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Yangian in the Twistor String

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Abstract

We study symmetries of the quantized open twistor string. In addition to global PSL(4|4) symmetry, we find non-local conserved currents. The associated non-local charges lead to Ward identities which show that these charges annihilate the string gluon tree amplitudes, and have the same form as symmetries of amplitudes in $\mathcal{N} = 4$ super conformal Yang Mills theory. We describe how states of the open twistor string form a realization of the PSL(4|4) Yangian superalgebra.

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1 Introduction

Twistor string theory [1]-[3] is equivalent to a massless field theory and provides a string structure to analyze four-dimensional massless scattering amplitudes with $N = 4$ supersymmetry. At tree level, the spectrum can describe states of both super Yang Mills theory and conformal supergravity. The world sheet theory has a target in super twistor space, and has been argued to have a Yangian extension of its $PSL(4|4)$ global symmetry [1].

In this paper we construct non-local conserved currents which lead to a Hopf algebra coproduct, show the associated charges annihilate the string gluon tree amplitudes and give a realization of the Yangian symmetry algebra for this world sheet action. We leave the conformal graviton amplitudes for future analysis. Explicit constructions of the amplitudes have revealed some of the string's framework [4]-[12]. How the Yangian acts on the various building blocks of the theory [13]-[17] should help further unravel the theory, at least in the planar limit. The string also relates to other twistor methods [18]-[22].

The Yangian symmetry is expected because of the string's close connection to Yang Mills theory, for which an infinite-dimensional symmetry for the planar theory has been displayed [23]-[39], and its appearance as dual conformal invariance was found explicitly for the amplitudes [40]-[49].

We review Yangians and their nature in field theory. A Yangian superalgebra is a superalgebra, that is an algebra that has bosonic and fermionic elements, and it is graded by the non-negative integers. The subalgebra of level zero is a Lie superalgebra and the grade one piece is its adjoint representation. The higher level parts of the superalgebra are generated by the algebra multiplication of elements of level one. They are subject to some restrictions which arise from the comultiplication. The comultiplication is an important additional structure that specifies how the elements of the Yangian superalgebra act on products of states. For an ordinary Lie superalgebra this is trivial

$$Q_0(A\,B) \;=\; Q_0(A)\,B + (-1)^{|Q_0||A|} A\,Q_0(B)\,. \tag{1.1}$$

But the elements of level one of the Yangian have non-trivial comultiplication. They act schematically as

$$Q_1(A\,B) \;=\; Q_1(A)\,B + (-1)^{|Q_1||A|} A\,Q_1(B) + Q_0(A)\,Q_0(B)\,. \tag{1.2}$$

Now one needs to require that this operation is compatible with the algebra product.

This can be ensured by the Serre relations. The question is how symmetries with such non-trivial action on products of states can arise in field theory. For a model specified by a Lagrangian plus a path integral measure, with both invariant under some global symmetry, usually we have corresponding conserved *local* Noether currents and charges. But the action of charges on products of states is then of the form (1.1). The way to realize a non-trivial action on products of fields is by introducing *non-local* conserved charges. Non-locality is then reflected in the comultiplication (1.2).

The world-sheet theory of the open twistor string is a two-dimensional conformal field theory, that is a local model. Introducing non-local fields thus seems strange. Nonetheless, in [50], it was shown how Yangian symmetry arises via non-local currents in Lie group Wess-Zumino-Witten models. We will follow their analysis in spirit, but we will use the non-local currents only as an intermediate step. We can do that due to the special nature of the global symmetry supergroup PSL(4|4) of the twistor string. Our strategy is as follows. We start by introducing non-local currents as in [50], [51]. We show that the associated charges satisfy the comultiplication of the level one elements of the psl(4|4) Yangian. Then we derive Ward identities for these charges. They simplify considerably due to the vanishing of the Killing form of psl(4|4). These Ward identities lead us to define new operators acting on the fields of the twistor string. These operators form a realization of the Yangian. In addition they form a symmetry because of the Ward identities we found, and the new level one charges do not have the properties of the old charges, namely construction from a non-local current.

In section 2 we review the twistor string, describe the ordinary $PSL(4|4)$ Noether currents and discuss their canonical quantization in the boundary conformal field theory. In section 3 we give the action of these ordinary currents and their charges on the fields and the tree amplitudes. The non-local Noether currents are described in section 4, and their action on the fields and the string tree amplitudes is shown in section 5. In section 6, we discuss the Serre relations and the representation of the Yangian of $PSL(4|4)$. In the appendices we review superalgebras and give the $PSL(4|4)$ structure constants and their properties, including the vanishing of the adjoint quadratic Casimir (the Killing form).

2 $PSL(4|4)$ Noether currents of the open twistor string

The open twistor string can be described by the action

$$S \;=\; S_{YZ} + S_G + S_{\text{ghost}}\,, \tag{2.1}$$

with S_G given by a conformal field theory of central charge $c = 28$, S_{ghost} is the standard $c = -26$ ghost system of the bosonic string, and S_{YZ} is the world sheet action for fields with a target of twistor superspace,

$$S_{YZ} = \int i \left[Y^{zI} D_z \overline{Z}_I + Y^{\bar{z}I} D_{\bar{z}} Z_I \right] \, g^{\frac{1}{2}} d^2 x \tag{2.2}$$

where $1 \leq I \leq 8$, $D_\alpha = \partial_\alpha - iA_\alpha$, and g is the determinant of the world sheet metric with Euclidean signature, $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$. The world sheet can be described as the upper half-plane, and thus has a boundary.

The equations of motion are

$$(\partial_{\bar{z}} - iA_{\bar{z}})Z_I = 0, \qquad (\partial_z - iA_z)\overline{Z}_I = 0, \qquad Y^{zI}_{;z} + iA_z Y^{zI} = 0, \qquad Y^{\bar{z}I}_{;\bar{z}} + iA_{\bar{z}} Y^{\bar{z}I} = 0. \tag{2.3}$$

The constraints from varying the world sheet gauge fields are

$$Y^{\bar{z}I} Z_I = Y^{zI} \overline{Z}_I = 0,$$

and the boundary conditions in upper half-plane world sheet coordinates are

$$\overline{Z}_I = U Z_I, \qquad Y^{zI} = U^{-1} Y^{\bar{z}I}, \tag{2.4}$$

where $U = e^{2i\alpha}$, for some function α, which varies and is real on the boundary, and is continuous up to multiples of π. The reality conditions are $\overline{Y^{zI}} = -Y^{\bar{z}I}$ for bosonic components ($1 \leq I \leq 4$), and $\overline{Y^{z,I}} = Y^{\bar{z}I}$ for fermionic components ($5 \leq I \leq 8$), and $\overline{A_z} = -A_{\bar{z}}$.

The twistor string has global PSL(4|4) symmetry. To write the Noether currents for PSL(4|4), we consider the following symmetry transformations of the world sheet action (2.1),

$$Z_I \to e^{\rho_a T^{aJ}_I} Z_J, \qquad \overline{Z}_I \to e^{\rho_a T^{aJ}_I} \overline{Z}_J,$$
$$Y^{\bar{z}I} \to Y^{\bar{z}J} e^{-\rho_a T^{aI}_J}, \qquad Y^{zI} \to Y^{zJ} e^{-\rho_a T^{aI}_J}, \tag{2.5}$$

where the generators of the Grassman envelope of the superalgebra psl(4|4), $\rho_a T^{aJ}_I$, are real and have zero trace and zero supertrace. The other fields in (2.1) are singlets of psl(4|4) and ultimately singlets of the Yangian, so from now on we will discuss symmetries of

(2.2). See appendix A for some background on the supergroup PSL(4|4) and superalgebra psl(4|4). The infinitesimal transformations are

$$\delta Z_I = \rho_a T_I^{aJ} Z_J, \qquad \delta \overline{Z}_I = \rho_a T_I^{aJ} \overline{Z}_J, \tag{2.6}$$

leaving the Lagrangian invariant. The Noether current is

$$J^z = g^{\frac{1}{2}} j^z, \qquad J^{\bar z} = g^{\frac{1}{2}} j^{\bar z}, \qquad \text{with} \quad j^z = iY^{zI} \rho_a T_I^{aJ} \overline{Z}_J, \qquad j^{\bar z} = iY^{\bar z I} \rho_a T_I^{aJ} Z_J, \tag{2.7}$$

where $\partial_\alpha J^\alpha = 0$ and j^α satisfies $j^\alpha_{;\alpha} = 0$ with use of the equations of motion (2.3), and also separately $j^z_{;z} = 0$ and $j^{\bar z}_{;\bar z} = 0$. So we also have

$$d\,j = 0. \tag{2.8}$$

In addition to (2.5), the identity matrix $T_I{}^J = \delta_I{}^J$ and the $U(1)$ R-symmetry matrix $T_I{}^J = (-1)^{\deg I} \delta_I{}^J$ also generate symmetries of the classical action. Here $\deg I = 0$ when I is a bosonic index, and $\deg I = 1$ when I is a fermionic index. For simplicity we will denote the degree by the index, $(-1)^{\deg I} \equiv (-1)^I$. The transformation associated with the identity matrix obeys the boundary condition (2.4) for any complex function ρ, and generates the $gl(1, \mathbb{C})$ gauge invariance which we use to choose $A_z = A_{\bar z} = 0$. The $U(1)$ R-symmetry corresponds to a current that has a conformal anomaly and is not conserved in the quantum theory.

We review the canonical quantization [52] of the conformal field theory (2.2). The commutation relations are found from

$$Z_I(z) Y^J(\zeta) =: Z_I(z) Y^J(\zeta) : + \frac{\delta_I^J}{z - \zeta}, \tag{2.9}$$

for $|z| > |\zeta|$, and where $Y^J \equiv g^{\frac{1}{2}} Y^{\bar z J}$ is holomorphic in the gauge where $A_z = A_{\bar z} = 0$. The Virasoro current relevant for the twistor fields is given by

$$L_{YZ}(z) = -\sum_J Y^J(z) \partial Z_J(z) \tag{2.10}$$

and its operator product expansion with the chiral currents $: Y^I(z) Z_J(z) :$ is

$$L_{YZ}(z) : Y^I(\zeta) Z_J(\zeta) : \sim -\frac{\delta_J^I\,(-1)^{\deg I}}{(z-\zeta)^3} + \frac{: Y^I(\zeta) Z_J(\zeta) :}{(z-\zeta)^2} + \frac{\partial : Y^I(\zeta) Z_J(\zeta) :}{(z-\zeta)}. \tag{2.11}$$

Here $\sim$ denotes equality up to the regular part. The anomalous term vanishes for the supertraceless currents, that is to say the psl(4|4) currents and the $gl(1,\mathbb{C})$ gauge current, so they are true primary fields and are conserved in the quantum theory.

In order to define Noether charges, we consider the boundary theory explicitly. In a boundary conformal field theory one specifies how fields are identified at the boundary, such as in (2.4),

$$-(-1)^I \bar{Y}^I = U^{-1}Y, \qquad \bar{Z} = UZ. \tag{2.12}$$

Our theory possesses a current superalgebra symmetry in addition to Virasoro invariance (see, e.g., [54]-[58].)

In such a situation one usually has boundary conditions of the form

$$J(z) \;=\; \Omega \bar{J}(\bar{z}) \qquad \text{for } z \;=\; \bar{z}\,. \tag{2.13}$$

Here Ω is a map on the space of fields specifing the boundary conditions. If this map is an automorphsim of the Lie superalgebra, then these gluing conditions preserve the current algebra symmetry. In this case, we can analytically continue the current $J(z)$ on the entire plane

$$J(z) \;=\; \begin{cases} J(z) & z \text{ in upper half plane} \\ \Omega \bar{J}(z) & z \text{ in lower half plane} \end{cases} \tag{2.14}$$

Then the charge is defined by an integral that runs from boundary (a) to boundary (b) of the open string, where

$$Q = \int d\sigma j^0 = \epsilon_{\mu\nu} \int dx^\mu j^\nu = \int_a^b dz j^{\bar{z}} - \int_a^b d\bar{z} j^z = \oint dz j^{\bar{z}},$$

and we include a factor of $\frac{1}{2\pi i}$ in $\oint$. See Figure 1.

In our case, $\Omega = -(-1)^{IJ+I}$ which is an automorphism of psl(4|4), and the conserved Noether charges associated with the transformations (2.6) can be written as

$$Q^I_{0J} = \oint dz J^I_{0J}(z) \tag{2.15}$$

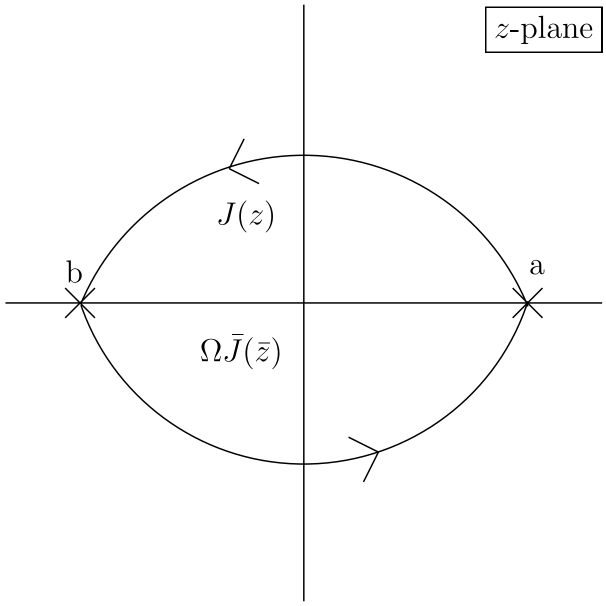

Figure 1: *Continuity of currents.*

where the contour encirles the origin, and the Nother currents for the psl(4|4) charges are

$$J^I_{0\,J}(z) \equiv Y^I(z)Z_J(z) - \frac{1}{8}(-1)^{I+E}\delta^I_J Y^E(z)Z_E(z) - \frac{1}{8}\delta^I_J Y^E(z)Z_E(z). \tag{2.16}$$

They have zero trace and zero supertrace and, from (2.9), the current algebra satisfies

$$J^I_{0\,J}(z)\, J^K_{0\,L}(\zeta) \sim (z-\zeta)^{-1}\left(f^{I\ K\ \ \ N}_{\ J\ LM}\, J^M_{0\,N}(\zeta) + \frac{1}{8}\Big(1-(-1)^{I+J}\Big)\delta^I_L\,\delta^K_J\ \ Y^E(\zeta)Z_E(\zeta)\right)$$
$$+ (z-\zeta)^{-2}\left(-(-1)^I\delta^I_L\delta^K_J + \frac{1}{8}((-1)^I+(-1)^K)\delta^I_J\delta^K_L\right), \tag{2.17}$$

where the psl(4|4) structure constants $f^{I\ K\ \ \ N}_{\ J\ LM}$ have vanishing Killing form

$$f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ \ f^{I\ K\ \ \ S}_{\ J\ LR} = 0, \tag{2.18}$$

and are given in appendix B, together with some of their properties. The charges (2.15) satisfy

$$[Q^I_{0\,J},\, Q^K_{0\,L}\} = f^{I\,K}_{J\;\,L\,M}{}^N\, Q^M_{0\,N} + \frac{1}{8}\Big(1-(-1)^{I+J}\Big)\delta^I_L\,\delta^K_J \quad \oint dzY^E(z)Z_E(z) \tag{2.19}$$

Since the central generator $\oint dzY^E(z)Z_E(z)$ acts as zero on gauge invariant quantities like the vertex operators, the twistor string has psl(4|4) global symmetry, and

$$[Q^I_{0\,J},\, Q^K_{0\,L}\} = f^{I\,K}_{J\;\,L\,M}{}^N\, Q^M_{0\,N} \tag{2.20}$$

on gauge invariant states. The mixed brackets in (2.19) denote an (anti) commutator for (odd) even generators,

$$[Q^I_{0\,J},\, Q^K_{0\,L}\} \equiv Q^I_{0\,J}\,Q^K_{0\,L} - (-1)^{(I+J)(K+L)}\,Q^K_{0\,L}Q^K_{0\,L}. \tag{2.21}$$

See appendix C for a familiar basis.

3 How the PSL(4|4) symmetry acts

3.1 *Action of PSL(4|4) Currents and Charges on Fields*

The ordinary charges generate infinitesimal transformation on the fields $Y^K(z)$ and $Z_K(z)$ and act on products of more general conformal fields via standard Lie algebra comultiplication, as follows.

The operator product of the current $J^I_{0\,J}(z)$ with fields $Z_K(\zeta)$ is

$$\begin{aligned} J^I_{0\,J}(z)\; Z_K(\zeta) &\sim (z-\zeta)^{-1}\left(-(-1)^{IJ+K}\delta^I_K Z_J(z) + \frac{1}{8}(-1)^I\delta^I_J Z_K(\zeta) + \frac{1}{8}\delta^I_J Z_K(\zeta)\right) \\ &\equiv (z-\zeta)^{-1}\; t^{\,I}_{\,J}[\zeta]\; Z_K(\zeta), \end{aligned} \tag{3.1}$$

for $|z| > |\zeta|$ where we have defined the differential operations

$$\begin{aligned} t^{\,I}_{\,J}[\zeta] \equiv &-(-1)^{IJ+I}\, Z_J(\zeta)\frac{\partial}{\partial Z_I(\zeta)} + \frac{1}{8}\delta^I_J(-1)^{I+E}Z_E(\zeta)\frac{\partial}{\partial Z_E(\zeta)} + \frac{1}{8}\delta^I_J Z_E(\zeta)\frac{\partial}{\partial Z_E(\zeta)} \\ &+ Y^I(\zeta)\frac{\partial}{\partial Y^J(\zeta)} - \frac{1}{8}\delta^I_J(-1)^{I+E}Y^E(\zeta)\frac{\partial}{\partial Y^E(\zeta)} - \frac{1}{8}\delta^I_J Y^E(\zeta)\frac{\partial}{\partial Y^E(\zeta)}. \end{aligned} \tag{3.2}$$

Since $J^I_{0\,J}(z)$ and $Z_K(\zeta)$ are respectively local, we find using the standard contour argument,

$$[Q^I_{0\,J}, Z_K(\zeta)\} = \oint_{\Gamma_0^>} dz\,(z-\zeta)^{-1} t^I_J[\zeta] Z_K(\zeta) - \oint_{\Gamma_0^<} dz\,(z-\zeta)^{-1} t^I_J[\zeta] Z_K(\zeta)$$

$$= \oint_{\Gamma_\zeta} dz\, t^I_J[\zeta] Z_K(\zeta) = t^I_J[\zeta] Z_K(\zeta), \tag{3.3}$$

where the contours $\Gamma_0^>, \Gamma_0^<$ encircle the origin clockwise with $|z| > |\zeta|, |z| < |\zeta|$ respectively, and Γ_ζ encircles ζ but not the origin. Also

$$[Q^I_{0\,J}, Y^K(\zeta)\} = t^I_J[\zeta] Y^K(\zeta) = \delta^K_J Y^I(\zeta) - \frac{1}{8}(-1)^{I+K}\delta^I_J Y^K(\zeta) - \frac{1}{8}\delta^I_J Y^K(\zeta). \tag{3.4}$$

(3.3),(3.4) correspond to infinitesimal field transformations that leave the twistor string world sheet action invariant, and are generated by the super commutator of the Noether charge with the field.

Similarly, for any field $V(\zeta) \equiv V(Z(\zeta), Y(\zeta))$ that is a function of $Z_K(\zeta)$ and or $Y^K(\zeta)$, such as the vertex operators, the operator product with the ordinary current is

$$J^I_{0J}(z) V(\zeta) \sim (z-\zeta)^{-1}\, t^I{}_J[\zeta]\; V(\zeta)\,, \tag{3.5}$$

From (3.2), it follows that

$$[\; t^I{}_J[\zeta],\; t^K_L[\zeta]\;\} = f^I{}_J{}^K{}_{L\,M}{}^N\; t^M_N[\zeta],$$

where the structure constants $f^I{}_J{}^K{}_{L\,M}{}^N$ are given (B.1). The charge acts

$$[Q^I_{0\,J}, V(\zeta)\} = t^I{}_J[\zeta]\; V(\zeta)\,, \tag{3.6}$$

and on a product of fields as

$$[Q^I_{0\,J}, V_1(\zeta_1) V_2(\zeta_2)\}$$

$$= [Q^I_{0\,J}, V_1(Z(\zeta_1))\}\;\; V_2(Z(\zeta_2)) + (-1)^{(I+J)\deg(V_1)} V_1(Z(\zeta_1))\;\; [Q^I_{0\,J}, V_2(Z(\zeta_2))\}$$

$$= t^I{}_J[\zeta_1] V_1(\zeta_1)\;\; V_2(\zeta_2) + (-1)^{(I+J)\deg(V_1)} V_1(\zeta_1)\;\; t^I{}_J[\zeta_2] V_2(\zeta_2). \tag{3.7}$$

This can be written as the standard Lie superalgebra comultiplication,

$$\Delta Q^I_{0\,J} = Q^I_{0\,J} \otimes 1 + (-1)^{(I+J)F} \otimes Q^I_{0\,J}, \tag{3.8}$$

which instructs one how to build the action of the charge on two fields from knowledge of the action of the charge on a single field, etc. The operator F is the fermion number operator. Clearly the standard comultiplication follows from the fact that the action of the charge on a single site is given by a commutator as in (3.6).

3.2 *Action of Ordinary Charges on the Tree Amplitudes*

It follows directly that the psl(4|4) symmetry charges annihilate the tree amplitudes, as we now show. The charges, which can be expressed in term of the modes as

$$Q^I_{0\,J} = \sum_n Y^I_{-n} Z_{J\,n} - \frac{1}{8}(-1)^{I+E}\delta^I_J \sum_n Y^E_{-n} Z_{E\,n} - \frac{1}{8}\delta^I_J \sum_n Y^E_{-n} Z_{E\,n}, \tag{3.9}$$

annihilate the vacuum,

$$Q^I_{0\,J}|0\rangle, \qquad \langle 0|(Q^I_{0\,J})^\dagger = \pm\langle 0|Q^I_{0\,J} = 0, \tag{3.10}$$

since the modes Y^I_n, Z^I_n satisfy the vacuum conditions [52]

$$Y^I_n|0\rangle = 0, \quad n \geq 0, \qquad Z^I_n|0\rangle = 0, \quad n \geq 1, \tag{3.11}$$

where $Z(z) = \sum Z_n z^{-n}$, and $Y(z) = \sum_n Y_n z^{-n-1}$, and the hermiticity conditions are

$$(Z_{J\,n})^\dagger = Z_{J\,-n}, \qquad (Y^I_n)^\dagger = -(-1)^I Y^I_{-n}. \tag{3.12}$$

This is in contrast to the $U(1)_R$ charge which does not annihilate the vacuum.

The tree amplitudes in twistor string theory are given by

$$\mathcal{A}^{\text{tree}}_n = \int \langle 0|e^{dq_0} V_1(z_1)V_2(z_2)\ldots V_n(z_n)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S \tag{3.13}$$

where $d\gamma_M$ is the invariant measure on the Möbius group, $d\gamma_S$ is the invariant measure on the group of scale transformations, d is the instanton number which is equal to one less than the number of negative helicities, $q_0 = \sum_{I=1}^{8} q_0^I$ the sum of the zero modes of X^I relevant for bosonizing $Y^I(z), Z_J(z)$; and the homogeneous conformal fields $V_i(z_i)$ are the vertex operators of the gluon or graviton supermultiplets [3, 52, 59].

Under a charge Q_0, the vertex operators transform as

$$V(z) \to V'(z) = e^{\rho Q_0} V(z) e^{-\rho Q_0} \simeq V(z) + \rho[Q_0, V(z)] + \emptyset(\rho^2). \tag{3.14}$$

The psl(4|4) charges Q_0 commute with e^{dq_0},

$$[Q_0, e^{dq_0}] = 0, \tag{3.15}$$

since the modes satisfy

$$Y^I_{n-d} e^{dq_0} = e^{dq_0} Y^I_n, \qquad Z^I_{n+d} e^{dq_0} = e^{dq_0} Z^I_n \qquad \text{for } 1 \leq I \leq 8. \tag{3.16}$$

Therefore the tree amplitudes are invariant under the symmetry transformations,

$$\begin{aligned} \mathcal{A}_n^{\text{tree}} &= \int \langle 0| e^{dq_0} V_1'(z_1) V_2'(z_2) \ldots V_n'(z_n) |0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S \\ &= \int \langle 0| e^{dq_0} e^{\rho Q_0} V_1(z_1) V_2(z_2) \ldots V_n(z_n) |0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S \\ &= \int \langle 0| e^{dq_0} V_1(z_1) V_2(z_2) \ldots V_n(z_n) |0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S \end{aligned} \tag{3.17}$$

from (3.9), (3.14), (3.15). Thus working to first order in ρ, we see that the charges Q_0 annihilate the tree amplitudes,

$$\int \langle 0| e^{dq_0} Q_0 V_1(z_1) V_2(z_2) \ldots V_n(z_n) |0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S = 0. \tag{3.18}$$

Furthermore we can evaluate the action of the charges on the amplitude as follows.

For simplicity, we consider the gluon tree amplitudes. The vertices for the negative and positive helicity gluons are

$$V_-^A(z) = \int dk k^3 \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\mu^{\dot{a}}(z)\bar{\pi}_{\dot{a}}} J^A(z)\psi^1(z)\psi^2(z)\psi^3(z)\psi^4(z),$$

$$V_+^A(z) = \int \frac{dk}{k} \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\mu^{\dot{a}}(z)\bar{\pi}_{\dot{a}}} J^A(z). \tag{3.19}$$

These are functions of $Z_J(z)$, whose components are labeled as $Z^I = (\lambda^a, \mu^{\dot{a}}, \psi^1, \psi^2, \psi^3, \psi^4)$. As an example, consider the action of the momentum operator on the vertices,

$$[p_a^{\ \dot{a}}, V_\pm^A(z)] = t^{\dot{a}}{}_a[z]\ V_\pm^A(z) = -\lambda_a(z)\frac{\partial}{\partial \mu_{\dot{a}}(z)}\ V_\pm^A(z) = ik\bar{\pi}^{\dot{a}}\lambda_a(z)\ V_\pm^A(z)$$

$$= i\pi_a\bar{\pi}^{\dot{a}}\ V_\pm^A(z). \tag{3.20}$$

Then from (3.18),

$$\int \langle 0|e^{dq_0} p_a^{\ \dot{a}} V_1(z_1)V_2(z_2)\dots V_n(z_n)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S$$

$$= \int \langle 0|e^{dq_0} V_1(z_1)\dots \sum_{r=1}^{n} [p_a^{\ \dot{a}}, V_r(z_r)] \dots\dots V_n(z_n)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S$$

$$= i\left(\sum_{r=1}^{n} \pi_{ra}\bar{\pi}_r^{\dot{a}}\right) \int \langle 0|e^{dq_0} V_1(z_1)V_2(z_2)\dots V_n(z_n)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S$$

$$= 0, \tag{3.21}$$

which is momentum conservation.

If we Fourier transform the gluon vertex operators from $\bar{\pi}_{\dot{a}}$ back to $\omega^{\dot{a}}$ which is a twistor space coordinate, and introduce the superspace coordinate θ^M to include all fields in the gluon supermultiplet, they become the wavefunction for the world sheet fields $Z^I(z)$ to be at a point $Z'^I = (\pi^a, \omega^{\dot{a}}, \theta^M)$ in super twistor space,

$$W^A(z) = \int \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a)\delta(k\mu^{\dot{a}}(z) - \omega^{\dot{a}}) \prod_{M=1}^{4} (k\psi^M(z) - \theta^M)\frac{dk}{k} J^A(z). \tag{3.22}$$

See for example [52]. Then the action of the charges on W^A is

$$[Q_{0\ J}^{I}, W^A(z)] = t^I{}_J[z]\ W^A(z) = t'^I{}_J\ W^A(z) \tag{3.23}$$

where $t'^I{}_J$ is now expressed in terms of *points in twistor space* rather than world sheet fields,

$$t'^{\ I}_{\ J} \equiv -(-1)^{IJ+I}\ Z'_J \frac{\partial}{\partial Z'_I} + \frac{1}{8}\delta^I_J(-1)^{I+E} Z'_E \frac{\partial}{\partial Z'_E} + \frac{1}{8}\delta^I_J Z'_E \frac{\partial}{\partial Z'_E}. \tag{3.24}$$

So if we consider the superspace tree amplitude, that is the tree constructed with vertices (3.22), then the psl(4|4) charges act as

$$\begin{aligned}
&\int \langle 0|e^{dq_0} Q_{0\ J}^{I} W_1(z_1)W_2(z_2)\ldots W_n(z_n)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S \\
&= \int \langle 0|e^{dq_0} W_1(z_1)\ldots \sum_{r=1}^{n}[Q_{0\ J}^{I}, W_r(z_r)]\ldots\ldots W_n(z_n)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S \\
&= \sum_{r=1}^{n} t'^I_{rJ} \int \langle 0|e^{dq_0} W_1(z_1)W_2(z_2)\ldots W_n(z_n)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S, \\
&= 0,
\end{aligned} \tag{3.25}$$

which is zero because the charge annihilates the vacuum as in (3.18). A similar formula for the gauge field theory appears in [47]. Indeed the transformation on generators t'^I_{rJ},

$$-(-1)^{IJ+I} Z'_{rJ} \frac{\partial}{\partial Z'_{rI}} \to Z'^I_r \frac{\partial}{\partial Z'^J_r} \tag{3.26}$$

(with trace and supertrace removed) leaves the psl(4|4) commutation relations invariant, and is the transformation of a matrix M^I_J, $M \to -(M)^{\text{supertranspose}}$ which is an outer automorphism of the superalgebra psl(4|4).

Note that the tree amplitude $\mathcal{M}_n(\pi^a,\omega^{\dot{a}},\theta^M)$ is the Fourier transform of $M_n(\pi^a,\overline{\pi}_{\dot{a}},\eta_M)$,

$$\mathcal{M}_n(\pi^a,\omega^{\dot{a}},\theta^M) \equiv \int \langle 0|e^{dq_0}W^{A_1}(z_1)W^{A_2}(z_2)\ldots W^{A_n}(z_n)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S$$

$$= \int \prod_{r=1}^{n}\prod_{\dot{a}=1}^{2}\prod_{M=1}^{4} d\eta_{rM}d\overline{\pi}_{r\dot{a}}e^{-i\omega_r^{\dot{a}}\overline{\pi}_{r\dot{a}}}e^{\theta_r^M\eta_{rM}}M_n(\pi^a,\overline{\pi}_{\dot{a}},\eta_M), \tag{3.27}$$

where $M_n(\pi^a,\overline{\pi}_{\dot{a}},\eta_M)$ is of Grassmann degree $4(d+1)$ and is expressed in terms of some of the conjugate twistor space points $W'_I = (\bar{\omega}_a,\overline{\pi}_{\dot{a}},\eta_M)$, e.g., for MHV trees $M_n^{\text{MHV}}(\pi^a,\overline{\pi}_{\dot{a}},\eta_M) = \delta^4(\sum_{r=1}^n \pi_r^a\overline{\pi}_{r\dot{a}})\delta^8(\sum_{r=1}^n \pi_r^a\eta_{rM})\frac{f^{A_1\ldots A_n}}{\langle 12\rangle\ldots\langle n1\rangle}$.

4 Non-local Noether currents and charges

In this section we compute non-local currents whose associated charges will also annihilate the scattering amplitudes. We show how these charges have the same comultiplication rules as the level one generators of the Yangian superalgebra of psl(4|4). In order to realize the coproduct of the Yangian, which is a Hopf algebra, it will be necessary to define currents in terms of a field that is not local with respect to the conformal fields [50], [51]. We introduce the non-local field ${\chi^I}_J(z)$,

$${\chi^I}_J(z) \equiv \int_P^z dw J^I_{0\,J}(w), \tag{4.1}$$

which not only is defined as an integral over a local field, but also is not local with respect to local fields, *e.g.*

$$(-1)^{R(I+J)}\ Z_R(\zeta)\ {\chi^I}_J(z) \cong {\chi^I}_J(z)\ Z_R(\zeta)\ +\ 2\pi i\,[Q^I_{0\,J}, Z_R(\zeta)\}, \tag{4.2}$$

where the left hand side of (4.2) is defined for $|\zeta| > |z|, |\zeta| > |P|$, the right hand side is defined for $|z| > |\zeta|, |P| > |\zeta|$ for an arbitrary point P, and the equality $\cong$ is meant in the sense of analytic continuation, see Figure 2.

The operator product of ${\chi^I}_J(z)$ with $Z_R(\zeta)$ is

$${\chi^I}_J(z)\ Z_R(\zeta) \sim \int_P^z dw(w-\zeta)^{-1}\, {t^I}_J[\zeta]\, Z_R(\zeta) = \ln\frac{(z-\zeta)}{(P-\zeta)}\ \ {t^I}_J[\zeta] Z_R(\zeta). \tag{4.3}$$

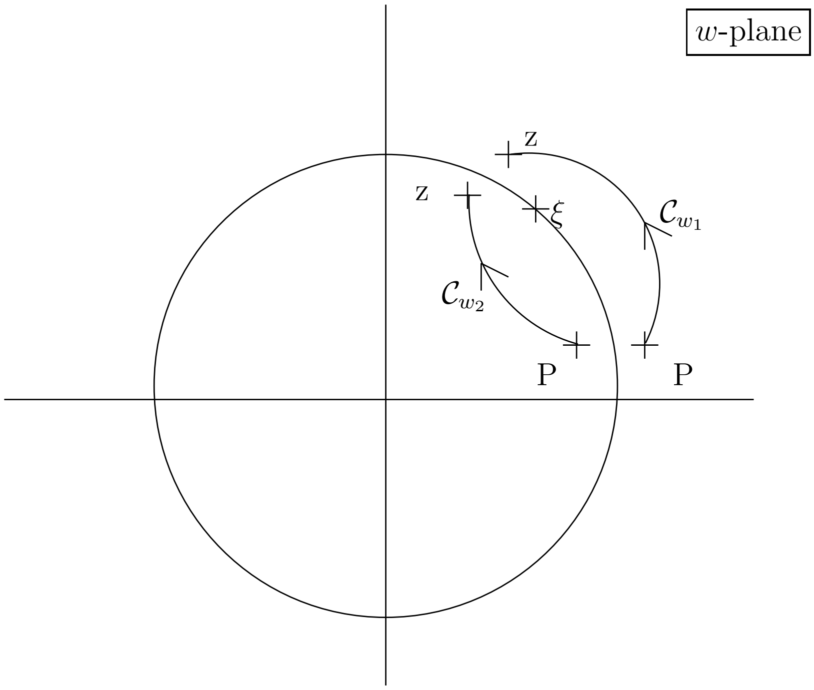

Figure 2: *The integral that defines* $\chi^I{}_J$ *is along* $\mathcal{C}_{w_1}$ *for the right-hand side of* (4.2) *and along* $\mathcal{C}_{w_2}$ *for the left-hand side.*

We define the non-local currents $J_{1\,N}^M(z)$ by

$$J_{1\,N}^{M}(z) = f^{M\ \ L\ \ J}_{\ \ NK\ \ I} \int_P^z dw J^I_{0\,J}(w)\ J^K_{0\,L}(z) = f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ \chi^I{}_J(z)\ J^K_{0\,L}(z). \tag{4.4}$$

It is not necessary to normal order because the singular terms vanish due to the properties of the structure constants. The $J^M_{1N}(z)$ are not local with respect to the fields,

$$(-1)^{R(M+N)}\ Z_R(\zeta)\ J^M_{1\,N}(z) \cong J^M_{1\,N}(z)\ Z_R(\zeta) + \\ + f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ (-1)^{R(K+L)}\ 2\pi i\, [Q^I_{0\,J}, Z_R(\zeta)\}\ J^K_{0\,L}(z). \tag{4.5}$$

The operator product with fields $Z_R(\zeta)$ is

$$J^{M}_{1\,N}(z)\; Z_R(\zeta) \sim (z-\zeta)^{-1}\; f^{M\;\;L\;\;J}_{\;\;NK\;\;I}\;\; :\chi^I{}_J(\zeta)\; t^{\;K}_{\;L}[\zeta]\, Z_R(\zeta):$$

$$+\ln\frac{(z-\zeta)}{(P-\zeta)}\;\; f^{M\;\;L\;\;J}_{\;\;NK\;\;I}\;(-1)^{(I+J)(K+L)}\;\; :J^{K}_{0\,L}(z)\;\; t^{\;I}_{\;J}[\zeta]\, Z_R(\zeta):. \quad (4.6)$$

The action of the non-local charge on the field $Z_R(\zeta)$, is

$$Q^{M}_{1\,N}\,(Z_R(\zeta)) = \oint_{\mathcal{C}_\zeta} dz J^{M}_{1\,N}(z)\; Z_R(\zeta) = 2\; f^{M\;\;L\;\;J}_{\;\;NK\;\;I}\;\; :\chi^I{}_J(\zeta)\; t^{\;K}_{\;L}[\zeta]\, Z_R(\zeta):, \quad (4.7)$$

where the cut for the logarithim extends from ζ passing through the point P, and the contour $\mathcal{C}_\zeta$ starts just above P, circles around ζ and stops just below P. See Figure 3.

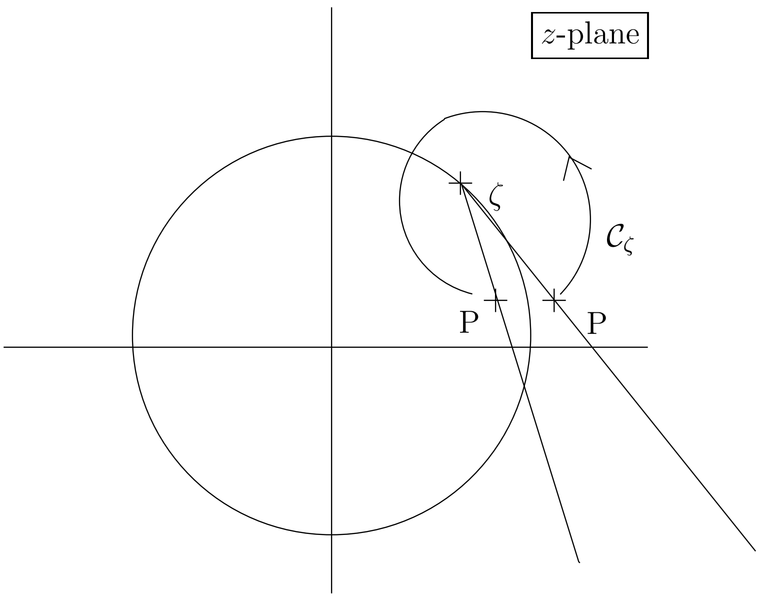

Figure 3: *The contour $\mathcal{C}_\zeta$ starts just above P, circles around ζ and stops just below P.*

To derive (4.7) we have used $\oint_{\mathcal{C}_\zeta} dz f(z) \ln\frac{(z-\zeta)}{(P-\zeta)} \;\; = -\int_P^\zeta dw f(w)$, for any $f(w)$ analytic on and inside the contour $\mathcal{C}_\zeta$ [50].

The expression (4.7) is the infinitesimal transformation on the field $Z_R(\zeta)$ generated by the non-local charge [51]. It is not merely $[Q^{M}_{1\,N}, Z_R(\zeta)\}$ due to the lack of locality in

(4.2),

$$Q^M_{1\,N}\,(Z_R(\zeta)) = [Q^M_{1\,N}, Z_R(\zeta)\} + f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ (-1)^{R(K+L)}\ 2\pi i\,[\widetilde{Q}^I_{0\,J}, Z_R(\zeta)\}\ Q^K_{0\,L}. \qquad (4.8)$$

This mismatch of the commutator and the infinitesimal field tranformation follows from

$$Q^M_{1\,N}\ Z_R(\zeta) = \oint_{\mathcal{C}_1:|z|>|\zeta|,|P|>|\zeta|} dz\ J^M_{1\,N}(z)\ Z_R(\zeta),$$

$$(-1)^{R(M+N)}\ Z_R(\zeta)\ Q^M_{1\,N} = \oint_{\mathcal{C}_2:|\zeta|>|z|,|\zeta|>|P|} dz\ (-1)^{R(M+N}\ Z_R(\zeta)\ J^M_{1\,N}(z)$$

$$= \oint_{\mathcal{C}_2:|\zeta|>|z|,|\zeta|>|P|} dz\ J^M_{1\,N}(z) Z_R(\zeta) + f^{M\ \ L\ \ J}_{\ \ NK\ \ I}(-1)^{R(K+L)}\ 2\pi i\,[Q^I_{0\,J}, Z_R(\zeta)\}\ Q^K_{0\,L}, \qquad (4.9)$$

so that

$$[Q^M_{1\,N}, Z_R(\zeta)\} = \oint_{\mathcal{C}_\zeta\equiv\mathcal{C}_1-\mathcal{C}_2} dz J^M_{1\,N}(z)\ Z_R(\zeta)$$

$$-\ \ f^{M\ \ L\ \ J}_{\ \ NK\ \ I}(-1)^{R(K+L)}\ 2\pi i\,[Q^I_{0\,J}, Z_R(\zeta)\}\ Q^K_{0\,L}, \qquad (4.10)$$

which is (4.8).

Therefore the operator product of the non-local current with any field $V(Z(\zeta))$ that is a function of $Z_R(\zeta)$, such as the gluon vertex operator, is

$$J^M_{1\,N}(z)\ V(Z(\zeta)) \sim (z-\zeta)^{-1}\ f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ \ :\chi^I{}_J(\zeta)\ t^{\ K}_{\ L}[\zeta]\, V(Z(\zeta)):$$

$$+ \ln\frac{(z-\zeta)}{(P-\zeta)}\ \ f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ (-1)^{(I+J)(K+L)}\ \ :J^K_{0\,L}(z)\ \ t^{\ I}_{\ J}[Z(\zeta)] V(Z(\zeta)):. \qquad (4.11)$$

The action of the first level Yangian charge on the field $V(Z(\zeta))$, is

$$Q^M_{1\,N}\,(V(Z(\zeta))) = \oint_{\mathcal{C}_\zeta} dz J^M_{1\,N}(z)\ V(Z(\zeta))$$

$$= 2\ \widetilde{f}^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ \ :\chi^I{}_J(\zeta)\ t^{\ K}_{\ L}[\zeta]\, V(Z(\zeta)): \qquad (4.12)$$

For simplicity, we consider a bosonic vertex operator $V(Z(\zeta))$, where

$$V(Z(\zeta))\,\chi^{I}{}_{J}(z) \cong \chi^{I}{}_{J}(z)\,V(Z(\zeta)) \;+\; 2\pi i\,[Q^{I}_{0\,J}, V(Z(\zeta))], \tag{4.13}$$

$$V(Z(\zeta))\,J^{M}_{1\,N}(z) \cong J^{M}_{1\,N}(z)\,V(Z(\zeta)) + f^{M}{}_{NK}{}^{L}{}_{I}{}^{J}\,2\pi i\,[Q^{I}_{0\,J}, V((\zeta))]\,J^{K}_{0\,L}(z), \tag{4.14}$$

$$Q^{M}_{1\,N}\,(V(Z(\zeta))) = [Q^{M}_{1\,N}, V(Z(\zeta))] + f^{M}{}_{NK}{}^{L}{}_{I}{}^{J}\,2\pi i\,[\widetilde{Q}^{I}_{0\,J}, V(Z(\zeta))]\,Q^{K}_{0\,L}. \tag{4.15}$$

Since the infinitesimal transformation (4.7) is not a simple (anti)commutator (4.8), the action of the non-local charge on a product of fields will not be as simple as for the ordinary generators (3.8), and leads to the Hopf superalgebra coproduct.

Comultiplication

The action of the non-local charge on two fields is

$$Q^{M}_{1\,N}\Big(V_1(Z(\zeta_1))\,V_2(Z(\zeta_2))\Big) = \oint_{\mathcal{C}_{\zeta_1,\zeta_2}} dz\,J^{M}_{1\,N}(z)\,V_1(Z(\zeta_1))\,V_2(Z(\zeta_2)), \tag{4.16}$$

where the contour $\mathcal{C}_{\zeta_1,\zeta_2}$ starts at P above both cuts (cut from ζ_1 to P, and cut from ζ_2 to P), encircles both ζ_1, ζ_2 and stops below both cuts at P, see Figure 4. For simplicity,

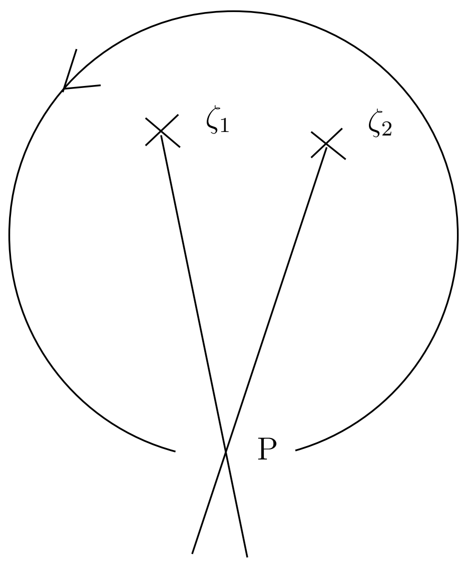

Figure 4: *The contour $\mathcal{C}_{\zeta_1,\zeta_2}$ starts at P above both cuts (cut from ζ_1 to P, and cut from ζ_2 to P), encircles both ζ_1, ζ_2 and stops below both cuts at P.*

we take both fields V_1 and V_2 to be bosonic. We have for $|z|, |P| > |\zeta_1|, |\zeta_2|$

$$J_{1\,N}^{M}(z)\, V_1(Z(\zeta_1))\, V_2(Z(\zeta_2)) = f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ \chi^{I}_{0\,J}(z)\ J^{K}_{0\,L}(z)\ V_1(Z(\zeta_1))\, V_2(Z(\zeta_2))\,, \tag{4.17}$$

and for $|\zeta_1|, |\zeta_2| > |z|, |P|$

$$\begin{aligned} V_1(Z(\zeta_1))\, V_2(Z(\zeta_2))\ J_{1\,N}^{M}(z) = {}& f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ \chi^{I}_{0\,J}(z)\ J^{K}_{0\,L}(z)\ V_1(Z(\zeta_1))\, V_2(Z(\zeta_2)), \\ & - f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ 2\pi i\, [Q^{I}_{0\,J}, V(Z(\zeta_1))]\ V(Z(\zeta_2))\ J^{K}_{0\,L}(z) \\ & - f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ V_1(Z(\zeta_1))\ 2\pi i\, [Q^{I}_{0\,J}, V_2(Z(\zeta_2))]\ J^{K}_{0\,L}(z), \end{aligned} \tag{4.18}$$

using (4.13) twice. Then

$$\begin{aligned} [Q_{1\,N}^{M},\, V_1(Z(\zeta_1))\, V_2(Z(\zeta_2))] = {}& \oint_{\mathcal{C}_{\zeta_1,\zeta_2}} dz\ J_{1\,N}^{M}(z)\, V_1(Z(\zeta_1))\, V_2(Z(\zeta_2)) \\ & - f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ 2\pi i\, [Q^{I}_{0\,J}, V_1(Z(\zeta_1))]\ V_2(Z(\zeta_2))\ Q^{K}_{0\,L} \\ & - f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ V_1(Z(\zeta_1))\ 2\pi i\, [Q^{I}_{0\,J}, V_2(Z(\zeta_2))]\ Q^{K}_{0\,L}, \end{aligned} \tag{4.19}$$

So from (4.16),

$$\begin{aligned} Q_{1\,N}^{M}\Big(V_1(Z(\zeta_1))\, V_2(Z(\zeta_2))\Big) = {}& [Q_{1\,N}^{M},\, V_1(Z(\zeta_1))\, V_2(Z(\zeta_2))] \\ & + f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ 2\pi i\, [Q^{I}_{0\,J}, V_1(Z(\zeta_1))]\ Q^{K}_{0\,L}\ V_2(Z(\zeta_2)) \\ & - f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ 2\pi i\, [Q^{I}_{0\,J}, V_1(Z(\zeta_1))]\ [Q^{K}_{0\,L}, V_2(Z(\zeta_2))] \\ & + f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ V_1(Z(\zeta_1))\ 2\pi i\, [Q^{I}_{0\,J}, V_2(Z(\zeta_2))]\ Q^{K}_{0\,L} \\ = {}& Q_{1\,N}^{M}\Big(V_1(Z(\zeta_1))\Big)\ V_2(Z(\zeta_2)) + V_1(Z(\zeta_1))\ Q_{1\,N}^{M}\Big(V_2(Z(\zeta_2))\Big) \\ & - f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ 2\pi i\ Q^{I}_{0\,J}\Big(V_1(Z(\zeta_1))\Big)\ Q^{K}_{0\,L}\Big(V_2(Z(\zeta_2))\Big), \end{aligned} \tag{4.20}$$

which can be written as the Hopf algebra comultiplication,

$$\Delta Q_{1\,N}^{M} = Q_{1\,N}^{M} \otimes 1 + (-1)^{(M+N)F} \otimes Q_{1\,N}^{M}\ - 2\pi i\ f^{M\ \ L\ \ J}_{\ \ NK\ \ I}\ Q^{I}_{0\,J} \otimes Q^{K}_{0\,L}. \tag{4.21}$$

We can reexpress the action of the non-local charge on the product of two fields as

$$Q^{M}_{1\,N}\Big(V(Z(\zeta_1))\,V(Z(\zeta_2))\Big) = [Q^{M}_{1\,N}, V(Z(\zeta_1))\,V(Z(\zeta_2))]$$

$$+\,2\pi i f^{M\ \ L\ \ J}_{\ \ NK\ \ I}[Q^{I}_{0\,J}, V(Z(\zeta_1))\,V(Z(\zeta_2))]\;Q^{K}_{0\,L}, \quad (4.22)$$

and will use this form to demonstrate how these non-local charges annihilate the tree amplitudes.

5 More symmetry in the open twistor string

We consider the action of the non-local charges on the tree amplitudes. We show that the non-local charges annihilate the super gluon amplitudes of the open twistor string, and hence give rise to interesting Ward identities which we compare with $\mathcal{N} = 4$ Yang Mills field theory.

The non-local charges formally annihilate the vacuum using (3.11),

$$Q^{M}_{1\,N}|0\rangle, \qquad \langle 0|(Q^{M}_{1\,N})^{\dagger} = \pm\langle 0|Q^{M}_{1\,N}, \qquad [Q^{M}_{1N}, e^{dq_0}] = 0. \qquad (5.1)$$

In analogy with the ordinary $PSL(4|4)$ case, we consider the action of the non-local charge on the tree amplitude using (4.22),

$$\int \langle 0|e^{dq_0} Q^{M}_{1\,N}\Big(W_1(z_1)W_2(z_2)\ldots W_n(z_n)\Big)|0\rangle \prod_{r=1}^{n} dz_r\Big/ d\gamma_M d\gamma_S$$

$$= \int \langle 0|e^{dq_0}[Q^{I}_{M\ N}, W_1(z_1)W_2(z_2)\ldots W_n(z_n)]|0\rangle \prod_{r=1}^{n} dz_r\Big/ d\gamma_M d\gamma_S$$

$$+ \int \langle 0|e^{dq_0} f^{M\ L\ \ \ J}_{\ N\ \ KI}\, 2\pi i\, [Q^{I}_{0\ J}, W_1(z_1)W_2(z_2)\ldots W_n(z_n)]\; Q^{K}_{0\ L}|0\rangle \prod_{r=1}^{n} dz_r\Big/ d\gamma_M d\gamma_S$$

$$= 0, \qquad (5.2)$$

where we have suppressed the group index. To display the action of the non-local charge on the amplitudes, we compute (5.2) directly, defining

$$W(z_1,\ldots,z_n) \;\equiv\; \langle 0|e^{dq_0}\Big(W_1(Z(z_1))\ldots W_n(Z(z_n))\Big)|0\rangle. \tag{5.3}$$

We use (4.20) to express (5.2),

$$\begin{aligned}
&\int \langle 0|e^{dq_0} Q_{1\ N}^{M}\Big(W_1(z_1)W_2(z_2)\ldots W_n(z_n)\Big)|0\rangle \prod_{r=1}^{n} dz_r\Big/ d\gamma_M d\gamma_S \\
&= \int \langle 0|e^{dq_0} \sum_{j=1}^{n} W_1(z_1)\ldots Q_{1\ J}^{I}\Big(W_j(z_j)\Big)\ \ldots W_n(z_n)|0\rangle \prod_{r=1}^{n} dz_r\Big/ d\gamma_M d\gamma_S \\
&\quad - \int \langle 0|e^{dq_0}\ 2\pi i\, f_{NK\ I}^{M\ L\ J} \sum_{1\le i<j\le n} W_1(z_1)\ldots W_{i-1}(z_{i-1})\ Q_{0\ J}^{I}\Big(W_i(z_i)\Big)\, W_{i+1}(z_{i+1})\ldots \\
&\qquad\qquad \times W_{j-1}(z_{j-1})\, Q_{0\,L}^{K}\Big(W_j(z_j)\Big) W_{j+1}(z_{j+1}\ \ldots W_n(z_n)|0\rangle \prod_{r=1}^{n} dz_r\Big/ d\gamma_M d\gamma_S \\
&= -2 f_{NK\ I}^{M\ L\ J} \sum_{1\le i<j\le n} t'^{I}_{i\ J}\, t'^{K}_{j\ L} \int \left(\ln\frac{(z_i-z_j)}{(P-z_j)} - \ln\frac{(z_j-z_i)}{(P-z_i)}\right) W(z_1,\ldots,z_n) \prod_{r=1}^{n} dz_r\Big/ d\gamma_M d\gamma_S \\
&\quad + (2\pi i)\, f_{NK\ I}^{M\ L\ J} \sum_{1\le i<j\le n} t'^{I}_{i\ J}\, t'^{K}_{j\ L} \int W(z_1,\ldots,z_n) \prod_{r=1}^{n} dz_r\Big/ d\gamma_M d\gamma_S
\end{aligned} \tag{5.4}$$

where we have replaced the $t^{I}_{iJ}[z_i]$ which appear in evaluating the operator products, by t'^{I}_{iJ} defined in (3.24), since the vertex operators $W(z)$ are delta functions given in (3.22). To show the independence of P, we write (5.4) as

$$\begin{aligned}
&-2 f_{NK\ I}^{M\ L\ J} \sum_{1\le i<j\le n} t'^{I}_{i\ J}\, t'^{K}_{j\ L} \int \left(\ln\frac{(z_i-z_j)}{(z_j-z_i)} - \ln\frac{(P-z_j)}{(P-z_i)}\right) W(z_1,\ldots,z_n) \prod_{r=1}^{n} dz_r\Big/ d\gamma_M d\gamma_S \\
&+ 2\pi i\, f_{NK\ I}^{M\ L\ J} \sum_{1\le i<j\le n} t'^{I}_{i\ J}\, t'^{K}_{j\ L} \int W(z_1,\ldots,z_n) \prod_{r=1}^{n} dz_r\Big/ d\gamma_M d\gamma_S,
\end{aligned} \tag{5.5}$$

where the P-dependent part is evaluated as

$$
\begin{aligned}
& f^{M\ L\ J}_{NK\ I} \sum_{1\leq i<j\leq n} t'^{I}_{i\ J}\, t'^{K}_{j\ L}\ \ln\frac{(P-z_j)}{(P-z_i)}\, W(z_1,\ldots,z_n) \\
&= -\ln(P-z_1)\ f^{M\ L\ J}_{NK\ I}\ \ t'^{I}_{1\ J} \sum_{j=2}^{n} t'^{K}_{j\ L}\, W(z_1,\ldots,z_n) \\
&\quad + \sum_{\ell=2}^{n-1} \ln(P-z_\ell)\ f^{M\ L\ J}_{NK\ I}\Big(\sum_{i=1}^{\ell-1} t'^{I}_{i\ J}\, t'^{K}_{\ell\ L} - t'^{I}_{\ell\ J} \sum_{j=\ell+1}^{n} t'^{K}_{j\ L}\Big)\ W(z_1,\ldots,z_n) \\
&\quad + \ln(P-z_n)\ f^{M\ L\ J}_{NK\ I} \sum_{i=1}^{n-1} t'^{I}_{i\ J}\, t'^{K}_{n\ L}\, W(z_1,\ldots,z_n).
\end{aligned}
\tag{5.6}
$$

We use that the ordinary symmetry annihilates the integrand of the amplitude (3.25), and

$$
f^{M\ L\ J}_{NK\ I}\left(t'^{I}_{i\ J}\, t'^{K}_{j\ L} + t'^{I}_{j\ J}\, t'^{K}_{i\ L}\right) = 0 \quad \text{for } i \neq j, \tag{5.7}
$$

and for each i,

$$
f^{M\ L\ J}_{NK\ I}\left(t'^{I}_{i\ J}\, t'^{K}_{i\ L}\right) = \frac{1}{2} f^{M\ L\ J}_{NK\ I}\ f^{I\ K\ \ \ S}_{J\ \ LR}\ t'^{R}_{i\ S} = 0, \tag{5.8}
$$

since the Killing form (B.2) vanishes. Then (5.6) becomes

$$
\begin{aligned}
&= \ln(P-z_1)\ f^{M\ L\ J}_{NK\ I}\ \ t'^{I}_{1\ J} t'^{K}_{1\ L}\, W(z_1,\ldots,z_n) \\
&\quad + \sum_{\ell=2}^{n-1} \ln(P-z_\ell)\ f^{M\ L\ J}_{NK\ I}\Big(\sum_{i=1}^{\ell-1} t'^{I}_{i\ J}\, t'^{K}_{\ell\ L} + t'^{I}_{\ell\ J}\, t'^{K}_{\ell\ L} + t'^{I}_{\ell\ J} \sum_{j=1}^{\ell-1} t'^{K}_{j\ L}\Big)\ W(z_1,\ldots,z_n) \\
&\quad + \ln(P-z_n)\ f^{M\ L\ J}_{NK\ I} \sum_{i=1}^{n-1} t'^{I}_{i\ J}\, t'^{K}_{n\ L}\, W(z_1,\ldots,z_n) \\
&= 0,
\end{aligned}
\tag{5.9}
$$

Therefore the P dependent part of (5.5) vanishes, and using $\ln\frac{(z_i-z_j)}{(z_j-z_i)} = \ln(-1) = \pi i + 2\pi i p$

for any integer p, we find (5.4) leads to

$$\int \langle 0|e^{dq_0} Q^I_{1\ J}\Big(W_1(z_1)\dots W_n(z_n)\Big)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S$$

$$= -4\pi i p\ f^{M\ L\ J}_{NK\ I} \sum_{1\le i<j\le n} t'^I_{i\ J}\, t'^K_{j\ L} \int \langle 0|e^{dq_0}\ W_1(Z(z_1))\dots\ W_n(Z(z_n)\,|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S$$

$$= 0, \tag{5.10}$$

for arbitrary integer p, so along with the ordinary symmetries (3.25), we have

$$f^{M\ L\ J}_{NK\ I} \sum_{1\le i<j\le n} t'^I_{i\ J}\, t'^K_{j\ L} \int \langle 0|e^{dq_0}\ W_1(Z(z_1))\dots W_n(Z(z_n)\,|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S = 0, \tag{5.11}$$

which reflects the non-local symmetry of the tree amplitudes of the twistor string. We note that up to central terms that separately annihilate the amplitudes, a similar formula appears in [47]-[49] as the first level Yangian generators acting on gauge theory amplitudes.

6 PSL(4|4) Yangian superalgebra symmetry

In this section, we show how the open twistor string carries a realization of the Yangian of psl(4|4). This realization will be independent of P, and is inspired from the observation that the Ward identities for the non-local charges are independent of P, (5.11).

Let us start by computing the OPE of our non-local currents (4.4) with the Virasoro field (2.10),

$$L_{YZ}(z) J^M_{1\ N}(\zeta) = \frac{J^M_{1\ N}(\zeta)}{(z-\zeta)^2} + \frac{\partial J^M_{1\ N}(\zeta)}{(z-\zeta)} + f^{M\ L\ J}_{NK\ I} \frac{J^I_{0\ J}(P)\ J^K_{0\ L}(w)}{(z-P)}\,. \tag{6.1}$$

Hence, the non-local current is a Virasoro primary only in the limit $P \to \infty$. Note that taking this limit does not commute with the integrals we take. But we take it as a hint for a P-indendent representation of the Yangian.

We keep the action of the level zero charges as before and choose the level one generators to act trivially on a single field

$$Q_{1\,N}^{M}\big(V(z)\big) \;=\; 0\,, \tag{6.2}$$

while the action of $Q_{1\,N}^{M}$ on products of fields is defined by the Yangian comultiplication (4.21), so that on a product of two fields,

$$Q_{1\,N}^{M}\Big(V(z_1)V(z_2)\Big) = -2\pi i\; f_{NK\;\;I}^{M\;\;L\;\;J}\; Q_{0\,J}^{I}\Big(V(z_1)\Big)\otimes Q_{0\,L}^{K}\Big(V(z_2)\Big). \tag{6.3}$$

The crucial observation is that this representation of the Yangian also provides a symmetry of the super gluon amplitudes of the open twistor string, as can be seen by (5.11).

We prove that the tree level representation in terms of twistor string fields for the PSL(4|4) charges (2.15), (2.16), (3.8), and the level one charges, (6.2), (6.3), (4.21), is consistent with the Serre relation, because it satisfies a useful criterion [24] as we now show.

Serre relation and the useful criterion

The Serre relation for the Yangian of $psl(4|4)$ can be written as

$$f_{N\;\;SK}^{M\;R\;\;\;L}[Q_{1\,J}^{I},\,Q_{1\,L}^{K}\} + (-1)^{(I+J)(M+N+R+S)}\, f_{S\;\;JK}^{R\;I\;\;\;L}[Q_{1\,N}^{M},\,Q_{1\,L}^{K}\}+$$

$$+\,(-1)^{(R+S)(I+J+M+N)}\, f_{J\;\;NK}^{I\;M\;\;\;L}[Q_{1\,S}^{R},\,Q_{1\,L}^{K}\} \;=$$

$$= h(-1)^{(C+D)(K+L)+(G+H)(U+V)+G+H}\;\times$$

$$\times\; f_{\;JC\;\;P}^{I\;\;D\;\;Q}\; f_{\;NE\;\;K}^{M\;\;F\;\;L}\; f_{S\;\;HU}^{R\;G\;\;\;V}\; f_{D\;\;FG}^{C\;E\;\;\;H}\; \Big\{Q_{0\,Q}^{P}\; Q_{0\,L}^{K}\; Q_{0\,V}^{U}\Big]\,, \tag{6.4}$$

where $\big\{\ldots\big]$ is the graded totally symmetrized product and h is a constant that depends on the normalization.

It is sufficient to check the Serre relation acting on one field, since then the coproduct (4.21) assures the relation for all higher sites. Clearly the left-hand side of (6.4) vanishes for one field from (6.2).

We show in appendix D that the right-hand side of the Serre relation acting on one field also vanishes, using the fact that our representation for the $gl(4|4)$ charge $\widetilde{Q}_{0\,J}^{I}$ on a single field,

$$[\widetilde{Q}^I_{0\,J}, V(Z(\zeta))] = [\oint dz : Y^I(z) Z_J(z) :, V(Z(\zeta))]$$

$$= \Big(-(-1)^{IJ+I} \Big(Z_J(\zeta) \frac{\partial}{\partial Z_I(\zeta)} \Big) + Y^I(\zeta) \frac{\partial}{\partial Y^J(\zeta)} \Big) \, V(Z(\zeta)) \equiv \widetilde{t}^I{}_J[\zeta] \, V(Z(\zeta)), \tag{6.5}$$

satisfies a useful criterion [24]:

$$(-1)^E \widetilde{Q}^M_{0\,E} \, \widetilde{Q}^E_{0\,J} \big(V(Z(\zeta)) \big) = \widetilde{Q}^M_{0\,J} \big(V(Z(\zeta)) \big). \tag{6.6}$$

Here we consider $V(Z(z)$ as a homogeneous function $Z_E(z) \frac{\partial}{\partial Z_E(z)} V(Z(z)) = 0$ which describes conformal fields $V(z)$ such as the vertex operators $W(z)$. This will simplify our discussion, although is not necessary since the central terms we drop would cancel in what follows. To derive (6.6),

$$(-1)^E \widetilde{Q}^M_{0\,E} \, \widetilde{Q}^E_{0\,J} \big(V(Z(\zeta)) \big) = (-1)^E \widetilde{t}^M_{0\,E} \, \widetilde{t}^E_{0\,J} \, V(Z(\zeta)$$

$$= (-1)^{ME+M+EJ} Z_E(\zeta) \frac{\partial}{\partial Z_M(\zeta)} \Big(Z_J(\zeta) \frac{\partial}{\partial Z_E(\zeta)} \, V(Z(\zeta)) \Big)$$

$$= (-1)^{M+EJ+MJ} Z_E(\zeta) Z_J(\zeta) \, \frac{\partial}{\partial Z_E(\zeta)} \frac{\partial}{\partial Z_M(\zeta)} \, V(Z(\zeta) = -(-1)^{M+MJ} Z_J(\zeta) \, \frac{\partial}{\partial Z_M(\zeta)} \, V(Z(\zeta)$$

$$= \widetilde{Q}^M_{0\,J} \big(V(Z(\zeta)) \big). \tag{6.7}$$

From (6.2) and (4.21) we also have that

$$[Q^I_{0\,J}, Q^K_{1\,L}\} = f^{I\,K}_{J\,\,L\,R}{}^{S} \, Q^R_{1\,S}, \tag{6.8}$$

acting on gauge invariant states. Therefore with the proof of (6.4) in appendix E, the Yangian symmetry algebra now follows from the defining relations (2.20), (6.8) and (6.4).

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A Some super analysis

We provide some properties of Lie superalgebras and Lie supergroups we use in our analysis. A good reference for Lie supergroups is [60], in this appendix we use the notation of [54].

A.1 *The Lie superalgebra psl(N|N)*

The Lie superalgebra sl(N|N) can be represented by matrices $(T)_{IJ} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, with Str (T) $\equiv$ TrM_1 - Tr $M_4 = 0$. This superalgebra possesses a one-dimensional ideal generated by the identity matrix. The quotient of sl(N|N) by this ideal is the Lie superalgebra psl(N|N). It is convenient to work with the Lie superalgebra sl(N|N), we simply have to remember that we divided out the central element.

We define the psl(4|4) structure constants by

$$[T^a, T^b\} = f^{ab}_{\ \ c} T^c = f^{abc} g_{cd} T^d \,, \tag{A.1}$$

where the brackets denote either commutators or anticommutators. There is an invariant, nondegenerate metric g_{ab} that is used to raise and lower indices, $g^{ab} = \frac{1}{2}\mathrm{Str}\, T^a T^b$ and $g_{ab} = \frac{1}{2}\mathrm{Str}\, T^b T^a$. The structure constants f_{abc} are totally antisymmetric with an additional minus sign under the interchange of two odd nearest neighbor indices.

It can be useful to rewrite the single index superalgebra generators J^a with a double index as $(E_{AB})_{IJ} = \delta_{AI}\delta_{BJ}$, where

$$[E_{AB}, E_{CD}] = \delta_{CB} E_{AD} - \delta_{AD} E_{CB},$$

$$[E_{AB}, E_{CD}\} = \delta_{CB} E_{AD} - (-1)^{(\deg E_{AB})\,(\deg E_{CD})} \delta_{AD} E_{CB},$$

where deg E_{AB} is 0 for bosonic generators, and 1 for fermionic generators.

A.2 *Lie supergroups*

The Lie supergroup is a Lie group over a Grassmann ring and it is obtained by exponentiating the Grassmannn envelope of the Lie superalgebra. An element of the Grassmann envelope has the following form

$$\rho \;=\; \rho_a T^a \tag{A.2}$$

where the ρ_a are Grassmann-even when T^a is bosonic and Grassmann-odd when T^a is fermionic. Hermitian conjugation is complex conjugation of the transpose in the bosonic case. For matrices generating a Lie superalgebra, it is super complex conjugation of the supertranspose,

$$\rho^{\ddagger} = \bar{\rho}_a(\overline{T}^a)^{st},$$

where the bar is ordinary complex conjugation for bosons, and satisfies[1]

$$\overline{c\theta} = \bar{c}\bar{\theta}, \qquad \bar{\bar{\theta}} = -\theta, \qquad \overline{\theta_1\theta_2} = \bar{\theta}_1\bar{\theta}_2,$$

for any Grassmann elements θ, θ_i and any complex number c. The supertranspose of T is defined

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}^{st} = \begin{pmatrix} M^t & -M_3^t \\ M_2^t & M_4^t \end{pmatrix}$$

so that $(TS)^{st} = S^{st}T^{st}$.

B Structure constants and their properties

The psl(4|4) structure constants are given by

$$f^{I\;K\;\;\;N}_{\;J\;\;LM} = \delta^K_J\delta^N_L\delta^I_M - (-1)^{(I+J)(K+L)}\,\delta^N_J\delta^I_L\delta^K_M + \frac{1}{8}((-1)^L - (-1)^K)(-1)^I\delta^I_J\delta^N_L\delta^K_M$$
$$- \frac{1}{8}((-1)^J - (-1)^I)(-1)^K\,\delta^N_J\delta^K_L\delta^I_M \quad - \frac{1}{8}\Big(1 - (-1)^{I+J}\Big)\,\delta^I_L\,\delta^K_J\,\delta^N_M. \tag{B.1}$$

[1]Lifting complex conjugation to the ring of Grassmann numbers is not unique. Our choice ensures that the adjoint operation for Lie supergroups is involutive.

These have vanishing Killing form

$$f^{M\ L\ J}_{\ NK\ I}\ f^{I\ K\ \ S}_{\ J\ LR} = 0, \tag{B.2}$$

and are traceless and supertraceless in all indices,

$$\delta^J_I\ f^{I\ K\ \ N}_{\ J\ L\,M} = 0, \quad (-1)^J\delta^J_I\ f^{I\ K\ \ N}_{\ J\ L\,M} = 0, \quad f^{I\ K\ \ N}_{\ J\ L\,M}\ \delta^M_N = 0, \quad f^{I\ K\ \ N}_{\ J\ L\,M}\ (-1)^M\delta^M_N = 0, \tag{B.3}$$

as well as totally anti supersymmetric in any pair of indices, *for eg.*

$$f^{I\ K\ \ N}_{\ J\ L\,M} = -(-1)^{(I+J)(K+L)} f^{K\ I\ \ N}_{\ L\ J\,M}.$$

They satisfy the Jacobi identity

$$f^{I\ K\ \ S}_{\ J\ LR}\, f^{R\ W\ \ V}_{\ S\ XU} + f^{K\ W\ \ S}_{\ L\ XR}\, f^{R\ I\ \ V}_{\ S\ JU}\,(-1)^{(I+J)(R+S)} + f^{W\ I\ \ S}_{\ X\ JR}\, f^{R\ K\ \ V}_{\ S\ LU}\,(-1)^{(W+X)(I+J+K+L)} = 0. \tag{B.4}$$

The inverted structure constants are

$$\begin{aligned} f^{M\ L\ J}_{\ NK\ I} &= (-1)^K \left(\delta^L_N\delta^M_I\delta^J_K - (-1)^{(M+N)(M+I)}\delta^M_K\delta^L_I\delta^J_N\right) \\ &\quad + \frac{1}{8}(1-(-1)^{I+J})(-1)^M\delta^M_N\delta^L_I\delta^J_K + \frac{1}{8}((-1)^I-(-1)^J)\delta^J_N\delta^L_K\delta^M_I \\ &\quad + \frac{1}{8}\Big((-1)^N-(-1)^M\Big)\,\delta^M_K\delta^L_N\delta^J_I. \end{aligned} \tag{B.5}$$

We 'raise' and 'lower' the index pairs on the structure constants by

$$g^{A\ C}_{\ B\ D} = (-1)^A\delta^A_D\,\delta^C_B, \qquad (g^{-1})_{A\ C}^{\ B\ D} = (-1)^B\delta^D_A\,\delta^B_C, \tag{B.6}$$

$$\begin{aligned} f^{M\ L\ J}_{\ NK\ I} &= f^{M\ P\ \ J}_{\ N\ Q\,I}\ (g^{-1})_{P\ K}^{\ Q\ L} = f^{M\ P\ \ J}_{\ N\ Q\,I}\ (-1)^Q\delta^Q_K\,\delta^L_P, \\ f^{M\ P\ \ J}_{\ N\ Q\,I} &= f^{M\ L\ J}_{\ NK\ I}\ \ g^{K\ P}_{\ L\ Q} = f^{M\ L\ J}_{\ NK\ I}\ \ (-1)^K\delta^P_L\,\delta^K_Q. \end{aligned} \tag{B.7}$$

C Real form psl(4|4)

The Noether charge generators of the real form $psl(4|4)$ can be written as

$$
\begin{aligned}
p_a{}^{\dot{a}} &= Q^{\dot{a}}_{0\,a} = -\oint dz\, Y^{\dot{a}}(z) Z_a(z), \\
k^a{}_{\dot{a}} &= Q^a_{0\,\dot{a}} = -\oint dz\, Y^a(z) Z_{\dot{a}}(z), \\
m^a{}_b &= Q^a_{0\,b} = -\oint dz\, \frac{1}{2}(Y_b(z) Z^a(z) + Y^a(z) Z_b(z)) \\
&= -\oint dz\, (Y^a(z) Z_b(z) - \tfrac{1}{2}\delta^a_b Y^c(z) Z_c(z)) \\
\widetilde{m}^{\dot{a}}{}_{\dot{b}} &= Q^{\dot{a}}_{0\,\dot{b}} = -\oint dz\, \frac{1}{2}(Y_{\dot{b}}(z) Z^{\dot{a}}(z) + Y^{\dot{a}}(z) Z_{\dot{b}}(z)) \\
&= -\oint dz\, \left(Y^{\dot{a}}(z) Z_{\dot{b}}(z) - \tfrac{1}{2}\delta^{\dot{a}}_{\dot{b}} Y^{\dot{c}}(z) Z_{\dot{c}}(z)\right), \\
d &= Q^{(d)}_0 = -\oint dz\, \frac{1}{2}(Y^a(z) Z_a(z) - Y^{\dot{a}}(z) Z_{\dot{a}}(z)), \\
r^A{}_B &= Q^A_{0\,B} = -\oint dz\, (Y^A(z) Z_B(z) - \tfrac{1}{4}\delta^A_B Y^C(z) Z_C(z)), \\
q_a{}^A &= Q^A_{0\,a} = \oint dz\, Y^A(z) Z_a(z), \\
\widetilde{q}^{\dot{a}}{}_A &= Q^{\dot{a}}_{0\,A} = -\oint dz\, Y^{\dot{a}}(z) Z_A(z), \\
s^a{}_A &= Q^a_{0\,A} = -\oint dz\, Y^a(z) Z_A(z), \\
\widetilde{s}^A{}_{\dot{a}} &= Q^A_{0\,\dot{a}} = \oint dz\, Y^A(z) Z_{\dot{a}}(z),
\end{aligned}
\tag{C.1}
$$

where $I = a, \dot{a}, A$, so $1 \leq a, \dot{a} \leq 2$ and $1 \leq A \leq 4$. All the generators in (C.1) are antihermitian, except for $q_a{}^A$ and $\widetilde{s}^A{}_{\dot{a}}$, which are hermitian.

The commutation relations for the generators (C.1) satisfy the real form $psl(4|4)$,

$$
\begin{aligned}
&[m^a{}_b, J_c] = \delta^a_c J_b - \tfrac{1}{2}\delta^a_b J_c, \qquad [m^a{}_b, J^c] = -\delta^c_b J^a + \tfrac{1}{2}\delta^a_b J^c, \\
&[\widetilde{m}^{\dot a}{}_{\dot b}, J_{\dot c}] = \delta^{\dot a}_{\dot c} J_{\dot b} - \tfrac{1}{2}\delta^{\dot a}_{\dot b} J_{\dot c}, \qquad [\widetilde{m}^{\dot a}{}_{\dot b}, J^{\dot c}] = -\delta^{\dot c}_{\dot b} J^{\dot a} + \tfrac{1}{2}\delta^{\dot a}_{\dot b} J^{\dot c}, \\
&[r^A{}_B, J_C] = \delta^A_C J_B - \tfrac{1}{4}\delta^A_B J_C, \qquad [r^A{}_B, J^C] = -\delta^C_B J^A + \tfrac{1}{4}\delta^A_B J^C, \\
&[s^a{}_A, p_b{}^{\dot b}] = \delta^a_b \widetilde{q}^{\dot b}{}_A, \qquad [k^a{}_{\dot a}, \widetilde{q}^{\dot b}{}_C] = -\delta^{\dot b}_{\dot a} s^a{}_C \;\Rightarrow\; [k^{a\dot a}, \widetilde{q}_{\dot b C}] = \delta^{\dot a}_{\dot b} s^a{}_C, \\
&[\widetilde{s}^A{}_{\dot a}, p_b{}^{\dot b}] = -\delta^{\dot b}_{\dot a} q^A{}_b \;\Rightarrow\; [\widetilde{s}^{A\dot a}, p_{b\dot b}] = \delta^{\dot a}_{\dot b} q^A{}_b, \qquad [k^a{}_{\dot a}, q_b{}^B] = \delta^a_b \widetilde{s}^B{}_{\dot a}, \\
&\{\widetilde{q}^{\dot a}{}_A, q^B{}_b\} = \delta^B_A p_b{}^{\dot a}, \qquad \{\widetilde{s}^A{}_{\dot a}, s^b{}_B\} = \delta^A_B k^b{}_{\dot a}, \\
&[k^a{}_{\dot a}, p_{\dot b}{}^{\dot b}] = -\delta^{\dot b}_{\dot a} m^a{}_b + \delta^a_b m^{\dot b}{}_{\dot a} - \delta^{\dot b}_{\dot a}\delta^a_b\, d \;\Rightarrow\; [k^{a\dot a}, p_{b\dot b}] = \delta^{\dot a}_{\dot b} m^a{}_b + \delta^a_b \widetilde{m}^{\dot a}{}_{\dot b} + \delta^{\dot b}_{\dot a}\delta^a_b\, d, \\
&\{s^a{}_A, q_b{}^B\} = \delta^B_A m^a{}_b + \delta^a_b r^B_A + \tfrac{1}{2}\delta^a_b \delta^B_A (d - \mathcal{C}), \\
&\{\widetilde{s}^A{}_{\dot a}, \widetilde{q}^{\dot b}{}_B\} = \delta^A_B \widetilde{m}^{\dot b}{}_{\dot a} - \delta^{\dot b}_{\dot a} r^A{}_B + \tfrac{1}{2}\delta^A_B \delta^{\dot b}_{\dot a}(d + \mathcal{C}), \\
&\qquad [d, J] = \mathrm{dim} J, \qquad [\mathcal{C}, J] = 0,
\end{aligned} \tag{C.2}
$$

where the central generator $\mathcal{C} \equiv \frac{1}{2}\oint dz Y^I(z) Z_I(z)$ acts as zero on gauge invariant states. We raise and lower the bosonic indices as $Z^a = \epsilon^{ab} Z_b$, $Z_a = \epsilon_{ab} Z^b$, and $Z^{\dot a} = \epsilon^{\dot a \dot b} Z_{\dot b}$, $Z_{\dot a} = \epsilon_{\dot a \dot b} Z^{\dot b}$, with $\epsilon^{12} = 1 = -\epsilon^{21}$, $\epsilon_{12} = -1 = -\epsilon_{21}$ for both the dotted and undotted indices.

D Vanishing of the right-hand side of the Serre relation (6.4) on one field

We evaluate the right hand side of the Serre relation first for $gl(4|4)$,

$$(-1)^{(C+D)(K+L)+(G+H)(U+V)+G+H}\ \widetilde{f}^{I\ \ D\ \ Q}_{\ JC\ \ P}\ \widetilde{f}^{M\ \ F\ \ L}_{\ NE\ \ K}\ \widetilde{f}^{R\ G\ \ \ \ V}_{S\ \ HU}\ \widetilde{f}^{C\ E\ \ \ \ H}_{D\ \ FG}\ \left\{\widetilde{Q}^{P}_{0\,Q}\,\widetilde{Q}^{K}_{0\,L}\,\widetilde{Q}^{U}_{0\,V}\right]$$

$$= \Bigg(-(-1)^{MJ+RN+S(M+J+R+N)+S+E}\ \delta^I_S \left\{\widetilde{Q}^{M}_{0\,E}\,\widetilde{Q}^{E}_{0\,J}\,\widetilde{Q}^{R}_{0\,N} - \widetilde{Q}^{M}_{0\,J}\,\widetilde{Q}^{R}_{0\,E}\,\widetilde{Q}^{E}_{0\,N}\right]$$

$$+ (-1)^{(M+J)(N+R)+N+E}\ \delta^I_N \left\{\widetilde{Q}^{R}_{0\,E}\,\widetilde{Q}^{E}_{0\,J}\,\widetilde{Q}^{M}_{0\,S} - \widetilde{Q}^{R}_{0\,J}\,\widetilde{Q}^{M}_{0\,E}\,\widetilde{Q}^{E}_{0\,S}\right]$$

$$+ (-1)^{(J+S)(R+N)+S+E}\ \delta^M_S \left\{\widetilde{Q}^{I}_{0\,E}\,\widetilde{Q}^{E}_{0\,N}\,\widetilde{Q}^{R}_{0\,J} - \widetilde{Q}^{I}_{0\,N}\,\widetilde{Q}^{R}_{0\,E}\,\widetilde{Q}^{E}_{0\,J}\right]$$

$$- (-1)^{JM+MS+JS+E}\ \delta^R_N \left\{\widetilde{Q}^{I}_{0\,E}\,\widetilde{Q}^{E}_{0\,S}\,\widetilde{Q}^{M}_{0\,J} - \widetilde{Q}^{I}_{0\,S}\,\widetilde{Q}^{M}_{0\,E}\,\widetilde{Q}^{E}_{0\,J}\right]$$

$$- (-1)^{IR+RN+NI+E}\ \delta^M_J \left\{\widetilde{Q}^{R}_{0\,E}\,\widetilde{Q}^{E}_{0\,N}\,\widetilde{Q}^{I}_{0\,S} - \widetilde{Q}^{R}_{0\,N}\,\widetilde{Q}^{I}_{0\,E}\,\widetilde{Q}^{E}_{0\,S}\right]$$

$$+ (-1)^{IS+SN+NJ+JM+MI+E}\ \delta^R_J \left\{\widetilde{Q}^{M}_{0\,E}\,\widetilde{Q}^{E}_{0\,S}\,\widetilde{Q}^{I}_{0\,N} - \widetilde{Q}^{M}_{0\,S}\,\widetilde{Q}^{I}_{0\,E}\,\widetilde{Q}^{E}_{0\,N}\right]\Bigg), \tag{D.1}$$

which vanishes for our single site representation from (6.6). This holds as well for $gl(n|n)$. Here the gl(4|4) structure constants are those of psl(4|4) but without the traces and supertraces removed,

$$\widetilde{f}^{I\ K\ \ \ \ N}_{\ J\ \ LM} = \delta^K_J \delta^N_L \delta^I_M - (-1)^{(I+J)(K+L)}\,\delta^N_J \delta^I_L \delta^K_M,$$

$$\widetilde{f}^{M\ \ L\ \ J}_{\ NK\ \ I} = (-1)^K \left(\delta^L_N \delta^M_I \delta^J_K - (-1)^{(M+N)(M+I)} \delta^M_K \delta^L_I \delta^J_N\right). \tag{D.2}$$

(D.1) implies that that the psl(4|4) expression

$$(-1)^{(C+D)(K+L)+(G+H)(U+V)+G+H}\ f^{I\ \ D\ \ Q}_{\ JC\ \ P}\ f^{M\ \ F\ \ L}_{\ NE\ \ K}\ f^{R\ G\ \ \ \ V}_{S\ \ HU}\ f^{C\ E\ \ \ \ H}_{D\ \ FG}\ \left\{Q^{P}_{0\,Q}\,Q^{K}_{0\,L}\,Q^{U}_{0\,V}\right] \tag{D.3}$$

also vanishes on one site, for if we view (D.1) as a tensor $\mathcal{M}^{IMR}_{JNS}$, then up to terms containing the central generator $\sum_C \widetilde{Q}^C_{0C}$, (D.3) is just the tensor $\mathcal{M}^{IMR}_{JNS}$ with the trace and supertrace removed in each pair of indices IJ, MN, RS.

This follows from the properties of the psl(n|n) structure constants given in appendix B,

$$(-1)^{(C+D)(K+L)+(G+H)(U+V)+G+H}\, f^{I\ \ D\ \ Q}_{\ JC\ \ P}\, f^{M\ \ F\ \ L}_{\ NE\ \ K}\, f^{R\ G\ \ \ \ V}_{\ S\ \ HU}\, f^{C\ E\ \ \ \ H}_{\ D\ \ FG}\, \Big\{ Q^{P}_{0\,Q}\, Q^{K}_{0\,L}\, Q^{U}_{0\,V} \Big]$$

$$= (-1)^{(C+D)(K+L)+(G+H)(U+V)+G+H}\, f^{I\ \ D\ \ Q}_{\ JC\ \ P}\, f^{M\ \ F\ \ L}_{\ NE\ \ K}\, f^{R\ G\ \ \ \ V}_{\ S\ \ HU}\, \widetilde{f}^{C\ E\ \ \ \ H}_{\ D\ \ FG}\, \Big\{ \widetilde{Q}^{P}_{0\,Q}\, \widetilde{Q}^{K}_{0\,L}\, \widetilde{Q}^{U}_{0\,V} \Big]$$

$$\sim (\widetilde{f} + h_1 + h_3)(\widetilde{f} + h_1 + h_3)(\widetilde{f} + h_1 + h_3)\, \widetilde{f}\, \widetilde{Q}\, \widetilde{Q}\, \widetilde{Q}, \tag{D.4}$$

where we have given separate labels to the supertrace and trace terms in (B.1) and (B.5) for the three pair of indices as h_1, h_2, h_3, $f = \widetilde{f} + h_1 + h_2 + h_3$, and the expression in the final term of (D.4), $(\widetilde{f} + h_1)(\widetilde{f} + h_1)(\widetilde{f} + h_1)\, \widetilde{f}\, \widetilde{Q}\, \widetilde{Q}\, \widetilde{Q}$ is the tensor $\mathcal{M}^{IMR}_{JNS}$ with the trace and supertrace removed in each pair of indices IJ, MN, RS.

Thus we have shown that the right hand side of the Serre relation (6.4) vanishes for charges acting on a single field, which together with (6.2) and the coproduct (4.21) implies that the Serre relation holds for any product of gauge invariant fields.

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