IFT-485-UNC

Renormalization Group Flow of Four-Fermi with Chern-Simon Interaction

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Abstract

We introduce Chern-Simons interaction into the three dimensional four-fermi theory, and suggest a possible line of non-Gaussian infrared stable fixed points of the four-fermi operator, which is characterized by the Chern-Simons coupling.

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1 Introduction

Four-fermi theory in d spacetime dimensions ($2 < d < 4$) in the low-energy regime is thought as *trivial* as the free theory, since the only infrared (IR) stable fixed point of four-fermi coupling is Gaussian. Evidently, this is attributed to the four-fermi operator being irrelevant or non-renormalizable. On the other hand, it was conjectured long ago [1][2] that the four-fermi operator could perhaps be relevant if one considered some non-trivial resummation of infinite numbers of diagrams. And there have been numerous recent efforts to explore new phenomena of the theory in the high-energy regime [3]-[8]. It turns out that if the four-fermi interaction is *attractive* and if the number of fermion species N is large, there exists a strong coupling ultraviolet (UV) stable fixed point and the perturbation expansion over the parameter 1/N is controllable. This UV fixed point relates to the dynamical symmetry breaking characterized by the vacuum expectation $< \psi^\dagger\psi >$ ($\neq 0$ for the symmetry broken phase). There have been also recent arguments that near certain strong critical couplings, the four-fermi operator might be relevant [9]. In this Paper, limiting to three dimensions, we introduce a Chern-Simons interaction, and suggest a possible line of non-trivial IR fixed points for the four-fermi operator, which is characterized by the Chern-Simons coupling.

Chern-Simons term appears naturally in planar systems. For instance, the massive fermion current-current correlation contains an anti-symmetric part (in the space-time indices), which corresponds to an induced Chern-Simons term in the effective Lagrangian [10]. On the other hand, charged matter fields can be coupled to a gauge potential who's dynamics is governed by a Chern-Simons term. Physically, this is to attach particles with 'magnetic' flux tubes. The composite particles carrying both charges and flux tubes are known as anyons [11][12], and have been generally thought responsible to phenomena in the quantum Hall systems [13]-[22]. Numerous efforts have been also made on using the idea of anyons to understand the high T_c superconductivity [23][24], and other strongly correlating planar systems like the quantum Heisenberg antiferromagnet [25].

The Chern-Simons coupling, denoted by a real number α, characterizes the statistics of the matter field, therefore it is also called the statistical parameter. As α varies from zero to one [see (2.1) for the convention], the minimally coupled matter changes continuously from the free fermion to free (hard corn) boson, via anyons - the very complicated many-body systems. This is known as the bose-fermi transmutation [26]-[28]. In relativistic systems, besides statistics transmutation there happens spin transmutation of matter fields [29]. That is, an integer part of α can be reabsorbed by changing the spin – the character of the Poincare representation – of matter fields.

Without a reference to the spacetime metric, and as a top form in three dimensional (or any odd dimensional) manifold, Chern-Simons terms are topological ones. This determines the Chern-Simons couplings to be insensitive to changes of the energy scale. In other words, the beta function of the Chern-Simons couplings is identically vanishing. Consequently, the Chern-Simons couplings may serve well as a controlling parameter in the perturbation expansion. It is known that the Chern-Simons coupling characterizes as well the scaling dimensions of the matter field operators and composite field operators [30][31]. These scaling dimensions determine the asymptotic behavior of the anyon systems. Amusingly, the effect of Chern-Simons couplings on the scaling dimensions of matter fields and a class of gauge invariant composite field operators is to drive these operators in the direction of *increased* relevance [31].

These observations on the Chern-Simons interaction are very appealing to the search for possible *IR* fixed points in the four-fermi theory in three dimensions. Let us couple the fermion minimally to the Chern-Simons gauge field. Then as the Chern-Simons coupling α goes up from 0 to 1, the spin-1/2 fermion system turns to be anyons and becomes a spin-0 (hard core) boson system at $\alpha = 1$. Particularly, the four-fermi interaction at $\alpha = 0$ turns to be a four-bose interaction at $\alpha = 1$. As is known, while the four-fermi theory has a Gaussian *IR* fixed point, the scalar ϕ^4 theory in three dimensions has a Gaussian *UV* fixed point and a non-Gaussian *IR* fixed point [32][33]. Then it is natural to expect, between the Chern-Simons couplings $\alpha = 0$ and 1, there exists a critical α_c so that as α approaches to α_c, a transition happens that the system changes from the fermion-like to boson-like or vise versa.

It is conceivable that this critical α_c has something to do with the critical scaling dimension of the four-fermi operator. As is known, the engineering dimension of four-fermi operator in three spacetime dimensions is four and so the operator is irrelevant. Now, interacting to the Chern-Simons field, the operator receives an anomalous dimension, which is a monotonically decreasing function of α [31]. Therefore the critical α_c can be so determined that at which the four-fermi operator becomes marginally irrelevant. We shall show in this Paper by the renormalization group approach that there exists indeed such a critical α_c, and when $\alpha > \alpha_c$ there exists a non-Gaussian

IR fixed points of the four-fermi coupling, besides a Gaussian *UV* fixed point. And when the critical Chern-Simons coupling α_c is approached from the boson-like side, *i.e.* $\alpha(>\alpha_c) \to \alpha_c$, the two fixed points of the four-fermi coupling coincide. Namely, the Gaussian fixed point at $\alpha = \alpha_c$ is a multi-fixed point.

The Paper is organized as follows: Section 2 is devoted to an analysis of the Chern-Simons coupling as a controlling parameter and an analysis of loop corrections to the two- and four-point correlation functions. In Section 3, we solve the renormalization group equation for the four-point function. The critical Chern-Simons coupling α_c is determined, and a picture for the renormalization group flows of the four-fermi interaction is drawn. Section 4 is for conclusions and discussions. In three appendixes, we present our conventions and calculation details.

2 CS Coupling as a Controlling Parameter

Let us introduce the topological Chern-Simons term and a minimal interaction between the Chern-Simons and fermion fields. The action in Euclidean space reads

$$S = \int d^3x \left(\psi^\dagger \gamma \cdot (\partial - iA)\psi + \frac{G}{2}(\psi^\dagger\psi)^2 + \frac{i}{4\pi\alpha}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right) . \tag{2.1}$$

Here, for simplicity, we have omitted a fermion mass term[1]. Namely, we concentrate on two parameters only, the Chern-Simons coefficient α and the four-fermi coupling G. With the gauge coupling, this theory is invariant under the gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \epsilon(x) , \tag{2.2}$$

$$\psi(x) \to e^{i\epsilon(x)}\psi(x) . \tag{2.3}$$

Without the Maxwell term, the Chern-Simons gauge field carries *no* dynamical degree of freedom. To see this, we vary (2.1) over A_μ field, and obtain the equation of motion

[1] Fermion mass is certainly an interesting issue in the four-fermi model. Unlike in 4d, there is no chiral symmetry to rule out a fermion mass term from the lagrangian in 3d. Instead, the symmetries a fermion mass term in 3d breaks are parity (P) and time reversal (T). However, since a Chern-Simons term breaks P and T, the CS coupling will induce a fermion mass term even the bare fermions are massless, besides the possible dynamical (mass) generation. Physically, omitting the fermion mass amounts to choosing the $M_r = 0$ slice on the phase diagram, with M_r denoting the renormalized fermion mass, as $M_r = 0$ is always a fixed point. This can be realized by introducing infinite and finite local counter terms to cancel any induced mass terms.

for the Chern-Simons field:

$$\frac{1}{2}\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda = 2\pi\alpha j_\mu \ , \tag{2.4}$$

where the conserved matter current $j_\mu = i\psi^\dagger\psi$. Now it is clear that without their own time evolution, the 'electric' and 'magnetic' fields associated with the Chern-Simons gauge potential A_μ are determined completely by the spatial and 'time' components of the matter current, respectively. (2.4) can be read in another way: Taking $\mu = 0$, one sees that a 'magnetic' flux $b = \frac{1}{2}\epsilon_{ij}\partial_i A_j$ in unit strength $2\pi\alpha$ is attached to a particle. This attachment of fluxes makes a change to the statistics the fermion matter fields obey. Namely that, when the positions of two identical particles are exchanged, their relative phase is no longer the exponential of $i\pi$ but $i(\alpha+1)\pi$. The real, dimensionless Chern-Simons gauge coupling α is thus called the statistical parameter. When α varies from zero to one, the matter field runs from the fermion limit through anyons to the boson limit, according to the statistics changes.

Formally, the Chern-Simons interaction is renormalizable and its perturbation expansion may contain logarithmic divergence. However, the topological nature of the Chern-Simons term allows only trivial, finite renormalization of the Chern-Simons term itself. It has been proven, based on a perturbation expansion [34]-[36], a non-renormalization theorem that says, there is no radiative correction to the (Abelian) Chern-Simons term from massive matters beyond one loop, and there is only finite correction to it from massless matters at even loops (two loops and beyond). This implies the Chern-Simons coupling is insensitive to changes of energy scale at all. Consequently, the beta function of the Chern-Simons coupling identically vanishes

$$\beta(\alpha) \equiv 0 \ . \tag{2.5}$$

Then, it is plausible to see that the Chern-Simons coupling serves as a controlling parameter in the renormalization of the four-fermi interaction (at least to the order we are interested in).

On the other hand, the four-fermi coupling constant G has dimension *one* in the unit of length (the inverse of mass). Introduce a large momentum cutoff, one can define a dimensionless four-fermi coupling constant g so that

$$G = g/\Lambda \ . \tag{2.6}$$

Accordingly, it is convenient to use the naive large momentum cutoff Λ as a regulator. This regularization procedure is of course neither gauge nor Lorentz invariant. We shall make up these in the renormalization. Considering Λ is to sent to infinity eventually to come back to the continuous theory, we shall introduce counterterms to cancel the terms which are Λ dependent to recover the symmetries.

The zero point renormalization will be chosen for convenience. Namely, the renormalized four-fermi coupling constant is defined at the external momenta $p_1 = p_2 = p_3 = p_4 = 0$. It is assumed that the asymptotic behavior of the theory and the existence of non-Gaussian fixed points are independent of the regularization and renormalization procedure, though these choices affect in general the values of fixed point couplings.

We shall consider perturbation theory over the four-fermi coupling G, and calculate the renormalization group functions of it. To each order in G, we perturb the theory to the order α^2, where one can have almost all non-trivial observations in the perturbation theory. We calculate *one particle irreducible* diagrams only, which are sufficient for the purpose of renormalization. We shall use the Landau gauge for fixing the $U(1)$ symmetry. As is well known [37], this gauge choice in the Chern-Simons theories is especially good at avoiding the infrared divergence in Feynman diagrams. Leaving the calculation details to the appendixes II and III, we shall outline the analysis and present the main results in the rest of the section.

Calculate the renormalization constant of ψ field, Z_ψ, first. Since we plan to consider the four-fermi vertex to order G^2, it is sufficient to have the fermion wave-function renormalization to order G^1. It is easy to check the fermion bubble of the fermion self-energy (see Fig. 3a) vanishes as $\mathrm{Tr}\gamma_\mu = 0$. Then at order $G\alpha^0$, $Z_\psi = 1$.

At order $G^0\alpha$, it appears a term linear in Λ in both the fermion and Chern-Simons self-energies. This requires mass renormalization. We simply remove them by introducing the mass counterterms to keep the theory massless.

At order $G^0\alpha^2$, the fermion self-energy is (see Appendix II)

$$\Sigma(\Lambda, p) = i \not{p} \frac{\alpha^2}{6} \left(ln(\frac{\Lambda}{p}) - \frac{4}{3} + \frac{3\pi^2}{32} \right) . \tag{2.7}$$

Here it shows up the logarithmic divergence, which requires a non-trivial fermion wave-function renormalization. And the value of the finite constant terms in the bracket is regularization dependent.

At orders $G\alpha$ and $G\alpha^2$, there is no contribution to the fermion wave-function renormalization, as the fermion self-energy takes a form

$$G\Lambda(\alpha + a\alpha^2)(\Lambda + b|p|) . \tag{2.8}$$

This obviously matters only the fermion mass (a and b are constants), which we send to zero by fine tuning as discussed above.

In summary, to the order $G^1\alpha^2$, only (2.7) is non-trivial to the fermion wave-function renormalization constant. The latter is

$$Z_\psi = 1 + \frac{\alpha^2}{6} ln(\frac{\Lambda}{\mu}) + \mathcal{O}(G^2) \ . \tag{2.9}$$

Let us turn to consider the loop corrections to the four-fermi vertex. First, without the Chern-Simons vertex, the fermion bubbles contribute to the vertex (see Fig. 3b and c) with

$$\Gamma(\Lambda, p_i) = G^2(\frac{2\Lambda}{\pi^2} - \frac{|p_1 - p_2| + |p_3 - p_4|}{8}) \ . \tag{2.10}$$

Upto order $G^2\alpha$, we obtain (see Appendix III) the vertex function at the renormalization point

$$\Gamma(\Lambda, p = 0) = -G(1 - i\pi\alpha n \cdot \gamma) + G^2(\frac{2\Lambda}{\pi^2})(1 - i\pi\alpha n \cdot \gamma) \ , \tag{2.11}$$

where $n \cdot \gamma$ is a unit Dirac vector, $n^2 = 1$. (2.11) carries three messages: 1) there is an induced non-local four-fermi vertex related to $p \cdot \gamma / |p|$; 2) the new vertex has the same coupling constant as that the original four-fermi vertex has, and so they have the same coupling constant renormalization. and (3) at the order α^1 there is no change comparing to the fixed point structure of the four-fermi theory without Chern-Simons interaction (see Appendix I).

The order $G\alpha^2$ is non-trivial in the sense of renormalization as the logarithmic divergence appears. Here the four-fermi vertex is:

$$\Gamma^{(1,2)}(\Lambda, p) = -\frac{5}{2} G\alpha^2 [ln(\frac{\Lambda}{p}) + finite] \ . \tag{2.12}$$

The *finite* part in (2.12) is obviously regularization and renormalization dependent.

At order $G^2\alpha^2$, the vertex function at the renormalization point takes the form

$$\Gamma^{(2,2)}(\Lambda, p = 0) = G^2\alpha^2\Lambda(b_1 + b_2 ln(\frac{\Lambda}{\mu})) \ . \tag{2.13}$$

Formally, there appears here the logarithmic divergence too. However, a diagram-by-diagram investigation shows that *all* the diagrams that contribute to the log term b_2 contain, as a sub-diagram, one of the diagrams with logarithmic divergence at the previous order $G\alpha^2$. Namely, at order $G^2\alpha^2$, the four-fermi vertex has no *primary* logarithmic divergence. If counterterms are introduced to remove divergence order by order, no logarithmic divergence will be seen at order $G^2\alpha^2$. Therefore, no contribution comes from this order to the vertex renormalization. Again, the finite correction b_1 is regularization and renormalization dependent.

3 RG Flows of Four-fermi Interaction

Now, we are ready to calculate the beta function of the four-fermi coupling. The renormalized coupling at scale μ is defined as

$$
\begin{aligned}
-G_r &= Z_\psi^2(\Lambda)\Gamma(\Lambda, p_i = 0) &(3.1)\\
&= -G\left(1 + a_1\alpha^2 + a_2\alpha^2 ln(\frac{\Lambda}{\mu})\right) + G^2\Lambda\left(b_0 + b_1\alpha^2 + b_2\alpha^2 ln(\frac{\Lambda}{\mu})\right)\ , &(3.2)
\end{aligned}
$$

where, from (2.9) and (2.12),

$$a_2 = \frac{17}{6}\ ; \tag{3.3}$$

$b_0 = 2/\pi^2$, as given in section 2; the other dimensionless constant a_1, b_1 and b_2 can be calculated from the Feynman diagrams. All these renormalization constants *except* a_2, however, are dependent of regularization and renormalization schemes, and thus are not very physically meaningful. Fortunately, as we shall see soon, these scheme dependent constants determine only the values of the fixed point couplings, which are *unnecessarily* universal.

Using the renormalization group equation on (3.2), we obtain

$$
\begin{aligned}
&-a_2G\alpha^2 + \left(b_0 + b_1\alpha^2 + b_2\alpha^2(1 + ln(\frac{\Lambda}{\mu}))\right)G^2\Lambda\\
=&\left(1 + a_1\alpha^2 + a_2\alpha^2 ln(\frac{\Lambda}{\mu}) - 2G\Lambda(b_0 + b_1\alpha^2 + b_2\alpha^2 ln(\frac{\Lambda}{\mu}))\right)\beta(G)\ , \qquad (3.4)
\end{aligned}
$$

where the beta function of the coupling G is defined as

$$\beta(G) \equiv \Lambda\frac{\partial G(\Lambda)}{\partial\Lambda}\ . \tag{3.5}$$

The solution of the RG equation (3.4) to the order $G^2\alpha^2$ is

$$\beta(G) = -b_0G^2\Lambda + \left(-a_2G + (b_1 - b_0A_1)G^2\Lambda\right)\alpha^2 + (b_2 - a_2b_0)G^2\Lambda\alpha^2 ln(\frac{\Lambda}{\mu})\ . \tag{3.6}$$

As the beta function shouldn't depend on the reference scale μ, the last term in the right hand side of (3.6) must vanish. This gives a consistent condition to the coefficients of the logarithmic terms at the order G and G^2:

$$b_2 = a_2 b_0 \ . \tag{3.7}$$

The condition is verified in the perturbative calculation by seeing that there is no primary logarithmic divergence in the four-fermi vertex at order G^2 (see the previous section and Appendix III).

We may also define a beta function for the dimensionless coupling $g = G\Lambda$:

$$\begin{aligned} \beta(g) &\equiv \Lambda \frac{\partial g(\Lambda)}{\partial \Lambda} \ , & (3.8) \\ &= (1 - a_2\alpha^2) g + \Big(b_0 + (b_1 - b_0 a_1 - b_0 a_2)\alpha^2 \Big) g^2 \ . & (3.9) \end{aligned}$$

From (3.3) and (3.9), we obtain the critical Chern-Simons coupling

$$\alpha_c^2 = \frac{1}{a_2} = \frac{6}{17} \ , \tag{3.10}$$

at which the four-fermi operator has scaling dimension three and thus becomes marginal. Please notice the critical coupling α_c is gauge choice independent as it is calculated from the anomalous dimensions.

Next, we discuss the nature of fixed points g^*, which by definition satisfy

$$\beta(g^*) = 0 \ . \tag{3.11}$$

Obviously, $g = 0$ is a fixed point as always. When $\alpha < \alpha_c$, the slop of the beta function $\beta'(0)$ is *positive*, and therefore the Gaussian fixed point is IR stable. Meanwhile, the four-fermi operator has a scaling dimension $(4 - \frac{17}{6}\alpha^2) > 3$, and so is irrelevant. Therefore the matters in the region $\alpha < \alpha_c$ are the fermion-like.

When $\alpha > \alpha_c$, matters fall in the boson-like phase. The slop of the beta function at $g = 0$ is *negative* and thus the Gaussian fixed point is IR *unstable*. This implies the renormalization group trajectory must flow to some *IR* stable fixed point as the energy scale is decreasing. Setting (3.9) zero, a non-Gaussian fixed point can be solved readily, which is a function of the Chern-Simons coupling α,

$$g^* = -\frac{1}{b_0^2}(b_0 + b_0 a_1 \alpha^2 - b_1 \alpha^2) \ . \tag{3.12}$$

As the slop of the beta function at the non-Gaussian fixed point is *positive*

$$\beta'(g^*) = a_2\alpha^2 - 1 > 0, \quad \text{when } \alpha > \alpha_c \ , \tag{3.13}$$

the non-Gaussian fixed point g^* is of the *IR* stable. By adjusting α ($> \alpha_c$), g^* can be as close to $g = 0$ as possible, so the perturbation expansion in g or G is well-defined and controllable.

The renormalization group flows of the four-fermi coupling are schematically depicted in Fig. 1.

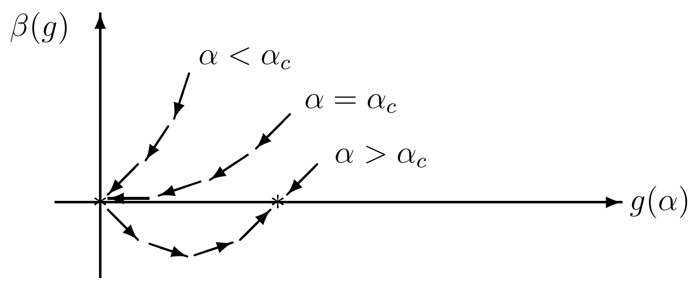


Fig. 1 Renormalization group flows for the Four-fermi coupling. Arrows point toward the infrared. The critical Chern-Simons coupling α_c divides the anyon matters into two types: the fermion-like ($\alpha < \alpha_c$) and the boson-like ($\alpha > \alpha_c$).

As is known, the slops of beta function at fixed points are the anomalous dimensions of the four-fermi operator, and these are the physical quantities that determine the asymptotical behavior of the system. Their determinations solely by only the regularization and renormalization *independent* quantity a_2 reflects the consistency of the quantization procedure used here.

4 Discussions

We have suggested the exitstence of a critical Chern-Simons coupling α_c, by using perturbation expansion and renormalization group method. For a given $\alpha > \alpha_c$, the four-fermi operator becomes relevant in the low-energy limit, and there exists a non-Gaussian *IR* fixed point for the four-fermi coupling g (or G). This critical

Chern-Simons coupling divides the matters into two type: the fermion-like and the boson-like, which have the fixed point structures of the three dimensional fermion and boson theories, respectively.

However, there is essential difference between the fermion-like and fermion systems, and between the boson-like and boson ones. First of all, when $\alpha \neq 0$ and 1, the systems describe anyons which are vary complicated many-body systems, and have not been understood so well as the fermion and boson systems have been. And secondly, as is seen, the Chern-Simons coupling characterizes the universality classes, and therefore all the anyon systems with different $\alpha' s$ have different critical exponents, and belong to different critical models.

The fixed point structure of the four-fermi operator is examined in this work in the perturbative expansion over G. The analysis here should be reliable, as the non-Gaussian IR fixed points g^* can be very small, as long as α ($> \alpha_c$) is sufficiently close to α_c. On the other hand, the theory has been perturbed in the Chern-Simons coupling α as well, the closer it is to the fermion end $\alpha = 0$, the more accurate the results are. As the bose-fermi transmutation strongly suggests the existence of a critical point α_c between $\alpha = 0$ and 1, one is tempted to extrapolate the perturbative results to the not-weak Chern-Simons couplings such as $\alpha^2 \sim a_c^2 = 6/17$, where qualitatively correct conclusions are drawn, as suggested in this work.

But one must be very careful with an extrapolation of large Chern-Simons couplings, as in the extreme case such as $\alpha \sim 1$, the perturbative results can not be trusted even qualitatively. To see this, let's consider three operators in the fermion Chern-Simons model: ψ, $\psi^\dagger\psi$, and $(\psi^\dagger\psi)^2$. Their scaling dimensions upto the second order in α are

$$1 - \frac{1}{6}\alpha^2, \quad 2 - \frac{8}{3}\alpha^2, \quad \text{and} \quad 4 - \frac{17}{6}\alpha^2 \,, \tag{4.1}$$

respectively [31]. Based on the bose-fermi transmutation, on the other hand, these "fermion" operators in the Chern-Simons fermion theory at $\alpha = 1$ correspond to the operators in the scalar theory ϕ, $\phi^*\phi$, and $(\phi^*\phi)^2$. These have scaling dimensions $1/2, 1$, and 2, respectively. Taking $\alpha = 1$ in (4.1), one sees large discrepancies between the perturbative results and that suggested by the bose-fermi transmutation. Particularly, the mass operator would have a negative dimension in the extrapolation! This provides an evidence for the inadequacies of the perturbation expansion beyond its valid regime. The case is unlike the $\epsilon \ = 4 - d$ expansion used in understanding the fixed point structure of the ϕ^4 theory in three dimensions [32][33], where the extrapolation of the controlling parameter ϵ from 0 to 1 gives reasonable values of exponents. To improve estimates, certain techniques can be used in perturbation series when an expansion parameter takes large values, such as Padé approximants which are in fact used in the ϵ expansion. For example, such approximant on the scaling dimension of the mass operator is $\frac{2}{1+4\alpha^2/3}$, which gives a positive value of 6/7 at $\alpha = 1$ (I thank the anonymous referee of the current work for pointing out the approximants). However, essentially, non-perturbative treatment to the extreme cases in the Chern-Simons theory is necessary.

On the other hand, as the Chern-Simons coupling has to be kept small for a reliable perturbative expansion, a resolution has been resently suggested for relativistic Chern-Simons matter theories with considerable large Chern-Simons coefficients. That is, one can map one Chern-Simons matter theory into another Chern-Simons matter theory, in the latter the matter field has higher spin but weaker Chern-Simons interaction [29].

Finally, our study on the renormalization of the four-fermi operator at lower orders seems to suggest the renormalizability of the Chern-Simons fermion theories with α not smaller than α_c, however, it needs further careful considerations for a general proof.

The author is grateful to I. Affleck, L. Dolan, M.P.A. Fisher, C. Itoi, A. Kovner, G.W. Semenoff, and Y.-S. Wu for discussions, and especially to J.W. Gan for his involvement in the early stage of the program. The work was supported in part by the U.S. DOE under contract No. DE-FG05-85ER-40219.

5 Appendix I: Conventions

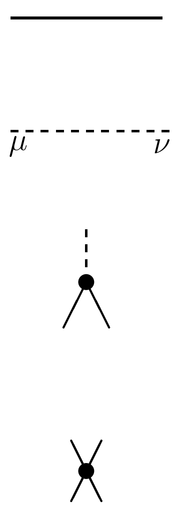
For the two-component fermions, we use Hermitean 2×2 Dirac Matrices:

$$\gamma^\mu = \sigma^\mu \ , \tag{5.1}$$

where σ^μ are Pauli matrices, so that

$$\begin{aligned} \gamma^\mu \gamma^\nu &= \delta^{\mu\nu} \mathbf{1} + i \epsilon^{\mu\nu\lambda} \gamma^\lambda \ , &(5.2) \\ \mathbf{Tr 1} &= 2 \ . &(5.3) \end{aligned}$$

The Feynman rules, read from (2.1) in the Landau gauge, are given in Fig. 2.



$$= 1/(i \not p) \ ,$$

$$= -(2\pi\alpha)\epsilon^{\mu\nu\lambda} p^\lambda / p^2 \ ,$$

$$= \quad i\gamma_\mu \ ,$$

$$= -G \ .$$

Fig. 2 The Feynman rules.

A naive large momentum cut-off Λ is chosen as a regulator. This is equivalent to setting up an energy scale for an effective theory, and so we may define a dimensionless (bare) four-fermi coupling relating to G via

$$g = G\Lambda \ . \tag{5.4}$$

In the rest of this appendix, we consider the four-fermi theory without the Chern-Simons interaction. The one loop fermion bubbles are depicted in Fig. 3.

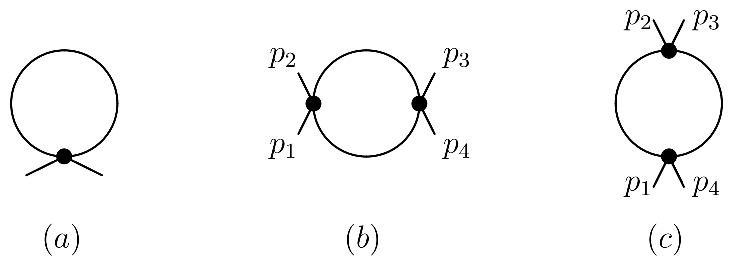


Fig. 3 One loop diagrams for four-fermi theory. The momenta p_1 and p_3 of particles in (b) and (c) are chosen entering into the loop, and p_2 and p_4 of anti-particles outgoing from the loop.

The vanishing of Fig. 3a is obvious as $\mathrm{Tr}\gamma_\mu = 0$. Therefore, the wave-function renormalization constant to this order is trivial

$$Z_\psi = 1 + \mathcal{O}(G^2) \; . \tag{5.5}$$

The correction to the four-fermi vertex from Fig. 1b and 1c is

$$\Gamma(\Lambda, p_i) = G^2\left(\frac{2\Lambda}{\pi^2} - \frac{|p_1 - p_2| + |p_1 - p_4|}{8}\right) \; . \tag{5.6}$$

We define the renormalized coupling constant g_r at a reference energy scale μ as

$$-\frac{g_r}{\mu} \equiv \Gamma(\Lambda, p_i = 0) = -\frac{1}{\Lambda}\left(g - \frac{2}{\pi^2}g^2\right) \; . \tag{5.7}$$

Then the Callan-Symanzik beta function for the dimensionless coupling g is

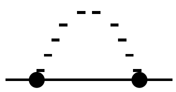
$$\beta(g) = g + \frac{2}{\pi^2}g^2 + \mathcal{O}(g^3) \; . \tag{5.8}$$

The renormalization group trajectory is schematically depicted in Fig. 1 (read $\alpha = 0 < \alpha_c$). It explicitly shows that near $g = 0$, the four-fermi coupling has only a Gaussian IR fixed point, therefore is *trivial*. This is necessarily consistent with the fact that the four-fermi operator in three dimensions is irrelevant or non-renormalizable.

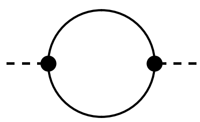
6 Appendix II: Wave-function Renormalization

We discuss in this Appendix the fermion self-energy, and calculate the fermion wave-function renormalization constant to the order $G\alpha^2$.

The fermion and Chern-Simons self-energies at order $G^0\alpha$ are readily to calculate



$$= i(2\alpha/\pi)\Lambda - i(\pi\alpha/4)|p| \ , \qquad (a)$$

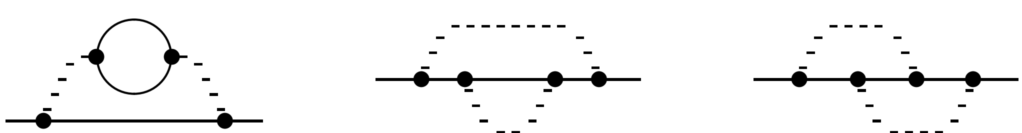


$$= (1/\pi^2)\Lambda\delta^{\mu\nu} - (/16)|p|(\delta^{\mu\nu} - p^\mu p^\nu/p^2) \ . \qquad (b)$$

Fig. 4 (a) Fermion and (b) Chern-Simons self-energies at order $\mathcal{O}(\alpha)$. There is no correction at this order to the wave-function renormalization constants.

Fig. 4 carries two messages: The first, there is a need of mass renormalization for the fermion and Chern-Simons fields because of the linear terms in Λ in 4a and 4b. We simply get rid of them by fine tuning with mass counterterms for both the fermion and Chern-Simons fields so that they are kept massless. And the second, there is no contribution to the wave-function renormalizations at this order, but it generates finite non-local terms to the effective theory, due to the second terms in 4a and 4b.

The fermion self-energy diagrams at order $G^0\alpha^2$ are given in Fig. 5.



(a) (b) (c)

Fig. 5 The fermion self-energy at $\mathcal{O}(\alpha^2)$. Logarithmic divergence appears here.

Calculating the diagrams, after one of the two loop integrations is done, we are left with

$$(f.5a) = -\frac{i(\pi\alpha)^2}{2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\frac{\not{k}(k+p)\cdot k}{(k+p)^2k^3}\ , \tag{6.1}$$

$$(f.5b) = -i(\pi\alpha)^2\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\frac{\not{k}}{|k+p|k^2}\ , \tag{6.2}$$

$$(f.5c) = -\frac{i(\pi\alpha)^2}{2}\frac{\not{p}}{p^2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3}[\frac{(k+p)\cdot p}{(k+p)^2k} - \frac{1}{2}\frac{k}{(k+p)^2} + \frac{1}{2}\frac{(k-p)\cdot k}{|k+p|k^2} - \frac{p}{|k+p|k} + \frac{p(k+\frac{1}{2}p)\cdot k}{(k+p)^2k^2}]\ . \tag{6.3}$$

Completing the integral over k, we obtain the fermion self-energy to this order

$$\Sigma(\Lambda,p) = i\ \not{p}\frac{\alpha^2}{6}\left(ln(\frac{\Lambda}{p}) - \frac{4}{3} + \frac{3\pi^2}{32}\right)\ . \tag{6.4}$$

Here it shows up first the logarithmic divergence, which obviously requires a non-trivial fermion wave-function renormalization.

To continue the discussion with the fermion self-energy to higher orders, let us consider first these three vertex corrections that have a fermion loop. The relevant diagrams are given in Fig. 6. Fig. 6a vanishes due to its tensor structure: the trace over the Dirac matrices in the fermion loop gives $Tr(\gamma_\mu\gamma_\nu\gamma_\lambda) = 2\epsilon_{\mu\nu\lambda}$, while its contraction with $p_\nu p_\lambda$ from the loop integration is zero. It is not difficult to check that Fig. 6b and 6c are proportional to p_μ, the momentum carried by the external Chern-Simons line.

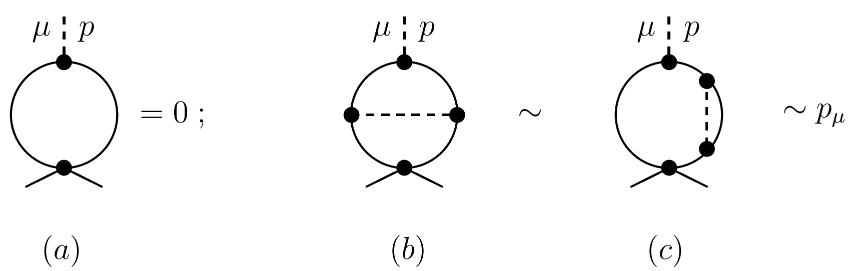
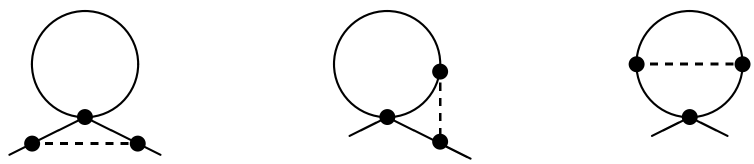


Fig. 6 The three vertex at $\mathcal{O}(G)$ and at $\mathcal{O}(G\alpha)$. (a) vanishes; (b) and (c) are proportional to p_μ.

The diagrams for the fermion self-energy at order $G\alpha$ can be obtained by inserting one Chern-Simons propagator as an internal line into the fermion self-energy diagram

Fig. 3a in all possible ways. These are given in Fig. 7. At this order, there is no correction to the fermion wave-function renormalization: Fig. 7a vanishes, because Fig. 3a is zero; Fig. 7b vanishes as one of its subdiagrams, shown in Fig. 6a, is zero; And finally, Fig. 7c is proportional to Λ and so contributes to the fermion mass, which, by fine tuning, is sent to zero.



(a) (b) (c)

Fig. 7 The fermion self-energy at $\mathcal{O}(G\alpha)$. There is no correction to the wave-function renormalization at this order.

Finally, at the order $G\alpha^2$ there is no contribution to the fermion wave-function renormalization either. This is because, besides many vanishing diagrams such as those with any of Fig. 3a, Fig. 6a, 6b and 6c, and Fig. 7a, 7b and 7c as a sub-diagram, all the non-vanishing diagrams are of form

$$G\alpha^2\Lambda(a\Lambda + b|p|) \, , \tag{6.5}$$

which matters only the fermion mass renormalization with a and b finite constants, as discussed above. A typical diagram of this type is drawn in Fig. 8.

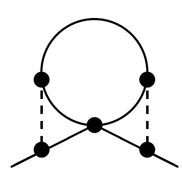


Fig. 8 A typical non-vanishing diagram of fermion self-energy at $\mathcal{O}(G\alpha^2)$. It takes a form $\Lambda(a\Lambda + b|p|)$.

In summary, the fermion wave-function renormalization constant is

$$Z_\psi = 1 + \frac{\alpha^2}{6} ln(\frac{\Lambda}{\mu}) + \mathcal{O}(G^2) \ . \tag{6.6}$$

7 Appendix III: Four-Fermion Vertex Function

In this Appendix, we consider the four-fermi vertex $\Gamma(\Lambda, p_1, p_2, p_3, p_4)$ to the order $G^2\alpha^2$. A representative diagram at order $G\alpha$ is depicted in Fig. 9.

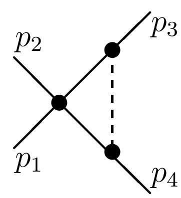


Fig. 9 Four-fermi vertex at $\mathcal{O}(G\alpha)$. It generates new finite term in the effective Lagrangian.

We calculate Fig. 9, and obtain ($i = 1, 2, 3, 4$)

$$\Gamma^{(1,1)}(\Lambda, p_i) = \frac{i\pi\alpha G}{8} \frac{p_3 + p_4 - |p_3 - p_4|}{p_3 + p_4 + |p_3 - p_4|} (\frac{\not{p}_3}{p_3} + \frac{\not{p}_4}{p_4}) + \text{permutation in } p_i \ . \tag{7.1}$$

Since (7.1) involves a unit Dirac vector $n \cdot \gamma$ ($n^2 = 1$) which is not seen in the bare Lagrangian, this means a generation of a new operator in the effective Lagrangian. An induced new interaction may probably need a new coupling constant and so its renormalization. However, it is not the case in the present model, at least to the second order. To see this, let us consider the order $G^2\alpha$. The diagrams are depicted in Fig. 10.

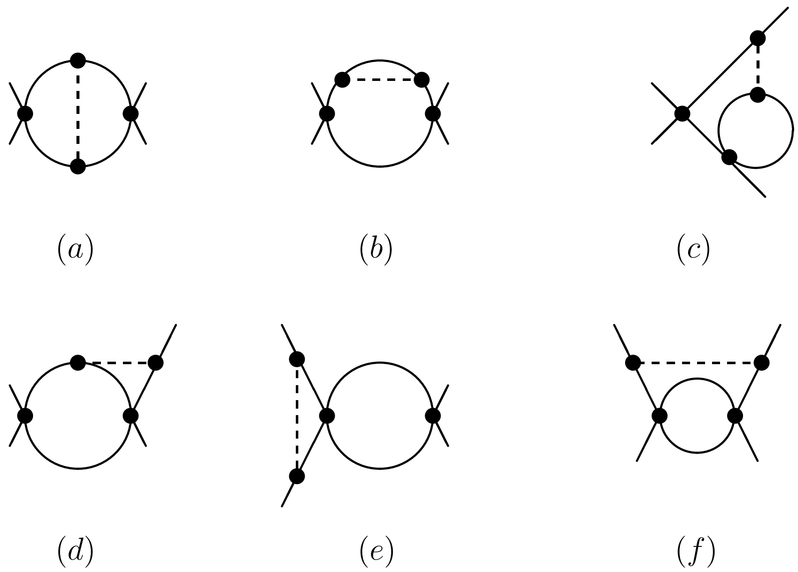


Fig. 10 Four-fermi vertex at $\mathcal{O}(G^2\alpha)$.

Not losing the generality, we set $p_1 = p_2 = p_3 = p_4$. It is easy to check

$$(f.10a) \ = \ (f.10b) \ = \ (f.10c) \ = \ (f.10d) \ = 0 \ , \tag{7.2}$$

$$(f.10e) \times 2 + (f.10f) \times 2 = -G^2(\frac{2\Lambda}{\pi^2})(i\pi\alpha n \cdot \gamma) \ . \tag{7.3}$$

Therefore, to the order $G^2\alpha$, the vertex function is

$$\Gamma(\Lambda) = -G(1 - i\pi\alpha n \cdot \gamma) + G^2(\frac{2\Lambda}{\pi^2})(1 - i\pi\alpha n \cdot \gamma) \ . \tag{7.4}$$

This implies that the coupling constant G and its renormalization is sufficient for both the original and the induced self-interactions. (7.4) also shows that at the order α^1, there is no non-trivial renormalization to the four-fermi coupling.

Let us come to the next order in α. The Feynman diagrams for the four-fermi vertex at order $G\alpha^2$ fall into two categories: the logarithmic divergent and the finite, depicted in Fig. 11 and 12, respectively.

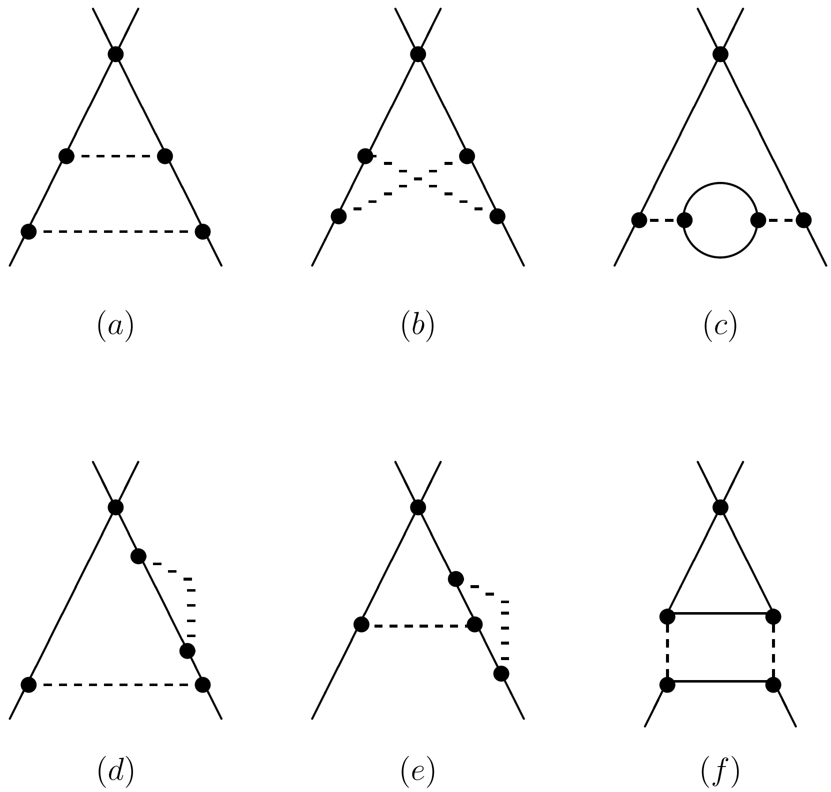


Fig. 11 Four-fermi vertex at $\mathcal{O}(G\alpha^2)$: the logarithmic divergent diagrams.

Fig. 12 Four-fermi vertex at $\mathcal{O}(G\alpha^2)$: the finite diagrams.

After integrating out one of the two Feynman integrations of the logarithmic divergent diagrams in Fig. 11, we arrive at

$$
\begin{aligned}
(f.11a) + (f.11d) \times 2 &= -G(\pi\alpha)^2 \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{(k+p)^2 k} , & (7.5)\\
(f.11b) + (f.11e) \times 2 &= -\frac{3}{2} G(\pi\alpha)^2 \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{(k+p)^2 k} , & (7.6)\\
(f.11c) + (f.11f) \times 2 &= -\frac{5}{2} G(\pi\alpha)^2 \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{(k+p)^2 k} . & (7.7)
\end{aligned}
$$

Then, completing the integral over k, we obtain

$$\Gamma^{(1,2)}(\Lambda, p) = -\frac{5}{2} G\alpha^2 [ln(\frac{\Lambda}{p}) + finite] , \tag{7.8}$$

where, the *finite* part, from Fig. 11 and 12, is regularization and renormalization dependent.

At the order $G^2\alpha^2$, most of the four-fermi vertex Feynman diagrams can be obtained by inserting one Chern-Simons propagator as an internal line into all the diagrams in Fig. 10. As the number of such diagrams is so huge, we decided to not draw them here but present the result of our analysis. These (three-loops) diagrams belong to one of the three types: (1) vanishing, (2) linear in Λ, and (3) proportional to $\Lambda ln(\frac{\Lambda}{\mu})$. The vertex function at the renormalization point takes the form

$$\Gamma^{(2,2)}(\Lambda, p = 0) = G^2\alpha^2\Lambda(b_1 + b_2 ln(\frac{\Lambda}{\mu})) . \tag{7.9}$$

A diagram-by-diagram investigation shows that all the diagrams that contribute to b_2 contain one of the diagrams in Fig. 11 as a sub-diagram. This implies the four-fermi vertex at order $G^2\alpha^2$ has no primary logarithmic divergence. Again, b_1 is regularization and renormalization dependent.

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