# Electric-magnetic Duality of Abelian Gauge Theory on the Four-torus, from the Fivebrane on $\mathbf{T^2} \times \mathbf{T^4}$, via their Partition Functions

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**Abstract**

We compute the partition function of four-dimensional abelian gauge theory on a general four-torus $T^4$ with flat metric using Dirac quantization. In addition to an $SL(4,\mathcal{Z})$ symmetry, it possesses $SL(2,\mathcal{Z})$ symmetry that is electromagnetic S-duality. We show explicitly how this $SL(2,\mathcal{Z})$ S-duality of the $4d$ abelian gauge theory has its origin in symmetries of the $6d$ $(2,0)$ tensor theory, by computing the partition function of a single fivebrane compactified on $T^2$ times $T^4$, which has $SL(2,\mathcal{Z}) \times SL(4,\mathcal{Z})$ symmetry. If we identify the couplings of the abelian gauge theory $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$ with the complex modulus of the $T^2$ torus $\tau = \beta^2 + i\frac{R_1}{R_2}$, then in the small $T^2$ limit, the partition function of the fivebrane tensor field can be factorized, and contains the partition function of the $4d$ gauge theory. In this way the $SL(2,\mathcal{Z})$ symmetry of the 6d tensor partition function is identified with the $S$-duality symmetry of the 4d gauge partition function. Each partition function is the product of zero mode and oscillator contributions, where the $SL(2,\mathcal{Z})$ acts suitably. For the 4d gauge theory, which has a Lagrangian, this product redistributes when using path integral quantization.

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## 1 Introduction

Four-dimensional $N=4$ Yang-Mills theory is conjectured to possess $S$-duality, which implies the theory with gauge coupling $g$, gauge group $G$, and theta parameter $\theta$ is equivalent to one with $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ transformed by modular transformations $SL(2,\mathcal{Z})$, and the group to $G^\vee$ [1]-[3], with the weight lattice of $G^\vee$ dual to that of $G$ . The conjecture has been tested by the Vafa-Witten partition function on various four-manifolds [4]. More recently, a computation of the $N=4$ Yang-Mills partition function on the four-sphere using the localization method for quantization, enables checking S-duality directly [5].

This duality is believed to have its origin in a certain superconformal field theory in six dimensions, the M5 brane $(2,0)$ theory. When the $6d$, $N=(2,0)$ theory is compactified on $T^2$, one obtains the $4d$, $N=4$ Yang-Mills theory, and the $SL(2,\mathcal{Z})$ group of the torus should imply the S-duality of the four-dimensional gauge theory [6]-[9].

In this paper, we compare the partition function of the $6d$ chiral tensor boson of one fivebrane compactified on $T^2 \times T^4$, with that of $U(1)$ gauge theory with a $\theta$ parameter, compactified on $T^4$. We use these to show explicitly how the $6d$ theory is the origin of S-duality in the gauge theory. Since the $6d$ chiral boson has a self-dual three-form field strength and thus lacks a Lagrangian [10], we will use the Hamiltonian formulation to compute the partition functions for both theories.

As motivated by [11], the four-dimensional $U(1)$ gauge partition function on $T^4$ is

$$Z^{4d,Maxwell} \equiv tr e^{-2\pi H^{4d} + i2\pi\gamma^\alpha P^{4d}_\alpha} = Z^{4d}_{\text{zero modes}} \cdot Z^{4d}_{\text{osc}}, \tag{1.1}$$

where the Hamiltonian and momentum are

$$\begin{aligned} H^{4d} &= \int_0^{2\pi} d^3\theta \Big( \frac{e^2}{4}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\Pi^\alpha\Pi^\beta + \frac{e^2}{32\pi^2}\sqrt{g}\left[\frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4}\right] g^{\alpha\beta}g^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta} + \frac{\theta e^2}{16\pi^2}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}F_{\gamma\delta}\Pi^\beta \Big), \\ P^{4d}_\alpha &= \int_0^{2\pi} d^3\theta \; \Pi^\beta F_{\alpha\beta} \end{aligned} \tag{1.2}$$

in terms of the gauge field strength tensor $F_{ij}(\theta^3,\theta^4,\theta^5,\theta^6)$, the conjugate momentum $\Pi^\alpha$, and the constant parameters $g_{\alpha\beta}, R_6$ and $\gamma^\alpha$ in the metric $G_{ij}$ of $T^4$. They will be derived from the abelian gauge theory Lagrangian, given here for Euclidean signature

$$I = \frac{1}{8\pi}\int_{T^4} d\theta_3 d\theta_4 d\theta_5 d\theta_6 \left( \frac{4\pi}{e^2}\sqrt{g}F^{ij}F_{ij} - \frac{i\theta}{4\pi}\epsilon^{ijkl}F_{ij}F_{kl} \right), \tag{1.3}$$

with $\epsilon^{3456} = 1$, $\epsilon_{ijkl} = g\epsilon^{ijkl}$, and $g = \det(G_{ij})$.

In contrast, the partition function of the abelian chiral two-form on $T^2 \times T^4$ is [12]

$$Z^{6d,chiral} = tr\, e^{-2\pi R_6 \mathcal{H} + i2\pi\gamma^\alpha \mathcal{P}_\alpha} = Z^{6d}_{\text{zero modes}} \cdot Z^{6d}_{\text{osc}},$$
$$\mathcal{H} = \frac{1}{12}\int_0^{2\pi} d\theta^1 \ldots d\theta^5 \sqrt{G_5} {G_5}^{mm'} {G_5}^{nn'} {G_5}^{pp'} H_{mnp}(\vec{\theta},\theta^6)\, H_{m'n'p'}(\vec{\theta},\theta^6),$$
$$\mathcal{P}_\alpha = -\frac{1}{24}\int_0^{2\pi} d\theta^1 ... d\theta^5 \epsilon^{mnprs} H_{mnp}(\vec{\theta},\theta^6)\, H_{\alpha rs}(\vec{\theta},\theta^6) \tag{1.4}$$

where $\theta^1$ and $\theta^2$ are the coordinates of the two one-cycles of $T^2$. The time direction $\theta^6$ is common to both theories, the angle between $\theta^1$ and $\theta^2$ is $\beta^2$, and ${G_5}^{mn}$ is the inverse metric of $G_{5mn}$, where $1 \leq m,n \leq 5$. The eight angles between the two-torus and the four-torus are set to zero.[1]

Section 2 is a list of our results; their derivations are presented in the succeeding sections. In section 3, the contribution of the zero modes to the partition function for the chiral theory on the manifold $M = T^2 \times T^4$ is computed as a sum over ten integer eigenvalues using the Hamiltonian formulation. The zero mode sum for the gauge theory on the same $T^4 \subset M$ is calculated with six integer eigenvalues. We find that once we identify the modulus of the $T^2$ contained in $M$, $\tau = \beta^2 + i\frac{R_1}{R_2}$, with the gauge couplings $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$, then the two theories are related by $Z^{6d}_{zero\,modes} = \epsilon Z^{4d}_{zero\,modes}$, where $\epsilon$ is due to the zero modes of the scalar field that arises in addition to $F_{ij}$ from the compactification of the $6d$ self-dual three-form. In section 4, the abelian gauge theory is quantized on a four-torus using Dirac constraints, and the Hamiltonian and momentum are computed in terms of oscillator modes. For small $T^2$, the Kaluza-Klein modes are removed from the partition function of the chiral two-form, and in this limit it agrees with the gauge theory result, up to the scalar field contribution. In Appendix A, we show the path integral quantization gives the same result for the 4d gauge theory partition function as canonical quantization. However, the zero and oscillator mode contributions differ in the two quantizations. In Appendix B, we show how the zero and oscillator mode contributions transform under $SL(2,\mathcal{Z})$ for the 6d theory, as well as for both quantizations of the 4d theory. We prove the partition functions in 4d and 6d are both $SL(2,\mathcal{Z})$ invariant. In Appendix C, the vacuum energy is regularized. In Appendix D, we introduce a complete set of $SL(4,\mathcal{Z})$ generators, and then prove the 4d and 6d partition functions are invariant under $SL(4,\mathcal{Z})$ transformations.

[1] A different consideration of the fivebrane on $T^2 \times T^4$ in [13] includes the time direction in $T^2$.

## 2 Statement of the main result

We compute partition functions for a chiral boson on $T^2 \times T^4$ and for a $U(1)$ gauge boson on the same $T^4$. The geometry of the manifold $T^2 \times T^4$ will be described by the line element,

$$\begin{aligned} ds^2 &= R_2^2(d\theta^2 - \beta^2 d\theta^1)^2 + R_1^2(d\theta^1)^2 \\ &+ \sum_{\alpha,\beta} g_{\alpha\beta}(d\theta^\alpha - \gamma^\alpha d\theta^6)(d\theta^\beta - \gamma^\beta d\theta^6) + R_6^2(d\theta^6)^2, \end{aligned} \tag{2.1}$$

with $0 \leq \theta^I \leq 2\pi$, $1 \leq I \leq 6$, and $3 \leq \alpha \leq 5$. $R_1$, $R_2$ are the radii for directions $I = 1, 2$ on $T^2$, and $\beta^2$ is the angle between them. $g_{\alpha\beta}$ fixes the metric for a $T^3$ submanifold of $T^4$, $R_6$ is the remaining radius, and $\gamma^\alpha$ is the angle between those. So, from (2.1) the metric is

$$\begin{aligned} &T^2: \quad G_{11} = {R_1}^2 + R_2^2\beta^2\beta^2, \quad G_{12} = -R_2^2\beta^2, \quad G_{22} = R_2^2; \\ &T^4: \quad G_{\alpha\beta} = g_{\alpha\beta}, \quad G_{\alpha 6} = -g_{\alpha\beta}\gamma^\beta, \quad G_{66} = {R_6}^2 + g_{\alpha\beta}\gamma^\alpha\gamma^\beta; \\ &\qquad\quad G_{\alpha 1} = G_{\alpha 2} = 0, \quad G_{16} = G_{26} = 0; \end{aligned} \tag{2.2}$$

and the inverse metric is

$$\begin{aligned} &T^2: \quad G^{11} = \frac{1}{{R_1}^2}, \quad G^{12} = \frac{\beta^2}{{R_1}^2}, \quad G^{22} = \frac{1}{R_2^2} + \frac{\beta^2\beta^2}{{R_1}^2} \equiv g^{22} + \frac{\beta^2\beta^2}{{R_1}^2}; \\ &T^4: \quad G^{\alpha\beta} = g^{\alpha\beta} + \frac{\gamma^\alpha\gamma^\beta}{{R_6}^2}, \quad G^{\alpha 6} = \frac{\gamma^\alpha}{{R_6}^2}, \quad G^{66} = \frac{1}{{R_6}^2} \\ &\qquad\quad G^{1\alpha} = G^{2\alpha} = 0, \quad G^{16} = G^{26} = 0. \end{aligned} \tag{2.3}$$

$\theta^6$ is chosen to be the time direction for both theories. In the $4d$ expression (1.3) the indices of the field strength tensor have $3 \leq i, j, k, l \leq 6$, whereas in (1.4), the Hamiltonian and momentum are written in terms of fields with indices $1 \leq m, n, p, r, s \leq 5$. The 5-dimensional inverse in directions 1, 2, 3, 4, 5 is ${G_5}^{mn}$,

$$\begin{aligned} &G_5^{11} = \frac{1}{{R_1}^2}, \quad G_5^{12} = \frac{\beta^2}{{R_1}^2}, \quad G_5^{22} = g^{22} + \frac{\beta^2\beta^2}{{R_1}^2} \\ &G_5^{\alpha\beta} = g^{\alpha\beta}, \quad G_5^{1\alpha} = 0, \quad G_5^{2\alpha} = 0. \end{aligned} \tag{2.4}$$

$g^{\alpha\beta}$ is the $3d$ inverse of $g_{\alpha\beta}$. The determinants are related by

$$\sqrt{G} = \sqrt{\det G_{IJ}} = R_1 R_2 \sqrt{g} = R_1 R_2 R_6 \sqrt{\tilde{g}} = R_6 \sqrt{G_5}, \tag{2.5}$$

where $G$ is the determinant for $6d$ metric $G_{IJ}$. $G_5$, $g$ and $\tilde{g}$ are the determinants for the $5d$ metric $G_{mn}$, $4d$ metric $G_{ij}$, and $3d$ metric $g_{\alpha\beta}$ respectively.

The zero mode partition function of the $6d$ chiral boson on $T^2 \times T^4$ with the metric (2.3) is

$$
\begin{aligned}
Z^{6d}_{\text{zero modes}} = &\sum_{n_8,n_9,n_{10}} \exp\{-\frac{\pi R_6}{R_1R_2}\sqrt{\tilde g} g^{\alpha\alpha'} H_{12\alpha}H_{12\alpha'}\}\\
&\cdot \sum_{n_7} \exp\{-\frac{\pi}{6}R_6R_1R_2\sqrt{\tilde g} g^{\alpha\alpha'}g^{\beta\beta'}g^{\delta\delta'}H_{\alpha\beta\delta}H_{\alpha'\beta'\delta'} - i\pi\gamma^\alpha\epsilon^{\gamma\beta\delta}H_{12\gamma}H_{\alpha\beta\delta}\}\\
&\cdot \sum_{n_4,n_5,n_6} \exp\{-\frac{\pi}{2}R_6R_1R_2\sqrt{\tilde g}(\frac{1}{R_2^2}+\frac{{\beta^2}^2}{{R_1}^2})g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'}\}\\
&\cdot \sum_{n_1,n_2,n_3} \exp\{-\pi\frac{R_6R_2}{R_1}\sqrt{\tilde g}\beta^2 g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'} + i\pi\gamma^\alpha\epsilon^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta}\\
&\qquad -\frac{\pi}{4}\frac{R_6R_2}{R_1}\sqrt{\tilde g}(g^{\alpha\alpha'}g^{\beta\beta'}-g^{\alpha\beta'}g^{\beta\alpha'})H_{1\alpha\beta}H_{1\alpha'\beta'}\}
\end{aligned}
\tag{2.6}
$$

where the zero mode eigenvalues of the field strength tensor are integers, and (2.6) factors into a sum on $H_{\alpha\beta\gamma}$ as $H_{345} = n_7$, $H_{12\alpha}$ as $H_{123} = n_8$, $H_{124} = n_9$, $H_{125} = n_{10}$; and a sum over $H_{1\alpha\beta}$ defined as $H_{134} = n_1$, $H_{145} = n_2$, $H_{135} = n_3$ and $H_{2\alpha\beta}$ as $H_{234} = n_4$, $H_{245} = n_5$, $H_{235} = n_6$, as we will show in section 3.

The zero mode partition function of the $4d$ gauge boson on $T^4$ with the metric (2.2) is

$$
\begin{aligned}
Z^{4d}_{\text{zero modes}} = &\sum_{n_4,n_5,n_6} \exp\{-\frac{e^2}{4}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\widetilde{\Pi}^\alpha\widetilde{\Pi}^\beta\} \cdot \sum_{n_1,n_2,n_3} \exp\{-\frac{\theta e^2}{8\pi}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\widetilde{F}_{\gamma\delta}\widetilde{\Pi}^\beta\}\\
&\cdot \exp\{-\frac{e^2\sqrt{g}}{8}(\frac{\theta^2}{4\pi^2}+\frac{16\pi^2}{e^4})g^{\alpha\beta}g^{\gamma\delta}\widetilde{F}_{\alpha\gamma}\widetilde{F}_{\beta\delta} + 2\pi i\gamma^\alpha\widetilde{\Pi}^\beta\widetilde{F}_{\alpha\beta}\},
\end{aligned}
\tag{2.7}
$$

where $\widetilde{\Pi}^\alpha$ take integer values $\widetilde{\Pi}^3 = n_4$, $\widetilde{\Pi}^4 = n_5$, $\widetilde{\Pi}^5 = n_6$, and $\widetilde{F}_{34} = n_1$, $\widetilde{F}_{35} = n_2$, $\widetilde{F}_{45} = n_3$, from section 3. We identify the integers

$$
H_{2\alpha\beta} = \widetilde{F}_{\alpha\beta} \qquad \text{and} \qquad H_{1\alpha\beta} = \frac{1}{\tilde g}\epsilon_{\alpha\beta\gamma}\widetilde{\Pi}^\gamma, \tag{2.8}
$$

where $\tilde g = g\, R_6^{-2}$ from (2.5), and the modulus

$$
\tau = \beta^2 + i\frac{R_1}{R_2} = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2},
$$

so that as shown in section 3, we have the factorization

$$
Z^{6d}_{\text{zero modes}} = \epsilon\, Z^{4d}_{\text{zero modes}}, \tag{2.9}
$$

where $\epsilon$ comes from the remaining four zero modes $H_{\alpha\beta\gamma}$ and $H_{12\alpha}$ due to the additional scalar that occurs in the compactification of the $6d$ self-dual three-form field strength,

$$\epsilon = \sum_{n_8,n_9,n_{10}} \exp\{-\frac{\pi R_6}{R_1R_2}\sqrt{\tilde{g}}g^{\alpha\alpha'}H_{12\alpha}H_{12\alpha'}\}$$
$$\cdot \sum_{n_7} \exp\{-\frac{\pi}{6}R_6R_1R_2\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}g^{\delta\delta'}H_{\alpha\beta\delta}H_{\alpha'\beta'\delta'} - i\pi\gamma^\alpha\epsilon^{\gamma\beta\delta}H_{12\gamma}H_{\alpha\beta\delta}\}. \quad (2.10)$$

From section 4, there is a similar relation between the oscillator partition functions

$$\lim_{R_1,R_2\to 0} Z^{6d}_{\text{osc}} = \epsilon' Z^{4d}_{\text{osc}}, \quad (2.11)$$

where

$$Z^{6d}_{\text{osc}} = \Big(e^{R_6\pi^{-3}\sum_{\vec{n}\neq\vec{0}}\frac{\sqrt{G_5}}{(G_{mp}n^m n^p)^3}} \prod_{\vec{p}\in\mathcal{Z}^5\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}p_\alpha p_\beta+\tilde{p}^2}+2\pi i\gamma^\alpha p_\alpha}}\Big)^3, \quad (2.12)$$

$$Z^{4d}_{\text{osc}} = \Big(e^{\frac{1}{2}R_6\pi^{-2}\sum_{\vec{n}\neq 0}\frac{\sqrt{g}}{(g_{\alpha\beta}n^\alpha n^\beta)^2}} \cdot \prod_{\vec{n}\in\mathcal{Z}^3\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}\Big)^2, \quad (2.13)$$

where $\tilde{p}^2 \equiv \frac{p_1^2}{R_1^2} + (\frac{1}{R_1^2} + \frac{\beta^2\beta^2}{R_2^2})p_2^2 + \frac{2\beta^2}{R_1^2}p_1p_2$, and $\epsilon'$ is the oscillator contribution from the additional scalar,

$$\epsilon' = e^{\frac{1}{2}R_6\pi^{-2}\sum_{\vec{n}\neq 0}\frac{\sqrt{g}}{(g_{\alpha\beta}n^\alpha n^\beta)^2}} \cdot \prod_{\vec{n}\in\mathcal{Z}^3\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}. \quad (2.14)$$

Therefore, in the limit of small $T^2$, we have

$$\lim_{R_1,R_2\to 0} Z^{6d,\,chiral} = \epsilon\,\epsilon'\,Z^{4d,\,Maxwell}. \quad (2.15)$$

We use this relation between the $6d$ and $4d$ partition functions to extract the S-duality of the latter from a geometric symmetry of the former. For $\tau = \beta^2 + i\frac{R_1}{R_2} = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$, under the $SL(2,\mathcal{Z})$ transformations

$$\tau \to -\frac{1}{\tau}; \quad \tau \to \tau - 1, \quad (2.16)$$

$Z^{6d}_{\text{zero modes}}$ and $Z^{6d}_{\text{osc}}$ are separately invariant, as are $Z^{4d}_{\text{zero modes}}$ and $Z^{4d}_{\text{osc}}$, which we will prove in Appendix B. In particular, $Z^{4d}_{\text{osc}}$ is independent of $e^2$ and $\theta$. A path integral computation agrees with our $U(1)$ partition function, as we review in Appendix A [14]. Nevertheless, in the path integral quantization the zero and non-zero mode contributions are rearranged, and although each is invariant under $\tau \to \tau - 1$, they transform differently under $\tau \to -\frac{1}{\tau}$, with $Z^{PI}_{\text{zero modes}} \to |\tau|^3 Z^{PI}_{\text{zero modes}}$ and $Z^{PI}_{\text{non-zero modes}} \to |\tau|^{-3} Z^{PI}_{\text{non-zero modes}}$. For a general spin manifold, the $U(1)$ partition function transforms as a modular form under S-duality [15], but in the case of $T^4$ which we consider in this paper the weight is zero.

## 3 Zero Modes

In this section, we show details for the computation of the zero mode partition functions. The $N=(2,0)$, $6d$ world volume theory of the fivebrane contains a chiral two-form $B_{MN}$, which has a self-dual three-form field strength $H_{LMN}=\partial_L B_{MN}+\partial_M B_{NL}+\partial_N B_{LM}$ with $1\le L,M,N\le 6$,

$$H_{LMN}(\vec{\theta},\theta^6)=\frac{1}{6\sqrt{-G}}G_{LL'}G_{MM'}G_{NN'}\epsilon^{L'M'N'RST}H_{RST}(\vec{\theta},\theta^6). \tag{3.1}$$

Since there is no covariant Lagrangian description for the chiral two-form, we compute its partition function from (1.4). As in [12],[16] the zero mode partition function of the $6d$ chiral theory is calculated in the Hamiltonian formulation similarly to string theory,

$$Z^{6d}_{\text{zero modes}}=tr\left(e^{-t\mathcal{H}+iy^l\mathcal{P}_l}\right) \tag{3.2}$$

where $t=2\pi R_6$ and $y^l=2\pi\frac{G^{l6}}{G^{66}}$, with $l=1,..5$. However, $y^1$ and $y^2$ are zero due to the metric (2.3). Neglecting the integrations and using the metric (2.4) in (1.4), we find

$$\begin{aligned}-t\mathcal{H}=&-\frac{\pi}{6}R_6R_1R_2\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}g^{\lambda\lambda'}H_{\alpha\beta\lambda}H_{\alpha'\beta'\lambda'}-\frac{\pi}{2}R_6\frac{R_1}{R_2}\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'}\\&-\frac{\pi}{2}\frac{R_6}{R_1}R_2{\beta^2}^2g^{\alpha\alpha'}g^{\beta\beta'}\sqrt{\tilde{g}}H_{2\alpha\beta}H_{2\alpha'\beta'}-\pi\frac{R_6}{R_1}R_2\beta^2\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'}\\&-\frac{\pi R_6}{R_1R_2}\sqrt{\tilde{g}}g^{\alpha\alpha'}H_{12\alpha}H_{12\alpha'}-\frac{\pi}{4}R_2\frac{R_6}{R_1}\sqrt{\tilde{g}}(g^{\alpha\alpha'}g^{\beta\beta'}-g^{\alpha\beta'}g^{\alpha'\beta})H_{1\alpha\beta}H_{1\alpha'\beta'},\end{aligned} \tag{3.3}$$

and the momentum components $3\le\alpha\le5$ are

$$\mathcal{P}_\alpha=-\frac{1}{2}\epsilon^{\gamma\beta\delta}H_{12\gamma}H_{\alpha\beta\delta}+\frac{1}{2}\epsilon^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta}, \tag{3.4}$$

where the zero modes of the ten fields $H_{lmp}$ are labeled by integers $n_1,\dots n_{10}$ [12]. Then (3.2) is given by (2.6).

Similarly, we compute the zero mode partition function for the 4d $U(1)$ theory from (1.1). We consider the charge quantization condition

$$n_I=\frac{1}{2\pi}\int_{\Sigma_2^I}F\equiv\frac{1}{2\pi}\int_{\Sigma_2^I}\frac{1}{2}F_{\alpha\beta}\,d\theta^\alpha\wedge d\theta^\beta,\qquad n_I\in\mathcal{Z},\ \text{for each } 1\le I\le 3. \tag{3.5}$$

as well as the commutation relation obtained from (4.17)

$$\left[\int_{\Sigma_1^\gamma}A_\alpha(\vec{\theta},\theta^6)d\theta^\alpha,\int\frac{d^3\theta'}{2\pi}\Pi^\beta(\vec{\theta}',\theta^6)\right]=\frac{i}{2\pi}\int_{\Sigma_1^\gamma}d\theta^\beta=i\,\delta^\beta_\gamma, \tag{3.6}$$

and use the standard argument [16],[17] to show that the field strength $F_{\alpha\beta}$ and momentum

$\Pi^\alpha$ zero modes have eigenvalues

$$F_{\alpha\beta} = \frac{n_{\alpha,\beta}}{2\pi}, \quad n_{\alpha,\beta} \in \mathcal{Z} \text{ for } \alpha < \beta, \qquad \text{and} \qquad \Pi^\alpha(\vec{\theta}', \theta^6) = \frac{n^{(\alpha)}}{(2\pi)^2}, \quad n^{(\alpha)} \in \mathcal{Z}^3. \quad (3.7)$$

Thus we define integer valued modes $\widetilde{F}_{\alpha\beta} \equiv 2\pi F_{\alpha\beta}$ and $\widetilde{\Pi}^\alpha \equiv (2\pi)^2 \Pi^\alpha$. Taking into account the spatial integrations $d\theta^\alpha$, (1.2) gives

$$\begin{aligned}
&- 2\pi H^{4d} + i2\pi\gamma^\alpha P^{4d}_\alpha \\
&= -\frac{e^2}{4}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\widetilde{\Pi}^\alpha\widetilde{\Pi}^\beta - \frac{e^2\sqrt{g}}{8}\left[\frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4}\right] g^{\alpha\beta}g^{\gamma\delta}\widetilde{F}_{\alpha\gamma}\widetilde{F}_{\beta\delta} - \frac{\theta e^2}{8\pi}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\widetilde{F}_{\gamma\delta}\widetilde{\Pi}^\beta \\
&\quad + 2\pi i\gamma^\alpha\widetilde{\Pi}^\beta\widetilde{F}_{\alpha\beta}, \qquad (3.8)
\end{aligned}$$

where (1.2) itself is derived in section 4. So from (3.8) and (1.1),

$$\begin{aligned}
Z^{4d}_{\text{zero modes}} = &\sum_{n_4,n_5,n_6} \exp\{-\frac{e^2}{4}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\widetilde{\Pi}^\alpha\widetilde{\Pi}^\beta\} \cdot \sum_{n_1,n_2,n_3} \exp\{-\frac{\theta e^2}{8\pi}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\widetilde{F}_{\gamma\delta}\widetilde{\Pi}^\beta\} \\
&\cdot \exp\{-\frac{e^2\sqrt{g}}{8}(\frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4}) g^{\alpha\beta}g^{\gamma\delta}\widetilde{F}_{\alpha\gamma}\widetilde{F}_{\beta\delta} + 2\pi i\gamma^\alpha\widetilde{\Pi}^\beta\widetilde{F}_{\alpha\beta}\}, \quad (3.9)
\end{aligned}$$

where $n_I$ are integers, with $\widetilde{F}_{34} = n_1$, $\widetilde{F}_{35} = n_2$, $\widetilde{F}_{45} = n_3$, and $\widetilde{\Pi}^3 = n_4$, $\widetilde{\Pi}^4 = n_5$, $\widetilde{\Pi}^6 = n_6$. (3.9) is the zero mode contribution to the $4d$ $U(1)$ partition function (1.1), and is (2.7).

If we identify the gauge couplings $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$ with the modulus of $T^2$, $\tau = \beta^2 + i\frac{R_1}{R_2}$, then

$$\frac{e^2}{4\pi} = \frac{R_2}{R_1}, \qquad \frac{\theta}{2\pi} = \beta^2, \qquad (3.10)$$

and (3.9) becomes

$$\begin{aligned}
Z^{4d}_{\text{zero modes}} = &\sum_{n_4,n_5,n_6} \exp\{-\pi\frac{R_2 R_6^2}{R_1\sqrt{g}} g_{\alpha\beta}\widetilde{\Pi}^\alpha\widetilde{\Pi}^\beta\} \cdot \sum_{n_1,n_2,n_3} \exp\{-\pi\beta^2\frac{R_2 R_6^2}{R_1\sqrt{g}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\widetilde{F}_{\gamma\delta}\widetilde{\Pi}^\beta\} \\
&\cdot \exp\{-\frac{\pi}{2}\frac{R_2}{R_1}\sqrt{g}({\beta^2}^2 + \frac{R_1^2}{R_2^2}) g^{\alpha\beta}g^{\gamma\delta}\widetilde{F}_{\alpha\gamma}\widetilde{F}_{\beta\delta} + 2\pi i\gamma^\alpha\widetilde{\Pi}^\beta\widetilde{F}_{\alpha\beta}\}.
\end{aligned}$$

(3.11)

Then the last four terms in the chiral boson zero mode sum (2.6) are equal to (2.7) since

$$
\begin{aligned}
-\frac{\pi}{2}\frac{R_2R_6\sqrt{\tilde g}}{R_1}(\frac{R_1^2}{R_2^2}+\beta^{2\,2})g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'} &= -\frac{\pi}{2}\frac{R_2R_6}{R_1}\sqrt{\tilde g}\,(\frac{R_1^2}{R_2^2}+\beta^{2\,2})g^{\alpha\beta}g^{\gamma\delta}\widetilde F_{\alpha\gamma}\widetilde F_{\beta\delta},\\
-\pi\frac{R_6R_2}{R_1}\sqrt{\tilde g}\,\beta^2 g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'} &= -\pi\beta^2\frac{R_6R_2}{R_1\sqrt{\tilde g}}g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\tilde F_{\gamma\delta}\tilde\Pi^\beta,\\
i\pi\gamma^\alpha\epsilon^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta} &= 2\pi i\gamma^\alpha\,\tilde\Pi^\beta\tilde F_{\alpha\beta},\\
-\frac{\pi}{2}\frac{R_6R_2}{R_1}\sqrt{\tilde g}g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{1\alpha'\beta'} &= -\pi\frac{R_6R_2}{R_1\sqrt{\tilde g}}g_{\alpha\beta}\tilde\Pi^\alpha\tilde\Pi^\beta,
\end{aligned}
\tag{3.12}
$$

when we identify the integers

$$
H_{2\alpha\beta}=\widetilde F_{\alpha\beta} \qquad \text{and} \qquad H_{1\alpha\beta}=\frac{1}{\tilde g}\epsilon_{\alpha\beta\gamma}\widetilde\Pi^\gamma, \tag{3.13}
$$

with $\tilde g = g\,R_6^{-2}$ from (2.5). Thus the $6d$ and $4d$ zero mode sums from (2.6) and (2.7) are related by

$$
Z^{6d}_{\text{zero modes}} = \epsilon\, Z^{4d}_{\text{zero modes}}, \tag{3.14}
$$

where

$$
\begin{aligned}
\epsilon = &\sum_{n_8,n_9,n_{10}}\exp\{-\frac{\pi R_6}{R_1R_2}\sqrt{\tilde g}g^{\alpha\alpha'}H_{12\alpha}H_{12\alpha'}\}\\
&\cdot\sum_{n_7}\exp\{-\frac{\pi}{6}R_6R_1R_2\sqrt{\tilde g}g^{\alpha\alpha'}g^{\beta\beta'}g^{\delta\delta'}H_{\alpha\beta\delta}H_{\alpha'\beta'\delta'}-i\pi\gamma^\alpha\epsilon^{\gamma\beta\delta}H_{12\gamma}H_{\alpha\beta\delta}\}.
\end{aligned}
\tag{3.15}
$$

## 4 Oscillator modes

To compute the oscillator contribution to the partition function (1.1), we quantize the $U(1)$ gauge theory with a theta term on the $T^4$ manifold using Dirac brackets. From (1.3), the equations of motion are $\partial^i F_{ij}=0$, since the theta term is a total divergence and does not contribute to them. So in Lorenz gauge, the gauge potential $A_i$ with field strength tensor $F_{ij}=\partial_iA_j-\partial_jA_i$ is obtained by solving the equation

$$
\partial^i\partial_iA_j=0, \qquad \text{with} \qquad \partial^iA_i=0. \tag{4.1}
$$

The potential has a plane wave solution

$$
A_i(\vec\theta,\theta^6)=\text{zero modes }+\sum_{\vec k\neq 0}(f_i(k)e^{ik\cdot\theta}+(f_i(k)e^{ik\cdot\theta})^*) \tag{4.2}
$$

with momenta satisfying the on shell condition and gauge condition

$$\widetilde{G}_L^{ij} k_i k_j = 0, \qquad k^i f_i = 0. \tag{4.3}$$

As in [11],[16] the Hamiltonian $H^{4d}$ and momentum $P_\alpha^{4d}$ are quantized with a Lorentzian signature metric that has zero angles with the time direction, $\gamma^\alpha = 0$. So we modify the metric on the four-torus (2.2), (2.3) to be

$$\begin{aligned} &\widetilde{G}_{L\,\alpha\beta} = g_{\alpha\beta}\,, \quad \widetilde{G}_{L\,66} = -{R_6}^2, \quad \widetilde{G}_{L\,\alpha 6} = 0 \\ &\widetilde{G}_L^{\alpha\beta} = g^{\alpha\beta}, \quad \widetilde{G}_L^{66} = -\frac{1}{R_6^2}, \quad \widetilde{G}_L^{\alpha 6} = 0, \quad \widetilde{G}_L = \det \widetilde{G}_{L\,ij} = -g. \end{aligned} \tag{4.4}$$

Solving for $k_6$ from (4.3) we find

$$k_6 = \frac{\sqrt{-\widetilde{G}_L^{66}}}{\widetilde{G}_L^{66}}\,|k|, \tag{4.5}$$

where $3 \leq \alpha, \beta \leq 5$, and $|k| \equiv \sqrt{g^{\alpha\beta} k_\alpha k_\beta}$. Employ the remaining gauge invariance $f_i \to f_i' = f_i + k_i \lambda$ to fix $f_6' = 0$, which is the gauge choice

$$A_6 = 0.$$

This reduces the number of components of $A_i$ from 4 to 3. To satisfy (4.3), we can use the $\partial^i F_{i6} = -\partial_6 \partial^\alpha A_\alpha = 0$ component of the equation of motion to eliminate $f_5$ in terms of $f_3, f_4$,

$$f_5 = -\frac{1}{p^5}(p^3 f_3 + p^4 f_4),$$

leaving just two independent polarization vectors corresponding to the physical degrees of freedom of a four-dimensional gauge theory.

From the Lorentzian Lagrangian and energy-momentum tensor given by

$$\begin{aligned} &\mathcal{L} = -\frac{1}{2e^2}\sqrt{-\widetilde{G}_L}\,\widetilde{G}_L^{ik}\widetilde{G}_L^{jl} F_{ij} F_{kl} + \frac{\theta}{32\pi^2}\epsilon^{ijkl} F_{ij} F_{kl}, \\ &\mathcal{T}^i{}_j = \frac{\delta \mathcal{L}}{\delta \partial_i A_k}\partial_j A_k - \delta^i_j \mathcal{L}, \end{aligned} \tag{4.6}$$

we obtain the Hamiltonian and momentum operators

$$H_c \equiv \int d^3\theta\, \mathcal{T}^6{}_6 = \int d^3\theta \Big( -\frac{\sqrt{g}}{e^2}\widetilde{G}_L^{66} g^{\alpha\beta}\, F_{6\alpha} F_{6\beta} + \frac{\sqrt{g}}{2e^2} g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha\beta} F_{\alpha'\beta'} - \partial_\alpha \Pi^\alpha\, A_6 \Big), \tag{4.7}$$

$$P_\alpha \equiv \int d^3\theta\, \mathcal{T}^6{}_\alpha = \int d^3\theta \Big( -\frac{2}{e^2}\sqrt{g}\,\widetilde{G}_L^{66} g^{\beta\gamma} F_{6\gamma} F_{\alpha\beta} - \partial_\beta \Pi^\beta\, A_\alpha + \Pi^6 \partial_\alpha A_6 \Big), \tag{4.8}$$

where we have integrated by parts; and the conjugate momentum is

$$\Pi^\alpha = \frac{\delta \mathcal{L}}{\delta \partial_6 A_\alpha} = -\frac{2}{e^2}\sqrt{g}\,\widetilde{G}_L^{66} g^{\alpha\beta} F_{6\beta} - \frac{\theta}{8\pi^2}\epsilon^{\alpha\beta\gamma} F_{\beta\gamma}, \qquad \Pi^6 = \frac{\delta \mathcal{L}}{\delta \partial_6 A_6} = 0. \tag{4.9}$$

Then we have

$$\begin{aligned} H_c - i\gamma^\alpha P_\alpha = \int d\theta^3 \Big( \frac{R_6^2}{4} \frac{e^2}{\sqrt{g}} g_{\alpha\beta} \big(\Pi^\alpha + \frac{\theta}{8\pi^2}\epsilon^{\alpha\gamma\delta} F_{\gamma\delta}\big)\big(\Pi^\beta + \frac{\theta}{8\pi^2}\epsilon^{\beta\rho\sigma} F_{\rho\sigma}\big) \\ + \frac{\sqrt{g}}{2e^2} g^{\alpha\tilde{\alpha}} g^{\beta\tilde{\beta}} F_{\alpha\beta} F_{\tilde{\alpha}\tilde{\beta}} - i\gamma^\alpha \big(\Pi^\beta + \frac{\theta}{8\pi^2}\epsilon^{\beta\gamma\delta} F_{\gamma\delta}\big) F_{\alpha\beta} \Big), \end{aligned} \tag{4.10}$$

up to terms proportional to $A_6$ and $\partial_\alpha \Pi^\alpha$ which vanish in Lorenz gauge. Note the term proportional to $\epsilon^{\beta\gamma\delta} F_{\gamma\delta} F_{\alpha\beta}$ vanishes identically. (4.10) is equal to $H^{4d} - i\gamma^\alpha P^{4d}_\alpha$ given in (1.2), and is used to compute the zero mode partition function in (2.7) via (3.8).

To compute the oscillator modes, the appearance of $\theta$ solely in the combination $\Pi^\alpha + \frac{\theta}{8\pi^2}\epsilon^{\alpha\gamma\delta} F_{\gamma\delta}$ in (4.10) suggests we make a canonical transformation on the oscillator fields $\Pi^\alpha(\vec{\theta}, \theta^6), A_\beta(\vec{\theta}, \theta^6)$ [18]. Consider the equal time quantum bracket, suppressing the $\theta^6$ dependence,

$$\Big[ \int d^3\theta' \epsilon^{\alpha\beta\delta} F_{\alpha\beta} A_\delta,\ \Pi^\gamma(\vec{\theta}) \Big] \;=\; 2i\epsilon^{\gamma\alpha\beta} F_{\alpha\beta}(\vec{\theta}), \tag{4.11}$$

and the canonical transformation

$$U(\theta) = \exp\{ i \frac{\theta}{32\pi^2} \int d^3\theta' \epsilon^{\alpha\beta\gamma} F_{\alpha\beta} A_\gamma \}, \tag{4.12}$$

under which $\Pi^\alpha(\vec{\theta}, \theta^6), A_\beta(\vec{\theta}, \theta^6)$ transform to $\widehat{\Pi}^\alpha(\vec{\theta}, \theta^6), \widehat{A}_\beta(\vec{\theta}, \theta^6)$,

$$\begin{aligned} \widehat{\Pi}^\alpha(\vec{\theta}) = U^{-1}(\theta)\, \Pi^\alpha(\vec{\theta})\, U(\theta) \;&=\; \Pi^\alpha(\vec{\theta}) + \frac{\theta}{8\pi^2}\epsilon^{\alpha\gamma\delta} F_{\gamma\delta}(\vec{\theta}) \\ \widehat{A}_\beta(\vec{\theta}) = U^{-1}(\theta)\, A_\beta(\vec{\theta})\, U(\theta) \;&=\; A_\beta(\vec{\theta}). \end{aligned} \tag{4.13}$$

Therefore the exponent (4.10) contains no theta dependence when written in terms of $\widehat{\Pi}^\alpha$, which now reads

$$\big(H_c - i\gamma^\alpha P_\alpha\big) = \int d\theta^3 \big( -\frac{R_6^2}{4} \frac{e^2}{\sqrt{g}} g_{\alpha\beta} \widehat{\Pi}^\alpha \widehat{\Pi}^\beta + \frac{\sqrt{g}}{2e^2} g^{\alpha\tilde{\alpha}} g^{\beta\tilde{\beta}} F_{\alpha\beta} F_{\tilde{\alpha}\tilde{\beta}} - i\gamma^\alpha \widehat{\Pi}^\beta F_{\alpha\beta} \big). \tag{4.14}$$

Thus, for the computation of the oscillator partition function we will quantize with $\theta = 0$. Note that had we done this for the zero modes, it would not be possible to pick the zero mode integer charges consistently. Since the zero and oscillator modes commute, we are free to canonically transform the latter and not the former.

In the discussion that follows we assume $\theta = 0$ and drop the hats. We directly quantize the Maxwell theory on the four-torus with the metric (4.4) in Lorenz gauge using Dirac constraints [19, 20]. The theory has a primary constraint $\Pi^6(\vec{\theta},\theta^6)\approx 0$. We can express the Hamiltonian (4.7) in terms of the conjugate momentum as

$$H_c=\int d\theta^3\frac{R_6^2}{4}\frac{e^2}{\sqrt{g}}g_{\alpha\beta}\Pi^\alpha\Pi^\beta+\frac{\sqrt{g}}{2e^2}g^{\alpha\tilde{\alpha}}g^{\beta\tilde{\beta}}F_{\alpha\beta}F_{\tilde{\alpha}\tilde{\beta}}. \tag{4.15}$$

The primary Hamiltonian is defined by

$$H_c=\int d\theta^3\Big(\frac{R_6^2}{4}\frac{e^2}{\sqrt{g}}g_{\alpha\beta}\Pi^\alpha\Pi^\beta+\frac{\sqrt{g}}{2e^2}g^{\alpha\tilde{\alpha}}g^{\beta\tilde{\beta}}F_{\alpha\beta}F_{\tilde{\alpha}\tilde{\beta}}-\partial_\alpha\Pi^\alpha A_6+\lambda_1\Pi^6\Big), \tag{4.16}$$

with $\lambda_1$ as a Lagrange multiplier. As in [16], we use the Dirac method of quantizing with constraints for the radiation gauge conditions $A_6\approx 0$, $\partial^\alpha A_\alpha\approx 0$, and find the equal time commutation relations:

$$\begin{aligned}&[\Pi^\beta(\vec{\theta},\theta^6),A_\alpha(\vec{\theta}',\theta^6)]=-i\Big(\delta_\alpha^\beta-g^{\beta\beta'}(\partial_\alpha\frac{1}{g^{\gamma\gamma'}\partial_\gamma\partial_{\gamma'}}\partial_{\beta'})\Big)\ \delta^3(\theta-\theta'),\\&[A_\alpha(\vec{\theta},\theta^6),A_\beta(\vec{\theta}',\theta^6)]=0,\qquad[\Pi^\alpha(\vec{\theta},\theta^6),\Pi^\beta(\vec{\theta}',\theta^6)]=0.\end{aligned} \tag{4.17}$$

In $A_6=0$ gauge, the vector potential on the torus is expanded as

$$A_\alpha(\vec{\theta},\theta^6)=\ \text{zero modes}\ +\sum_{\vec{k}\neq 0,\vec{k}\in\mathcal{Z}_3}(f_\alpha^\kappa a_{\vec{k}}^\kappa e^{ik\cdot\theta}+f_\alpha^{\kappa *}a_{\vec{k}}^{\kappa\dagger}e^{-ik\cdot\theta}),$$

where $1\leq\kappa\leq 2$, $3\leq\alpha\leq 5$ and $k_6$ defined in (4.5). The sum is on the dual lattice $\vec{k}=k_\alpha\in\mathcal{Z}_3\neq\vec{0}$. Here we only consider the oscillator modes expansion of the potential and the conjugate momentum in (4.9) with vanishing $\theta$ angle

$$\begin{aligned}A_\alpha(\vec{\theta},\theta^6)&=\sum_{\vec{k}\neq 0}(a_{\vec{k}\,\alpha}e^{ik\cdot\theta}+a_{\vec{k}\,\alpha}^\dagger e^{-ik\cdot\theta}),\\ \Pi^\beta(\vec{\theta},\theta^6)&=-i\frac{2\sqrt{g}}{e^2}\tilde{G}_L^{66}g^{\beta\beta'}\sum_{\vec{k}}k_6\,(a_{\vec{k}\,\beta'}e^{ik\cdot\theta}-a_{\vec{k}\,\beta'}^\dagger e^{-ik\cdot\theta}).\end{aligned} \tag{4.18}$$

and the polarizations absorbed in

$$a_{\vec{k}\,\alpha}=f_\alpha^\kappa a_{\vec{k}}^\kappa. \tag{4.19}$$

From (4.17), the commutator in terms of the oscillators is

$$\int\frac{d^3\theta d^3\theta'}{(2\pi)^6}e^{-ik_\alpha\theta^\alpha}e^{-ik'_\alpha\theta'^\alpha}[A_\alpha(\vec{\theta},0),A_\beta(\vec{\theta}',0)]=[(a_{\vec{k}\,\alpha}+a_{-\vec{k}\,\alpha}^\dagger),(a_{\vec{k}'\,\beta}+a_{-\vec{k}'\,\beta}^\dagger)]=0. \tag{4.20}$$

We consider the Fourier transform (4.20) of all the commutators (4.17), so the commutator of the oscillators is found to be:

$$[a_{\vec{k}\,\alpha}, a^{\dagger}_{\vec{k}'\,\beta}] = \frac{e^2}{2\sqrt{g}\widetilde{G}^{66}_L k_6}\frac{1}{2(2\pi)^3}\big(g_{\alpha\beta} - \frac{k_\alpha k_\beta}{g^{\gamma\gamma'}k_\gamma k_{\gamma'}}\big)\delta_{\vec{k},\vec{k}'},$$
$$[a_{\vec{k}\,\alpha}, a_{\vec{k}'\,\beta}] = 0, \qquad [a^{\dagger}_{\vec{k}\,\alpha}, a^{\dagger}_{\vec{k}'\,\beta}] = 0. \tag{4.21}$$

In $A_6 = 0$ gauge, we use (4.18) and (4.21) to evaluate the Hamiltonian and momentum in (4.7) and (4.8)

$$H_c = \int d^3\theta\, \frac{2\sqrt{g}}{e^2}\Big( -\frac{1}{2}\widetilde{G}^{66}_L g^{\alpha\alpha'}\,\partial_6 A_\alpha \partial_6 A_{\alpha'} + \frac{1}{4} g^{\alpha\alpha'} g^{\beta\beta'}\, F_{\alpha\beta}F_{\alpha'\beta'}\Big),$$
$$P_\alpha = \frac{2}{R_6^2 e^2}\int_0^{2\pi} d\theta^3 d\theta^4 d\theta^5 \sqrt{g}\; g^{\beta\beta'}\, F_{6\beta'}F_{\alpha\beta}. \tag{4.22}$$

With (4.18), (4.22) can be expressed in terms of the oscillator modes where time-dependent terms cancel,

$$H_c = (2\pi)^3\frac{2\sqrt{g}}{e^2}\sum_{\vec{k}\in\mathcal{Z}^3\neq\vec{0}} g^{\alpha\alpha'}|k|^2\;(a_{\vec{k}\,\alpha}a^{\dagger}_{\vec{k}\,\alpha'} + a^{\dagger}_{\vec{k}\,\alpha}a_{\vec{k}\,\alpha'}),$$
$$P_\alpha = -\frac{2\sqrt{g}}{e^2}\widetilde{G}^{66}_L g^{\beta\beta'}(2\pi)^3\sum_{\vec{k}\in\mathcal{Z}^3\neq\vec{0}} k_6 k_\alpha\,\big(a_{\vec{k}\,\beta'}a^{\dagger}_{\vec{k}\,\beta} + a^{\dagger}_{\vec{k}\,\beta'}a_{\vec{k}\,\beta}\big). \tag{4.23}$$

and we have used the on-shell condition $\widetilde{G}^{66}_L k_6 k_6 + |k|^2 = 0$, and the transverse condition $k^\alpha a_{\vec{k}\,\alpha} = k^\alpha a^{\dagger}_{\vec{k}\,\alpha} = 0$. Then,

$$-H_c + i\gamma^\alpha P_\alpha = -i\frac{1}{R_6}\frac{2\sqrt{g}}{e^2}\,(2\pi)^3\sum_{\vec{k}\in\mathcal{Z}^3\neq\vec{0}}|k|\big(-iR_6|k| + \gamma^\alpha k_\alpha\big)g^{\beta\beta'}\big(a_{\vec{k}\,\beta}a^{\dagger}_{\vec{k}\,\beta'} + a^{\dagger}_{\vec{k}\,\beta}a_{\vec{k}\,\beta'}\big). \tag{4.24}$$

Inserting the polarizations as $a_{\vec{k}\,\alpha} = f^\kappa_\alpha a^\kappa_{\vec{k}}$ and $a^{\dagger}_{\vec{k}\,\alpha} = f^{\lambda *}_\alpha a^{\lambda\dagger}_{\vec{k}}$ from (4.19) in the commutator (4.21) gives

$$[a_{\vec{k}\,\alpha}, a^{\dagger}_{\vec{k}'\,\beta}] = \frac{e^2}{4\sqrt{g}}\frac{R_6}{|k|}\frac{1}{(2\pi)^3}\Big(g_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2}\Big)\delta_{\vec{k},\vec{k}'} = f^\kappa_\alpha f^{\lambda *}_\beta [a^\kappa_{\vec{k}}, a^{\lambda\dagger}_{\vec{k}}], \tag{4.25}$$

where we choose the normalization

$$[a^\kappa_{\vec{k}}, a^{\lambda\dagger}_{\vec{k}'}] = \delta^{\kappa\lambda}\delta_{\vec{k},\vec{k}'}, \tag{4.26}$$

with $1 \leq \kappa, \lambda \leq 2$. Then the polarization vectors satisfy

$$f^\kappa_\alpha f^{\lambda *}_\beta \delta^{\kappa\lambda} = \frac{e^2}{4\sqrt{g}} \frac{R_6}{|k|} \frac{1}{(2\pi)^3} \Big( g_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2} \Big), \qquad g^{\beta\beta'} f^\kappa_\beta f^{\lambda *}_{\beta'} \delta^{\kappa\lambda} = \frac{e^2}{4\sqrt{g}} \frac{R_6}{|k|} \frac{1}{(2\pi)^3} \cdot 2,$$

$$g^{\beta\beta'} f^\kappa_\beta f^{\lambda *}_{\beta'} = \delta^{\kappa\lambda} \frac{e^2}{4\sqrt{g}} \frac{R_6}{|k|} \frac{1}{(2\pi)^3}.$$

So the exponent in (1.1) is given by

$$\begin{aligned} -H_c + i\gamma^\alpha P_\alpha &= -iR_6 \, \frac{2\sqrt{g}}{e^2} (2\pi)^3 \sum_{\vec{k} \in \mathcal{Z}^3 \neq \vec{0}} |k| \big( -iR_6 |k| + \gamma^\alpha k_\alpha \big) g^{\beta\beta'} \big( 2a^\dagger_{\vec{k}\,\beta} a_{\vec{k}\,\beta'} + [a_{\vec{k}\,\beta}, a^\dagger_{\vec{k}\,\beta'}] \big) \\ &= -i \sum_{\vec{k} \in \mathcal{Z}^3 \neq \vec{0}} \big( \gamma^\alpha k_\alpha - iR_6 |k| \big) \, a^{\kappa\dagger}_{\vec{k}} a^\kappa_{\vec{k}} \quad - \frac{i}{2} \sum_{\vec{k} \in \mathcal{Z}^3 \neq \vec{0}} \big( -iR_6 |k| \big) \, \delta^{\kappa\kappa}. \end{aligned} \tag{4.27}$$

The $U(1)$ partition function is

$$Z^{4d, Maxwell} \equiv tr \, \exp\{2\pi(-H_c + i\gamma^i P_i)\} = Z^{4d}_{\text{zero modes}} \, Z^{4d}_{\text{osc}}, \tag{4.28}$$

so from (4.27),

$$Z^{4d}_{\text{osc}} = tr \; e^{-2\pi i \sum_{\vec{k} \in \mathcal{Z}^3 \neq \vec{0}} \big( \gamma^\alpha k_\alpha - iR_6 |k| \big) \, a^{\kappa\dagger}_{\vec{k}} a^\kappa_{\vec{k}} \; - \pi R_6 \sum_{\vec{k} \in \mathcal{Z}^3 \neq \vec{0}} |k| \, \delta^{\kappa\kappa}}. \tag{4.29}$$

From the usual Fock space argument

$$tr \; \omega^{\sum_p p a^\dagger_p a_p} = \prod_p \sum_{k=0}^{\infty} \langle k | \omega^{p a^\dagger_p a_p} | k \rangle = \prod_p \frac{1}{1 - \omega^p},$$

we perform the trace on the oscillators,

$$Z^{4d}_{\text{osc}} = \Big( e^{-\pi R_6 \sum_{\vec{n} \in \mathcal{Z}^3} \sqrt{g^{\alpha\beta} n_\alpha n_\beta}} \prod_{\vec{n} \in \mathcal{Z}^3 \neq \vec{0}} \frac{1}{1 - e^{-i2\pi(\gamma^\alpha n_\alpha - iR_6 \sqrt{g^{\alpha\beta} n_\alpha n_\beta})}} \Big)^2, \tag{4.30}$$

$$Z^{4d, Maxwell} = Z^{4d}_{\text{zero modes}} \cdot \Big( e^{-\pi R_6 \sum_{\vec{n} \in \mathcal{Z}^3} \sqrt{g^{\alpha\beta} n_\alpha n_\beta}} \prod_{\vec{n} \in \mathcal{Z}^3 \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha\beta} n_\alpha n_\beta} - 2\pi i \gamma^\alpha n_\alpha}} \Big)^2, \tag{4.31}$$

where $Z^{4d}_{\text{zero modes}}$ is given in (2.7). (4.31) and (4.37) are each manifestly $SL(3, \mathcal{Z})$ invariant due to the underlying $SO(3)$ invariance we have labeled as $\alpha = 3, 4, 5$. We use the $SL(3, \mathcal{Z})$ invariant regularization of the vacuum energy reviewed in Appendix C to obtain

$$Z^{4d, Maxwell} = Z^{4d}_{\text{zero modes}} \cdot \Big( e^{\frac{1}{2} R_6 \pi^{-2} \sum_{\vec{n} \neq 0} \frac{\sqrt{g}}{(g_{\alpha\beta} n^\alpha n^\beta)^2}} \prod_{\vec{n} \in \mathcal{Z}^3 \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha\beta} n_\alpha n_\beta} - 2\pi i \gamma^\alpha n_\alpha}} \Big)^2, \tag{4.32}$$

which leads to (2.13).

On the other hand, one can evaluate the oscillator trace for the 6$d$ chiral boson from (1.4) as in[12],[16]. The exponent in the trace is

$$-2\pi R_6\mathcal{H}+i2\pi\gamma^iP_i=\frac{i\pi}{12}\int_0^{2\pi}d^5\theta H_{lrs}\epsilon^{lrsmn}H_{6mn}=\frac{i\pi}{2}\int_0^{2\pi}d^5\theta\sqrt{-G}H^{6mn}H_{6mn}$$
$$=-i\pi\int_0^{2\pi}d^5\theta(\Pi^{mn}H_{6mn}+H_{6mn}\Pi^{mn})$$
$$=-2i\pi\sum_{\vec{p}\neq 0}p_6\mathcal{C}^{\kappa\dagger}_{\vec{p}}B^{\kappa}_{\vec{p}}-i\pi\sum_{\vec{p}\neq 0}p_6\delta^{\kappa\kappa}, \tag{4.33}$$

where $\Pi^{mn}=-\frac{\sqrt{-G}}{4}\Pi^{6mn}$, and $\Pi^{6mn}$ is the momentum conjugate to $B_{MN}$. In the gauge $B_{6n}=0$, the normal mode expansion for the free quantum fields $B_{mn}$ and $\Pi^{mn}$ on a torus is given in terms of oscillators $B^{\kappa}_{\vec{p}}$ and $\mathcal{C}^{\kappa\dagger}_{\vec{p}}$ defined in [12], with the commutation relations

$$[B^{\kappa}_{\vec{p}},\mathcal{C}^{\lambda\dagger}_{\vec{p}'}]=\delta^{\kappa\lambda}\ \delta_{\vec{p},\vec{p}'} \tag{4.34}$$

where $1\leq\kappa,\lambda\leq 3$ labels the three physical degrees of freedom of the chiral two-form, and $\vec{p}=(p_1,p_2,p_\alpha)$ lies on the integer lattice $\mathcal{Z}^5$. From the on-shell condition $G^{LM}p_Lp_M=0$,

$$p_6=-\gamma^\alpha p_\alpha-iR_6\sqrt{g^{\alpha\beta}p_\alpha p_\beta+\frac{p_1^2}{R_1^2}+(\frac{1}{R_2^2}+\frac{{\beta^2}^2}{R_1^2})p_2^2+2\frac{\beta^2}{R_1^2}p_1p_2}. \tag{4.35}$$

Thus the oscillator partition function of the chiral two-form on $T^2\times T^4$ is obtained by tracing over the oscillators

$$Z^{6d}_{\rm osc}=tr\ e^{-2i\pi\sum_{\vec{p}\neq 0}p_6\mathcal{C}^{\kappa\dagger}_{\vec{p}}B^{\kappa}_{\vec{p}}-i\pi\sum_{\vec{p}\neq 0}p_6\delta^{\kappa\kappa}}$$
$$=\Big(e^{-\pi R_6\sum_{\vec{p}}\sqrt{g^{\alpha\beta}p_\alpha p_\beta+\tilde{p}^2}}\prod_{\vec{p}\neq 0}\frac{1}{1-e^{-2\pi i p_6}}\Big)^3$$
$$=\Big(e^{-\pi R_6\sum_{\vec{p}\in\mathcal{Z}^5}\sqrt{g^{\alpha\beta}p_\alpha p_\beta+\tilde{p}^2}}\prod_{\vec{p}\in\mathcal{Z}^5\neq\vec{0}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}p_\alpha p_\beta+\tilde{p}^2}+2\pi i\gamma^\alpha p_\alpha}}\Big)^3, \tag{4.36}$$

where $\tilde{p}^2\equiv\frac{p_1^2}{R_1^2}+(\frac{1}{R_2^2}+\frac{{\beta^2}^2}{R_1^2})p_2^2+2\frac{\beta^2}{R_1^2}p_1p_2$. Regularizing the vacuum energy in the oscillator sum [12] yields

$$Z^{6d,chiral}=Z^{6d}_{\text{zero modes}}\cdot\Big(e^{R_6\pi^{-3}\sum_{\vec{n}\neq\vec{0}}\frac{\sqrt{G_5}}{(G_{mp}n^mn^p)^3}}\prod_{\vec{p}\in\mathcal{Z}^5\neq\vec{0}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}p_\alpha p_\beta+\tilde{p}^2}+2\pi i\gamma^\alpha p_\alpha}}\Big)^3, \tag{4.37}$$

where $\vec{n}\in\mathcal{Z}^5$ is on the dual lattice, $G_{mp}$ is defined in (2.2), and $Z^{6d}_{\text{zero modes}}$ is given in (2.6).

Comparing the 4$d$ and 6$d$ oscillator traces (4.31) and (4.36), the 6$d$ chiral boson sum has a cube rather than a square, corresponding to one additional polarization, and it contains

Kaluza-Klein modes. In Appendix D, we prove that the product of the zero mode and the oscillator mode partition function for the $4d$ theory in (4.32) is $SL(4,\mathcal{Z})$ invariant. In (D.48) we give an equivalent expression,

$$Z^{4d,Maxwell} = Z^{4d}_{\text{zero modes}} \cdot \Big( e^{\frac{\pi R_6}{6R_3}} \prod_{n_3\neq 0} \frac{1}{1-e^{-2\pi\frac{R_6}{R_3}|n_3|+2\pi i\gamma^3 n_3}}\Big)^2$$
$$\cdot \Big( \prod_{(n_a)\in\mathcal{Z}^2\neq(0,0)} e^{-2\pi R_6 <H>_{p_\perp}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}+2\pi i\gamma^\alpha n_\alpha}}\Big)^2, \tag{4.38}$$

where $4\leq a\leq 5$, with $< H >_{p_\perp}$ defined in (C.3).

In Appendix D, we also prove the $SL(4,\mathcal{Z})$ invariance of the $6d$ chiral partition function (4.37), using the equivalent form (D.65),

$$Z^{6d,chiral} = Z^{6d}_{\text{zero modes}} \cdot \big( e^{\frac{\pi R_6}{6R_3}} \prod_{n_3\in\mathcal{Z}\neq 0} \frac{1}{1-e^{-2\pi\frac{R_6}{R_3}|n_3|+2\pi i\,\gamma^3\,n_3\,)}}\big)^3$$
$$\cdot \big( \prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+\tilde{n}^2}\,+i2\pi\gamma^\alpha n_\alpha}}\big)^3, \tag{4.39}$$

with $< H >^{6d}_{p_\perp}$ in (D.64), and $\tilde{n}^2 = \frac{(n_1)^2}{R_1^2} + (\frac{1}{R_2^2} + \frac{(\beta^2)^2}{R_1^2})n_2^2 + 2\frac{\beta^2}{R_1^2}n_2 n_1$. In the limit when $R_1$ and $R_2$ are small with respect to the metric parameters $g_{\alpha\beta}, R_6$ of the four-torus, the contribution from each polarization in (4.38) and (4.39) is equivalent. To see this limit, we can separate the product on $n_\perp = (n_1, n_2, n_a) \neq 0_\perp$ in (4.39), into $(n_1 = 0, n_2 = 0, n_a \neq (0,0))$, $(n_1 \neq 0, n_2 \neq 0$, all $n_a)$, $(n_1 = 0, n_2 \neq 0$, all $n_a)$, $(n_1 \neq 0, n_2 = 0$, all $n_a))$ to find, at fixed $n_3$,

$$\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+\frac{(n_1)^2}{R_1^2}+(\frac{1}{R_2^2}+\frac{(\beta^2)^2}{R_1^2})n_2^2+2\frac{\beta^2}{R_1^2}n_2 n_1}+2\pi i\gamma^\alpha n_\alpha}}$$
$$= \prod_{n_a\in\mathcal{Z}^2\neq(0,0)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}+2\pi i\gamma^\alpha n_\alpha}}$$
$$\cdot \prod_{n_1\neq0,n_2\neq0,(n_a\in\mathcal{Z}^2)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+\frac{(n_1)^2}{R_1^2}+(\frac{1}{R_2^2}+\frac{(\beta^2)^2}{R_1^2})n_2^2+\frac{\beta^2}{R_1^2}n_2 n_1}+2\pi i\gamma^\alpha n_\alpha}}$$
$$\cdot \prod_{n_1=0,n_2\neq0,(n_a\in\mathcal{Z}^2)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+(\frac{1}{R_2^2}+\frac{(\beta^2)^2}{R_1^2})n_2^2}+2\pi i\gamma^\alpha n_\alpha}}$$
$$\cdot \prod_{n_2=0,n_1\neq0,(n_a\in\mathcal{Z}^2)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+\frac{(n_1)^2}{R_1^2}}+2\pi i\gamma^\alpha n_\alpha}} \tag{4.40}$$

Thus for $T^2$ smaller than $T^4$, the last three products reduce to unity, so

$$\prod_{n_\perp\in\mathcal{Z}^4\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+\tilde{n}^2}+2\pi i\gamma^\alpha n_\beta}} \xrightarrow{R_1,R_2\to 0} \prod_{n_a\in\mathcal{Z}^2\neq(0,0)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}+2\pi i\gamma^\alpha n_\alpha}}. \tag{4.41}$$

The regularized vacuum energies in (C.3) and (D.64),

$$\begin{aligned} < H >_{p_\perp\neq 0} &= -\pi^{-1}\ |p_\perp| \sum_{n=1}^{\infty} \cos(p_a\kappa^a 2\pi n)\frac{K_1(2\pi n R_3|p_\perp|)}{n}, \quad \text{for} \quad |p_\perp|\equiv\sqrt{\widetilde{g}^{ab}n_a n_b}, \\ < H >^{6d}_{p_\perp\neq 0} &= -\pi^{-1}\ |p_\perp| \sum_{n=1}^{\infty} \cos(p_a\kappa^a 2\pi n)\frac{K_1(2\pi n R_3|p_\perp|)}{n}, \quad \text{for} \quad |p_\perp|\equiv\sqrt{\tilde{n}^2+\widetilde{g}^{ab}n_a n_b}, \end{aligned} \tag{4.42}$$

have the same form of spherical Bessel function, but the argument differs by modes $(p_1, p_2)$. Again separating the product on $n_\perp = (n_1, n_2, n_a)$ in (4.39), into $(n_1 = 0, n_2 = 0, n_a \neq (0,0))$, $(n_1 \neq 0, n_2 \neq 0\,\text{all}\ n_a)$, $(n_1 = 0, n_2 \neq 0,\ \text{all}\ n_a)$, $(n_1 \neq 0, n_2 = 0,\ \text{all}\ n_a))$ we have

$$\begin{aligned} \prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} &= (\prod_{n_a\in\mathcal{Z}^2\neq(0,0)} e^{-2\pi R_6 <H>^{4d}_{p_\perp}}) \cdot (\prod_{n_1\neq 0, n_2\neq 0, n_a\in\mathcal{Z}^2} e^{-2\pi R_6 <H>^{6d}_{p_\perp}}) \\ &\cdot (\prod_{n_1\neq 0, n_2=0, n_a\in\mathcal{Z}^2} e^{-2\pi R_6 <H>^{6d}_{p_\perp}}) \cdot (\prod_{n_1=0, n_2\neq 0, n_a\in\mathcal{Z}^2} e^{-2\pi R_6 <H>^{6d}_{p_\perp}}) \end{aligned} \tag{4.43}$$

In the limit $R_1, R_2 \to 0$, the last three products are unity. For example, the second is unity because for $n_1, n_2 \neq 0$,

$$\begin{aligned} &\lim_{R_1,R_2\to 0}\sqrt{\tilde{n}^2+\widetilde{g}^{\alpha\beta}n_\alpha n_\beta} \sim \sqrt{\tilde{n}^2}, \\ &\lim_{R_1,R_2\to 0}(|p_\perp|\ K_1(2\pi n R_3|p_\perp|) = \lim_{R_1,R_2\to 0}\sqrt{\tilde{n}^2}\ K_1\Big(2\pi n R_3\big(\sqrt{\tilde{n}^2}\big)\Big) = 0, \end{aligned} \tag{4.44}$$

since $\lim_{x\to\infty} xK_1(x) \sim \sqrt{x}\ e^{-x} \to 0$ [21]. So (4.43) leads to

$$\lim_{R_1,R_2\to 0} \prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} = \prod_{n_a\in\mathcal{Z}^2\neq(0,0)} e^{-2\pi R_6 <H>_{p_\perp}}. \tag{4.45}$$

Thus in the limit when $T^2$ is small with respect to $T^4$,

$$\lim_{R_1,R_2\to 0} \prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{(g^{\alpha\beta}n_\alpha n_\beta+\frac{n_1^2}{R_1^2}+(\frac{1}{R_2^2}+\frac{(\beta^2)^2}{R_1^2})n_2^2+2\frac{\beta^2}{R_1^2}n_2n_1}+i2\pi\gamma^\alpha n_\alpha}}$$
$$= \prod_{n_a\in\mathcal{Z}^2\neq(0,0)} e^{-2\pi R_6<H>_{p_\perp}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}+2\pi i\gamma^\alpha n_\alpha}}. \tag{4.46}$$

So we have shown the partition functions of the chiral theory on $T^2\times T^4$ and of gauge theory on $T^4$, agree in the small $T^2$ limit upon neglecting the less interesting contribution $\epsilon'$,

$$\lim_{R_1,R_2\to 0} Z^{6d}_{osc} = \epsilon'\cdot Z^{4d}_{osc}, \tag{4.47}$$

which is (2.11). Again, $\epsilon'$ is equivalently the oscillator contribution from one polarization, that is

$$\epsilon' = \Big(e^{\frac{1}{8}R_6\pi^{-2}\sum_{\vec{n}\neq 0}\frac{\sqrt{g}}{(g_{\alpha\beta}n^\alpha n^\beta)^2}}\cdot \prod_{\vec{n}\in\mathcal{Z}^3\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}\Big) \tag{4.48}$$

The relation between the $4d$ gauge and $6d$ tensor partition function is shown in the small $T^2$ limit,

$$\lim_{R_1,R_2\to 0} Z^{6d,chiral} = \epsilon\epsilon'\cdot Z^{4d,Maxwell}, \tag{4.49}$$

which is (2.15). $\epsilon\epsilon'$ is the partition function of a real scalar field in 4d, and is independent of the gauge coupling $\tau$.

## 5 S-duality of $Z^{4d,Maxwell}$ from $Z^{6d,chiral}$

In Appendices B and D we show explicitly how the $SL(2,\mathcal{Z})\times SL(4,\mathcal{Z})$ symmetry of the partition function of the $6d$ tensor field of the M-fivebrane of $N=(2,0)$ theory compactified on $T^2\times T^4$ implies the $SL(2,\mathcal{Z})$ S-duality of the $4d$ $U(1)$ gauge field partition function. These computations use the Hamiltonian formulation. In Appendix A we review the path integral formalism for the $4d$ zero and non-zero mode partition functions, and give their relations to the quantities computed in the Hamiltonian formulation. The results are summarized here.

$$Z^{4d}_{\text{zero modes}} = (\text{Im}\,\tau)^{\frac{3}{2}}\frac{g^{\frac{1}{4}}}{R_6^2}\,Z^{PI}_{\text{zero modes}}. \tag{5.1}$$

$$Z^{4d}_{osc} = (\text{Im}\,\tau)^{-\frac{3}{2}}g^{-\frac{1}{4}}R_6^2 Z^{PI}_{osc}. \tag{5.2}$$

$$
\begin{array}{llr}
Z^{4d}_{\text{zero modes}} \longrightarrow Z^{4d}_{\text{zero modes}}, \qquad Z^{PI}_{\text{zero modes}} \longrightarrow |\tau|^3 Z^{PI}_{\text{zero modes}} & \text{under } S & \\
Z^{4d}_{\text{zero modes}} \longrightarrow Z^{4d}_{\text{zero modes}}, \qquad Z^{PI}_{\text{zero modes}} \longrightarrow Z^{PI}_{\text{zero modes}} & \text{under } T & (5.3)
\end{array}
$$

and

$$
\begin{array}{llr}
Z^{4d}_{\text{osc}} \longrightarrow Z^{4d}_{\text{osc}}, \qquad Z^{PI}_{\text{non-zero modes}} \longrightarrow |\tau|^{-3} Z^{PI}_{\text{non-zero modes}} & \text{under } S & \\
Z^{4d}_{\text{osc}} \longrightarrow Z^{4d}_{\text{osc}}, \qquad Z^{PI}_{\text{non-zero modes}} \longrightarrow Z^{PI}_{\text{non-zero modes}} & \text{under } T. & (5.4)
\end{array}
$$

$S$ and $T$ are the generators of the duality symmetry $SL(2,\mathcal{Z})$, $S:\tau\to -\frac{1}{\tau}$, $T:\tau\to\tau-1$, where $\tau=\frac{\theta}{2\pi}+i\frac{4\pi}{e^2}$ is also given by the modulus of the two-torus, $\tau=\beta^2+i\frac{R_1}{R_2}$.

## 6 Conclusions and Discussion

We computed the partition function of the abelian gauge theory on a general four-dimensional torus $T^4$ and the partition function of a chiral boson compactified on $T^2\times T^4$. The coupling for the $4d$ gauge theory, $\tau=\frac{\theta}{2\pi}+i\frac{4\pi}{e^2}$, is identified with the complex modulus $\tau=\beta^2+i\frac{R_1}{R_2}$ of the two-torus $T^2$ in directions 1 and 2. Assuming the metric of $T^2$ is much smaller than $T^4$, the $6d$ partition function factorizes to a partition function for gauge theory on $T^4$ and a contribution from the extra scalar arising from compactification.[2] The $6d$ partition function has a manifest $SL(2,\mathcal{Z})\times SL(4,\mathcal{Z})$ symmetry. Therefore the $SL(2,\mathcal{Z})$ symmetry with the group action on the coupling, $\tau=\frac{\theta}{2\pi}+i\frac{4\pi}{e^2}$, known as S-duality becomes manifest in the $4d$ Maxwell theory. Presumably this happens for an arbitrary four manifold, but we chose $T^4$ in order to generate explicit formulas, *i.e.* explicit functions of $\tau$ and the 4d metric.

The $6d$ chiral two-form has no Lagrangian, so we use the Hamiltonian approach to compute both the $4d$ and $6d$ partition functions. For $4d$ gauge theory, the integration of the electric and magnetic fields as observables around one- and two-cycles respectively take integer values due to charge quantization. We sum over all possible integers to get the zero mode partition function. For the oscillator modes, we quantize the gauge theory using the Dirac method with constraints. In $6d$, the partition function follows from [12],[16].

We have also given the path integral result for the $4d$ partition function. It agrees with the partition function obtained in the Hamiltonian formulation. However, the path integral factors into zero modes and oscillator modes differently, which leads to different $SL(2,\mathcal{Z})$ transformation properties for the components. The $6d$ and $4d$ partition functions share the same $SL(2,\mathcal{Z})\times SL(4,\mathcal{Z})$ symmetry.

If we consider supersymmetry, compactification of the $6d$ theory on $T^2$ leads to $N=4$ gauge theory in the limit of small $T^2$. On the other hand, an $N=2$ theory of class S [22],[23] arises when the $6d$, $(2,0)$ theory is compactified on a punctured Riemann surface with genus $g$. Here the mapping class group of the Riemann surfaces acts as a generalized S-duality on 4d super-Yang-Mills theory [24]-[26].

[2] The Lagrangian for this single $4d$ scalar with a Lorentzian signature metric is $\mathcal{L}=\frac{R_6\sqrt{g}}{R_1R_2}\left(\frac{1}{2R_6^2}\partial_6\phi\partial_6\phi-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)$.

In another direction, we can study the $2d$ field theory present when $6d$ theory is compactified on a four-dimensional manifold. A $2d$-$4d$ correspondence, relating a generalized gauge theory partition function and a 2d correlation function, acting between $N = 4$ gauge theory on $S^4$ and a $2d$ Toda-Liouville conformal theory on $T^2$, holds for the radius of $S^4$ fixed to 1 [24],[5],[27]-[29]. It is difficult to see how the $2d$-$4d$ correspondence works for the gauge field on $T^4$ because the $4d$ oscillator partition function is most naturally viewed as that of a $2d$ theory on a $T^2$ in directions 3 and 6, whereas the $4d$ zero mode sum is equivalent to a $2d$ zero mode partition function on a $T^2$ in directions 1 and 2. For an arbitrary $4d$ metric, the theory may be too rich for a $2d$-$4d$ pairing. A $2d$-$4d$ relation can also be analyzed from a topological point of view [4],[30],[31]. Finding explicit results, such as we have derived for $T^2 \times T^4$, for these more general investigations would be advantageous.

## A Comparison of the $4d$ $U(1)$ partition function in the Hamiltonian and path integral formulations

For convenience in comparing the $4d$ gauge theory with the $6d$ chiral theory in sections 2 and 3, we quantized both using canonical quantization. Since a Lagrangian exists for the $4d$ gauge theory, it is useful to verify that its path integral quantization agrees with canonical quantization. We find the two quantizations distribute zero and oscillator mode contributions differently, and thus these factors transform differently under the action of $SL(2,\mathcal{Z})$. We summarize the path integral quantization results from [9], [14], [15], [32]. Following [9], [15], the two-form zero mode part, $\frac{F}{2\pi}$ is the harmonic representative and can be expanded in terms of the basis $\alpha_I = \frac{1}{(2\pi)^2} d\theta^1 \wedge d\theta^2$, etc., $I = 1, 2, .., 6$ namely

$$\frac{F}{2\pi} \equiv m = \sum_I m_I \alpha_I, \tag{A.1}$$

where $m_I$ are integers. Define $(m, n)$ to be the intersection form such that $(m, n) = \int m \wedge n$, and thus

$$\begin{aligned} (m, m) &= \frac{1}{16\pi^2} \int d^4\theta \epsilon^{ijkl} F_{ij} F_{kl} \\ (m, *m) &= \frac{1}{8\pi^2} \int d^4\theta \sqrt{g} F^{ij} F_{ij}. \end{aligned} \tag{A.2}$$

So the action (1.3) is given as

$$I = \frac{4\pi^2}{e^2}(m, *m) - \frac{i\theta}{2}(m, m) = \frac{1}{2e^2} \int d^4\theta \sqrt{g} F^{ij} F_{ij} - \frac{i\theta}{32\pi^2} \int d^4\theta \epsilon^{ijkl} F_{ij} F_{kl}. \tag{A.3}$$

The zero mode partition function from the path integral formalism can be expressed as a lattice sum over the integral basis of $m_I$ [9], [15],

$$\begin{aligned} Z^{PI}_{\text{zero modes}} &= \sum_{m_I \in \mathcal{Z}^6} \exp\Big[ -\frac{4\pi^2}{e^2}(m, *m) + \frac{i\theta}{2}(m, m) \Big] \\ &= \sum_{m_I \in \mathcal{Z}^6} \exp\Big[ \frac{i\pi}{2}\tau\Big( (m, m) + (m, *m) \Big) - \frac{i\pi}{2}\bar{\tau}\Big( -(m, m) + (m, *m) \Big) \Big], \end{aligned} \tag{A.4}$$

where $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$, and we have chosen the $\theta$ dependence of the action as in [9]. Alternatively the zero mode sum be can written in terms of the metric using (A.3)

$$\begin{aligned} Z^{PI}_{\text{zero modes}} = \sum_{\widetilde{F}_{ij} \in \mathcal{Z}^6} \exp\Big\{ \Big[ &-\frac{\pi}{2} R_6 \sqrt{\tilde{g}} g^{\alpha\beta} g^{\gamma\delta} \widetilde{F}_{\alpha\gamma} \tilde{F}_{\beta\delta} - \pi \frac{\sqrt{\tilde{g}}}{R_6} g^{\delta\delta'} \widetilde{F}_{\delta\beta} \gamma^\beta \widetilde{F}_{\delta'\beta'} \gamma^{\beta'} - \pi \frac{\sqrt{\tilde{g}}}{R_6} g^{\alpha\beta} \widetilde{F}_{6\alpha} \widetilde{F}_{6\beta} \\ &+ 2\pi \frac{\sqrt{\tilde{g}}}{R_6} g^{\alpha\delta} \widetilde{F}_{6\alpha} \widetilde{F}_{\delta\beta} \gamma^\beta - i\frac{\theta e^2}{8\pi} \epsilon^{\alpha\beta\gamma} \widetilde{F}_{6\alpha} \tilde{F}_{\beta\gamma} \Big] \frac{4\pi}{e^2} \Big\} \end{aligned} \tag{A.5}$$

where $\widetilde{F}_{ij} = 2\pi F_{ij} = m_I$ are integers due to the charge quantization (A.1), and where we have taken into account the integrations $\int d^4\theta = (2\pi)^4$ in (A.5). To compare the zero mode partition functions from the Hamiltonian and path integral formalisms, we rewrite the Hamiltonian formulation result (2.7) as

$$
\begin{aligned}
&Z^{4d}_{\text{zero modes}} \\
&= \sum_{\widetilde{\Pi}^\alpha, \widetilde{F}_{\alpha\beta}} \exp\Big[ -\frac{e^2 R_6}{4\sqrt{g}} g_{\alpha\beta} \big(\widetilde{\Pi}^\alpha + i\frac{4\pi\sqrt{g}}{e^2 R_6} g^{\alpha\delta} \widetilde{F}_{\delta\lambda}\gamma^\lambda + \frac{\theta\epsilon^{\alpha\gamma\delta}}{4\pi} \widetilde{F}_{\gamma\delta}\big) \cdot \big(\widetilde{\Pi}^\beta + i\frac{4\pi\sqrt{g}}{e^2 R_6} g^{\beta\delta'} \widetilde{F}_{\delta'\lambda'}\gamma^{\lambda'} + \frac{\theta\epsilon^{\beta\gamma'\delta'}}{4\pi} \widetilde{F}_{\gamma'\delta'}\big) \\
&\qquad - \frac{4\pi^2}{e^2} \frac{\sqrt{g}}{R_6} g^{\delta\delta'} \widetilde{F}_{\delta\beta}\gamma^\beta \widetilde{F}_{\delta'\beta'}\gamma^{\beta'} - \frac{2\pi^2}{e^2} \sqrt{g} g^{\alpha\beta} g^{\gamma\delta} \widetilde{F}_{\alpha\gamma} \widetilde{F}_{\beta\delta}\Big]. \qquad \text{(A.6)}
\end{aligned}
$$

After Poisson resummation,

$$
\sum_{n\in\mathcal{Z}^3} \exp[-\pi(n+x)\cdot\mathcal{A}\cdot(n+x)] = (\det\mathcal{A})^{-\frac{1}{2}} \sum_{\mathrm{n}\in\mathcal{Z}^3} \mathrm{e}^{-\pi \mathrm{n}\cdot\mathcal{A}^{-1}\cdot\mathrm{n}} \mathrm{e}^{2\pi \mathrm{i} \mathrm{n}\cdot\mathrm{x}}, \qquad \text{(A.7)}
$$

where $\mathcal{A}_{\alpha\beta} \equiv \frac{e^2 R_6}{4\pi\sqrt{g}} g_{\alpha\beta}$ and $x^\alpha \equiv i\frac{4\pi\sqrt{g}}{e^2 R_6} g^{\alpha\delta} \widetilde{F}_{\delta\lambda}\gamma^\lambda + \frac{\theta}{4\pi}\epsilon^{\alpha\gamma\delta}\widetilde{F}_{\gamma\delta}$, we get the Hamiltonian expression as

$$
\begin{aligned}
Z^{4d}_{\text{zero modes}} &= (\frac{e^2}{4\pi})^{-\frac{3}{2}} \frac{\widetilde{g}^{\frac{1}{4}}}{R_6{}^{\frac{3}{2}}} \sum_{\hat{\Pi}_\alpha, \tilde{F}_{\alpha\beta}} \exp\{ -\frac{4\pi^2\sqrt{g}}{e^2 R_6} g^{\alpha\beta} \hat{\Pi}_\alpha \hat{\Pi}_\beta - i\frac{\theta}{2} \hat{\Pi}_\alpha \epsilon^{\alpha\gamma\delta} \tilde{F}_{\gamma\delta} + \frac{8\pi^2\sqrt{g}}{e^2 R_6} g^{\alpha\beta} \hat{\Pi}_\alpha \tilde{F}_{\beta\delta}\gamma^\delta \\
&\qquad - \frac{4\pi^2}{e^2}\frac{\sqrt{g}}{R_6} g^{\delta\delta'} \tilde{F}_{\delta\beta}\gamma^\beta \tilde{F}_{\delta'\beta'}\gamma^{\beta'} - \frac{2\pi^2 R_6}{e^2} \sqrt{\tilde{g}} g^{\alpha\beta} g^{\gamma\delta} \tilde{F}_{\alpha\gamma} \tilde{F}_{\beta\delta} \} \\
&= (\operatorname{Im}\tau)^{\frac{3}{2}} \frac{\widetilde{g}^{\frac{1}{4}}}{R_6{}^{\frac{3}{2}}} Z^{PI}_{\text{zero modes}},
\end{aligned}
\qquad \text{(A.8)}
$$

where $\hat{\Pi}_\alpha$ is the integer value of $\widetilde{\Pi}^\alpha$, and we identify $\hat{\Pi}_\alpha$ with $\widetilde{F}_{6\alpha}$ in (A.5). Then

$$
Z^{PI}_{\text{zero modes}} = (\operatorname{Im}\tau)^{-\frac{3}{2}} \frac{R_6^2}{g^{\frac{1}{4}}} Z^{4d}_{\text{zero modes}}, \qquad \text{(A.9)}
$$

which is (5.1).

We review from [14] how the non-zero mode partition function is defined by a path integral,

$$
Z^{PI}_{\text{non-zero modes}} = \int_A DA^\mu e^{-I}. \qquad \text{(A.10)}
$$

Performing the functional integration with the Fadeev-Popov approach, [14] regularizes the path integral by

$$Z^{PI}_{\rm non-zero\,modes}=\frac{1}{(2\pi)^{\frac{b_1-1}{2}}}\left(\frac{g}{{\rm vol}T^4}\right)^{\frac{1}{2}}\left[\det(\Delta_0)\frac{\det(2\pi{\rm Im}\tau\Delta_0)}{\det(2\pi{\rm Im}\tau\Delta_1)}\right]^{\frac{1}{2}}=(\frac{g}{(2\pi)^4\sqrt{g}})^{\frac{1}{2}}\,(2\pi{\rm Im}\tau)^{\frac{b_1-1}{2}}\ \ \frac{\det\Delta_0}{\det\Delta_1^{\frac{1}{2}}},\tag{A.11}$$

where $b_1=4$ is the dimension of the group $H^1(T^4)$. $\Delta_p=(d^\dagger d+dd^\dagger)_p$ is the Laplacian operator acting on the $p$-form. $g=\det G_{ij}$. So $\Delta_0=-G^{ij}\partial_i\partial_j$, and $\det(\Delta_1)=\det(\Delta_0)^4$. Thus

$$Z^{PI}_{\rm non-zero\,modes}=\frac{g^{\frac{1}{4}}}{\sqrt{2\pi}}({\rm Im}\tau)^{\frac{3}{2}}\ \det\Delta_0^{-1}.\tag{A.12}$$

The determinant can be computed

$$\det\Delta_0^{-\frac{1}{2}}=\exp\{-\frac{1}{2}{\rm trln}A\},\tag{A.13}$$

$$\begin{aligned}\exp\{-\frac{1}{2}{\rm trln}\Delta_0\}&=\exp\Big(-\frac{1}{2}{\rm trln}\big(-{\rm G}^{66}\partial_6^2-2{\rm G}_{6\alpha}\partial_6\partial_\alpha-{\rm G}^{\alpha\beta}\partial_\alpha\partial_\beta\big)\Big)\\&=\exp\Big(-\frac{1}{2}\sum_{{\rm n}_\alpha\neq\tilde{0}}\sum_{{\rm n}_6}\ln\big(\frac{1}{{\rm R}_6^2}{\rm n}_6^2+2\frac{\gamma^\alpha}{{\rm R}_6^2}{\rm n}_\alpha{\rm n}_6+{\rm G}^{\alpha\beta}{\rm n}_\alpha{\rm n}_\beta\big)\Big)\\&=\exp\Big(-\frac{1}{2}\sum_{{\rm n}_\alpha\neq\tilde{0}}\sum_{{\rm n}_6}\ln\big(\frac{1}{{\rm R}_6^2}({\rm n}_6+\gamma^\alpha{\rm n}_\alpha)^2+{\rm g}^{\alpha\beta}{\rm n}_\alpha{\rm n}_\beta\big)\Big).\end{aligned}\tag{A.14}$$

Let $\mu(E)\equiv\sum_{n_6}\ln\big(\frac{1}{R_6^2}(n_6+\gamma^\alpha n_\alpha)^2+E^2\big)$, where $E^2\equiv g^{\alpha\beta}n_\alpha n_\beta$, $\rho=2\pi R_6$,

$$\begin{aligned}\frac{\partial\mu(E)}{\partial E}&=\sum_{n_6}\frac{2E}{\frac{1}{R_6^2}(n_6+\gamma^\alpha n_\alpha)^2+E^2}=\frac{\rho\sinh(\rho E)}{\cosh(\rho E)-\cos(2\pi\gamma^\alpha n_\alpha)}\\&=\partial_E\ln\big[\cosh(\rho E)-\cos(2\pi\gamma^\alpha n_\alpha)\big].\end{aligned}\tag{A.15}$$

After integration, we have

$$\mu(E)=\ln\Big[\cosh(\rho E)-\cos(2\pi\gamma^\alpha n_\alpha)\Big]+\ln\left(R_6^2\sqrt{\frac{2}{\pi}}\right).\tag{A.16}$$

where the constant $\ln\Big(R_6^2\sqrt{\frac{2}{\pi}}\Big)$ maintains $SL(4,\mathcal{Z})$ invariance of the partition function. So,

$$\begin{aligned}
\det\Delta_0{}^{-\frac{1}{2}} = \exp\Big(-\frac{1}{2}\mathrm{trln}\Delta_0\Big) &= e^{-\frac{1}{2}\sum_{n_\alpha\neq\vec{0}}\mu(E)} \\
&= \frac{(2\pi)^{\frac{1}{4}}}{R_6}\prod_{n_\alpha\in\mathcal{Z}^3\neq\vec{0}}\frac{1}{\sqrt{2}\sqrt{\cosh(\rho E)-\cos(2\pi\gamma^\alpha n_\alpha)}} \\
&= \frac{(2\pi)^{\frac{1}{4}}}{R_6}\prod_{n_\alpha\in\mathcal{Z}^3\neq\vec{0}}\frac{e^{-\frac{\rho E}{2}}}{1-e^{-\rho E+2\pi i\gamma^\alpha n_\alpha}}.
\end{aligned} \tag{A.17}$$

Therefore, using (A.12), we have

$$Z^{PI}_{\text{non-zero modes}} = (\mathrm{Im}\,\tau)^{\frac{3}{2}}\frac{g^{\frac{1}{4}}}{R_6^2}\,Z^{4d}_{osc}, \tag{A.18}$$

which is (5.2).

Together with (A.9), the partition functions from the two quantizations agree but they factor differently into zero and oscillator modes,

$$Z^{4d,Maxwell} = Z^{4d}_{\text{zero modes}}Z^{4d}_{\text{osc}} = Z^{PI}_{\text{zero modes}}Z^{PI}_{\text{non-zero modes}}. \tag{A.19}$$

## B $SL(2,\mathcal{Z})$ invariance of the $Z^{6d,chiral}$ and $Z^{4d,Maxwell}$ partition functions

The S-duality group $SL(2,\mathcal{Z})$ group has two generators $S$ and $T$ which act on the parameter $\tau$ to give

$$S:\tau\to-\frac{1}{\tau},\qquad T:\tau\to\tau-1. \tag{B.1}$$

Since $\tau=\beta^2+i\frac{R_1}{R_2}=\frac{\theta}{2\pi}+i\frac{4\pi}{e^2}$, the transformation $S$ corresponds to

$$R_1\to R_1|\tau|^{-1},\qquad R_2\to R_2|\tau|,\qquad \beta^2\to-|\tau|^{-2}\beta^2, \tag{B.2}$$

and $T$ corresponds to

$$\beta^2\to\beta^2-1. \tag{B.3}$$

Or equivalently

$$\begin{aligned}
S:&\qquad \frac{4\pi}{e^2}\to\frac{4\pi}{e^2}|\tau|^{-2},\quad \theta\to-\theta|\tau|^{-2} \\
T:&\qquad \theta\to\theta-2\pi,
\end{aligned} \tag{B.4}$$

which for $\theta=0$ is the familiar electromagnetic duality transformation $\frac{e^2}{4\pi}\to\frac{4\pi}{e^2}$.

*6d partition function*

The $6d$ chiral boson zero mode partition function (2.6),

$$
\begin{aligned}
Z^{6d}_{\text{zero modes}} = &\sum_{n_8,n_9,n_{10}} \exp\{-\frac{\pi R_6}{R_1R_2}\sqrt{\tilde{g}}g^{\alpha\alpha'}H_{12\alpha}H_{12\alpha'}\} \\
&\cdot\sum_{n_7}\exp\{-\frac{\pi}{6}R_6R_1R_2\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}g^{\delta\delta'}H_{\alpha\beta\delta}H_{\alpha'\beta'\delta'} - i\pi\gamma^{\alpha}\epsilon^{\gamma\beta\delta}H_{12\gamma}H_{\alpha\beta\delta}\} \\
&\cdot\sum_{n_4,n_5,n_6}\exp\{-\frac{\pi}{2}R_6R_1R_2\sqrt{\tilde{g}}(\frac{1}{R_2^2}+\frac{{\beta^2}^2}{{R_1}^2})g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'}\} \\
&\cdot\sum_{n_1,n_2,n_3}\exp\{-\pi\frac{R_6R_2}{R_1}\sqrt{\tilde{g}}\beta^2g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'} + i\pi\gamma^{\alpha}\epsilon^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta} \\
&\qquad -\frac{\pi}{4}\frac{R_6R_2}{R_1}\sqrt{\tilde{g}}(g^{\alpha\alpha'}g^{\beta\beta'} - g^{\alpha\beta'}g^{\beta\alpha'})H_{1\alpha\beta}H_{1\alpha'\beta'}\}
\end{aligned} \tag{B.5}
$$

where $H_{134} = n_1$,$H_{145} = n_2$, $H_{135} = n_3$, $H_{234} = n_4$, $H_{245} = n_5$, $H_{235} = n_6$, $H_{345} = n_7$, $H_{123} = n_8$, $H_{124} = n_9$, $H_{125} = n_{10}$, is invariant under both $S$ and $T$. To show the invariance using (B.2,B.3) we group the exponents in (B.5) into two sets,

$$
-\frac{\pi R_6}{R_1R_2}\sqrt{\tilde{g}}g^{\alpha\alpha'}H_{12\alpha}H_{12\alpha'} - \frac{\pi}{6}R_6R_1R_2\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}g^{\delta\delta'}H_{\alpha\beta\delta}H_{\alpha'\beta'\delta'} - i\pi\gamma^{\alpha}\epsilon^{\gamma\beta\delta}H_{12\gamma}H_{\alpha\beta\delta}, \tag{B.6}
$$

and

$$
\begin{aligned}
&-\frac{\pi}{2}R_6R_1R_2\sqrt{\tilde{g}}(\frac{1}{R_2^2}+\frac{{\beta^2}^2}{{R_1}^2})g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'} - \pi\frac{R_6}{R_1}\,R_2\sqrt{\tilde{g}}\beta^2g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'} \\
&+i\pi\gamma^{\alpha}\epsilon^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta} - \frac{\pi}{2}\frac{R_6R_2}{R_1}\sqrt{\tilde{g}}\,g^{\alpha\alpha'}g^{\beta\beta'}\,H_{1\alpha\beta}H_{1\alpha'\beta'}.
\end{aligned} \tag{B.7}
$$

(B.6) has no $\beta^2$ dependence and therefore is invariant under $T$. (B.7) transforms under $T$ to become

$$
\begin{aligned}
&-\frac{\pi}{2}R_6R_1R_2\sqrt{\tilde{g}}(\frac{1}{R_2^2}+\frac{{\beta^2}^2}{{R_1}^2})g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'} - \pi\frac{R_6}{R_1}\,R_2\sqrt{\tilde{g}}\beta^2g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'} \\
&+i\pi\gamma^{\alpha}\epsilon^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta} - \frac{\pi}{2}\frac{R_6}{R_1R_2}\sqrt{\tilde{g}}\,g^{\alpha\alpha'}g^{\beta\beta'}\,H_{1\alpha\beta}H_{1\alpha'\beta'} \\
&+\pi\frac{R_6}{R_1}\sqrt{\tilde{g}}R_2\beta^2g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'} - \frac{\pi}{2}\frac{R_6}{R_1}\sqrt{\tilde{g}}R_2g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'} + \pi\frac{R_6}{R_1}R_2\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'},
\end{aligned} \tag{B.8}
$$

which is equivalent to (B.7) in the sum where we shift the integer zero mode field strength $H_{1\alpha\beta}$ to $H_{1\alpha\beta} - H_{2\alpha\beta}$.

Under $S$, we see (B.6) as a function of $R_1R_2$ is invariant, and find (B.7) transforms to

$$-\frac{\pi}{2}\frac{R_6R_2}{R_1}\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'}+\pi\frac{R_6}{R_1}R_2\sqrt{\tilde{g}}\beta^2g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'}$$
$$+i\pi\gamma^{\alpha}\epsilon^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta}-\frac{\pi}{2}R_1R_6R_2\sqrt{\tilde{g}}(g^{22}+\frac{{\beta^2}^2}{R_1^2})g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{1\alpha'\beta'}. \tag{B.9}$$

So by shifting the integer field strength tensors $H_{1\alpha\beta}\to H_{2\alpha\beta}$ and $H_{2\alpha\beta}\to -H_{1\alpha\beta}$, the sum on (B.7) is left invariant by $S$. Thus we have proved $SL(2,\mathcal{Z})$ invariance of the $6d$ zero mode partition function (2.6), and that its factors $\epsilon$ and $Z^{4d}_{\text{zeromodes}}$ in (2.9) are separately $SL(2,\mathcal{Z})$ invariant.

For the oscillator modes (4.36), the only term that transforms in the sum and product is

$$\tilde{p}^2\equiv\frac{{p_1}^2}{{R_1}^2}+(g^{22}+\frac{{\beta^2}^2}{{R_1}^2})p_2^2+\frac{2\beta^2}{{R_1}^2}p_1p_2, \tag{B.10}$$

which is invariant under $T$ by shifting the momentum $p_1\to p_1+p_2$. With the $S$ transformation, $\tilde{p}^2$ becomes

$${p_1}^2(g^{22}+\frac{{\beta^2}^2}{{R_1}^2})+\frac{1}{{R_1}^2}p_2^2-\frac{2\beta^2}{{R_1}^2}p_1p_2, \tag{B.11}$$

and by also exchanging the momentum $p_1\to p_2$ and $p_2\to -p_1$, the term remains the same. So the 6d oscillator partition function (4.36) is $SL(2\mathcal{Z})$ invariant, which holds also for regularized vacuum energy as given in (4.37).

*$4d$ $U(1)$ partition function*

In the Hamiltonian formulation, $SL(2,\mathcal{Z})$ leaves invariant the $U(1)$ oscillator partition function (4.30), since it is independent of $e^2$ and $\theta$. We have also checked above, starting from $6d$, that the zero mode $4d$ partition function (2.7) is invariant. Thus the $U(1)$ partition function from the Hamiltonian formalism is S-duality invariant.

The S-duality transformations on the corresponding quantities in the path integral quantization can be derived from (A.9) and (A.18). Since $\text{Im}\,\tau\to\frac{1}{|\tau|^2}\text{Im}\tau$ under S, and is invariant under T, we have

$$\begin{aligned}
&Z^{4d}_{\text{zero modes}}\longrightarrow Z^{4d}_{\text{zero modes}}, &&Z^{PI}_{\text{zero modes}}\longrightarrow|\tau|^3Z^{PI}_{\text{zero modes}} &&\text{under } S\\
&Z^{4d}_{\text{zero modes}}\longrightarrow Z^{4d}_{\text{zero modes}}, &&Z^{PI}_{\text{zero modes}}\longrightarrow Z^{PI}_{\text{zero modes}} &&\text{under } T
\end{aligned} \tag{B.12}$$

and

$$\begin{aligned}
&Z^{4d}_{\text{osc}}\longrightarrow Z^{4d}_{\text{osc}}, &&Z^{PI}_{\text{osc}}\longrightarrow|\tau|^{-3}Z^{PI}_{\text{osc}} &&\text{under } S\\
&Z^{4d}_{\text{osc}}\longrightarrow Z^{4d}_{\text{osc}}, &&Z^{PI}_{\text{osc}}\longrightarrow Z^{PI}_{\text{osc}} &&\text{under } T,
\end{aligned} \tag{B.13}$$

which is (5.3) and (5.4).

*A 2d-4d correspondence for $Z^{PI}_{\text{zero modes}}$*

We remark here that the zero mode contribution to the partition function for the 4d Maxwell field is equivalent to the zero mode contribution to the partition function for a 2d worldsheet action [9],

$$S = \int d^2\sigma [\sqrt{h} h^{\mu\nu} G_{\alpha\beta} \partial_\mu X^\alpha \partial_\nu X^\beta + \epsilon^{\mu\nu} B_{\alpha\beta} \partial_\mu X^\alpha \partial_\nu X^\beta], \tag{B.14}$$

where $1 \leq \mu, \nu \leq 2$ and $3 \leq \alpha, \beta \leq 5$.

$$\begin{aligned} Z^{2d}_{\text{zero modes}} &= \sum_{(p_l, p_r) \in \Gamma_{3,3}} e^{i\pi\tau (p_L)^2 - i\pi\bar{\tau}(p_R)^2} = \sum_{(p_l, p_r) \in \Gamma_{3,3}} e^{i\frac{\theta}{2}((p_L)^2 - (p_r)^2) - \frac{4\pi^2}{e^2}((p_L)^2 + (p_r)^2)} \\ &= \sum_{n_\alpha \in \mathcal{Z}^3, m^\beta \in \mathcal{Z}^3} e^{i\theta n_\alpha m^\alpha - \frac{4\pi^2}{e^2}(m^\alpha m^\beta G_{\alpha\beta} + (n_\alpha - B_{\alpha\rho} m^\rho) G^{\alpha\beta} (n_\beta - B_{\beta\sigma} m^\sigma))} \\ &= Z^{PI}_{\text{zero modes}} \end{aligned}$$

where $Z^{PI}_{\text{zero modes}}$ is given in (A.5). The 2d metric is $h_{11} = R_1^2 + R_2^2 \beta^2 \beta^2$, $h_{12} = -R_2^2 \beta^2$, $h_{22} = R_2^2$, and $\tau = \beta^2 + i\frac{R_1}{R_2} = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$. The nine parameters of the moduli space $\frac{O(3,3)}{O(3)\times O(3)}$ of the Lorentzian lattice $\Gamma_{3,3}$ are given by the 4d gauge theory metric as

$$G_{\alpha\beta} = \frac{R_6}{\sqrt{\tilde{g}}} g_{\alpha\beta}, \qquad B_{\alpha\beta} = \frac{\epsilon_{\alpha\beta\lambda}\gamma^\lambda}{\tilde{g}}.$$

The integers are identified with the Maxwell field components as

$$\widetilde{F}_{6\alpha} = n_\alpha, \qquad \widetilde{F}_{\alpha\beta} = \frac{\epsilon_{\alpha\beta\rho} m^\rho}{\tilde{g}}.$$

The points $(p_{L\gamma}, p_{R\delta})$ on the Lorentzian lattice $\Gamma_{d,d}$ are [33]

$$\begin{aligned} &p_{L\gamma} = n_\alpha e^{*\alpha}_\gamma + m^\alpha (B_{\alpha\beta} + G_{\alpha\beta}) e^{*\beta}_\gamma, \qquad p_{R\gamma} = n_\alpha e^{*\alpha}_\gamma + m^\alpha (B_{\alpha\beta} - G_{\alpha\beta}) e^{*\beta}_\gamma, \\ &p_L^2 = \sum_{\gamma=1}^d p_{L\gamma} p_{L\gamma}, \quad p_R^2 = \sum_{\gamma=1}^d p_{R\gamma} p_{R\gamma}, \quad \sum_{\gamma=1}^d e^{*\alpha}_\gamma e^{*\beta}_\gamma = \frac{1}{2} G^{\alpha\beta}, \quad p_L^2 - p_R^2 = 2 n_\alpha m^\alpha, \\ &p_L^2 + p_R^2 = m^\alpha m^\beta G_{\alpha\beta} + (n_\alpha - B_{\alpha\rho} m^\rho) G^{\alpha\beta} (n_\beta - B_{\beta\sigma} m^\sigma), \end{aligned}$$

for $1 \leq \alpha, \beta, \gamma, \delta \leq d$.

However, the non-zero mode partition function of the 2d theory (B.14)

$$Z^{2d}_{\text{non-zero modes}} = (\eta(\tau)\bar{\eta}(\bar{\tau}))^{-3} \tag{B.15}$$

is not the $4d$ non-zero mode partition function (A.18), although they both transform in the same way under the $SL(2, \mathcal{Z})$ duality transformation. Indeed (A.18) is more naturally described by a 2d scalar theory with massless and massive modes on a two-torus in the directions 3 and 6, as we show in Appendix D.

## C Regularization of the vacuum energy for $4d$ Maxwell theory

The sum in (4.30) is divergent. We regularize the vacuum energy following [12],[16]. For $< H >= \frac{1}{2}\sum_{p_\alpha\in\mathcal{Z}^3}\sqrt{g^{\alpha\beta}p_\alpha p_\beta}$, the $SL(3,\mathcal{Z})$ invariant regularized vacuum energy becomes

$$< H >= -\frac{1}{4\pi^3}\sqrt{\tilde{g}}\sum_{n^\alpha\in\mathcal{Z}^3\neq 0}\frac{1}{(g_{\alpha\beta}n^\alpha n^\beta)^2} = -4\pi\sqrt{\tilde{g}}\sum_{\vec{n}\in\mathcal{Z}^3\neq 0}\frac{1}{|2\pi\vec{n}|^4}. \tag{C.1}$$

For the proof of $SL(4,\mathcal{Z})$ invariance in Appendix D, it is also useful to write the regularized sum (C.1), as

$$< H > = \sum_{p_\perp\in\mathcal{Z}^2} < H >_{p_\perp} = < H >_{p_\perp=0} + \sum_{p_\perp\in\mathcal{Z}^2\neq 0} < H >_{p_\perp}, \tag{C.2}$$

where $p_\perp = p_a \in \mathcal{Z}^2,\ a = 4,5$, and

$$\begin{aligned} < H >_{p_\perp=0} &= \frac{1}{2}\sum_{p_3\in\mathcal{Z}}\sqrt{g^{33}p_3p_3} = \frac{1}{R_3}\sum_{n=1}^{\infty} n = \frac{1}{R_3}\zeta(-1) = -\frac{1}{12R_3}; \\ < H >_{p_\perp\neq 0} &= |p_\perp|^2 R_3\sum_{n=1}^{\infty}\cos(p_a\kappa^a 2\pi n)\big[K_2(2\pi n R_3|p_\perp|) - K_0(2\pi n R_3|p_\perp|)\big]. \end{aligned} \tag{C.3}$$

$|p_\perp| = \sqrt{p_a p_b \tilde{g}^{ab}}$, using the 2d inverse metric as defined in Appendix D.

## D $SL(4,\mathcal{Z})$ invariance of $Z^{4d,\text{Maxwell}}$ and $Z^{6d,\text{chiral}}$

*Rewriting the 4d metric (3,4,5,6)*

From (2.2) the metric on the four-torus, for $\alpha,\beta = 3,4,5$, is

$$G_{\alpha\beta} = g_{\alpha\beta}, \qquad G_{\alpha 6} = -g_{\alpha\beta}\gamma^\beta, \qquad G_{66} = R_6^2 + g_{\alpha\beta}\gamma^\alpha\gamma^\beta. \tag{D.1}$$

We can rewrite this metric using $a, b = 4,5$,

$$g_{33} \equiv R_3^2 + g_{ab}\kappa^a\kappa^b, \qquad g_{a3} \equiv -g_{ab}\kappa^b, \qquad g_{ab} \equiv g_{ab}, \qquad (\gamma^3)\kappa^a - \gamma^a \equiv -\widetilde{\gamma}^a, \tag{D.2}$$

$$\begin{aligned} &G_{33} = R_3^2 + g_{ab}\kappa^a\kappa^b, \qquad G_{36} = -(\gamma^3)R_3^2 + g_{ab}\kappa^b\widetilde{\gamma}^a, \qquad G_{3a} = -g_{ab}\kappa^b, \\ &G_{ab} = g_{ab}, \qquad G_{a6} = -g_{ab}\widetilde{\gamma}^b, \qquad G_{66} = R_6^2 + (\gamma^3)^2R_3^2 + g_{ab}\,\widetilde{\gamma}^a\widetilde{\gamma}^b. \end{aligned} \tag{D.3}$$

The $3d$ inverse of $g_{\alpha\beta}$ is

$$g^{ab} = \widetilde{g}^{ab} + \frac{\kappa^a\kappa^b}{R_3^2}, \qquad g^{a3} = \frac{\kappa^a}{R_3^2}, \qquad g^{33} = \frac{1}{R_3^2}, \tag{D.4}$$

where $\widetilde{g}^{ab}$ is the $2d$ inverse of $g_{ab}$.

$$g \equiv \det G_{ij} = R_6^2 \ \det g_{\alpha\beta} \equiv R_6^2 \, \widetilde{g} = R_6^2 R_3^2 \ \det g_{ab} \equiv R_6^2 R_3^2 \, \bar{g}.$$

The line element can be written as

$$\begin{aligned} ds^2 &= R_6^2 (d\theta^6)^2 + \sum_{\alpha,\beta=3,4,5} g_{\alpha\beta}(d\theta^\alpha - \gamma^\alpha d\theta^6)(d\theta^\beta - \gamma^\beta d\theta^6) \\ &= R_3^2(d\theta^3 - (\gamma^3) d\theta^6)^2 + R_6^2 (d\theta^6)^2 \\ &\quad + \sum_{a,b=4,5} g_{ab}(d\theta^a - \widetilde{\gamma}^a d\theta^6 - \kappa^a d\theta^3)\,(d\theta^b - \widetilde{\gamma}^b d\theta^6 - \kappa^b d\theta^3). \end{aligned} \tag{D.5}$$

We define

$$\widetilde{\tau} \equiv \gamma^3 + i \frac{R_6}{R_3}. \tag{D.6}$$

The 4d inverse is

$$\begin{aligned} &\widetilde{G}_4^{33} = \frac{|\widetilde{\tau}|^2}{R_6^2} = \widetilde{G}_4^{66} |\widetilde{\tau}|^2, \qquad \widetilde{G}_4^{66} = \frac{1}{R_6^2}, \qquad \widetilde{G}_4^{36} = \frac{\gamma^3}{R_6^2}, \qquad \widetilde{G}_4^{3a} = \frac{\kappa^a |\widetilde{\tau}|^2}{R_6^2} + \frac{\gamma^3 \widetilde{\gamma}^a}{R_6^2}, \\ &\widetilde{G}_4^{ab} = \widetilde{g}^{ab} + \frac{\kappa^a \kappa^b}{R_6^2} |\widetilde{\tau}|^2 + \frac{\widetilde{\gamma}^a \widetilde{\gamma}^b}{R_6^2} + \frac{\gamma^3 (\widetilde{\gamma}^a \kappa^b + \kappa^a \widetilde{\gamma}^b)}{R_6^2}, \qquad \widetilde{G}_4^{6a} = \frac{\gamma^a}{R_6^2} = \frac{\gamma^3 \kappa^a + \widetilde{\gamma}^a}{R_6^2}. \end{aligned} \tag{D.7}$$

*Generators of* $GL(n, \mathcal{Z})$

The $GL(n, \mathcal{Z})$ unimodular group can be generated by three matrices [34]. For $GL(4, \mathcal{Z})$ these can be taken to be $U_1, U_2$ and $U_3$,

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \qquad U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \qquad U_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{D.8}$$

so that every matrix $M$ in $GL(4, \mathcal{Z})$ can be written as a product $U_1^{n_1} U_2^{n_2} U_3^{n_3} U_1^{n_4} U_2^{n_5} U_3^{n_6} \ldots$, for integers $n_i$. Matrices $U_1$, $U_2$ and $U_3$ act on the basis vectors of the four-torus $\vec{\alpha}_i$ where $\vec{\alpha}_i \cdot \vec{\alpha}_j \equiv \alpha_i^k \alpha_j^l G_{kl} = G_{ij}$,

$$\begin{aligned} \vec{\alpha}_3 &= (1, 0, 0, 0) \\ \vec{\alpha}_6 &= (0, 1, 0, 0) \\ \vec{\alpha}_4 &= (0, 0, 1, 0) \\ \vec{\alpha}_5 &= (0, 0, 0, 1). \end{aligned} \tag{D.9}$$

For our metric (D.3), the $U_2$ transformation

$$\begin{pmatrix} \vec{\alpha}'_3 \\ \vec{\alpha}'_6 \\ \vec{\alpha}'_4 \\ \vec{\alpha}'_5 \end{pmatrix} = U_2 \begin{pmatrix} \vec{\alpha}_3 \\ \vec{\alpha}_6 \\ \vec{\alpha}_4 \\ \vec{\alpha}_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{D.10}$$

results in $\vec{\alpha}'_3 \cdot \vec{\alpha}'_3 \equiv \alpha'^i_3 \alpha'^j_3 G_{ij} = G_{33} = G'_{33}$, $\vec{\alpha}'_3 \cdot \vec{\alpha}'_6 \equiv \alpha'^i_3 \alpha'^j_6 G_{ij} = G_{33} + G_{36} = G'_{36}$, etc. So $U_2$ corresponds to

$$R_3 \to R_3,\ R_6 \to R_6,\ \gamma^3 \to \gamma^3 - 1,\ \kappa^a \to \kappa^a,\ \widetilde{\gamma}^a \to \widetilde{\gamma}^a + \kappa^a,\ g_{ab} \to g_{ab}, \tag{D.11}$$

or equivalently

$$R_6 \to R_6,\ \gamma^3 \to \gamma^3 - 1,\ g_{\alpha\beta} \to g_{\alpha\beta},\ \gamma^a \to \gamma^a, \tag{D.12}$$

which leaves invariant the line element (D.5) if $d\theta^3 \to d\theta^3 - d\theta^6$, $d\theta^6 \to d\theta^6$, $d\theta^a \to d\theta^a$. $U_2$ is the generalization of the usual $\widetilde{\tau} \to \widetilde{\tau} - 1$ modular transformation. The 3d inverse metric $g^{\alpha\beta} \equiv \{g^{ab}, g^{a3}, g^{33}\}$ does not change under $U_2$. It is easily checked that $U_2$ is an invariance of the 4d Maxwell partition function (4.32) as well as the $6d$ chiral boson partition function (4.37). It leaves the zero mode and oscillator contributions invariant separately.

The other generator, $U_1$ is related to the $SL(2,\mathcal{Z})$ transformation $\widetilde{\tau} \to -(\widetilde{\tau})^{-1}$ that we discuss as follows:

$$U_1 = U' M_3 \tag{D.13}$$

where $M_3$ is a $GL(3,\mathcal{Z})$ transformation given by

$$M_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{D.14}$$

and $U'$ is the matrix corresponding to the transformation on the metric parameters (D.16),

$$U' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{D.15}$$

Under $U'$, the metric parameters transform as

$$R_3 \to R_3|\widetilde{\tau}|, \quad R_6 \to R_6|\widetilde{\tau}|^{-1}, \quad \gamma^3 \to -\gamma^3|\widetilde{\tau}|^{-2}, \quad \kappa^a \to \widetilde{\gamma}^a, \quad \widetilde{\gamma}^a \to -\kappa^a, \quad g_{ab} \to g_{ab}.$$
$$\widetilde{\tau} \to -\frac{1}{\widetilde{\widetilde{\tau}}}. \qquad \text{Or equivalently,}$$
$$G_{ab} \to G_{ab}, \quad G_{a3} \to G_{a6}, \quad G_{a6} \to -G_{a3}, \quad G_{33} \to G_{66}, \quad G_{66} \to G_{33}, \quad G_{36} \to -G_{36},$$
$$\widetilde{G}_4^{ab} \to \widetilde{G}_4^{ab}, \quad \widetilde{G}_4^{a3} \to \widetilde{G}_4^{a6}, \quad \widetilde{G}_4^{a6} \to -\widetilde{G}_4^{a3}, \quad \widetilde{G}_4^{33} \to \frac{\widetilde{G}_4^{33}}{|\widetilde{\tau}|^2}, \quad \widetilde{G}_4^{66} \to |\widetilde{\tau}|^2\widetilde{G}_4^{66}, \quad \widetilde{G}_4^{36} \to -\widetilde{G}_4^{36}, \tag{D.16}$$

where $4 \le a, b \le 5$, and

$$\widetilde{\tau} \equiv \gamma^3 + i\frac{R_6}{R_3}, \qquad |\widetilde{\tau}|^2 = (\gamma^3)^2 + \frac{R_6^2}{R_3^2}. \tag{D.17}$$

The transformation (D.16) leaves invariant the line element (D.5) when $d\theta^3 \to d\theta^6$, $d\theta^6 \to -d\theta^3$, $d\theta^a \to d\theta^a$. The generators have the property $\det U_1 = -1, \det U_2 = 1$, $\det U_3 = -1$, $\det U' = 1$, $\det M_3 = -1$.

Under $M_3$, the metric parameters transform as

$$R_6 \to R_6, \qquad \gamma^3 \to -\gamma^4, \quad \gamma^a \to \gamma^{a+1}, \quad g_{ab} \to g_{a+1,b+1}, \quad g_{a3} \to -g_{a+1,4}, \quad g_{33} \to g_{44},$$
$$g^{ab} \to g^{a+1,b+1}, \quad g^{a3} \to -g^{a+1,4}, \quad g^{33} \to g^{44}, \quad \det g_{\alpha\beta} = \tilde{g}, \quad \tilde{g} \to \tilde{g}. \qquad \text{Or equivalently,}$$

$$G_{ab} \to G_{a+1,b+1}, \quad G_{a3} \to -G_{a+1,4}, \quad G_{a6} \to G_{a+1,6}, \quad G_{33} \to G_{44}, \quad G_{66} \to G_{66}, \quad G_{36} \to -G_{46},$$
$$\widetilde{G}_4^{ab} \to \widetilde{G}_5^{a+1,b+1}, \quad \widetilde{G}_4^{a3} \to -\widetilde{G}_5^{a+1,4}, \quad \widetilde{G}_4^{a6} \to \widetilde{G}_4^{a+1,6}, \quad \widetilde{G}_4^{33} \to \widetilde{G}_4^{44}, \quad \widetilde{G}_4^{36} \to -\widetilde{G}_4^{46}, \quad \widetilde{G}_4^{66} \to \widetilde{G}_4^{66},$$
$$\det \widetilde{G}_4 = R_6\,\tilde{g}, \qquad \det \widetilde{G}_4 \to \det \widetilde{G}_4, \tag{D.18}$$

where $4 \le a, b \le 5$, and $a+1 \equiv 3$ for $a = 5$. We see that $M_3$ takes $Z^{4d}_{\text{zero modes}}$ to its complex conjugate as follows. Letting the $M_3$ transformation (D.18) act on (2.7), we find that the three subterms in the exponent

$$-\frac{e^2}{8}R_6\sqrt{\tilde{g}}(\frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4})\Big(g^{aa'}g^{bb'}\tilde{F}_{ab}\tilde{F}_{a'b'} + 4g^{aa'}g^{b3}\tilde{F}_{ab}\tilde{F}_{a'3} + 2g^{aa'}g^{33}\tilde{F}_{a3}\tilde{F}_{a'3} - 2g^{a3}g^{a'3}\tilde{F}_{a3}\tilde{F}_{a'3}\Big),$$
$$-\frac{e^2R_6}{4\sqrt{\tilde{g}}}\tilde{\Pi}^\alpha g_{\alpha\beta}\tilde{\Pi}^\beta,$$
$$-\frac{\theta e^2 R_6}{8\pi^2\sqrt{\tilde{g}}}g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\tilde{F}_{\gamma\delta}\tilde{\Pi}^\beta, \tag{D.19}$$

are separately invariant under (D.18) if we replace the integers $\widetilde{F}_{\alpha\beta} \in \mathcal{Z}^3, \tilde{\Pi}^\alpha \in \mathcal{Z}^3$ by

$$\tilde{F}_{ab} \to \tilde{F}_{a+1,b+1}, \quad \tilde{F}_{a3} \to -\tilde{F}_{a+1,4}, \quad \tilde{\Pi}^3 \to \tilde{\Pi}^4, \qquad \tilde{\Pi}^a \to -\tilde{\Pi}^{a+1}. \tag{D.20}$$

However, acted on by $M_3$ with the field shift (D.20), the term

$$2\pi i\gamma^\alpha\tilde{\Pi}^\beta\widetilde{F}_{\alpha\beta} \rightarrow -2\pi i\gamma^\alpha\tilde{\Pi}^\beta\widetilde{F}_{\alpha\beta} \tag{D.21}$$

changes sign. Thus we have

$$M_3: \qquad Z^{4d}_{\text{zero modes}} \rightarrow Z^{4d\ *}_{\text{zero modes}} \tag{D.22}$$

*The action of $U'$ on $Z^{4d}_{\text{zero modes}}$*

Next we show that under $U'$, $Z^{4d}_{\text{zero modes}}$ transforms to $|\widetilde{\tau}|^2\, Z^{4d}_{\text{zero modes}}$. From (A.5) and (A.9), we have

$$Z^{4d}_{\text{zero modes}} = (\frac{4\pi}{e^2})^{-\frac{3}{2}}\frac{\tilde{g}^{\frac{1}{4}}}{R_6^{\frac{3}{2}}}\sum_{\tilde{F}_{ij}\in\mathcal{Z}^6}\exp\{-\frac{2\pi^2}{e^2}R_6\sqrt{\tilde{g}}g^{ij}g^{i'j'}\tilde{F}_{ii'}\tilde{F}_{jj'} - \frac{i}{2}\theta\epsilon^{\alpha\beta\gamma}\tilde{F}_{6\alpha}\tilde{F}_{\beta\gamma}\}, \tag{D.23}$$

from which it will be easy to see how it transforms under the $U'$ transformation. Under $U'$ from (D.16), the coefficient transforms as

$$U': \qquad (\frac{4\pi}{e^2})^{-\frac{3}{2}}\frac{\tilde{g}^{\frac{1}{4}}}{R_6^{\frac{3}{2}}} \rightarrow (\frac{4\pi}{e^2})^{-\frac{3}{2}}\frac{\tilde{g}^{\frac{1}{4}}}{R_6^{\frac{3}{2}}}\ |\widetilde{\tau}|^2. \tag{D.24}$$

The Euclidean action for the zero mode computation is invariant under $U'$, as we show next by first summing $i = \{3, a, 6\}$, with $4 \leq a \leq 5$.

$$\begin{aligned}
&-\frac{2\pi^2 R_6\sqrt{\tilde{g}}}{e^2}\frac{R_1}{R_2}g^{ij}g^{i'j'}\tilde{F}_{ii'}\tilde{F}_{jj'}\\
&= -\frac{2\pi^2 R_6\sqrt{\tilde{g}}}{e^2}\Big(\widetilde{G}_4^{aa'}\widetilde{G}_4^{bb'}\widetilde{F}_{ab}\tilde{F}_{a'b'} + 4\widetilde{G}_4^{aa'}\widetilde{G}_4^{b3}\tilde{F}_{ab}\tilde{F}_{a'3} + 4\widetilde{G}_4^{aa'}\widetilde{G}_4^{b6}\tilde{F}_{ab}\widetilde{F}_{a'6} + 2\widetilde{G}_4^{aa'}\widetilde{G}_4^{33}\tilde{F}_{a3}\tilde{F}_{a'3}\\
&\qquad - 2\widetilde{G}_4^{a3}\widetilde{G}_4^{a'3}\tilde{F}_{a3}\widetilde{F}_{a'3} + 4\widetilde{G}_4^{aa'}\widetilde{G}_4^{36}\tilde{F}_{a3}\widetilde{F}_{a'6} - 4\widetilde{G}_4^{a6}\widetilde{G}_4^{a'3}\tilde{F}_{a3}\tilde{F}_{a'6} + 4\widetilde{G}_4^{a3}\widetilde{G}_4^{b6}\tilde{F}_{ab}\tilde{F}_{36}\\
&\qquad + 2\widetilde{G}_4^{aa'}\widetilde{G}_4^{66}\tilde{F}_{a6}\tilde{F}_{a'6} - 2\widetilde{G}_4^{a6}\widetilde{G}_4^{a'6}\tilde{F}_{a6}\tilde{F}_{a'6} + 4\widetilde{G}_4^{a3}\widetilde{G}_4^{36}\tilde{F}_{a3}\tilde{F}_{36} - 4\widetilde{G}_4^{a6}\widetilde{G}_4^{33}\tilde{F}_{a3}\tilde{F}_{36}\\
&\qquad + 4\widetilde{G}_4^{a3}\widetilde{G}_4^{66}\tilde{F}_{a6}\tilde{F}_{36} - 4\widetilde{G}_4^{a6}\widetilde{G}_4^{36}\tilde{F}_{a6}\tilde{F}_{36} - 2\widetilde{G}_4^{36}\widetilde{G}_4^{36}\tilde{F}_{36}\tilde{F}_{36} + 2\widetilde{G}_4^{33}\widetilde{G}_4^{66}\tilde{F}_{36}\tilde{F}_{36}\Big).
\end{aligned} \tag{D.25}$$

Letting the $U'$ transformation (D.16) act on (D.25), we see the first term in the exponent of (D.23) changes to

$$
\begin{aligned}
&-\frac{2\pi^2 R_6\sqrt{\tilde g}}{e^2}\Big(\widetilde G_4^{aa'}\widetilde G_4^{bb'}\tilde F_{ab}\tilde F_{a'b'}+4\widetilde G_4^{aa'}\widetilde G_4^{b6}\tilde F_{ab}\tilde F_{a3}-4\widetilde G_4^{aa'}\widetilde G_4^{b3}\tilde F_{ab}\tilde F_{a'6}+\frac{2}{|\widetilde\tau|^2}\widetilde G_4^{aa'}\widetilde G_4^{33}\tilde F_{a3}\tilde F_{a'3}\\
&\qquad -2\widetilde G_4^{a6}\widetilde G_4^{a'6}\tilde F_{a3}\tilde F_{a'3}-4\widetilde G_4^{aa'}\widetilde G_4^{36}\tilde F_{a3}\tilde F_{a'6}+4\widetilde G_4^{a3}\widetilde G_4^{a'6}\tilde F_{a3}\tilde F_{a'6}-4\widetilde G_5^{\alpha 6}\widetilde G_5^{\alpha' 3}\tilde F_{aa'}\tilde F_{36}\\
&\qquad +2|\widetilde\tau|^2\widetilde G_4^{aa'}\widetilde G_4^{66}F_{a6}F_{a'6}-2\widetilde G_4^{a6}\widetilde G_4^{a'3}\tilde F_{a6}\tilde F_{a'6}-4\widetilde G_4^{a6}\widetilde G_4^{36}\tilde F_{a3}\tilde F_{36}+\frac{4}{|\widetilde\tau|^2}\widetilde G_4^{a3}\widetilde G_4^{33}\tilde F_{a3}\tilde F_{36}\\
&\qquad +4|\widetilde\tau|^2\widetilde G_4^{a6}\widetilde G_4^{66}\tilde F_{a6}\tilde F_{36}-4\widetilde G_4^{a3}\widetilde G_4^{36}\tilde F_{a6}\tilde F_{36}-2\widetilde G_4^{36}\widetilde G_4^{36}\tilde F_{36}\tilde F_{36}+2\widetilde G_4^{33}\widetilde G_4^{66}\tilde F_{36}\tilde F_{36}\Big).
\end{aligned}
\tag{D.26}
$$

The second term in the exponential of (D.23) is a topological term, and is left invariant under the action of $U'$ by inspection. If we replace the integers $\widetilde F_{3a}\to\widetilde F_{6a}$ and $\widetilde F_{a6}\to-\widetilde F_{a3}$, the two terms are left invariant, so the sum

$$
\sum_{\tilde F_{ij}\in\mathcal Z^6} e^{-\frac{2\pi^2\sqrt{\tilde g}}{e^2}g^{ij}g^{i'j'}\tilde F_{ii'}\tilde F_{jj'}+i\frac{\theta}{2}\epsilon^{\alpha\beta\gamma}\tilde F_{6\alpha}\tilde F_{\beta\gamma}} \tag{D.27}
$$

is invariant. Thus we have shown that under the $U'$ transformation (D.16),

$$
Z^{4d}_{\text{zero modes}}(R_3|\widetilde\tau|,R_6|\widetilde\tau|^{-1},g_{ab},-\gamma^3|\widetilde\tau|^{-2},\widetilde\gamma^a,-\kappa^a)=|\widetilde\tau|^2\ Z^{4d}_{\text{zero modes}}(R_3,R_6,g_{ab},\gamma^3,\kappa^a,\widetilde\gamma^a). \tag{D.28}
$$

Also from (D.23), we can write (D.22) as

$$
M_3:\qquad Z^{4d}_{\text{zero modes}}(e^2,\theta,G_{ij})\to Z^{4d}_{\text{zero modes}}(e^2,-\theta,G_{ij}). \tag{D.29}
$$

and thus under the $GL(4,\mathcal Z)$ generator $U_1$,

$$
Z^{4d}_{\text{zero modes}}\to|\widetilde\tau|^2\left(Z^{4d}_{\text{zero modes}}\right)^*. \tag{D.30}
$$

The residual factor $|\widetilde\tau|^2$ is sometimes referred to as an $SL(2,\mathcal Z)$ anomaly of the zero mode partition function, because $U'$ includes the $\widetilde\tau\to-\frac{1}{\widetilde\tau}$ transformation. Finally we will show how this anomaly is canceled by the oscillator contribution.

Under $U_3$, the metric parameters transform as

$$
\begin{aligned}
&R_6\to R_6,\quad \gamma^3\to-\gamma^3,\quad \gamma^a\to\gamma^a,\quad g_{ab}\to g_{ab},\quad g_{a3}\to-g_{a3},\quad g_{33}\to g_{33},\\
&g^{ab}\to g^{ab},\quad g^{a3}\to-g^{a3},\quad g^{33}\to g^{33},\quad \det g_{\alpha\beta}=\tilde g,\quad \tilde g\to\tilde g.\qquad \text{Or equivalently,}
\end{aligned}
$$

$$
\begin{aligned}
&G_{ab}\to G_{ab},\quad G_{a3}\to-G_{a3},\quad G_{a6}\to G_{a6},\quad G_{33}\to G_{33},\quad G_{66}\to G_{66},\quad G_{36}\to-G_{36},\\
&\widetilde G_4^{ab}\to\widetilde G_4^{ab},\quad \widetilde G_4^{a3}\to-\widetilde G_4^{a3},\quad \widetilde G_4^{a6}\to\widetilde G_4^{a6},\quad \widetilde G_4^{33}\to\widetilde G_4^{33},\quad \widetilde G_4^{36}\to-\widetilde G_4^{36},\quad \widetilde G_4^{66}\to\widetilde G_4^{66},\\
&\det\widetilde G_4=R_6\,\tilde g,\qquad \det\widetilde G_4\to\det\widetilde G_4,
\end{aligned}
\tag{D.31}
$$

where $4 \leq a, b \leq 5$ and $\widetilde{G}^{\alpha\beta}$ is the 3d inverse. We can check that $Z^{4d}_{\text{zero modes}}$ becomes its complex conjugate under $U_3$ given in (D.31) as follows. Letting the $U_3$ transformation (D.31) act on (2.7), we find that three of the terms in the exponent

$$
\begin{aligned}
&-\frac{e^2 R_6 \sqrt{\tilde{g}}}{8}\left(\frac{\theta^2}{4\pi^2}+\frac{16\pi^2}{e^4}\right)\Big(g^{aa'}g^{bb'}F_{ab}F_{bb'}+4g^{aa'}g^{b3}F_{ab}F_{a'3}+2g^{aa'}g^{33}F_{a3}F_{a'3}-2g^{a3}g^{a'3}F_{a3}F_{a'3}\Big),\\
&-\frac{e^2 R_6}{4\sqrt{\tilde{g}}}\tilde{\Pi}^\alpha g_{\alpha\beta}\tilde{\Pi}^\beta,\\
&-\frac{\theta e^2 R_6}{8\pi\sqrt{\tilde{g}}}g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\tilde{F}_{\gamma\delta}\tilde{\Pi}^\beta,
\end{aligned}
\tag{D.32}
$$

are separately invariant under (D.18), if we replace the the integers $\widetilde{F}_{\alpha\beta} \in \mathcal{Z}^3, \tilde{\Pi}^\alpha \in \mathcal{Z}^3$ by

$$
\widetilde{F}_{ab} \to \widetilde{F}_{ab}, \quad \tilde{F}_{a3} \to -\tilde{F}_{a3}, \quad \tilde{\Pi}^3 \to \tilde{\Pi}^3, \qquad \tilde{\Pi}^a \to -\tilde{\Pi}^a, \tag{D.33}
$$

However the subterm

$$
2\pi i\gamma^\alpha \tilde{\Pi}^\beta \tilde{F}_{\alpha\beta} \to -2\pi i\gamma^\alpha \tilde{\Pi}^\beta \tilde{F}_{\alpha\beta} \tag{D.34}
$$

acted by $U_3$ with the field shift in (D.33). Therefore the zero mode partition function goes to its complex conjugate under $U_3$.

*Appropriate generators for $SL(4, \mathcal{Z})$*

We claim that $U_1^2$, $U_2$ and $U_1 U_3$ generate the group $SL(4, \mathcal{Z})$ since $GL(n, \mathcal{Z})$ is generated by $U_1$, $U_2$ and $U_3$ or alternatively $R_1 = U_1$, $R_2 = U_3^{-1} U_2$ and $R_3 = U_3$, i.e., any element in $GL(n, \mathcal{Z})$ $U$ can be written as

$$
U = {R_1}^{n_1} {R_2}^{n_2} {R_3}^{n_3} {R_1}^{n_4} {R_2}^{n_5} {R_3}^{n_6} \ldots. \tag{D.35}
$$

It is understood that $SL(n, \mathcal{Z})$ is generated by even numbers of $R_1$, $R_2$ and $R_3$. Thus, the possible set of group generators for $SL(n, \mathcal{Z})$ is

$$
R_1^2, R_2^2, R_3^2, \quad R_1 R_2, R_2 R_3, R_3 R_1, \quad R_2 R_1, R_3 R_2, R_1 R_3 \tag{D.36}
$$

with the properties that $R_2^2 = 1$ and $R_3^2 = 1$. A smaller set of the $SL(4, \mathcal{Z})$ generators is

$$
R_1^2, R_1 R_3, R_2 R_3, \tag{D.37}
$$

since other generators in (D.36) can be expressed with the generators in (D.39) through the following relations

$$
\begin{aligned}
&R_1 R_2 = R_1 R_3 (R_2 R_3)^{-1}, \qquad R_2 R_1 = (R_1 R_2)^{-1} R_1^2\\
&R_3 R_2 = (R_2 R_3)^{-1}, \qquad R_3 R_1 = (R_1 R_3)^{-1} R_1^2.
\end{aligned}
\tag{D.38}
$$

Notice that

$$\{R_1^2, R_1R_3, R_2R_3\} = \{U_1^2, U_1U_3, U_2^{-1}\}. \tag{D.39}$$

*These three matrices generate $SL(4,\mathcal{Z})$.* They can be shown to generate Trott's twelve generators $B_{ij}$[35].

Since we have tested the invariance of the zero mode partition function under $U_2$, we only need to check invariance under $U_1U_3$ and $U_1^2$. For $U_1U_3$, as previously we separate $U_1$ into $U'$ and $M_3$,

$$U_1U_3 = U'M_3U_3 = U'(M_3U_3). \tag{D.40}$$

Since both $M_3$ and $U_3$ take $Z^{4d}_{\text{zero modes}}$ to its complex conjugate, $M_3U_3$ is an invariance of the zero mode partition function. Thus from (D.28),

$$U_1U_3: \qquad Z^{4d}_{zero\ modes} \to |\widetilde{\tau}|^2 Z^{4d}_{zero\ modes}. \tag{D.41}$$

*${U_1}^2$ acts on $Z^{4d}_{\text{zero modes}}$*

Since we have shown before

$$U_1: \qquad Z^{4d}_{zero\ modes} \to |\widetilde{\tau}|^2 Z^{4d\,*}_{zero\ modes}, \tag{D.42}$$

then

$${U_1}^2: \qquad Z^{4d}_{zero\ modes} \to Z^{4d}_{zero\ modes}. \tag{D.43}$$

To summarize, we have

$$\begin{aligned} U_2 &: Z^{4d}_{\text{zero modes}} \to Z^{4d}_{\text{zero modes}}, \\ U_1U_3 &: Z^{4d}_{\text{zero modes}} \to |\widetilde{\tau}|^2 Z^{4d}_{\text{zero modes}}, \\ {U_1}^2 &: Z^{4d}_{\text{zero modes}} \to Z^{4d}_{\text{zero modes}}. \end{aligned} \tag{D.44}$$

One can derive a similar transformation property for $Z^{6d}_{\text{zero modes}}$ using (2.9),

$$\begin{aligned} U_2 &: Z^{6d}_{\text{zero modes}} \to Z^{6d}_{\text{zero modes}}, \\ U_1U_3 &: Z^{6d}_{\text{zero modes}} \to |\widetilde{\tau}|^3 Z^{6d}_{\text{zero modes}}, \\ {U_1}^2 &: Z^{6d}_{\text{zero modes}} \to Z^{6d}_{\text{zero modes}}, \end{aligned} \tag{D.45}$$

which follows from transformations on the factor $\epsilon$, given in (2.10). By inspection $\epsilon$ is invariant under $U_2$ and $M_3$, and transforms as

$$U': \epsilon \to |\widetilde{\tau}|\epsilon. \tag{D.46}$$

This can be seen by Poisson resummation since $\epsilon$ can be written as

$$\epsilon = \sum_{n_a} \exp\{-\frac{\pi R_6\sqrt{\tilde{g}}}{R_1R_2} g^{ab} n_a n_b - \frac{\pi R_6\sqrt{\tilde{g}}}{R_3R_1R_2|\tilde{\tau}|^2}\tilde{\gamma}^a\tilde{\gamma}^b n_a n_b\} \sum_{m,n_3} \exp\{-\pi(N+x)\cdot A\cdot(N+x)\},$$
$$= |\tilde{\tau}|^{-1}\, U'\, \epsilon, \tag{D.47}$$

where

$$H_{12\alpha} = n_\alpha, \qquad H_{\alpha\beta\delta} = \frac{\epsilon_{\alpha\beta\delta}}{\tilde{g}}\, m, \qquad m, n_\alpha \in \mathcal{Z}^4,$$
$$A = \begin{pmatrix} \frac{R_6\sqrt{\tilde{g}}}{R_3R_1R_2} & i\gamma^3 \\ i\gamma^3 & \frac{R_6R_1R_2}{R_3\sqrt{\tilde{g}}} \end{pmatrix}, \qquad \det A = |\tilde{\tau}|^2, \qquad N = \begin{pmatrix} n_3 \\ m \end{pmatrix}, \qquad x = \begin{pmatrix} \kappa^a n_a + \frac{\gamma^3\tilde{\gamma}^a n_a}{|\tilde{\tau}|^2} \\ i\frac{R_6\sqrt{\tilde{g}}\,\tilde{\gamma}^3 n_a}{R_3R_1R_2|\tilde{\tau}|^2}, \end{pmatrix}.$$

*$U'$ acts on $Z^{4d}_{\rm osc}$*

To derive how $U'$ acts on $Z^{4d}_{\rm osc}$, we first separate the product on $\vec{n} = (n, n_a) \neq \vec{0}$ into a product on (all $n$, but $n_\alpha \neq (0,0)$) and on ($n \neq 0$, $n_a = (0,0)$). Then using the regularized vacuum energy (C.1) expressed as sum over zero and non-zero transverse momenta $p_\perp = n_a$ in (C.2), we find that (4.32) becomes

$$Z^{4d,Maxwell} = Z^{4d}_{\text{zero modes}} \cdot \Big( e^{\frac{\pi R_6}{6R_3}} \prod_{n\neq 0} \frac{1}{1 - e^{-2\pi\frac{R_6}{R_3}|n| - 2\pi i\gamma^3 n}} \Big)^2$$
$$\cdot \Big( \prod_{n_a\in\mathcal{Z}^2\neq(0,0)} e^{-2\pi R_6 <H>_{p_\perp}} \prod_{n^3\in\mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta} - 2\pi i\gamma^\alpha n_\alpha}} \Big)^2. \tag{D.48}$$

As in [12] we observe the middle expression above can be written in terms of the Dedekind eta function $\eta(\tilde{\tau}) \equiv e^{\frac{\pi i\tilde{\tau}}{12}} \prod_{n\in\mathcal{Z}\neq 0}(1 - e^{2\pi i\tilde{\tau}})$, with $\tilde{\tau} = \gamma^3 + i\frac{R_6}{R_3}$,

$$\Big( e^{\frac{\pi R_6}{6R_3}} \prod_{n\neq 0} \frac{1}{1 - e^{-2\pi\frac{R_6}{R_3}|n| - 2\pi i\gamma^3 n}} \Big)^2 = (\eta(\tilde{\tau})\bar{\eta}(\bar{\tilde{\tau}}))^{-2}. \tag{D.49}$$

This transforms under $U'$ in (D.16) as

$$(\eta(-\tilde{\tau}^{-1})\bar{\eta}(-\bar{\tilde{\tau}}^{-1}))^{-2} = |\tilde{\tau}|^{-2}\; (\eta(\tilde{\tau})\bar{\eta}(\bar{\tilde{\tau}}))^{-2}, \tag{D.50}$$

where $\eta(-\tilde{\tau}^{-1}) = (i\tilde{\tau})^{\frac{1}{2}}\eta(\tilde{\tau})$. In this way the anomaly of the zero modes in (D.28) is canceled by (D.50). Lastly we demonstrate the third expression in (D.48) is invariant under $U'$,

$$\Big(\prod_{n_a\in\mathcal{Z}^2\neq(0,0)} e^{-2\pi R_6<H>_\perp}\prod_{n_3\in\mathcal{Z}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}\Big)^2 = PI, \tag{D.51}$$

where $PI$ is the modular invariant $2d$ path integral of two massive scalar bosons of mass $\sqrt{\widetilde{g}^{ab}n_a n_b}$, coupled to a worldsheet gauge field, on a two-torus in directions 3,6. Following [12], with more detail in (D.68), we extract from (4.30)

$$Z^{4d}_{\rm osc} = (e^{-\pi R_6\sum_{\vec{n}\in\mathcal{Z}^3}\sqrt{g^{\alpha\beta}n_\alpha n_\beta}}\prod_{\vec{n}\in\mathcal{Z}^3\neq\vec{0}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}\Big)^2 \tag{D.52}$$

the $2d$ path integral of free massive bosons coupling to the gauge field, where $n_a$ is fixed and non-zero,

$$\begin{aligned}
(PI)^{\frac{1}{2}} &\equiv e^{-\pi R_6\sum_{n_3\in\mathcal{Z}}\sqrt{g^{\alpha\beta}n_\alpha n_\beta}}\prod_{n_3\in\mathcal{Z}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}+2\pi i\gamma^\alpha n_\alpha}}\\
&= \prod_{s\in\mathcal{Z}}\frac{e^{-\frac{\beta' E}{2}}}{1-e^{-\beta' E+2\pi i(\gamma^3 s+\gamma^a n_a)}}\qquad \text{where } s\equiv n_3,\quad E\equiv\sqrt{g^{\alpha\beta}n_\alpha n_\beta},\quad \beta'\equiv 2\pi R_6\\
&= \prod_{s\in\mathcal{Z}}\frac{1}{\sqrt{2}\,\sqrt{\cosh\beta' E-\cos 2\pi(\gamma^3 s+\gamma^a n_a)}}\qquad \text{for } n_a\to -n_a\\
&= e^{-\frac{1}{2}\sum_{s\in\mathcal{Z}}\big(\ln[\cosh\beta' E-\cos 2\pi(\gamma^3 s+\gamma^a n_a)]+\ln 2\big)}\equiv e^{-\frac{1}{2}\sum_{s\in\mathcal{Z}}\ \nu(E)},
\end{aligned} \tag{D.53}$$

where

$$\begin{aligned}
\sum_{s\in\mathcal{Z}}\nu(E) &\equiv \sum_{s\in\mathcal{Z}}\big(\ln[\cosh\beta' E-\cos 2\pi(\gamma^3 s+\gamma^a n_a)]+\ln 2\big)\\
&= \sum_{s\in\mathcal{Z}}\sum_{r\in\mathcal{Z}}\ln\,[\frac{4\pi^2}{\beta'^2}(r+\gamma^3 s+\gamma^a n_a)^2+E^2].
\end{aligned} \tag{D.54}$$

We can show directly that (D.54) is invariant under $U'$, since

$$\begin{aligned}
&E^2 = g^{\alpha\beta}n_\alpha n_\beta = g^{33}s^2+2g^{3a}sn_a+g^{ab}n_a n_b = \frac{1}{R_3^2}(s+\kappa^a n_a)^2+\widetilde{g}^{ab}n_a n_b,\\
&\frac{4\pi^2}{\beta'^2}(r+\gamma^3 s+\gamma^a n_a)^2 = \frac{1}{R_6^2}(r+\widetilde{\gamma}^a n_a+\gamma^3(s+\kappa^a n_a))^2,
\end{aligned} \tag{D.55}$$

then

$$\begin{aligned}
&\frac{4\pi^2}{\beta'^2}(r+\gamma^3 s+\gamma^a n_a)^2+E^2\\
&= \frac{1}{R_6^2}(s+\kappa^a n_a)^2\,|\widetilde{\tau}|^2+\frac{1}{R_6^2}(r+\widetilde{\gamma}^a n_a)^2+\frac{2\gamma^3}{R_6^2}(r+\widetilde{\gamma}^a n_a)(s+\kappa^a n_a)+\widetilde{g}^{ab}n_a n_b.
\end{aligned} \tag{D.56}$$

So we see the transformation $U'$ given in (D.16) leaves (D.56) invariant if $s \to r$ and $r \to -s$. Therefore (D.54) is invariant under $U'$, so that $(PI)^{\frac{1}{2}}$ given in (D.53) is invariant under $U'$.

*$M_3$ acts on $Z^{4d}_{\rm osc}$*

$M_3$ leaves the $Z^{4d}_{osc}$ invariant as can be seen from (D.48) by shifting the integer $n_\alpha$ as

$$n_3 \to -n_4, \qquad n_a \to n_{a+1}. \tag{D.57}$$

So, under $U_1 = U'M_3$,

$$Z^{4d}_{\rm osc} \to |\widetilde{\tau}|^{-2} Z^{4d}_{\rm osc}. \tag{D.58}$$

$U_2$ is an invariance of the oscillator partition function by inspection.

*$U_3$ acts on $Z^{4d}_{\rm osc}$*

$U_3$ leaves the $Z^{4d}_{osc}$ invariant as can be seen from (D.48) by shifting the integers $n_\alpha$ as

$$n_3 \to -n_3, \qquad n_a \to n_a. \tag{D.59}$$

Thus, the oscillator partition function transforms under the $SL(4,\mathcal{Z})$ generators $\{U_1^2, U_1U_3, U_2\}$ as

$$\begin{aligned} U_2 : Z^{4d}_{\rm osc} &\to Z^{4d}_{\rm osc}, \\ U_1U_3 : Z^{4d}_{\rm osc} &\to |\widetilde{\tau}|^{-2} Z^{4d}_{\rm osc}, \\ {U_1}^2 : Z^{4d}_{\rm osc} &\to Z^{4d}_{\rm osc}. \end{aligned} \tag{D.60}$$

So together with (D.44) we have established invariance under (D.39), and thus proved the partition function for the $4d$ Maxwell theory on $T^4$, given alternatively by (4.32) or (D.48), is invariant under $SL(4,\mathcal{Z})$, the mapping class group of $T^4$.

*$U'$ acts on $Z^{6d}_{osc}$*

For the $6d$ chiral theory on $T^2 \times T^4$, where $< H >^{6d} \equiv \frac{1}{2}\sum_{\vec{p}\in\mathcal{Z}^5}\sqrt{G_5^{lm}p_l p_m}$ appears in (4.36), the $SL(3,\mathcal{Z})$ invariant regularized vacuum energy [12] becomes,

$$\begin{aligned} < H >^{6d} &= -\frac{1}{2\pi^4}\sqrt{G_5}\sum_{\vec{n}\neq\vec{0}}\frac{1}{(G_{lm}n^l n^m)^3} \\ &= -32\pi^2\sqrt{G_5}\sum_{\vec{n}\neq\vec{0}}\frac{1}{(2\pi)^6\Big(g_{\alpha\beta}n^\alpha n^\beta + (R_1^2 + R_2^2\beta^2\beta^2)(n^1)^2 - 2\beta^2 R_2^2 n^1 n^2 + R_2^2(n^2)^2\Big)^3} \end{aligned} \tag{D.61}$$

and can be decomposed similarly to (C.2),

$$< H >^{6d} = \sum_{p_\perp \in \mathcal{Z}^4} < H >^{6d}_{p_\perp} = < H >^{6d}_{p_\perp = 0} + \sum_{p_\perp \in \mathcal{Z}^4 \neq 0} < H >^{6d}_{p_\perp}, \tag{D.62}$$

where

$$< H >^{6d}_{p_\perp} = -32\pi^2 \sqrt{G_5} \frac{1}{(2\pi)^4} \int d^4 y_\perp e^{-ip_\perp \cdot y_\perp} \sum_{n^3 \in \mathcal{Z} \neq 0} \frac{1}{|2\pi n^3 + y_\perp|^6}, \tag{D.63}$$

with denominator $|2\pi n^3 + y_\perp|^2 = G_{33}(2\pi n^3)^2 + 2(2\pi n^3) G_{3k} y^k_\perp + G_{kk'} y^k_\perp y^{k'}_\perp$, with $k = 1, 2, 4, 5$,

$$\begin{aligned}
< H >^{6d}_{p_\perp = 0} &= -\frac{1}{12 R_3}, \\
< H >^{6d}_{p_\perp \neq 0} &= |p_\perp|^2 R_3 \sum_{n=1}^{\infty} \cos(p_a \kappa^a 2\pi n) \big[ K_2(2\pi n R_3 |p_\perp|) - K_0(2\pi n R_3 |p_\perp|) \big] \\
&= -\pi^{-1} |p_\perp| R_3 \sum_{n=1}^{\infty} \cos(p_a \kappa^a 2\pi n) \frac{K_1(2\pi n R_3 |p_\perp|)}{n},
\end{aligned} \tag{D.64}$$

with $p_\perp = (p_1, p_2, p_a) = n_\perp = (n_1, n_2, n_a) = (n_1, n_2, n_4, n_5) \in \mathcal{Z}^4$,
$|p_\perp| = \sqrt{\frac{(n_1)^2}{R_1^2} + 2\frac{\beta^2}{R_1^2} + (\frac{1}{R_2^2} + \frac{{\beta^2}^2}{R_1^2}) n_2^2 + \widetilde{g}^{ab} n_a n_b}$.

The $U'$ invariance of (4.37) follows when we separate the product on $\vec{n} \in \mathcal{Z}^5 \neq \vec{0}$ into a product on ($n_3 \neq 0$, $n_\perp \equiv (n_1, n_2, n_4, n_5) = (0,0,0,0)$), and on (all $n_3$, but $n_\perp = (n_1, n_2, n_4, n_5) \neq (0,0,0,0)$). Then

$$\begin{aligned}
Z^{6d}_{\rm osc} &= \big( e^{\frac{\pi R_6}{6 R_3}} \prod_{n_3 \in \mathcal{Z} \neq 0} \frac{1}{1 - e^{2\pi i\, (\gamma^3\, n_3 + i \frac{R_6}{R_3} |n_3|\,)}} \big)^3 \\
&\quad \cdot \big( \prod_{n_\perp \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 < H >^{6d}_{p_\perp}} \prod_{n_3 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{\widetilde{n}^2 + g^{\alpha\beta} n_\alpha n_\beta} + i 2\pi \gamma^\alpha n_\alpha}} \big)^3 \\
&= \big( \eta(\widetilde{\tau})\, \bar{\eta}(\bar{\widetilde{\tau}}) \big)^{-3} \\
&\quad \cdot \big( \prod_{(n_1, n_2, n_4, n_5) \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 < H >^{6d}_{p_\perp}} \prod_{n_3 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha\beta} n_\alpha n_\beta + \widetilde{n}^2} + i 2\pi \gamma^\alpha n_\alpha}} \big)^3,
\end{aligned} \tag{D.65}$$

where $\widetilde{\tau} = \gamma^3 + i\frac{R_6}{R_3}$, and $\widetilde{n}^2 \equiv \frac{n_1^2}{R_1^2} + 2\frac{\beta^2}{R_1^2} n_1 n_2 + (\frac{1}{R_2^2} + \frac{{\beta^2}^2}{R_1^2}) n_2^2$. Under $U'$,

$$\eta(\widetilde{\tau})\, \bar{\eta}(\bar{\widetilde{\tau}}) \to |\widetilde{\tau}|\, \eta(\widetilde{\tau})\, \bar{\eta}(\bar{\widetilde{\tau}}). \tag{D.66}$$

$U'$ leaves invariant the part of the $6d$ oscillator partition function (D.65) at fixed $n_\perp \neq 0$, since

$$e^{-2\pi R_6 <H>^{6d}_{n_\perp \neq 0}} \prod_{n_3 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha\beta} n_\alpha n_\beta + \frac{n_1^2}{R_1^2} + 2\frac{\beta^2}{R_1^2} n_1 n_2 + (\frac{1}{R_2^2} + \frac{\beta^{2\,2}}{R_1^2}) n_2^2} + i2\pi\gamma^\alpha n_\alpha}} \tag{D.67}$$

is the square root of the partition function on $T^2$ (now in the directions 3,6) of a massive complex scalar with $m^2 \equiv G^{11} n_1^2 + G^{22} n_2^2 + 2G^{12} n_1 n_2 + \widetilde{g}^{ab} n_a n_b$, $4 \leq a, b \leq 5$, that couples to a constant gauge field $A^\mu \equiv iG^{\mu i} n_i$ with $\mu, \nu = 3, 6$; $i, j = 1, 2, 4, 5$. The metric on this $T^2$ is $h_{33} = R_3^2\,, h_{66} = R_6^2 + (\gamma^3)^2 R_3^2\,,\ h_{36} = -\gamma^3 R_3^2$. Its inverse is $h^{33} = \frac{1}{R_3^2} + \frac{(\gamma^3)^2}{R_6^2}$, $h^{66} = \frac{1}{R_6^2}$ and $h^{36} = \frac{\gamma^3}{R_6^2}$. The manifestly $SL(2, \mathcal{Z})$ invariant path integral is

$$\begin{aligned}
\text{P.I.} &= \int d\phi\, d\bar{\phi}\ e^{-\int_0^{2\pi} d\theta^3 \int_0^{2\pi} d\theta^6\ h^{\mu\nu}(\partial_\mu + A_\mu)\bar{\phi}(\partial_\nu - A_\nu)\phi + m^2 \bar{\phi}\phi} \\
&= \int d\bar{\phi}\, d\phi e^{-\int_0^{2\pi} d\theta^3 \int_0^{2\pi} d\theta^6 \bar{\phi}(-(\frac{1}{R_3^2} + \frac{(\gamma^3)^2}{R_6^2})\partial_3^2 - (\frac{1}{R_6})^2 \partial_6^2 - 2\frac{\gamma^3}{R_6^2}\partial_3\partial_6 + 2A^3\partial_3 + 2A^6\partial_6 + G^{11}n_1n_1 + G^{22}n_2n_2 + 2G^{12}n_1n_2 + G^{ab}n_an_b)\phi} \\
&= \det\Big([-(\tfrac{1}{R_3^2} + (\tfrac{\gamma^3}{R_6})^2)\partial_3^2 - (\tfrac{1}{R_6})^2\partial_6^2 - 2\gamma^3(\tfrac{1}{R_6})^2\partial_3\partial_6 + G^{11}n_1n_1 + G^{22}n_2n_2 \\
&\qquad + 2G^{12}n_1n_2 + G^{ab}n_an_b + 2iG^{3a}n_a\partial_3 + 2iG^{6a}n_a\partial_6]\Big)^{-1} \\
&= e^{-\mathrm{trln}\Big[-(\frac{1}{R_3^2} + (\frac{\gamma^3}{R_6})^2)\partial_3^2 - (\frac{1}{R_6})^2\partial_6^2 - 2\gamma^3(\frac{1}{R_6})^2\partial_3\partial_6 + G^{11}n_1n_1 + G^{22}n_2n_2 + 2G^{12}n_1n_2 + G^{ab}n_an_b + 2iG^{3a}n_a\partial_3 + 2iG^{6a}n_a\partial_6\Big]} \\
&= e^{-\sum_{s\in\mathcal{Z}}\sum_{r\in\mathcal{Z}}\Big[\ln(\frac{4\pi^2}{\beta'^2}r^2 + (\frac{1}{R_3^2} + (\frac{\gamma^3}{R_6})^2)s^2 + 2\gamma^3(\frac{1}{R_6})^2 rs + G^{11}n_1n_1 + G^{22}n_2n_2 + 2G^{12}n_1n_2 + G^{ab}n_an_b + 2G^{3a}n_a\, s + 2G^{6a}n_a\, r)\Big]} \\
&= e^{-\sum_{s\in\mathcal{Z}} \nu(E)},
\end{aligned} \tag{D.68}$$

where from (2.3), $G^{11} = \frac{1}{R_1^2}$, $G^{22} = \frac{1}{R_2^2} + \frac{\beta^{2\,2}}{R_1^2}$, $G^{12} = \frac{\beta^2}{R_1^2}$, $G^{ab} = g^{ab} + \frac{\gamma^a\gamma^b}{{R_6}^2}$,
$G^{3a} = g^{3a} + \frac{\gamma^3\gamma^a}{R_6^2}$, $G^{6a} = \frac{\gamma^a}{R_6^2}, G^{63} = \frac{\gamma^3}{R_6^2}$, and $\partial_3\phi = -is\phi$, $\partial_6\phi = -ir\phi$, $s = n_3$, and $\beta' = 2\pi R_6$. The sum on $r$ is

$$\nu(E) = \sum_{r\in\mathcal{Z}} \ln\Big[\frac{4\pi^2}{\beta'^2}(r + \gamma^3 s + \gamma^a n_a)^2 + E^2\Big]\,, \tag{D.69}$$

with $E^2 \equiv G_5^{lm} n_l n_m = G_5^{11} n_1 n_1 + G_5^{22} n_2 n_2 + G_5^{21} n_2 n_1 + G_5^{ab} n_a n_b + 2G_5^{a3} n_a n_3 + G_5^{33} n_3 n_3$, and $G_5^{11} = \frac{1}{R_1^2}$, $G_5^{12} = \frac{\beta^2}{R_1^2}$, $G_5^{22} = \frac{1}{R^2} + \frac{\beta^2\beta^2}{R_1^2}$, $G_5^{1\alpha} = G_5^{2\alpha} = 0$, $G_5^{3a} = g^{3a} = \frac{\kappa^a}{R_3^2}$,
$G_5^{33} = g^{33} = \frac{1}{R_3^2}$, $G_5^{ab} = g^{ab} = \widetilde{g}^{ab} + \frac{\kappa^a\kappa^b}{R_3^2}$. We evaluate the divergent sum $\nu(E)$ on $r$ by

$$\begin{aligned}
\frac{\partial\nu(E)}{\partial E} &= \sum_r \frac{2E}{\frac{4\pi^2}{\beta'^2}(r + \gamma^3 s + \gamma^a n_a)^2 + E^2} \\
&= \partial_E \ln\Big[\cosh\beta' E - \cos 2\pi\big(\gamma^3 s + \gamma^a n_a\big)\Big],
\end{aligned} \tag{D.70}$$

using the sum $\sum_{n\in\mathcal{Z}} \frac{2y}{(2\pi n+z)^2+y^2} = \frac{\sinh y}{\cosh y-\cos z}$. Then integrating (D.70), we choose the integration constant to maintain modular invariance of (D.68),

$$\nu(E) = \ln\big[\cosh\beta' E - \cos 2\pi\big(\gamma^3 s + \gamma^a n_a\big)\big] + \ln 2. \tag{D.71}$$

It follows for $s = n_3$ that (D.68) gives

$$\begin{aligned}
(\text{P.I.})^{\frac{1}{2}} &= \prod_{s\in\mathcal{Z}} \frac{1}{\sqrt{2}\sqrt{\cosh\beta' E - \cos 2\pi(\gamma^3 s + \gamma^a n_a)}} \\
&= \prod_{s\in\mathcal{Z}} \frac{e^{-\frac{\beta' E}{2}}}{1 - e^{-\beta' E + 2\pi i(\gamma^3 s + \gamma^a n_a)}} \\
&= e^{-\pi R_6 \sum_{s\in\mathcal{Z}} \sqrt{G_5^{lm} n_l n_m}} \prod_{s\in\mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6\sqrt{G_5^{lm} n_l n_m} + 2\pi i\gamma^3 s + 2\pi i\gamma^a n_a}} \\
&= e^{-2\pi R_6 < H >_{n\perp}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6\sqrt{G_5^{lm} n_l n_m} + 2\pi i\gamma^3 n_3 + 2\pi i\gamma^a n_a}}\,,
\end{aligned} \tag{D.72}$$

which is (D.67). Its invariance under $U'$ follows from the $U'$ invariance of (D.54), which differs from (D.69) only by an additional contribution of $\widetilde{n}^2$ to the mass $m^2$.

Hence (D.72) and thus (D.67) are invariant under $U'$.

Furthermore $Z^{6d}_{\text{osc}}$ is invariant under $M_3, U_2, U_3$ by inspection.

Using the same approach for proving $SL(4,\mathcal{Z})$ symmetry of the $4d$ partition function, we have shown the $6d$ oscillator partition function for the chiral boson given by (4.36), or equivalently (D.65), transforms as

$$\begin{aligned}
U_2 : Z^{6d}_{\text{osc}} &\to Z^{6d}_{\text{osc}}, \\
U_1 U_3 : Z^{6d}_{\text{osc}} &\to |\widetilde{\tau}|^{-3} Z^{6d}_{\text{osc}}, \\
{U_1}^2 : Z^{6d}_{\text{osc}} &\to Z^{6d}_{\text{osc}}.
\end{aligned} \tag{D.73}$$

Together with (D.45), the $6d$ partition function $Z^{6d,chiral} \equiv Z^{6d}_{\text{zero modes}}\, Z^{6d}_{\text{osc}}$ is $SL(4,\mathcal{Z})$ invariant.

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