillator contribution to the partition function for the 6d theory (4.8),

$$\begin{aligned}
&-2\pi R_6\mathcal{H} + i2\pi\gamma^i\mathcal{P}_i \\
&= -2\pi \sum_{\vec{p}\in\mathcal{Z}^5\neq 0} \Big(- i\gamma^i p_i + R_6\sqrt{g^{ij}p_ip_j + \frac{p_1^2}{R_1^2}}\Big)\, \mathcal{C}^{\kappa\dagger}_{\vec{p}} B^{\kappa}_{\vec{p}} - \pi R_6 \sum_{\vec{p}\in\mathcal{Z}^5} \sqrt{g^{ij}p_ip_j + \frac{p_1^2}{R_1^2}}\ \delta^{\kappa\kappa},
\end{aligned} \tag{4.17}$$

and for the gauge theory (3.35),

$$-2\pi H^{5d} + 2\pi i\gamma^i P^{5d}_i = -2\pi \sum_{\vec{k}\in\mathcal{Z}^4\neq 0} \Big(i\gamma^i k_i + R_6\sqrt{g^{ij}k_ik_j}\Big)\ a^{\kappa\dagger}_{\vec{k}}\, a^{\kappa}_{\vec{k}} - \pi R_6 \sum_{\vec{k}\in\mathcal{Z}^4} \sqrt{g^{ij}k_ik_j}\, \delta^{\kappa\kappa}, \tag{4.18}$$

differ only by the sum on the Kaluza-Klein modes p_1 of S^1 since for the chiral case $\vec{p}\in\mathcal{Z}^5$, and for the Maxwell case $\vec{k}\in\mathcal{Z}^4$. Both theories have three polarizations, $1\leq\kappa\leq 3$, and from (4.7), (3.34) the oscillators have the same commutation relations,

$$[B^{\lambda}_{\vec{p}}, \mathcal{C}^{\lambda\dagger}_{\vec{p}}] = \delta^{\kappa\lambda}\ \delta_{\vec{p},\vec{p}'}, \qquad [a^{\kappa}_{\vec{k}}, a^{\lambda\dagger}_{\vec{k}'}] = \delta^{\kappa\lambda}\ \delta_{\vec{k},\vec{k}'}. \tag{4.19}$$

If we discard the Kaluza-Klein modes p_1^2 in the usual limit [27] as the radius of the circle R_1 is very small with respect to the radii and angles g_{ij}, R_6, of the five-torus, then the oscillator products in (4.16) and (4.15) are equivalent. This holds as a precise limit since we can separate the product on $n_\perp = (n_1, n_\alpha)\neq 0_\perp$ in (4.16), into $(n_1 = 0, n_\alpha\neq(0,0,0))$ and $(n_1\neq 0, \text{all } n_\alpha)$, to find at fixed n_2,

$$\begin{aligned}
&\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j + \frac{(n_1)^2}{R_1^2}} + 2\pi i\gamma^i n_i}} \\
&\quad = \prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j} + 2\pi i\gamma^i n_i}} \cdot \prod_{n_1\neq 0, n_\alpha\in\mathcal{Z}^3} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j + \frac{(n_1)^2}{R_1^2}} + 2\pi i\gamma^i n_i}}.
\end{aligned} \tag{4.20}$$

In the limit of small R_1 the last product reduces to unity, thus for S^1 smaller than T^5

$$\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j + \frac{(n_1)^2}{R_1^2}} + 2\pi i\gamma^i n_i}} \to \prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j} + 2\pi i\gamma^i n_i}}. \tag{4.21}$$

Inspecting the regularized vacuum energies $< H >_{p_\perp}$ and $< H >^{6d}_{p_\perp}$ in (C.13),(D.44),

$$< H >_{p_\perp\neq 0} = -\pi^{-1}\,|p_\perp|\sum_{n=1}^{\infty}\cos(p_\alpha\kappa^\alpha 2\pi n)\frac{K_1(2\pi n R_2|p_\perp|)}{n},\quad \text{for}\quad |p_\perp|\equiv\sqrt{\widetilde{g}^{\alpha\beta}n_\alpha n_\beta},$$

$$< H >^{6d}_{p_\perp\neq 0} = -\pi^{-1}\,|p_\perp|\sum_{n=1}^{\infty}\cos(p_\alpha\kappa^\alpha 2\pi n)\frac{K_1(2\pi n R_2|p_\perp|)}{n},\quad \text{for}\quad |p_\perp|\equiv\sqrt{\frac{(n_1)^2}{R_1^2}+\widetilde{g}^{\alpha\beta}n_\alpha n_\beta}, \tag{4.22}$$

we see they have the same form of spherical Bessel functions, but the argument differs by Kaluza-Klein modes. Again separating the product on $n_\perp = (n_1, n_\alpha)$ in (4.16), into $(n_1 = 0, n_\alpha \neq (0,0,0))$ and $(n_1 \neq 0, \text{all } n_\alpha)$ we have

$$\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6 < H>^{6d}_{p_\perp}} = \Big(\prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} e^{-2\pi R_6 < H>_{p_\perp}}\Big)\cdot\Big(\prod_{n_1\neq 0, n_\alpha\in\mathcal{Z}^3} e^{-2\pi R_6 < H>^{6d}_{p_\perp}}\Big). \tag{4.23}$$

In the limit $R_1 \to 0$, the last product is unity because for $n_1 \neq 0$,

$$\lim_{R_1\to 0}\sqrt{\frac{(n_1)^2}{R_1^2}+\widetilde{g}^{\alpha\beta}n_\alpha n_\beta}\sim\frac{|n_1|}{R_1},$$
$$\lim_{R_1\to 0}|p_\perp|\;K_1(2\pi n R_2|p_\perp|) = \lim_{R_1\to 0}\frac{|n_1|}{R_1}\;K_1\big(2\pi n R_2\frac{|n_1|}{R_1}\big) = 0, \tag{4.24}$$

since $\lim_{x\to\infty} xK_1(x) \sim \sqrt{x}\, e^{-x} \to 0$. [33]. So (4.23) leads to

$$\lim_{R_1\to 0}\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6 < H>^{6d}_{p_\perp}} = \prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} e^{-2\pi R_6 < H>_{p_\perp}}. \tag{4.25}$$

Thus in the limit where the radius of the circle S^1 is small with respect to T^5, which is the limit of weak coupling g^2_{5YM}, we have proved

$$\lim_{R_1\to 0}\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6 < H>^{6d}_{p_\perp}}\prod_{n_2\in\mathcal{Z}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_i n_j+\frac{n_1^2}{R_1^2}}+i2\pi\gamma^i n_i}}$$
$$= \prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} e^{-2\pi R_6 < H>_{p_\perp}}\prod_{n_2\in\mathcal{Z}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_i n_j}+2\pi i\gamma^i n_i}}. \tag{4.26}$$

So together with (1.3), we have shown the partition functions of the chiral theory on $S^1 \times T^5$ and of Maxwell theory on T^5, which we computed in (4.16) and (4.15), are equal only in the weak coupling limit,

$$\lim_{R_1\to 0} Z^{6d,chiral} = Z^{5d,Maxwell}. \tag{4.27}$$

5 Discussion and Conclusions

We have addressed a conjecture of the quantum equivalence between the six-dimensional conformally invariant $N = (2, 0)$ theory compactified on a circle and the five-dimensional maximally supersymmetric Yang-Mills theory. In this paper we consider an abelian case without supersymmetry when the five-dimensional manifold is a twisted torus. We compute the partition functions for the chiral tensor field B_{LN} on $S^1 \times T^5$, and for the Maxwell field $A_{\tilde{m}}$ on T^5. We prove the two partition functions are each $SL(5, \mathcal{Z})$ invariant, but are equal only in the limit of weak coupling g^2_{5YM}, a parameter which is proportional to R_1, the radius of the circle S^1.

To carry out the computations we first restricted an earlier calculation [3] of the chiral partition function on T^6 to $S^1 \times T^5$. Then we used an operator quantization to compute the Maxwell partition on T^5 as defined in (1.1) which inserts non-zero γ^i as the coefficient of P_i^{5d}, but otherwise quantizes the theory in a 5d Lorentzian signature metric that has zero angles with its time direction, *i.e.* $\gamma^i = 0,\ 2 \leq i \leq 5$, [13]. We used this metric and form (1.1) to derive both the zero mode and oscillator contributions. The Maxwell field theory was thus quantized on T^5, with the Dirac method of constraints resulting in the commutation relations in (3.24).

Comparing the partition function of the Maxwell field on a twisted five-torus T^5 with that of a two-form potential with a self-dual three-form field strength on $S^1 \times T^5$, where the radius of the circle is $R_1 \equiv g^2_{5YM}/4\pi^2$, we find the two theories are not equivalent as quantum theories, but are equal only in the limit where R_1 is small relative to the metric parameters of the five-torus, a limit which effectively removes the Kaluza-Klein modes from the 6d partition sum. How to incorporate these modes rigorously in the 5d theory, possibly interpreted as instantons in the non-abelian version of the gauge theory with appropriate dynamics remains difficult [34]-[37], suggesting that the 6d finite conformal $N = (2, 0)$ theory on a circle is an ultraviolet completion of the 5d maximally supersymmetric gauge theory rather than an exact quantum equivalence.

Furthermore, it would be compelling to find how expressions for the partition function of the 6d $N = (2, 0)$ conformal quantum theory computed on various manifolds using localization should reduce to the expression in [3] in an appropriate limit, providing a check that localization is equivalent to canonical quantization.

Acknowledgments

We are grateful to Michael Douglas, Peter Goddard, Sergei Gukov and Edward Witten for discussions. LD thanks the Institute for Advanced Study at Princeton for its hospitality. LD and YS are partially supported by the U.S. Department of Energy, Grant No. DE-FG02-06ER-4141801, Task A.

A Dirac Method of Quantization with Constraints

The 5d Maxwell theory on a five-torus with metric (3.3) has the Hamiltonian (3.11),

$$H_p = \int d^4\theta \Big(- \frac{2\pi^2 R_1}{R_6 \sqrt{g} \widetilde{G}_L^{66}} \, g_{ii'} \Pi^i \, \Pi^{i'} + \frac{R_6 \sqrt{g}}{16\pi^2 R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'} - \partial_i \Pi^i \, A_6 + \lambda_1 \Pi^6 \Big), \qquad (A.1)$$

with λ_1 as a Lagrange multiplier. To quantize and derive the commutation relations, we start with the equal-time *canonical Poisson brackets*

$$\begin{aligned} \{\Pi^{\tilde{m}}(\vec{\theta}, \theta^6), A_{\tilde{n}}(\vec{\theta}', \theta^6)\} &= -\{A_{\tilde{n}}(\vec{\theta}', \theta^6), \Pi^{\tilde{m}}(\vec{\theta}, \theta^6)\} = -\delta^4(\vec{\theta} - \vec{\theta}') \, \delta^{\tilde{m}}_{\tilde{n}}, \\ \{\Pi^{\tilde{m}}(\vec{\theta}, \theta^6), \Pi^{\tilde{n}}(\vec{\theta}', \theta^6)\} &= \{A_{\tilde{m}}(\vec{\theta}, \theta^6), A_{\tilde{n}}(\vec{\theta}', \theta^6)\} = 0. \end{aligned} \qquad (A.2)$$

The constraints are required to be time-independent, so for $\phi^1(\theta) \equiv \Pi^6(\vec{\theta}, \theta^6)$,

$$\partial_6 \phi^1(\vec{\theta}, \theta^6) = \{\phi^1(\vec{\theta}, \theta^6), H_p\} = -\int d^4\theta' \{\Pi^6(\theta), A_6(\theta')\} \, \partial_i \Pi^i(\theta') = \partial_i \Pi^i(\theta) \approx 0. \qquad (A.3)$$

Thus the secondary constraint is

$$\phi^2(\theta) \equiv \partial_i \Pi^i(\vec{\theta}, \theta^6) \approx 0, \qquad (A.4)$$

which is time-independent from the contribution

$$\partial_6 \phi^2(\vec{\theta}, \theta^6) = \{\phi^2(\vec{\theta}, \theta^6), H_p\} = \frac{R_6 \sqrt{g}}{16\pi^2 R_1} g^{ii'} g^{jj'} \int d^4\theta' \{\partial_k \Pi^k(\theta), F_{ij}(\theta') F_{i'j'}(\theta')\} = 0. \qquad (A.5)$$

The two constraints ϕ^1, ϕ^2 are first class constraints since they have vanishing Poisson bracket,

$$\{\Pi^6(\theta), \partial_i \Pi^i(\theta')\} = 0. \qquad (A.6)$$

We introduce the gauge conditions

$$\phi^3(\theta) \equiv A_6(\theta) \approx 0, \qquad \phi^4(\theta) \equiv \partial^i A_i(\theta) = g^{ij} \partial_j A_i \approx 0. \qquad (A.7)$$

These convert all four constraints to second class, *i.e.* all now have at least one non-vanishing Poisson bracket with each other, where the non-vanishing brackets are

$$\begin{aligned} &\{\phi^1(\theta), \phi^3(\theta')\} = \{\Pi^6(\theta), A_6(\theta')\} = -\delta^4(\theta - \theta') = -\{A_6(\theta), \Pi^6(\theta')\}, \\ &\{\phi^2(\theta), \phi^4(\theta')\} = \{\partial_i \Pi^i(\theta), g^{jj'} \partial_{j'} A_j(\theta')\} = g^{ij} \frac{\partial}{\partial \theta^i} \frac{\partial}{\partial \theta^j} \delta^4(\theta - \theta') = -\{g^{jj'} \partial_{j'} A_j(\theta), \partial_i \Pi^i(\theta')\}. \end{aligned} \qquad (A.8)$$

Furthermore, there are no new constraints since $\partial_6 \phi^A(\vec{\theta}, \theta^6) = \{\phi^A(\vec{\theta}, \theta^6), H\} \approx 0$, when all $\phi^A \approx 0, \ 1 \le A \le 4$, and $\lambda_1 = \partial_6 A_6$. We can write (A.8) as a matrix

$C^{AB}(\theta,\theta') \equiv \{\phi^A(\theta), \phi^B(\theta')\}$,

$$C^{AB} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & g^{ij}\frac{\partial}{\partial\theta^i}\frac{\partial}{\partial\theta^j} \\ 1 & 0 & 0 & 0 \\ 0 & -g^{ij}\frac{\partial}{\partial\theta^i}\frac{\partial}{\partial\theta^j} & 0 & 0 \end{pmatrix} \delta^4(\theta-\theta'). \tag{A.9}$$

The inverse matrix is

$$(C_{AB})^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{g^{kk'}\frac{\partial}{\partial\theta^k}\frac{\partial}{\partial\theta^{k'}}} \\ -1 & 0 & 0 & 0 \\ 0 & \frac{1}{g^{kk'}\frac{\partial}{\partial\theta^k}\frac{\partial}{\partial\theta^{k'}}} & 0 & 0 \end{pmatrix} \delta^4(\theta-\theta'). \tag{A.10}$$

The Dirac bracket is defined to vanish with any constraint,

$$\begin{aligned} \{A_{\tilde m}(\theta), \Pi^{\tilde n}(\theta')\}_D = \{A_{\tilde m}(\theta), \Pi^{\tilde n}(\theta')\} - \int d^4\rho d^4\rho' \Big(&\{A_{\tilde m}(\theta), \Pi^6(\rho)\} C^{-1}_{13} \{A_6(\rho'), \pi^{\tilde n}(\theta')\} \\ &+ \{A_{\tilde m}(\theta), \partial_i \Pi^i(\rho)\} C^{-1}_{24} \{\partial^j A_j(\rho'), \Pi^{\tilde n}(\theta')\} \\ &+ \{A_{\tilde m}(\theta), A_6(\rho)\} C^{-1}_{31} \{\Pi^6(\rho'), \pi^{\tilde n}(\theta')\} \\ &+ \{A_{\tilde m}(\theta), \partial^j A_j(\rho)\} C^{-1}_{42} \{\partial_i \Pi^i(\rho'), \Pi^{\tilde n}(\theta')\}. \Big) \end{aligned} \tag{A.11}$$

So

$$\begin{aligned} \{A_i(\theta), \Pi^j(\theta')\}_D &= \{A_i(\theta), \pi^j(\theta')\} - \int d^4\rho d^4\rho' \Big(\{A_i(\theta), \partial_k \Pi^k(\rho)\}\; C^{-1}_{24}\; \{\partial^{k'} A_{k'}(\rho'), \pi^j(\theta')\} \Big) \\ &= \Big(\delta_i^j - g^{jj'} \partial_i \frac{1}{g^{kk'}\partial_k \partial_{k'}} \partial_{j'} \Big)\; \delta^4(\theta - \theta'), \end{aligned} \tag{A.12}$$

where here all ∂_j are with respect to θ^j. So promoting the Dirac Poisson bracket to a quantum commutator, we derive the equal time commutation relations

$$[\Pi^j(\vec\theta, \theta^6), A_i(\vec\theta', \theta^6)] = -i\Big(\delta_i^j - g^{jj'} (\partial_i \frac{1}{g^{kk'}\partial_k\partial_{k'}} \partial_{j'}) \Big)\; \delta^4(\theta-\theta'), \tag{A.13}$$

and similarly,

$$[A_i(\vec\theta, \theta^6), A_j(\vec\theta', \theta^6)] = 0, \qquad [\Pi^i(\vec\theta, \theta^6), \Pi^j(\vec\theta', \theta^6)] = 0. \tag{A.14}$$

Furthermore we can check explicitly that Dirac brackets with a constraint vanish, for example

$$\begin{aligned}\{\Pi^j(\theta), \partial^i A_i(\theta')\}_D &= \{\Pi^j(\theta), g^{ik}\partial_k A_i(\theta') - g_{ik}\gamma^k\Pi^i(\theta')\\ &= \widetilde{G}_L^{jk}\frac{\partial}{\partial\theta^k}\delta^4(\theta-\theta') - \widetilde{G}_L^{jl}\frac{\partial}{\partial\theta^l}\delta^4(\theta-\theta') = 0 = [\Pi^j(\theta), \partial^i A_i(\theta')],\end{aligned} \tag{A.15}$$

and

$$[\partial_j\Pi^j(\theta), A_i(\theta')] = \partial_j\Big(\delta_i^j - g^{jj'}(\partial_i\frac{1}{g^{kk'}\partial_k\partial_{k'}}\partial_{j'})\Big)\,\delta^4(\theta-\theta') = 0. \tag{A.16}$$

B Equations of Motion

We check that the Hamiltonian gives the correct equations of motion for $A_6 = 0$ which are derived from $\mathcal{L}$ given in (3.5):

$$\begin{aligned}\partial^{\tilde m}F_{\tilde m\tilde n} &= \partial^{\tilde m}\partial_{\tilde m}A_{\tilde n} - \partial_{\tilde n}\partial^{\tilde m}A_{\tilde m}\\ &\Rightarrow g^{ij}\partial_i\partial_j A_k + \widetilde{G}_L^{66}\partial_6\partial_6 A_k - \partial_k g^{ij}\partial_j A_i = 0, \qquad \text{for } \widetilde{n} = k,\end{aligned} \tag{B.1}$$

$$\Rightarrow g^{ki'}\partial_{i'}\partial_6 A_k = 0, \qquad \text{for } \widetilde{n} = 6. \tag{B.2}$$

Hamilton's equations are

$$\partial_6 A_k(\theta) = \{A_k(\theta), H_p\} = -\frac{4\pi^2 R_1}{R_6\sqrt{g}\widetilde{G}_L^{66}}g_{ki}\Pi^i(\theta) + \partial_k A_6(\theta), \tag{B.3}$$

$$\partial_6\Pi^k(\theta) = \{\Pi^k(\theta), H_p\} = \frac{R_6\sqrt{g}}{4\pi^2 R_1}g^{ii'}g^{kj'}\partial_i F_{i'j'}(\theta). \tag{B.4}$$

where regular Poisson brackets are used to compute the time evolution as in (A.5). (B.3) is simply the definition of Π^i in (3.9). (B.4) is Faraday's law,

$$-\partial_6 F^{6k} = \partial_i F^{ik} \tag{B.5}$$

which is (B.1). Gauss' law (B.2) follows from the constraint condition $\phi^4 = \partial^i A_i \approx 0$.

C Regularization of the Vacuum Energy for 5d Maxwell Theory

The Fourier transform of powers of a radial function is

$$|\vec{p}|^{\alpha-n} = \frac{c_\alpha}{(2\pi)^n}\int d^n y\sqrt{G_n}\, e^{-i\vec{p}\cdot\vec{y}}\frac{1}{|\vec{y}|^\alpha}, \qquad \text{where} \qquad c_\alpha \equiv \frac{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}. \tag{C.1}$$

This formula holds by analytic continuation, since for general n, α, where the area of the unit sphere S_{n-2} is

$$\omega_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \equiv \int_0^\pi d\theta_1 d\theta_2 \dots d\theta_{n-3} \, \sin\theta_1 \, \sin^2\theta_2 \dots \sin^{n-3}\theta_{n-3} \int_0^{2\pi} d\phi, \tag{C.2}$$

the Fourier integral is

$$\begin{aligned}
\int d^n y \sqrt{G_n} \, e^{-i\vec{p}\cdot\vec{y}} \frac{1}{|\vec{y}|^\alpha} &= \int_0^\infty dy \, y^{n-1-\alpha} \int_0^\pi d\theta \sin^{n-2}\theta \, e^{-i|\vec{p}|y\cos\theta} \, \omega_{n-2} \\
&= \int_0^\infty dy \, y^{n-1-\alpha} \, \frac{(2\pi)^{\frac{n}{2}}}{(|\vec{p}|y)^{\frac{n-2}{2}}} \, J_{\frac{n-2}{2}}(|\vec{p}|y) \\
&= |\vec{p}|^{\alpha-n} \, (2\pi)^{\frac{n}{2}} \, \frac{2^{\frac{n}{2}-\alpha}\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})},
\end{aligned} \tag{C.3}$$

where the last expression is valid for the integral when $-\frac{n}{2} < \frac{n}{2} - \alpha < \frac{1}{2}$, but can be analytically continued for all $\alpha \neq -n, -n-1, \dots$

So expressing $|\vec{p}|$ in terms of its 4d Fourier transform,

$$\begin{aligned}
|\vec{p}| &= -\frac{3}{4\pi^2} \int d^4 y \sqrt{g} \, e^{-i\vec{p}\cdot\vec{y}} \frac{1}{|\vec{y}|^5}, \\
< H > &= \frac{1}{2} \sum_{\vec{p}\in\mathcal{Z}^4} |\vec{p}| \, e^{i\vec{p}\cdot\vec{x}}|_{\vec{x}=0} = \frac{1}{2} \sum_{\vec{p}\in\mathcal{Z}^4} \sqrt{g^{ij} p_i p_j},
\end{aligned} \tag{C.4}$$

we have for the sum on the dual lattice, $p_i \in \mathcal{Z}^4$,

$$\begin{aligned}
&\sum_{\vec{p}\in\mathcal{Z}^4} |\vec{p}| e^{i\vec{p}\cdot\vec{x}} = -\frac{3}{4\pi^2}\sqrt{g} \int d^4 y \frac{1}{|\vec{y}|^5} \sum_{\vec{p}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \\
&= -\frac{3}{4\pi^2}\sqrt{g} \int d^4 y \frac{1}{|\vec{y}|^5} (2\pi)^4 \sum_{\vec{n}\neq 0} \delta^4(\vec{x} - \vec{y} + 2\pi\vec{n}) = -12\pi^2\sqrt{g} \sum_{\vec{n}\in\mathcal{Z}^4\neq 0} \frac{1}{|\vec{x} + 2\pi\vec{n}|^5}
\end{aligned} \tag{C.5}$$

where the regularization consists of removing the $\vec{n} = 0$ term from the equality,

$$\sum_{\vec{p}\in\mathcal{Z}^4} e^{i\vec{p}\cdot\vec{x}} = (2\pi)^4 \sum_{\vec{n}\in\mathcal{Z}^4} \delta^4(\vec{x} + 2\pi\vec{n}) \tag{C.6}$$

and the sum on $\vec{n}$ is on the original lattice $\vec{n} = n^i \in \mathcal{Z}^4$. The regularized vacuum energy is

$$< H > = -\frac{3}{16\pi^3}\sqrt{g} \sum_{\vec{n}\in\mathcal{Z}^4\neq 0} \frac{1}{(g_{ij} n^i n^j)^{\frac{5}{2}}} = -6\pi^2\sqrt{g} \sum_{\vec{n}\in\mathcal{Z}^4\neq 0} \frac{1}{|2\pi\vec{n}|^5} \, . \tag{C.7}$$

For the discussion of $SL(5, \mathcal{Z})$ invariance in Appendix D, it is also useful to write the regularized sum (C.7), as

$$< H > = \sum_{p_\perp \in \mathcal{Z}^3} < H >_{p_\perp} = < H >_{p_\perp=0} + \sum_{p_\perp \in \mathcal{Z}^3 \neq 0} < H >_{p_\perp}, \tag{C.8}$$

where $p_\perp = p_\alpha \in \mathcal{Z}^3,\ \alpha = 3, 4, 5$, and

$$< H >_{p_\perp=0} = \frac{1}{2} \sum_{p_2 \in \mathcal{Z}} \sqrt{g^{22} p_2 p_2} = \frac{1}{R_2} \sum_{n=1}^{\infty} n = \frac{1}{R_2} \zeta(-1) = -\frac{1}{12 R_2} \tag{C.9}$$

by zeta function regularization. For general $p_\perp$, we express (C.7) as a sum of terms at fixed transverse momentum [3],

$$< H >_{p_\perp} = -6\pi^2 \sqrt{g} \frac{1}{(2\pi)^3} \int d^3 z_\perp e^{-i p_\perp \cdot z_\perp} \sum_{\vec{n} \in \mathcal{Z}^4 \neq 0} \frac{1}{|2\pi \vec{n} + z_\perp|^5}, \tag{C.10}$$

using the equality for the periodic delta function, $\sum_{p_\alpha \in \mathcal{Z}^3} e^{ip \cdot z} = (2\pi)^3 \sum_{n^\alpha \in \mathcal{Z}^3} \delta^3(\vec{z} + 2\pi \vec{n})$. Changing variables $z^\alpha \to y^\alpha + 2\pi n^\alpha$, (C.10) becomes

$$< H >_{p_\perp} = -6\pi^2 \sqrt{g} \frac{1}{(2\pi)^3} \int d^3 y_\perp e^{-i p_\perp \cdot y_\perp} \sum_{n \in \mathcal{Z} \neq 0} \frac{1}{|2\pi n + y_\perp|^5} \tag{C.11}$$

where n is the n^2 component on the original lattice, and the denominator is $|2\pi n + y_\perp|^2 \equiv [(2\pi n)^2 G_{22} + 2(2\pi n) G_{2\alpha} y_\perp^\alpha + y_\perp^\alpha y_\perp^\beta G_{\alpha\beta}] = [(2\pi n)^2 (R_2^2 + g_{\alpha\beta} \kappa^\alpha \kappa^\beta) - 2(2\pi n) g_{\alpha\beta} \kappa^\beta y_\perp^\alpha + y_\perp^\alpha y_\perp^\beta g_{\alpha\beta}]$. We can extract the $p_\perp = 0$ part of (C.11) to verify (C.9),

$$\begin{aligned} < H >_{p_\perp=0} &= -6\pi^2 \sqrt{g} \frac{1}{(2\pi)^3} \sum_{n \in \mathcal{Z} \neq 0} \int d^3 y_\perp \frac{1}{|2\pi n + y_\perp|^5} \\ &= -6\pi^2 \sqrt{g} \frac{1}{(2\pi)^3} \sum_{n \in \mathcal{Z} \neq 0} \frac{4\pi}{3} \frac{1}{(2\pi)^2 R_2^2} \frac{1}{n^2} \frac{1}{\sqrt{\tilde{g}}} = -\frac{\zeta(2)}{2\pi^2 R_2} = -\frac{1}{12 R_2}, \end{aligned} \tag{C.12}$$

by performing the y integrations. For general $p_\perp \in \mathcal{Z}^3 \neq 0$, (C.11) integrates to give the spherical Bessel functions,

$$\begin{aligned} < H >_{p_\perp \neq 0} &= |p_\perp|^2 R_2 \sum_{n=1}^{\infty} \cos(p_\alpha \kappa^\alpha 2\pi n) \big[K_2(2\pi n R_2 |p_\perp|) - K_0(2\pi n R_2 |p_\perp|) \big] \\ &= -\pi^{-1}\ |p_\perp| R_2 \sum_{n=1}^{\infty} \cos(p_\alpha \kappa^\alpha 2\pi n) \frac{K_1(2\pi n R_2 |p_\perp|)}{n}, \end{aligned} \tag{C.13}$$

where $|p_\perp| = \sqrt{\tilde{g}^{\alpha\beta} n_\alpha n_\beta}$ can be viewed as the mass of three scalar bosons [3].

For a d-dimensional lattice sum, the general formula used in (C.4) for regulating the divergent sum is [3],

$$|\vec{p}| = 2\pi^{-\frac{d}{2}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(-\frac{1}{2})} \int d^d y \sqrt{G_d}\, e^{-i\vec{p}\cdot\vec{y}} \frac{1}{|\vec{y}|^{d+1}},$$

$$< H > = \frac{1}{2} \sum_{\vec{p}\in\mathcal{Z}^d} |\vec{p}|\, e^{i\vec{p}\cdot\vec{x}}|_{\vec{x}=0} = \frac{1}{2} \sum_{\vec{p}\in\mathcal{Z}^d} \sqrt{g^{\alpha\beta} p_\alpha p_\beta}$$

$$= 2^d \pi^{\frac{d}{2}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(-\frac{1}{2})} \sqrt{G_d} \sum_{\vec{n}\in Z^d \neq \vec{0}} \frac{1}{|2\pi\vec{n}|^{d+1}}. \tag{C.14}$$

D $SL(5,\mathcal{Z})$ invariance

Rewriting the 5d metric (2,3,4,5,6)

From (2.3) the metric on the five-torus, for $i, j = 2, 3, 4, 5$, is

$$G_{ij} = g_{ij}, \qquad G_{i6} = -g_{ij}\gamma^j, \qquad G_{66} = R_6^2 + g_{ij}\gamma^i\gamma^j,$$
$$\widetilde{G}_5 \equiv \det G_{\tilde{m}\tilde{n}} = R_6^2 \det g_{ij} \equiv R_6^2\, g. \tag{D.1}$$

We can rewrite this metric using $\alpha, \beta = 3, 4, 5$,

$$g_{22} \equiv R_2^2 + \widetilde{g}_{\alpha\beta}\kappa^\alpha\kappa^\beta, \qquad g_{\alpha 2} \equiv -\widetilde{g}_{\alpha\beta}\kappa^\beta, \qquad g_{\alpha\beta} \equiv \widetilde{g}_{\alpha\beta}, \qquad (\gamma^2)\kappa^\alpha - \gamma^\alpha \equiv -\widetilde{\gamma}^\alpha, \tag{D.2}$$

$$G_{22} = R_2^2 + \widetilde{g}_{\alpha\beta}\kappa^\alpha\kappa^\beta, \qquad G_{26} = -(\gamma^2)R_2^2 + \widetilde{g}_{\alpha\beta}\kappa^\beta\widetilde{\gamma}^\alpha, \qquad G_{2\alpha} = -\widetilde{g}_{\alpha\beta}\kappa^\beta,$$
$$G_{\alpha\beta} = \widetilde{g}_{\alpha\beta}, \qquad G_{\alpha 6} = -\widetilde{g}_{\alpha\beta}\widetilde{\gamma}^\beta, \qquad G_{66} = R_6^2 + (\gamma^2)^2 R_2^2 + \widetilde{g}_{\alpha\beta}\,\widetilde{\gamma}^\alpha\widetilde{\gamma}^\beta. \tag{D.3}$$

The $4d$ inverse of g_{ij} is

$$g^{\alpha\beta} = \widetilde{g}^{\alpha\beta} + \frac{\kappa^\alpha\kappa^\beta}{R_2^2}, \qquad g^{\alpha 2} = \frac{\kappa^\alpha}{R_2^2}, \qquad g^{22} = \frac{1}{R_2^2}, \tag{D.4}$$

where $\widetilde{g}^{\alpha\beta}$ is the $3d$ inverse of $\widetilde{g}_{\alpha\beta}$.

$$g \equiv \det g_{ij} = R_2^2 \det \widetilde{g}_{\alpha\beta} \equiv R_2^2\, \widetilde{g}.$$

The line element can be written as

$$ds^2 = R_6^2 (d\theta^6)^2 + \sum_{i,j=2,\ldots,5} g_{ij}(d\theta^i - \gamma^i d\theta^6)(d\theta^j - \gamma^j d\theta^6)$$
$$= R_2^2 (d\theta^2 - (\gamma^2) d\theta^6)^2 + R_6^2 (d\theta^6)^2$$
$$+ \sum_{\alpha,\beta=3,4,5} \widetilde{g}_{\alpha\beta}(d\theta^\alpha - \widetilde{\gamma}^\alpha d\theta^6 - \kappa^\alpha d\theta^2)\,(d\theta^\beta - \widetilde{\gamma}^\beta d\theta^6 - \kappa^\beta d\theta^2). \tag{D.5}$$

We define

$$\widetilde{\tau} \equiv \gamma^2 + i\frac{R_6}{R_2}. \tag{D.6}$$

The 5d inverse is

$$\widetilde{G}_5^{22} = \frac{|\widetilde{\tau}|^2}{R_6^2} = \widetilde{G}_5^{66}|\widetilde{\tau}|^2, \qquad \widetilde{G}_5^{66} = \frac{1}{R_6^2}, \qquad \widetilde{G}_5^{26} = \frac{\gamma^2}{R_6^2}, \qquad \widetilde{G}_5^{2\alpha} = \frac{\kappa^\alpha|\widetilde{\tau}|^2}{R_6^2} + \frac{\gamma^2\widetilde{\gamma}^\alpha}{R_6^2},$$
$$\widetilde{G}_5^{\alpha\beta} = \widetilde{g}^{\alpha\beta} + \frac{\kappa^\alpha\kappa^\beta}{R_6^2}|\widetilde{\tau}|^2 + \frac{\widetilde{\gamma}^\alpha\widetilde{\gamma}^\beta}{R_6^2} + \frac{\gamma^2(\widetilde{\gamma}^\alpha\kappa^\beta + \kappa^\alpha\widetilde{\gamma}^\beta)}{R_6^2}, \qquad \widetilde{G}_5^{6\alpha} = \frac{\gamma^\alpha}{R_6^2} = \frac{\gamma^2\kappa^\alpha + \widetilde{\gamma}^\alpha}{R_6^2}. \tag{D.7}$$

Generators of $SL(n,\mathcal{Z})$

The $SL(n,\mathcal{Z})$ unimodular groups can be generated by two matrices [38]. For $SL(5,\mathcal{Z})$ these can be taken to be U_1, U_2,

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \qquad U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{D.8}$$

so that every matrix M in $SL(5,\mathcal{Z})$ can be written as a product $U_1^{n_1}U_2^{n_2}U_1^{n_3}\ldots$. Therefore to prove the $SL(5,\mathcal{Z})$ invariance of (4.14), we will show it is invariant under U_1 and U_2. Matrices U_1 and U_2 act on the basis vectors of the five-torus $\vec{\alpha}_{\tilde{m}}$ where $\vec{\alpha}_{\tilde{m}} \cdot \vec{\alpha}_{\tilde{n}} \equiv \alpha^{\tilde{p}}_{\tilde{m}}\alpha^{\tilde{q}}_{\tilde{n}}G_{\tilde{p}\tilde{q}} = G_{\tilde{m}\tilde{n}}$,

$$\begin{aligned} \vec{\alpha}_2 &= (1,0,0,0,0) \\ \vec{\alpha}_6 &= (0,1,0,0,0) \\ \vec{\alpha}_3 &= (0,0,1,0,0) \\ \vec{\alpha}_4 &= (0,0,0,1,0) \\ \vec{\alpha}_5 &= (0,0,0,0,1). \end{aligned} \tag{D.9}$$

For our metric (D.3), the U_2 transformation

$$\begin{pmatrix} \vec{\alpha}'_2 \\ \vec{\alpha}'_6 \\ \vec{\alpha}'_3 \\ \vec{\alpha}'_4 \\ \vec{\alpha}'_5 \end{pmatrix} = U_2 \begin{pmatrix} \vec{\alpha}_2 \\ \vec{\alpha}_6 \\ \vec{\alpha}_3 \\ \vec{\alpha}_4 \\ \vec{\alpha}_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{D.10}$$

results in $\vec{\alpha}'_2 \cdot \vec{\alpha}'_2 \equiv \alpha'^{\tilde{p}}_2\alpha'^{\tilde{q}}_2 G_{\tilde{p}\tilde{q}} = G_{22} = G'_{22}$, $\vec{\alpha}'_2 \cdot \vec{\alpha}'_6 \equiv \alpha'^{\tilde{p}}_2\alpha'^{\tilde{q}}_6 G_{\tilde{p}\tilde{q}} = G_{22} + G_{26} = G'_{26}$, etc. So U_2 corresponds to

$$R_2 \to R_2,\; R_6 \to R_6,\; \gamma^2 \to \gamma^2 - 1,\; \kappa^\alpha \to \kappa^\alpha,\; \widetilde{\gamma}^\alpha \to \widetilde{\gamma}^\alpha + \kappa^\alpha,\; \widetilde{g}_{\alpha\beta} \to \widetilde{g}_{\alpha\beta}, \tag{D.11}$$

or equivalently

$$R_6 \to R_6,\ \gamma^2 \to \gamma^2 - 1,\ g_{ij} \to g_{ij},\ \gamma^\alpha \to \gamma^\alpha, \tag{D.12}$$

which leaves invariant the line element (D.5) if $d\theta^2 \to d\theta^2 - d\theta^6$, $d\theta^6 \to d\theta^6$, $d\theta^\alpha \to d\theta^\alpha$. U_2 is the generalization of the usual $\widetilde{\tau} \to \widetilde{\tau} - 1$ modular transformation. The 4d inverse metric $g^{ij} \equiv \{g^{\alpha\beta}, g^{\alpha 2}, g^{22}\}$ does not change under U_2. It is easily checked that U_2 is an invariance of the 5d Maxwell partition function (4.13) as well as the chiral partition function (4.10). It leaves the zero mode and oscillator contributions invariant separately.

The other generator, U_1 is related to the $SL(2,\mathcal{Z})$ transformation $\widetilde{\tau} \to -(\widetilde{\tau})^{-1}$ that we discuss as follows:

$$U_1 = U' M_4 \tag{D.13}$$

where M_4 is an $SL(4,\mathcal{Z})$ transformation given by

$$M_4 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{D.14}$$

and U' is the matrix corresponding to the transformation on the metric parameters (D.16),

$$U' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{D.15}$$

Under U', the metric parameters transform as

$$\begin{gathered} R_2 \to R_2|\widetilde{\tau}|, \quad R_6 \to R_6|\widetilde{\tau}|^{-1}, \quad \gamma^2 \to -\gamma^2|\widetilde{\tau}|^{-2}, \quad \kappa^\alpha \to \widetilde{\gamma}^\alpha, \quad \widetilde{\gamma}^\alpha \to -\kappa^\alpha, \quad \widetilde{g}_{\alpha\beta} \to \widetilde{g}_{\alpha\beta}. \\ \widetilde{\tau} \to -\frac{1}{\widetilde{\tau}}. \qquad \text{Or equivalently,} \\ G_{\alpha\beta} \to G_{\alpha\beta}, \quad G_{\alpha 2} \to G_{\alpha 6}, \quad G_{\alpha 6} \to -G_{\alpha 2}, \quad G_{22} \to G_{66}, \quad G_{66} \to G_{22}, \quad G_{26} \to -G_{26}, \\ \widetilde{G}_5^{\alpha\beta} \to \widetilde{G}_5^{\alpha\beta}, \quad \widetilde{G}_5^{\alpha 2} \to \widetilde{G}_5^{\alpha 6}, \quad \widetilde{G}_5^{\alpha 6} \to -\widetilde{G}_5^{\alpha 2}, \quad \widetilde{G}_5^{22} \to \frac{\widetilde{G}_5^{22}}{|\widetilde{\tau}|^2}, \quad \widetilde{G}_5^{66} \to |\widetilde{\tau}|^2 \widetilde{G}_5^{66}, \quad \widetilde{G}_5^{26} \to -\widetilde{G}_5^{26}, \end{gathered} \tag{D.16}$$

where $3 \le \alpha, \beta \le 5$, and

$$\widetilde{\tau} \equiv \gamma^2 + i\frac{R_6}{R_2}, \qquad |\widetilde{\tau}|^2 = (\gamma^2)^2 + \frac{R_6^2}{R_2^2}. \tag{D.17}$$

The transformation (D.16) leaves invariant the line element (D.5) when $d\theta^2 \to d\theta^6$, $d\theta^6 \to -d\theta^2$, $d\theta^1 \to d\theta^1$, $d\theta^\alpha \to d\theta^\alpha$. The generators have the property $\det U_1 = 1$,

$\det U_2 = 1,\ \det U' = 1,\ \det M_4 = 1.$

Under M_4, the metric parameters transform as

$$R_6 \to R_6, \qquad \gamma^2 \to -\gamma^3, \quad \gamma^\alpha \to \gamma^{\alpha+1}, \quad g_{\alpha\beta} \to g_{\alpha+1,\beta+1}, \quad g_{\alpha 2} \to -g_{\alpha+1,3}, \quad g_{22} \to g_{33},$$
$$g^{\alpha\beta} \to g^{\alpha+1,\beta+1}, \quad g^{\alpha 2} \to -g^{\alpha+1,3}, \quad g^{22} \to g^{33}, \quad \det g_{ij} = g, \quad g \to g. \qquad \text{Or equivalently,}$$

$$G_{\alpha\beta} \to G_{\alpha+1,\beta+1}, \quad G_{\alpha 2} \to -G_{\alpha+1,3}, \quad G_{\alpha 6} \to G_{\alpha+1,6}, \quad G_{22} \to G_{33}, \quad G_{66} \to G_{66}, \quad G_{26} \to -G_{36},$$
$$\widetilde{G}_5^{\alpha\beta} \to \widetilde{G}_5^{\alpha+1,\beta+1}, \quad \widetilde{G}_5^{\alpha 2} \to -\widetilde{G}_5^{\alpha+1,3}, \quad \widetilde{G}_5^{\alpha 6} \to \widetilde{G}_5^{\alpha+1,6}, \quad \widetilde{G}_5^{22} \to \widetilde{G}_5^{33}, \quad \widetilde{G}_5^{26} \to -\widetilde{G}_5^{36}, \quad \widetilde{G}_5^{66} \to \widetilde{G}_5^{66},$$
$$\det \widetilde{G}_5 = R_6\, g, \qquad \det \widetilde{G}_5 \to \det \widetilde{G}_5, \tag{D.18}$$

where $3 \le \alpha, \beta \le 5$, and $\alpha + 1 \equiv 2$ for $\alpha = 5$.

We can check that $Z^{5d}_{\text{zero modes}}$ is invariant under M_4 given in (D.14) as follows. Letting the M_4 transformation (D.18) act on (2.17), we find that the three subterms in the exponent

$$\begin{aligned}
&-2\pi^3 \frac{R_6\sqrt{g}}{R_1}\Big(g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha\beta} F_{\alpha'\beta'} + 4 g^{\alpha\alpha'} g^{\beta 2} F_{\alpha\beta} F_{\alpha' 2} + 2 g^{\alpha\alpha'} g^{22} F_{\alpha 2} F_{\alpha' 2} - 2 g^{\alpha 2} g^{\alpha' 2} F_{\alpha 2} F_{\alpha' 2}\Big),\\
&-\pi \frac{R_1 R_6}{\sqrt{g}} m^i g_{ij} m^j,\\
&\ i4\pi^2 \gamma^i m^j F_{ij}
\end{aligned} \tag{D.19}$$

are separately invariant under (D.18), if we replace the the integers $2\pi F_{ij} \in \mathcal{Z}^6, m^i \in \mathcal{Z}^4$ by

$$2\pi F_{\alpha\beta} \to 2\pi F_{\alpha+1,\beta+1}, \quad 2\pi F_{\alpha 2} \to -2\pi F_{\alpha+1,3}, \quad m^2 \to -m^3, \qquad m^\alpha \to m^{\alpha+1}, \tag{D.20}$$

where $m^i \equiv \frac{2\pi\sqrt{g}}{R_1 R_6} g^{ii'} F_{6i'}$ relabels $(n^7, n^8, n^9, n^{10}) = (m^2, m^3, m^4, m^5)$.
Therefore under M_4, for the zero mode contribution,

$$\sum_{n_1,\ldots,n_6,n^7,\ldots n^{10}} e^{-2\pi H^{5d} + i2\pi\gamma^i P_i^{5d}} \to \sum_{n_1,\ldots,n_6,n^7,\ldots n^{10}} e^{-2\pi H^{5d} + i2\pi\gamma^i P_i^{5d}}. \tag{D.21}$$

So $Z^{5d}_{\text{zero modes}}$ is invariant under M_4. The origin of this is the $SO(4)$ invariance in the coordinate space labeled by $i = 2, 3, 4, 5$.

Next we show under U' that $Z^{5d}_{\text{zero modes}}$ transforms to $|\widetilde{\tau}|^3\, Z^{5d}_{\text{zero modes}}$. From (2.17),

$$\begin{aligned}
Z^{5d}_{\text{zero modes}} &= \sum_{n_1\ldots n_6} \exp\{-2\pi^3 \frac{R_6\sqrt{g}}{R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'}\} \sum_{m^2\ldots m^5} \exp\{-\pi \frac{R_1 R_6}{\sqrt{g}} m^i g_{ij} m^j + i4\pi^2 \gamma^i m^j\, F_{ij}\}\\
&= \sum_{n_1\ldots n_6} \exp\{-2\pi^3 \frac{R_6\sqrt{g}}{R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'}\} \sum_{m^2\ldots m^5} \exp\{-\pi m\cdot A^{-1}\cdot m + 2\pi i\, m\cdot x\},
\end{aligned} \tag{D.22}$$

where $A^{-1}_{ij} = \frac{R_1 R_6}{\sqrt{g}} g_{ij}$ and $x_j = 2\pi\gamma^i F_{ij}$. Using a generalization of the Poisson summation formula

$$\sum_{m\in\mathcal{Z}^p} e^{-\pi m\cdot A^{-1}\cdot m} e^{2\pi i m\cdot x} = (\det A)^{\frac{1}{2}} \sum_{m\in\mathcal{Z}^p} e^{-\pi(m+x)\cdot A\cdot(m+x))}$$

we obtain from (D.22),

$$Z^{5d}_{\text{zero modes}} = (\det A)^{\frac{1}{2}} \sum_{n_1\ldots n_6\in\mathcal{Z}^6} \exp\{-2\pi^3 \frac{R_6\sqrt{g}}{R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'}\}$$
$$\cdot \sum_{m_2\ldots m_5\in\mathcal{Z}^4} \exp\{-\pi \frac{\sqrt{g}}{R_1R_6} g^{jj'} (m_j + \gamma^i 2\pi F_{ij})(m_{j'} + \gamma^{i'} 2\pi F_{i'j'})\}, \quad \text{(D.23)}$$

where

$$A^{jj'} = \frac{\sqrt{g}}{R_1R_6} g^{jj'}, \qquad \det A = (\det A^{-1})^{-1} = \frac{g}{(R_1R_6)^4}. \quad \text{(D.24)}$$

To check how this transforms under U' as given in (D.16), it is convenient to express (D.23) in terms of the metric $\widetilde{G}_5^{\tilde{l}\tilde{m}}$ found in (2.6),

$$Z^{5d}_{\text{zero modes}} = \frac{\sqrt{g}}{(R_1R_6)^2} \sum_{n_1\ldots n_6\in\mathcal{Z}^6} \exp\{-\frac{\pi}{2} \frac{R_6\sqrt{g}}{R_1} \widetilde{G}_5^{ii'} \widetilde{G}_5^{jj'} (2\pi F_{ij})(2\pi F_{i'j'})\}$$
$$\cdot \sum_{m_2\ldots m_5\in\mathcal{Z}^4} \exp\Big\{ - 2\pi \frac{\sqrt{g}R_6}{R_1} \widetilde{G}_5^{6i'} \widetilde{G}_5^{jj'} m_{j'} (2\pi F_{ij}) - \pi \frac{R_6\sqrt{g}}{R_1 1} g^{jj'} m_j m_{j'} \Big\}. \quad \text{(D.25)}$$

Curiously we can identify the exponent in (D.25) as the Euclidean action, if we relabel the integers m_i by f_{6i}, and the $2\pi F_{ij}$ by f_{ij}; and neglect the integrations. In this form it will be easy to study its U' transformation, where (D.25) and (2.17) can also be written as

$$Z^{5d}_{\text{zero modes}} = \frac{\sqrt{g}}{(R_1R_6)^2} \sum_{f_{\tilde{m}\tilde{n}}\in\mathcal{Z}^{10}} \exp\Big\{ - 2\pi \frac{\sqrt{\widetilde{G}_5}}{4R_1} \widetilde{G}_5^{\tilde{m}\tilde{m}'} \widetilde{G}_5^{\tilde{n}\tilde{n}'} f_{\tilde{m}\tilde{n}} f_{\tilde{m}'\tilde{n}'} \Big\}. \quad \text{(D.26)}$$

Under U' from (D.16), the coefficient transforms as

$$U': \qquad \frac{\sqrt{g}}{(R_1R_6)^2} \to \frac{\sqrt{g}}{(R_1R_6)^2}\, |\widetilde{\tau}|^3, \quad \text{(D.27)}$$

since $\frac{\sqrt{g}}{(R_1R_6)^2} = \frac{R_2\sqrt{\tilde{g}}}{(R_1R_6)^2}$. The Euclidean action for the zero mode computation is invariant under U', as we show next by first summing $\tilde{m} = \{2, \alpha, 6\}$, with $3 \le \alpha \le 5$.

$$
\begin{aligned}
&-2\pi\frac{\sqrt{\widetilde{G}_5}}{4R_1}\widetilde{G}^{\tilde{m}\tilde{m}'}\widetilde{G}^{\tilde{n}\tilde{n}'}f_{\tilde{m}\tilde{n}}f_{\tilde{m}'\tilde{n}'}\\
&=-\frac{\pi R_2R_6\sqrt{\widetilde{g}}}{2R_1}\Big(\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{\beta\beta'}f_{\alpha\beta}f_{\alpha'\beta'}+4\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{\beta2}f_{\alpha\beta}f_{\alpha'2}+4\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{\beta6}f_{\alpha\beta}f_{\alpha'6}+2\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{22}f_{\alpha2}f_{\alpha'2}\\
&\qquad-2\widetilde{G}_5^{\alpha2}\widetilde{G}_5^{\alpha'2}f_{\alpha2}f_{\alpha'2}+4\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{26}f_{\alpha2}f_{\alpha'6}-4\widetilde{G}_5^{\alpha6}\widetilde{G}_5^{\alpha'2}f_{\alpha2}f_{\alpha'6}+4\widetilde{G}_5^{\alpha2}\widetilde{G}_5^{\alpha'6}f_{\alpha\alpha'}f_{26}\\
&\qquad+2\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{66}f_{\alpha6}f_{\alpha'6}-2\widetilde{G}_5^{\alpha6}\widetilde{G}_5^{\alpha'6}f_{\alpha6}f_{\alpha'6}+4\widetilde{G}_5^{\alpha2}\widetilde{G}_5^{26}f_{\alpha2}f_{26}-4\widetilde{G}_5^{\alpha6}\widetilde{G}_5^{22}f_{\alpha2}f_{26}\\
&\qquad+4\widetilde{G}_5^{\alpha2}\widetilde{G}_5^{66}f_{\alpha6}f_{26}-4\widetilde{G}_5^{\alpha6}\widetilde{G}_5^{26}f_{\alpha6}f_{26}-2\widetilde{G}_5^{26}\widetilde{G}_5^{26}f_{26}f_{26}+2\widetilde{G}_5^{22}\widetilde{G}_5^{66}f_{26}f_{26}\Big).
\end{aligned}
\tag{D.28}
$$

Letting the U' transformation (D.16) act on (D.28), we see (D.28) changes to

$$
\begin{aligned}
&\Big(-2\pi\frac{\sqrt{\widetilde{G}_5}}{4R_1}\widetilde{G}^{\tilde{m}\tilde{m}'}\widetilde{G}^{\tilde{n}\tilde{n}'}f_{\tilde{m}\tilde{n}}f_{\tilde{m}'\tilde{n}'}\Big)'\\
&=-\frac{\pi R_2R_6\sqrt{\widetilde{g}}}{2R_1}\Big(\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{\beta\beta'}f_{\alpha\beta}f_{\alpha'\beta'}+4\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{\beta6}f_{\alpha\beta}f_{\alpha'2}-4\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{\beta2}f_{\alpha\beta}f_{\alpha'6}+\frac{2}{|\widetilde{\tau}|^2}\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{22}f_{\alpha2}f_{\alpha'2}\\
&\qquad-2\widetilde{G}_5^{\alpha6}\widetilde{G}_5^{\alpha'6}f_{\alpha2}f_{\alpha'2}-4\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{26}f_{\alpha2}f_{\alpha'6}+4\widetilde{G}_5^{\alpha2}\widetilde{G}_5^{\alpha'6}f_{\alpha2}f_{\alpha'6}-4\widetilde{G}_5^{\alpha6}\widetilde{G}_5^{\alpha'2}f_{\alpha\alpha'}f_{26}\\
&\qquad+2|\widetilde{\tau}|^2\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{66}F_{\alpha6}F_{\alpha'6}-2\widetilde{G}_5^{\alpha6}\widetilde{G}_5^{\alpha'2}f_{\alpha6}f_{\alpha'6}-4\widetilde{G}_5^{\alpha6}\widetilde{G}_5^{26}f_{\alpha2}f_{26}+\frac{4}{|\widetilde{\tau}|^2}\widetilde{G}_5^{\alpha2}\widetilde{G}_5^{22}f_{\alpha2}f_{26}\\
&\qquad+4|\widetilde{\tau}|^2\widetilde{G}_5^{\alpha6}\widetilde{G}_5^{66}f_{\alpha6}f_{26}-4\widetilde{G}_5^{\alpha2}\widetilde{G}_5^{26}f_{\alpha6}f_{26}-2\widetilde{G}_5^{26}\widetilde{G}_5^{26}F_{26}F_{26}+2\widetilde{G}_5^{22}\widetilde{G}_5^{66}f_{26}f_{26}\Big).
\end{aligned}
\tag{D.29}
$$

In the partition sum $\sum_{f_{\tilde{m}\tilde{n}}\in\mathcal{Z}^{10}} e^{-2\pi\left(\frac{\sqrt{\widetilde{G}_5}}{4R_1}\widetilde{G}^{\tilde{m}\tilde{m}'}\widetilde{G}^{\tilde{n}\tilde{n}'}f_{\tilde{m}\tilde{n}}f_{\tilde{m}'\tilde{n}'}\right)'}$, we can replace the integers as follows: $f_{\alpha2}\to f_{\alpha6},\ f_{\alpha6}\to -f_{\alpha2}$. Then using (D.7), we have

$$
\sum_{f_{\tilde{m}\tilde{n}}\in\mathcal{Z}^{10}} e^{-2\pi\left(\frac{\sqrt{\widetilde{G}_5}}{4R_1}\widetilde{G}^{\tilde{m}\tilde{m}'}\widetilde{G}^{\tilde{n}\tilde{n}'}f_{\tilde{m}\tilde{n}}f_{\tilde{m}'\tilde{n}'}\right)'}=\sum_{f_{\tilde{m}\tilde{n}}\in\mathcal{Z}^{10}} e^{-2\pi\left(\frac{\sqrt{\widetilde{G}_5}}{4R_1}\widetilde{G}^{\tilde{m}\tilde{m}'}\widetilde{G}^{\tilde{n}\tilde{n}'}f_{\tilde{m}\tilde{n}}f_{\tilde{m}'\tilde{n}'}\right)}. \tag{D.30}
$$

So we have proved that under the U' transformation (D.16),

$$
Z^{5d}_{\text{zero modes}}(R_2|\widetilde{\tau}|,R_6|\widetilde{\tau}|^{-1},\widetilde{g}_{\alpha\beta},-\gamma^2|\widetilde{\tau}|^2,\widetilde{\gamma}^\alpha,-\kappa^\alpha)=|\widetilde{\tau}|^3\ Z^{5d}_{\text{zero modes}}(R_2,R_6,\widetilde{g}_{\alpha\beta},\gamma^2,\kappa^\alpha,\widetilde{\gamma}^\alpha); \tag{D.31}
$$

and thus under the $SL(5,\mathcal{Z})$ generator U_1, $Z^{5d}_{\text{zero modes}}$ transforms to $|\widetilde{\tau}|^3\ Z^{5d}_{\text{zero modes}}$. (D.31) also holds for $Z^{6d}_{\text{zero modes}}$, from (2.18). This is sometimes referred to as an $SL(2,\mathcal{Z})$ anomaly of the zero mode partition function, because U' includes the $\widetilde{\tau}\to-\frac{1}{\widetilde{\tau}}$ transformation. Finally we will show how this anomaly is canceled by the oscillator contribution. The 5d and 6d oscillator contributions are not equal, as given in (4.12) and (4.9). By inspection each is invariant under M_4, (D.18).

U' acts on $Z^{5d}_{\rm osc}$

To derive how U' acts on $Z^{5d}_{\rm osc}$, we first separate the product on $\vec{n} = (n, n_\alpha) \neq \vec{0}$ into a product on (all n, but $n_\alpha \neq (0,0,0)$) and on ($n \neq 0$, $n_\alpha = (0,0,0)$). Then using the regularized vacuum energy (C.7) expressed as sum over zero and non-zero transverse momenta $p_\perp = n_\alpha$ in (C.8),(C.9),(C.13), we find that (4.12) becomes

$$Z^{5d,Maxwell} = Z^{5d}_{\text{zero modes}} \cdot \Big(e^{\frac{\pi R_6}{6R_2}} \prod_{n\neq 0} \frac{1}{1 - e^{-2\pi \frac{R_6}{R_2}|n| - 2\pi i \gamma^2 n}}\Big)^3$$
$$\cdot \Big(\prod_{n_\alpha \in \mathcal{Z}^3 \neq (0,0,0)} e^{-2\pi R_6 <H>_{p_\perp}} \prod_{n_2 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j} - 2\pi i \gamma^i n_i}} \Big)^3. \tag{D.32}$$

As in [3] we observe the middle expression above can be written in terms of the Dedekind eta function $\eta(\widetilde{\tau}) \equiv e^{\frac{\pi i \widetilde{\tau}}{12}} \prod_{n \in Z \neq 0} (1 - e^{2\pi i \widetilde{\tau}})$, with $\widetilde{\tau} = \gamma^2 + i \frac{R_6}{R_2}$,

$$\Big(e^{\frac{\pi R_6}{6R_2}} \prod_{n\neq 0} \frac{1}{1 - e^{-2\pi \frac{R_6}{R_2}|n| - 2\pi i \gamma^2 n}}\Big)^3 = (\eta(\widetilde{\tau})\bar{\eta}(\bar{\widetilde{\tau}}))^{-3}. \tag{D.33}$$

This transforms under U' in (D.16) as

$$(\eta(-\widetilde{\tau}^{-1})\bar{\eta}(-\bar{\widetilde{\tau}}^{-1}))^{-3} = |\widetilde{\tau}|^{-3} \, (\eta(\widetilde{\tau})\bar{\eta}(\bar{\widetilde{\tau}}))^{-3}, \tag{D.34}$$

where $\eta(-\widetilde{\tau}^{-1}) = (i\widetilde{\tau})^{\frac{1}{2}} \eta(\widetilde{\tau})$. In this way the anomaly of the zero modes in (D.31) is canceled by the massless part of the oscillator partition function (D.34). Lastly we demonstrate the third expression in (D.32) is invariant under U',

$$\Big(\prod_{n_\alpha \in \mathcal{Z}^3 \neq (0,0,0)} e^{-2\pi R_6 <H>_\perp} \prod_{n_2 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j} - 2\pi i \gamma^i n_i}} \Big)^3 = (PI)^{\frac{3}{2}} \tag{D.35}$$

where $(PI)^{\frac{3}{2}}$ is the modular invariant 2d partition function of three massive scalar bosons of mass $\sqrt{\widetilde{g}^{\alpha\beta} n_\alpha n_\beta}$, coupled to a worldsheet gauge field following [3]. From (4.13),

$$Z^{5d}_{\rm osc} = (e^{-\pi R_6 \sum_{\vec{n} \in \mathcal{Z}^4} \sqrt{g^{ij} n_i n_j}} \prod_{\vec{n} \in \mathcal{Z}^4 \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j}}}\Big)^3 \tag{D.36}$$

we can extract for fixed $n_\alpha \neq 0$,

$$
\begin{aligned}
(PI)^{\frac{1}{2}} &\equiv e^{-\pi R_6 \sum_{n_2\in\mathcal{Z}} \sqrt{g^{ij}n_in_j}} \quad \prod_{n_2\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}+2\pi i\gamma^i n_i}} \\
&= \prod_{s\in\mathcal{Z}} \frac{e^{-\frac{\beta' E}{2}}}{1-e^{-\beta' E+2\pi i(\gamma^2 s+\gamma^\alpha n_\alpha)}} \qquad \text{where } s\equiv n_2, \quad E\equiv\sqrt{g^{ij}n_in_j}, \quad \beta'\equiv 2\pi R_6 \\
&= \prod_{s\in\mathcal{Z}} \frac{1}{\sqrt{2}\,\sqrt{\cosh\beta' E-\cos 2\pi(\gamma^2 s+\gamma^\alpha n_\alpha)}} \qquad \text{for } n_\alpha\to -n_\alpha \\
&= e^{-\frac{1}{2}\sum_{s\in\mathcal{Z}}\left(\ln[\cosh\beta' E-\cos 2\pi(\gamma^2 s+\gamma^\alpha n_\alpha)]+\ln 2\right)} \equiv e^{-\frac{1}{2}\sum_{s\in\mathcal{Z}}\ \nu(E)},
\end{aligned}
\tag{D.37}
$$

where

$$
\begin{aligned}
\sum_{s\in\mathcal{Z}}\nu(E) &\equiv \sum_{s\in\mathcal{Z}}\left(\ln[\cosh\beta' E-\cos 2\pi(\gamma^2 s+\gamma^\alpha n_\alpha)]+\ln 2\right) \\
&= \sum_{s\in\mathcal{Z}}\sum_{r\in\mathcal{Z}} \ln\,[\frac{4\pi^2}{\beta'^2}(r+\gamma^2 s+\gamma^\alpha n_\alpha)^2+E^2].
\end{aligned}
\tag{D.38}
$$

(D.38) follows in a similar way to steps (B.3)-(B.3) in [3], thus confirming its U' invariance due to the modular invariance of the massive 2d partition function, which we discuss further in the next section. We can also show directly that (D.38) is invariant under U', since

$$
\begin{aligned}
&E^2 = g^{ij}n_in_j = g^{22}s^2+2g^{2\alpha}sn_\alpha+g^{\alpha\beta}n_\alpha n_\beta = \frac{1}{R_2^2}(s+\kappa^\alpha)^2+\widetilde{g}^{\alpha\beta}n_\alpha n_\beta, \\
&\frac{4\pi^2}{\beta'^2}(r+\gamma^2 s+\gamma^\alpha n_\alpha)^2 = \frac{1}{R_6^2}(r+\widetilde{\gamma}^\alpha n_\alpha+\gamma^2(s+\kappa^\alpha n_\alpha))^2,
\end{aligned}
\tag{D.39}
$$

then

$$
\begin{aligned}
&\frac{4\pi^2}{\beta'^2}(r+\gamma^2 s+\gamma^\alpha n_\alpha)^2+E^2 \\
&= \frac{1}{R_6^2}(s+\kappa^\alpha n_\alpha)^2\,|\widetilde{\tau}|^2+\frac{1}{R_6^2}(r+\widetilde{\gamma}^\alpha n_\alpha)^2+\frac{2\gamma^2}{R_6^2}(r+\widetilde{\gamma}^\alpha n_\alpha)(s+\kappa^\alpha n_\alpha)+\widetilde{g}^{\alpha\beta}n_\alpha n_\beta.
\end{aligned}
\tag{D.40}
$$

So we see the transformation U' given in (D.16) leaves (D.40) invariant if $s\to r$ and $r\to -s$. Therefore (D.38) is invariant under U', so that $(PI)^{\frac{1}{2}}$ given in (D.37) is invariant under U'.

In this way, we have established invariance under U_1 and U_2, and thus proved the partition function for the 5d Maxwell theory on T^5, given alternatively by (4.14) or (D.32), is invariant under $SL(5,\mathcal{Z})$, the mapping class group of T^5.

U' acts on Z^{6d}_{osc}

For the 6d chiral theory on $S^1 \times T^5$, the regularized vacuum energy from (4.10) or (C.14),

$$< H >^{6d} = -32\pi^2 \sqrt{G_5} \sum_{\vec{n} \neq \vec{0}} \frac{1}{(2\pi)^6 (g_{ij} n^i n^j + R_1^2 (n^1)^2)^3} \tag{D.41}$$

can be decomposed similarly to (C.8),

$$< H >^{6d} = \sum_{p_\perp \in \mathcal{Z}^3} < H >^{6d}_{p_\perp} = < H >^{6d}_{p_\perp = 0} + \sum_{p_\perp \in \mathcal{Z}^3 \neq 0} < H >^{6d}_{p_\perp}, \tag{D.42}$$

where

$$< H >^{6d}_{p_\perp} = -32\pi^2 \sqrt{G_5} \frac{1}{(2\pi)^4} \int d^4 y_\perp e^{-ip_\perp \cdot y_\perp} \sum_{n^2 \in \mathcal{Z} \neq 0} \frac{1}{|2\pi n^2 + y_\perp|^6}, \tag{D.43}$$

with denominator $|2\pi n^2 + y_\perp|^2 = G_{22}(2\pi n^2)^2 + 2(2\pi n^2) G_{2k} y^k_\perp + G_{kk'} y^k_\perp y^{k'}_\perp$,

$$\begin{aligned}
< H >^{6d}_{p_\perp = 0} &= -\frac{1}{12 R_2}, \\
< H >^{6d}_{p_\perp \neq 0} &= |p_\perp|^2 R_2 \sum_{n=1}^{\infty} \cos(p_\alpha \kappa^\alpha 2\pi n) \big[K_2(2\pi n R_2 |p_\perp|) - K_0(2\pi n R_2 |p_\perp|) \big] \\
&= -\pi^{-1} \; |p_\perp| R_2 \sum_{n=1}^{\infty} \cos(p_\alpha \kappa^\alpha 2\pi n) \frac{K_1(2\pi n R_2 |p_\perp|)}{n},
\end{aligned} \tag{D.44}$$

where $p_\perp = (p_1, p_\alpha) = n_\perp = (n_1, n_\alpha) = (n_1, n_3, n_4, n_5) \in \mathcal{Z}^4$, $|p_\perp| = \sqrt{\frac{(n_1)^2}{R_1^2} + \widetilde{g}^{\alpha\beta} n_\alpha n_\beta}$.

The U' invariance of (4.10) follows when we separate the product on $\vec{n} \in \mathcal{Z}^5 \neq \vec{0}$ into a product on ($n_2 \neq 0$, $n_\perp \equiv (n_1, n_3, n_4, n_5) = (0,0,0,0)$,) and on (all n_2, but $n_\perp = (n_1, n_3, n_4, n_5) \neq (0,0,0,0)$). Then

$$\begin{aligned}
Z^{6d,chiral} &= Z^{6d}_{\text{zero modes}} \cdot \Big(e^{\frac{\pi R_6}{6 R_2}} \prod_{n_2 \in \mathcal{Z} \neq 0} \frac{1}{1 - e^{2\pi i \, (\gamma^2 \, n_2 + i \frac{R_6}{R_2} |n_2| \,)}} \Big)^3 \\
&\quad \cdot \Big(\prod_{n_\perp \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 < H >^{6d}_{p_\perp}} \prod_{n_2 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{n_1^2}{R_1^2}} + i 2\pi \gamma^i n_i}} \Big)^3 \\
&= Z^{6d}_{\text{zero modes}} \cdot \big(\eta(\widetilde{\tau}) \; \bar{\eta}(\bar{\widetilde{\tau}}) \big)^{-3} \\
&\quad \cdot \Big(\prod_{(n_1, n_3, n_4, n_5) \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 < H >^{6d}_{p_\perp}} \prod_{n_2 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{n_1^2}{R_1^2}} + i 2\pi \gamma^i n_i}} \Big)^3,
\end{aligned} \tag{D.45}$$

where $\widetilde{\tau} = \gamma^2 + i\frac{R_6}{R_2}$. So from the previous section together with (1.3), U' leaves invariant

$$Z^{6d}_{\text{zero modes}} \cdot \big(\eta(\widetilde{\tau})\,\bar{\eta}(\bar{\widetilde{\tau}})\big)^{-3}. \tag{D.46}$$

The part of the 6d partition function (D.45) at fixed $n_\perp \neq 0$,

$$e^{-2\pi R_6 <H>_{n_\perp \neq 0}} \prod_{n_2 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{n_1^2}{R_1^2}} + i2\pi\gamma^i n_i}} \tag{D.47}$$

corresponds to massive bosons on a two-torus and is invariant under the $SL(2,\mathcal{Z})$ transformation U' given in (D.16), as follows [3]. Each term with fixed $n_\perp \neq 0$ given in (D.47) is the square root of the partition function on T^2 (in the directions 2,6) of a massive complex scalar with $m^2 \equiv G^{11} n_1^2 + \widetilde{g}^{\alpha\beta} n_\alpha n_\beta$, $3 \leq \alpha, \beta \leq 5$, that couples to a constant gauge field $A^\mu \equiv iG^{\mu i} n_i$ with $\mu, \nu = 2, 6$; $i, j = 1, 3, 4, 5$. The metric on T^2 is $h_{22} = R_2^2\,, h_{66} = R_6^2 + (\gamma^2)^2 R_2^2\,, h_{26} = -\gamma^2 R_2^2$. Its inverse is $h^{22} = \frac{1}{R_2^2} + \frac{(\gamma^2)^2}{R_6^2}$, $h^{66} = \frac{1}{R_6^2}$ and $h^{26} = \frac{\gamma^2}{R_6^2}$. The manifestly $SL(2,\mathcal{Z})$ invariant *path integral* on the two-torus is

$$\begin{aligned}
\text{P.I.} &= \int d\phi\, d\bar{\phi}\; e^{-\int_0^{2\pi} d\theta^2 \int_0^{2\pi} d\theta^6\; h^{\mu\nu}(\partial_\mu + A_\mu)\bar{\phi}(\partial_\nu - A_\nu)\phi + m^2\bar{\phi}\phi} \\
&= \int d\bar{\phi}\, d\phi e^{-\int_0^{2\pi} d\theta^2 \int_0^{2\pi} d\theta^6 \bar{\phi}(-(\frac{1}{R_2^2} + \frac{(\gamma^2)^2}{R_6^2})\partial_2^2 - (\frac{1}{R_6})^2\partial_6^2 - 2\frac{\gamma^2}{R_6^2}\partial_2\partial_6 + 2A^2\partial_2 + 2A^6\partial_6 + G^{11}n_1n_1 + G^{\alpha\beta}n_\alpha n_\beta)\phi} \\
&= \det\Big([-(\tfrac{1}{R_2^2} + (\tfrac{\gamma^2}{R_6})^2)\partial_2^2 - (\tfrac{1}{R_6})^2\partial_6^2 - 2\gamma^2(\tfrac{1}{R_6})^2\partial_2\partial_6 + G^{11}n_1n_1 + G^{\alpha\beta}n_\alpha n_\beta + 2iG^{2\alpha}n_\alpha\partial_2 + 2iG^{6\alpha}n_\alpha\partial_6]\Big)^{-1} \\
&= e^{-\text{tr}\ln\Big[-(\frac{1}{\mathrm{R}_2^2} + (\frac{\gamma^2}{\mathrm{R}_6})^2)\partial_2^2 - (\frac{1}{\mathrm{R}_6})^2\partial_6^2 - 2\gamma^2(\frac{1}{\mathrm{R}_6})^2\partial_2\partial_6 + \mathrm{G}^{11}\mathrm{n}_1\mathrm{n}_1 + \mathrm{G}^{\alpha\beta}\mathrm{n}_\alpha\mathrm{n}_\beta + 2\mathrm{i}\mathrm{G}^{2\alpha}\mathrm{n}_\alpha\partial_2 + 2\mathrm{i}\mathrm{G}^{6\alpha}\mathrm{n}_\alpha\partial_6\Big]} \\
&= e^{-\sum_{s\in\mathcal{Z}}\sum_{r\in\mathcal{Z}}\Big[\ln(\frac{4\pi^2}{\beta'^2}r^2 + (\frac{1}{R_2^2} + (\frac{\gamma^2}{R_6})^2)s^2 + 2\gamma^2(\frac{1}{R_6})^2 rs + G^{11}n_1n_1 + G^{\alpha\beta}n_\alpha n_\beta + 2G^{1\alpha}n_\alpha\, s + 2G^{6\alpha}n_\alpha\, r)\Big]} \\
&= e^{-\sum_{s\in\mathcal{Z}}\nu(E)}
\end{aligned} \tag{D.48}$$

where from (2.4), $G^{11} = \frac{1}{R_1^2}, G^{\alpha\beta} = g^{\alpha\beta} + \frac{\gamma^\alpha\gamma^\beta}{R_6^2}, G^{2\alpha} = g^{2\alpha} + \frac{\gamma^2\gamma^\alpha}{R_6^2}, G^{6\alpha} = \frac{\gamma^\alpha}{R_6^2}$, and $\beta' \equiv 2\pi R_6$, and $\partial_2\phi = -is\phi$; $\partial_6\phi = -ir\phi$, and $n_2 \equiv s$. The sum on r is

$$\nu(E) = \sum_{r\in\mathcal{Z}} \ln\Big[\frac{4\pi^2}{\beta'^2}(r + \gamma^2 s + \gamma^\alpha n_\alpha)^2 + E^2\Big]\,, \tag{D.49}$$

with $E^2 \equiv G_5^{lm} n_l n_m = G_5^{11} n_1 n_1 + G_5^{\alpha\beta} n_\alpha n_\beta + 2G_5^{\alpha 2} n_\alpha n_2 + G_5^{22} n_2 n_2$, and $G_5^{11} = \frac{1}{R_1^2}$, $G_5^{12} = 0, G_5^{1\alpha} = 0, G_5^{2\alpha} = g^{2\alpha} = \frac{\kappa^\alpha}{R_2^2}, G_5^{22} = g^{22} = \frac{1}{R_2^2}, G_5^{\alpha\beta} = g^{\alpha\beta} = \widetilde{g}^{\alpha\beta} + \frac{\kappa^\alpha\kappa^\beta}{R_2^2}$. We evaluate the divergent sum $\nu(E)$ on r by

$$\frac{\partial\nu(E)}{\partial E} = \sum_r \frac{2E}{\frac{4\pi^2}{\beta'^2}(r+\gamma^2 s+\gamma^\alpha n_\alpha)^2+E^2}$$
$$= \partial_E \ln\Big[\cosh\beta' E - \cos 2\pi\big(\gamma^2 s+\gamma^\alpha n_\alpha\big)\Big], \qquad \text{(D.50)}$$

using the sum $\sum_{n\in\mathcal{Z}} \frac{2y}{(2\pi n+z)^2+y^2} = \frac{\sinh y}{\cosh y-\cos z}$. Then integrating (D.50), we choose the integration constant to maintain modular invariance of (D.48),

$$\nu(E) = \ln\big[\cosh\beta' E - \cos 2\pi\big(\gamma^2 s+\gamma^\alpha n_\alpha\big)\big] + \ln 2. \qquad \text{(D.51)}$$

It follows for $n_2 \equiv s$ we have that (D.48) is

$$(\text{P.I.})^{\frac{1}{2}} = \prod_{s\in\mathcal{Z}} \frac{1}{\sqrt{2}\sqrt{\cosh\beta E - \cos 2\pi(\gamma^2 s+\gamma^\alpha n_\alpha)}}$$
$$= \prod_{s\in\mathcal{Z}} \frac{e^{-\frac{\beta E}{2}}}{1-e^{-\beta E+2\pi i(\gamma^2 s+\gamma^\alpha n_\alpha)}}$$
$$= e^{-\pi R_6 \sum_{s\in\mathcal{Z}} \sqrt{G_5^{lm} n_l n_m}} \prod_{s\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{G_5^{lm}n_l n_m}+2\pi i\gamma^2 s+2\pi i\gamma^\alpha n_\alpha}}$$
$$= e^{-2\pi R_6 <H>_{n\perp}} \prod_{n_2\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{G_5^{lm}n_l n_m}+2\pi i\gamma^2 n_2+2\pi i\gamma^\alpha n_\alpha}}, \qquad \text{(D.52)}$$

which is (D.47). Its invariance under U' follows since (D.16) is an $SL(2,\mathcal{Z})$ transformation on T^2 combined with a gauge transformation on the 2d gauge field, $A_\mu \equiv h_{\mu\nu} i n_i G^{\nu i}$ where $\mu,\nu = 2,6$, $A_\mu \to A_\mu + \partial_\mu\lambda$, and $\phi \to e^{i\lambda}$, $\bar\phi \to e^{-i\lambda}$,

$$\lambda(\theta^1,\theta^6) = \theta^2\, i(\tilde\gamma^\alpha - \kappa^\alpha) - \theta^6\, i(\tilde\gamma^\alpha + \kappa^\alpha) \qquad \text{(D.53)}$$

since $A_2 = i\kappa^\alpha n_\alpha,\ A_6 = i\,\tilde\gamma^\alpha\, n_\alpha$. Hence (D.52) and thus (D.47) are invariant under U'. So we have proved the 6d partition function for the chiral field on $S^1\times T^5$, given by (4.10) or equivalently (D.45), is invariant under U_1 and U_2 and is hence $SL(5,\mathcal{Z})$ invariant.

References

[1] M. R. Douglas, *On D=5 super Yang-Mills theory and (2,0) theory,* JHEP **1102**, 011 (2011) [arXiv:1012.2880 [hep-th]].

[2] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, *M5-Branes, D4-Branes and quantum 5D super-Yang-Mills,* JHEP **1101**, 083 (2011) [arXiv:1012.2882 [hep-th]]; *M-Theory and maximally supersymmetric gauge theories,* [arXiv:1203.4244 [hep-th]]; N. Lambert, H. Nastase and C. Papageorgakis, *5D Yang-Mills instantons from ABJM Monopoles,* Phys. Rev. D **85**, 066002 (2012) [arXiv:1111.5619 [hep-th]].

[3] L. Dolan and C. R. Nappi, *A modular invariant partition function for the five-brane,* Nucl. Phys. B **530**, 683 (1998) [arXiv:hep-th/9806016]; *The Ramond-Ramond self-dual five form's partition function on T**10,* Mod. Phys. Lett. A **15**, 1261 (2000) [arXiv:hep-th/0005074].

[4] C. Beasley and E. Witten, *Non-abelian localization for Chern-Simons theory,* J. Diff. Geom. **70**, 183 (2005) [arXiv:hep-th/0503126]].

[5] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,* Commun. Math. Phys. **313**, 71 (2012) [arXiv:0712.2824 [hep-th]].

[6] A. Kapustin, B. Willett and I. Yaakov, *Tests of Seiberg-like duality in three dimensions,* [arXiv:1012.4021 [hep-th]].

[7] T. Dimofte, D. Gaiotto and S. Gukov, *3-Manifolds and 3d Indices,* [arXiv:1112.5179 [hep-th]].

[8] A. Gadde, L. Rastelli, S. S. Razamat and W. Yan, *The 4d superconformal index from q-deformed 2d Yang-Mills,* Phys. Rev. Lett. **106**, 241602 (2011) [arXiv:1104.3850 [hep-th]].

[9] Y. Tachikawa, *4d partition function on $S^1 \times S^3$ and 2d Yang-Mills with nonzero area,* [arXiv:1207.3497 [hep-th]].

[10] D. Bak and A. Gustavsson, *M5/D4 brane partition function on a circle bundle,* JHEP **1212**, 099 (2012) [arXiv:1209.4391].

[11] A. Gustavsson, *A preliminary test of abelian D4-M5 duality,* Phys. Lett. B **706**, 225 (2011) [arXiv:1111.6339 [hep-th]]; *On the holomorphically factorized partition function for abelian gauge theory in six-dimensions,* Int. J. Mod. Phys. A **17**, 383 (2002) [arXiv:hep-th/0008161]].

[12] R. Dijkgraaf, E. P. Verlinde and M. Vonk, *On the partition sum of the NS five-brane,* [arXiv:hep-th/0205281]].

[13] M.B. Green, J. H. Schwarz and E. Witten, Superstring Theory, vol. I and II, Cambridge University Press: Cambridge, U.K. 1987. See vol. II p40.

[14] H. -C. Kim, S. Kim, E. Koh, K. Lee and S. Lee, *On instantons as Kaluza-Klein modes of M5-branes,* JHEP **1112**, 031 (2011) [arXiv:1110.2175 [hep-th]]; S. Bolognesi and K. Lee, *Instanton partons in 5-dim SU(N) gauge theory,* Phys. Rev. D **84**, 106001 (2011) [arXiv:1106.3664 [hep-th]].

[15] B. Collie and D. Tong, *The partonic nature of instantons,* JHEP **0908**, 006 (2009) [arXiv:0905.2267 [hep-th]].

[16] H. Singh, *Super-Yang-Mills and M5-branes,* JHEP **1108**, 136 (2011) [arXiv:1107.3408 [hep-th]].

[17] P. -M. Ho, K. -W. Huang and Y. Matsuo, *A non-abelian self-dual gauge theory in 5+1 dimensions,* JHEP **1107**, 021 (2011) [arXiv:1104.4040 [hep-th]].

[18] H. Samtleben, *Actions for non-abelian twisted self-duality,* Nucl. Phys. B **851**, 298 (2011) [arXiv:1105.3216 [hep-th]].

[19] C. -S. Chu and S. -L. Ko, *Non-abelian action for multiple five-branes with self-dual tensors,* JHEP **1205**, 028 (2012) [arXiv:1203.4224 [hep-th]].

[20] K. -M. Lee and J. -H. Park, *5-D actions for 6-D selfdual tensor field theory,* Phys. Rev. D **64**, 105006 (2001) [arXiv:hep-th/0008103]].

[21] N. Nekrasov, *Instanton partition functions and M-theory*, Japanese Journal of Mathematics, **4**, 63 (2009) [springerlink]; N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting,* Adv. Theor. Math. Phys. **7**, 831 (2004) [arXiv:hep-th/0206161].

[22] J. Källén, J. Qiu and M. Zabzine, *The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere,* [arXiv:1206.6008 [hep-th]].

[23] J. Källén, J. A. Minahan, A. Nedelin and M. Zabzine, *N^3-behavior from 5D Yang-Mills theory,* [arXiv:1207.3763 [hep-th]].

[24] H. -C. Kim and S. Kim, *M5-branes from gauge theories on the 5-sphere,* [arXiv:1206.6339 [hep-th]].

[25] H. Linander and F. Ohlsson, *(2,0) theory on circle fibrations,* JHEP **1201**, 159 (2012) [arXiv:1111.6045 [hep-th]].

[26] B. Haghighat and S. Vandoren, *Five-dimensional gauge theory and compactification on a torus,* JHEP **1109**, 060 (2011) [arXiv:1107.2847 [hep-th]].

[27] E. Witten, *Conformal field theory in four and six dimensions,* [arXiv:0712.0157 [math.RT]]; *Some comments on string dynamics*, In *Los Angeles 1995, Future perspectives in string theory* 501-523 [arXiv:hep-th/9507121].

[28] E. Witten, *On S-duality in abelian gauge theory,* Selecta Math. **1**, 383 (1995) [arXiv:hep-th/9505186].

[29] E. P. Verlinde, *Global aspects of electric - magnetic duality,* Nucl. Phys. B **455**, 211 (1995) [arXiv:hep-th/9506011].

[30] M. Henningson, *The quantum Hilbert space of a chiral two form in d = (5+1)-dimensions,* JHEP **0203**, 021 (2002) [arXiv:hep-th/0111150].

[31] P .A. M. Dirac, <u>Lectures on quantum mechanics</u>, New York: Belfer Graduate School of Science, Yeshiva University (1964) 87p.

[32] A. Das, <u>Lectures on quantum field theory</u>, Hackensack, USA: World Scientific (2008) 775 p.

[33] M. Abramowitz and I. Stegun, <u>Handbook of Mathematical Functions</u>, New York: Dover Publications (1972), p378.

[34] M. Berkooz, M. Rozali and N. Seiberg, *Matrix description of M theory on T^4 and T^5,* Phys. Lett. B **408**, 105 (1997) [arXiv:hep-th/9704089].

[35] N. Seiberg, *Notes on theories with 16 supercharges,* Nucl. Phys. Proc. Suppl. **67**, 158 (1998) [arXiv:hep-th/9705117].

[36] C. M. Hull, *BPS supermultiplets in five-dimensions,* JHEP **0006**, 019 (2000) [arXiv:hep-th/0004086].

[37] Z. Bern, J. J. Carrasco, L. J. Dixon, M. R. Douglas, M. von Hippel and H. Johansson, *$D = 5$ maximally supersymmetric Yang-Mills theory diverges at six loops,* Phys. Rev. D **87**, 025018 (2013) [arXiv:hep-th/1210.7709].

[38] H. Coxeter and W. Moser, Generators and Relations for Discrete Groups, New York: Springer Verlag (1980), p85.