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Flavor Symmetry for Quarks and Leptons

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Abstract

Present data on neutrino masses and mixing favor the highly symmetric tribimaximal neutrino mixing matrix which suggests an underlying flavor symmetry. A systematic study of non-abelian finite groups of order $g \leq 31$ reveals that tribimaximal mixing can be derived not only from the well known tetrahedral flavor symmetry $T \equiv A_4$, but also by using the binary tetrahedral symmetry $T^{'} \equiv SL_2(F_3)$ which does not contain the tetrahedral group as a subgroup. $T^{'}$ has the further advantage that it can also neatly accommodate the quark masses including a heavy top quark.

In this letter we propose a flavor symmetry for quarks and leptons.

We shall consider only three left-handed neutrinos at first. The Majorana mass matrix $\mathcal{M}$ is a 3×3 unitary symmetric matrix and without CP violation has six real parameters. Let write the diagonal form as $\mathbf{M} = \text{diag}(m_1, m_2, m_3)$, related to the flavor basis $\mathcal{M}$ by $\mathbf{M} = U^T \mathcal{M} U$ where U is orthogonal. It is the form of $\mathcal{M} = U\mathbf{M}U^T$ and U which are the targets of lepton flavor physics. One technique for analysis of $\mathcal{M}$ is to assume texture zeros [1–3] in $\mathcal{M}$ and this gives rise to relationships between the mass eigenvalues m_i and the mixing angles θ_{ij}. For example, it was shown in [2] that $\mathcal{M}$ cannot have as many as three texture zeros out of a possible six but can have two. A quite different interesting philosophy is that neutrino masses may arise from breaking of lorentz invariance [4]. Clearly, a wide range of approaches is being aimed at the problem.

In the present study we focus on a symmetric texture for $\mathcal{M}$ with only four independent parameters, of the form

$$\mathcal{M} = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix} \tag{1}$$

The $\mathcal{M}$ can be reached from a diagonal $\mathbf{M}$ by the orthogonal transformation

$$U = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12}/\sqrt{2} & \cos\theta_{12}/\sqrt{2} & -1/\sqrt{2} \\ -\sin\theta_{12}/\sqrt{2} & \cos\theta_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{2}$$

where one commits to a relationship between θ_{12} and the four parameters in Eq.(1), namely

$$\tan 2\theta_{12} = 2\sqrt{2B(A - C - D)^{-1}} \tag{3}$$

Written in the standard PMNS form [5]

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} e^{i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13} e^{-1\delta} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

this ansatz requires that $\theta_{23} = \pi/4$ and $\theta_{13} = 0$, both of which are consistent with present data. These values of maximal θ_{23} and vanishing θ_{13} are presumably only approximate but departures therefrom, if they show up in future experiments, could be accommodated by higher order corrections.

To arrive at tribimaximal mixing [6–11], one more parameter θ_{12} in Eq. (2) is assigned such that the entries of the second column are equal, *i.e.* $\sin\theta_{12} = \cos\theta_{12}/\sqrt{2}$ which implies that $\tan^2\theta_{12} = 1/2$. Experimentally θ_{12} is non-zero and over 5σ from a maximal $\pi/4$. The present value [12] is $\tan^2\theta_{12} = 0.452^{+0.088}_{-0.070}$, so the tribimaximal value is within the allowed range. With this identification Eq.(2) becomes [10]

$$U_{HPS} = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & -1/\sqrt{2} \\ -\sqrt{1/6} & \sqrt{1/3} & 1/\sqrt{2} \end{pmatrix} \tag{5}$$

This ensures that the three mixing angles θ_{ij} are consistent with present data, although more accurate experiments may require corrections to U_{HPS}.

The data allow a normal or inverted hierarchy, or a degenerate spectrum. The tribimaximal mixing, U_{HPS} of Eq.(5), can accommodate all three of these neutrino mass patterns and so makes no prediction in that regard.

The success of U_{HPS} tribimaximal neutrino mixing has precipitated many studies of its group theoretic basis [8, 9, 11] and the tetrahedral group $T \equiv A_4$ has emerged. The present analysis was prompted by earlier work of the present authors in systematically studying *all* non-abelian finite groups of order $g \leq 31$ both for a quark flavor group [14] and for orbifold compactification in string theory [15]. Our question is whether or not T is singled out from these as the neutrino flavor symmetry?

Character Table of $T \equiv A_4$

$\omega = \exp(2\pi i/3)$

	1_1	1_2	1_3	3
C_1	1	1	1	3
C_2	1	1	1	-1
C_3	1	ω	ω^2	0
C_4	1	ω^2	ω	0

Kronecker Products for Irreducible Representations of $T \equiv A_4$

	1_1	1_2	1_3	3
1_1	1_1	1_2	1_3	3
1_2	1_2	1_3	1_1	3
1_3	1_3	1_1	1_2	3
3	3	3	3	$1_1 + 1_2 + 1_3 + 3 + 3$

The Kronecker products for irreducible representations for all the fourty-five non-abelian finite groups with order $g \leq 31$ are explicitly tabulated in the Appendix of [15], where the presentation is adapted to a style aimed at model builders in theoretical physics rather than at mathematicians as in [13].

Study of [15] shows that a promising flavor group is $\equiv SL_2(F_3)$. The Kronecker products are identical to those of $T \equiv A_4$ if the doublet representations are omitted and so the group $SL_2(F_3)$ can reproduce successes of T model building. The use of $SL_2(F_3)$ as a flavor group first appeared in [14] and then analysed in more details in [16].

$SL_2(F_3)$ has an advantage over T in extension to the quark sector because the doublets of $SL_2(F_3)$, absent in T, allow the implementation of a $(2+1)$ structure to the three quark families, thus permitting the third heavy family to be treated differently as espoused in [14, 17, 18]

Character Table of $T' \equiv SL_2(F_3)$

$\omega = \exp(2\pi i/6)$

	1_1	1_2	1_3	2_1	2_2	2_3	3
C_1	1	1	1	2	2	2	3
C_2	1	1	1	-2	-2	-2	3
C_3	1	ω^2	ω^4	-1	ω^5	ω	0
C_4	1	ω^4	ω^2	-1	ω	ω^5	0
C_5	1	1	1	0	0	0	-1
C_6	1	ω^2	ω^4	1	ω^2	ω^4	0
C_7	1	ω^4	ω^2	1	ω^4	ω^2	0

Kronecker Products for Irreducible Representations of $T' \equiv SL_2(F_3)$

	1_1	1_2	1_3	2_1	2_2	2_3	3
1_1	1_1	1_2	1_3	2_1	2_2	2_3	3
1_2	1_2	1_3	1_1	2_2	2_3	2_1	3
1_3	1_3	1_1	1_2	2_3	2_1	2_2	3
2_1	2_1	2_2	2_3	1_1+3	1_2+3	1_3+3	$2_1+2_2+2_3$
2_2	2_2	2_3	2_1	1_2+3	1_3+3	1_1+3	$2_1+2_2+2_3$
2_3	2_3	2_1	2_2	1_3+3	1_1+3	1_2+3	$2_1+2_2+2_3$
3	3	3	3	$2_1+2_2+2_3$	$2_1+2_2+2_3$	$2_1+2_2+2_3$	$1_1+1_2+1_3+3+3$

It is important to remark that $T' \equiv SL_2(F_3)$ does not contain $T \equiv A_4$ as a subgroup [13] so our discussion about quark masses does not merely extend T, but replaces it.

The philosophy used for $SL_2(F_3)$ is reminiscent of much earlier work in [17, 18] where the dicyclic group Q_6 was used with doublets and singlets for the (1st, 2nd) and (3rd) families to transform as $(\mathbf{2}+\mathbf{1})$ respectively. On the other hand, Q_6 is not suited for tribimaximal neutrino mixing because like all dicyclic groups Q_{2n} it has no triplet representation. Recall that when the work on Q_6 was done, experiments had not established neutrino mixing for the reason explained in our first paragraph.

To discuss the model building using $SL_2(F_3)$ we must recall from the A_4 model building [8, 9, 11] that the leptons can be assigned[#1] to singlets and triplets as follows:

$$\left.\begin{array}{l}\begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L\\ \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L\\ \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L\end{array}\right\} 3 \quad \begin{array}{ll}\tau_R^- & 1_1\\ \\ \mu_R^- & 1_2\\ \\ e_R^- & 1_3\end{array} \tag{6}$$

[#1] An alternative assignment is in [19].

The symmetry breaking pattern of most interest is [13]

$$SL_2(F_3) \longrightarrow Q \longrightarrow Z_4 \longrightarrow Z_2 \longrightarrow \text{no symmetry} \tag{7}$$

where Q is the quarternionic group so the first discussion concerns the vacuum alignment will cause the symmetry to break according to the pattern (7). Recall that the irreps of $SL_2(F_3)$ are $1_1, 1_2, 1_3, 2_1, 2_2, 2_3, 3$.

By study of the character tables of these groups, we can ascertain the VEVS (Vacuum Expectation Values) which generate the required spontaneous symmetry breakdown.

The irreps of Q are $1_1, 1_2, 1_3, 1_4, 2$. Concerning the crucial first stage of symmetry breaking $SL_2(F_3) \longrightarrow Q$, the irreps are related by

$$\begin{aligned}
1_1 &\longrightarrow 1_1 \\
1_2 &\longrightarrow 1_1 \\
1_3 &\longrightarrow 1_1 \\
2_1 &\longrightarrow 2 \\
2_2 &\longrightarrow 2 \\
2_3 &\longrightarrow 2 \\
3 &\longrightarrow 1_2 + 1_3 + 1_4
\end{aligned} \tag{8}$$

so the breaking requires a VEV in 1_2 or 1_3 of $SL_2(F_3)$. We therefore assign the left-handed quarks, consistent with the $2+1$ philosophy and the third family treated differently [17,18] as follows:

$$\begin{array}{ll}
\begin{pmatrix} t \\ b \end{pmatrix}_L & 1_1 \\
\left.\begin{array}{l} \begin{pmatrix} c \\ s \end{pmatrix}_L \\ \begin{pmatrix} u \\ d \end{pmatrix}_L \end{array}\right\} & 2_1
\end{array} \tag{9}$$

and similarly the right-handed quarks are assigned as:

$$\begin{array}{ll}
t_R & 1_1 \\
\left.\begin{array}{l} c_R \\ u_R \end{array}\right\} & 2_2 \\
b_R & 1_2 \\
\left.\begin{array}{l} s_R \\ d_R \end{array}\right\} & 2_3
\end{array} \tag{10}$$

whereupon the mass matrices are:

$$U = \left(\begin{array}{c|c} <1_3+3> & <2_1> \\ \hline <2_3> & <1_1> \end{array} \right) \tag{11}$$

and

$$D = \left(\begin{array}{c|c} <1_2+3> & <2_3> \\ \hline <2_2> & <1_3> \end{array} \right) \tag{12}$$

To implement a hierarchy requires first a VEV to a $SU(2)_L$-doublet Higgs, H_{1_1} which is in the trivial singlet representation of $SL_2(F_3)$, thus giving a heavy mass to t without breaking $SL_2(F_3)$. This mass is naturally of order the weak scale $\sim v/\sqrt{2} \sim 175$ GeV.

A VEV to a Higgs H_{1_2} breaks $SL_2(F_3)$ to Q and can give masses to the b quark and c quarks. More explicitly, starting from the lagrangian for the $SL_2(F_3)$ model, we have the Yukawa terms involving the top, bottom, and charm quarks:

$$Y_t \begin{pmatrix} t \\ b \end{pmatrix}_{1_1} t_{1_1} H_{1_1} \tag{13}$$

$$Y_b \begin{pmatrix} t \\ b \end{pmatrix}_{1_1} b_{1_2} H_{1_3} \tag{14}$$

and

$$Y_c \left[\begin{pmatrix} c \\ s \end{pmatrix}_{2_1} c_{2_2} + \begin{pmatrix} u \\ d \end{pmatrix}_{2_1} u_{2_2} \right] H_{1_3} + Y_c' \left[\begin{pmatrix} c \\ s \end{pmatrix}_{2_1} c_{2_2} + \begin{pmatrix} u \\ d \end{pmatrix}_{2_1} u_{2_2} \right] H_3 \tag{15}$$

where the subscripts on the quark representations and Higgs doublets are the $Sl_2(F_3)$ irreps where they live.

Hence, as stated above, the VEV for the H_{1_1} gives a mass to the top quark, but does not break $SL_2(F_3)$. The bottom and, in part, charm quark get their masses from a VEV for the Higgs H_{1_3} transforming according to the 1_3 irrep of $SL_2(F_3)$. Thus giving a VEV to H_{1_3} gives masses to these next heaviest quarks. This causes the family group to break from $SL_2(F_3)$ to the quarternionic group Q. (As we shall show below all quarks can acquire a Q invariant mass). The b/c mass ratio is then simply

$$\frac{m_b}{m_c} = \frac{Y_b}{Y_c}. \tag{16}$$

The Yukawa couplings $Y_{b,c}$ are free parameters and we can therefore get any m_b/m_c ratio we want. The theory is not predictive at this stage, but it is at least tunable. We can proceed this way to get the other quark mass ratios.

The remaining quark masses are generated from the following Yukawa terms

$$Y_s \left[\begin{pmatrix} c \\ s \end{pmatrix}_{2_1} s_{2_3} + \begin{pmatrix} u \\ d \end{pmatrix}_{2_1} d_{2_3} \right] H_{1_2} + Y_s' \left[\begin{pmatrix} c \\ s \end{pmatrix}_{2_1} s_{2_3} + \begin{pmatrix} u \\ d \end{pmatrix}_{2_1} d_{2_3} \right] H_3 \tag{17}$$

where s, d can get a mass from an H_{1_2} VEV and keep Q unbroken.

Hence, all the quarks can get masses from H_{1_2} and H_{1_3} VEVs that leave Q unbroken. Note that VEVs for H_3's also contribute to masses of the first two quark generations, but not to t and b quark masses. More Higgses are needed to fill out all the off diagonal terms in the quark mass matrix. For instance, an H_{2_2} is needed to avoid a texture zero for m_{ut} if this is desired.

$$Y_{ut} \begin{pmatrix} t \\ b \end{pmatrix}_{1_1} u_{2_2} H_{2_3} \tag{18}$$

However, such a contribution is known phenomenologically to be very small.

However, it is important to observe that the only leptonic mass terms are from H_3 VEVs, but under $SL_2(F_3) \to Q$ we have

$$3 \to 1_2 + 1_3 + 1_4, \tag{19}$$

so a VEV for an H_3 breaks Q, and giving multiple H_3 VEVs can break Q to Z_4, Z_2, or the trivial group of one element.

If we were to ignore $< H_3 >$ and off diagonal terms the hierarchy of quark masses becomes $m_b/(m_c = m_u) = Y_b/Y_c$ and $m_s = m_d$ and, because these relations are unsatisfactory, $< H_3 >$ must be significant which is interesting because, as mentioned above, it controls also the lepton masses. Although five quark masses remain encoded in parametric Yukawa couplings, the advantage over the minimal standard model is that the top quark mass is naturally large. As for all flavor groups, including the present one, the proliferation of Yukawa couplings Y_k is the principal obstacle to quantitative calculation of the quark masses.

To see that a reasonable lowest-order CKM matrix can be achieved rewrite $m_i \equiv Y_k v_i$ where i is a T' representation and Y_k the appropriate Yukawa coupling. Then the quark mass matrices in Eq.(11) amd Eq.(12) must be diagonalized. We work for simplicity in the limit

$$m^U_{2_1} = m^U_{2_3} = m^D_{2_2} = m^D_{2_3} = 0 \tag{20}$$

which corresponds to taking $V_{ub} = V_{cb} = V_{td} = V_{ts} = 0$ in the CKM matrix. In this case all we need to do is diagonalize the (2×2) sub-matrices,

$$\tilde{\mathcal{M}}_U = \left[\begin{array}{c|c} im_3^U & m_{1_3}^U + \frac{m_3^U e^{-i\frac{\pi}{4}}}{\sqrt{2}} \\ \hline -m_{1_3}^U + \frac{m_3^U e^{-i\frac{\pi}{4}}}{\sqrt{2}} & m_3^U \end{array}\right], \tag{21}$$

$$\tilde{\mathcal{M}}_D = \left[\begin{array}{c|c} im_3^D & m_{1_2}^D + \frac{m_3^D e^{-i\frac{\pi}{4}}}{\sqrt{2}} \\ \hline -m_{1_2}^D + \frac{m_3^D e^{-i\frac{\pi}{4}}}{\sqrt{2}} & m_3^D \end{array}\right]. \tag{22}$$

We begin by diagonalizing $\tilde{\mathcal{M}}_U$, which can be achieved by introducing two unitary matrices U_L, and U_R satisfying

$$U_L^\dagger \cdot \tilde{\mathcal{M}}_U \cdot U_R = \begin{bmatrix} m_u & 0 \\ 0 & m_c \end{bmatrix}, \tag{23}$$

or equivalently,

$$U_L^\dagger \cdot \left(\tilde{\mathcal{M}}_U \tilde{\mathcal{M}}_U^\dagger\right) \cdot U_L = \begin{bmatrix} m_u^2 & 0 \\ 0 & m_c^2 \end{bmatrix}, \tag{24}$$

where

$$\tilde{\mathcal{M}}_U \tilde{\mathcal{M}}_U^\dagger = \left[\begin{array}{c|c} \frac{3}{2}[m_3^U]^2 + [m_{1_3}^U]^2 + m_{1_3}^U m_3^U & \sqrt{2} m_3^U m_{1_3}^U e^{-i\frac{\pi}{4}} \\ \hline \sqrt{2} m_3^U m_{1_3}^U e^{i\frac{\pi}{4}} & \frac{3}{2}[m_3^U]^2 + [m_{1_3}^U]^2 - m_{1_3}^U m_3^U \end{array}\right]. \tag{25}$$

From Eqs.(25) and (24) the eigenvalues are calculated:

$$m_{u,c}^2 = \frac{(3[m_3^U]^2 + 2[m_{1_3}^U]^2) \mp 2\sqrt{3} m_3^U m_{1_3}^U}{2}. \tag{26}$$

which indicates how the quark mass spectrum can be successfully accommodated. The unitary matrix U_L takes the form:

$$U_L = \frac{1}{\sqrt{1+A^2}} \begin{bmatrix} 1 & e^{-i\frac{\pi}{4}} A \\ -e^{i\frac{\pi}{4}} A & 1 \end{bmatrix}, \tag{27}$$

with $A = (\sqrt{3}+1)/\sqrt{2}$. We note that U_L is independent of quark masses.

In the down-sector, since the mass matrix $\tilde{\mathcal{M}}_D$ takes the same form as $\tilde{\mathcal{M}}_U$, a matrix $(\tilde{\mathcal{M}}_D \tilde{\mathcal{M}}_D^\dagger)$ is diagonalized by using the same unitary matrix U_L of Eq.(27). Hence we reach the result for this special case

$$V_{\rm CKM} = \left[\begin{array}{c|c} U_L^\dagger D_L & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} U_L^\dagger U_L & 0 \\ \hline 0 & 1 \end{array}\right] = \mathbf{1}. \tag{28}$$

which is an acceptable first approximation to the CKM matrix.

For the neutrinos, as in earlier work on T [8], the masses are not uniquely predicted but the tribimaximal mixing angles [10] are. All these three neutrino mixing angles are consistent with existing measurements.

The Higgs VEVs with a commonality between quarks and leptons are in the H_3 of $T^{'} \equiv SL_2(F_3)$ which has a simple decomposition under the quarternionic subgroup Q which is likely to play a key role in the goal of linking lepton masses with quark masses.

In summary, while $T \equiv A_4$ is one candidate for a lepton flavor group which naturally gives rise to tribimaximal mixing, it is not unique among the non abelian finite groups in this regard. The choice $T^{'} \equiv SL_2(F_3)$, also known as the binary tetrahedral group [20], satisfies the requirement equally well, and because it has doublet representations can thereby begin to accommodate the quark mass spectrum, particularly the anomalously heavy third family [21]. If our choice is the correct flavor symmetry, it remains to understand why Nature chooses the triplet representations for leptons and the doublet representations for quarks. Quantitative results for masses will require a relationship between the Yukawa parameters from our proposed symmetry.

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[21] Our $SL_2(F_3)$ model appears to us as a promising framework worthy of further study with the goal to obtain quantitative results for fermion masses. When we first discussed $SL_2(F_3)$ as a family symmetry in 1994 in [14], the state of neutrino physics at that time was insufficient to extend the model to the lepton sector. Subsequent data has now made this a realistic objective. Recent theoretical work [22] was a first step in this direction. A similar model has recently appeared [23] where tribimaximal neutrino mixing is included in an $SL_2(F_3) \times Z_3$ family symmetry model with somewhat different assignments of quarks to $SL_2(F_3)$ irreps, and different vacuum allignment. This model is also supersymmetric, so contains considerably more states and parameters. A grand unified theory based on $T^{'}$ is in [24]

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