

ALGEBRAIC BETHE ANSATZ FOR THE ELLIPTIC QUANTUM GROUP $E_{\tau,\eta}(sl_2)$

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ABSTRACT. To each representation of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ is associated a family of commuting transfer matrices. We give common eigenvectors by a version of the algebraic Bethe ansatz method. Special cases of this construction give eigenvectors for IRF models, for the eight-vertex model and for the two-body Ruijsenaars operator. The latter is a q -deformation of Hermite's solution of the Lamé equation.

1. INTRODUCTION

The Bethe ansatz is a method to construct common eigenvectors of commuting families of operators (transfer matrices) occurring in two-dimensional models of statistical mechanics. Faddeev and Takhtadzhan [TF] reformulated the problem into a question of representation theory: commuting families of transfer matrices are associated to representations of certain algebras with quadratic relations (now called quantum groups). Eigenvectors are constructed by properly acting with algebra elements on "highest weight vectors". In this form, the Bethe ansatz is called algebraic Bethe ansatz.

Whereas this construction has been very successful in rational and trigonometric integrable models, its extension to elliptic models has been problematic, although the Bethe ansatz, in the case of the eight-vertex model, is known since Baxter's work [B]. For elliptic models associated to sl_N , an algebra with quadratic relations has been introduced by Sklyanin [S], but the notion of highest weight vector is not defined for its representations, which makes a direct application of the algebraic Bethe ansatz impossible.

Recently, a definition of elliptic quantum groups $E_{\tau,\eta}(\mathfrak{g})$ associated to any simple classical Lie algebra \mathfrak{g} was given [F]. It is related to a q -deformation of the Knizhnik–Zamolodchikov–Bernard equation on tori. The representation theory of $E_{\tau,\eta}(sl_2)$ was described in [FV3].

In this paper we describe the algebraic Bethe ansatz for $E_{\tau,\eta}(sl_2)$. The construction is very close to the construction done in the trigonometric case in [TF]. The main difference is that transfer matrices act on spaces of vector-valued functions rather than on finite dimensional vector spaces.

The basic results in this paper are Theorems 4 and 5. We present them in Sect. 3 after a summary of the representation theory of $E_{\tau,\eta}(sl_2)$ in Sect.

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2. In Sect. 4 we formulate our result in the discrete case, which gives a construction of eigenfunctions for interaction-round-a-face models (Sect. 5).

In Sect. 6 we introduce a version of Baxter's vertex-IRF transformation. With the help of this, we show how to obtain eigenvectors of the eight-vertex model transfer matrix from our eigenvectors. From this point of view, our construction is very reminiscent to Baxter's transformation of the eight-vertex model into an inhomogeneous six-vertex model [B]. Also, this vertex-IRF transformation shows a very close relation of $E_{\tau,\eta}$ with Sklyanin's algebra in the sl_2 case. Note however that Sklyanin's algebra can only be defined for sl_N , whereas the elliptic quantum group exists (at least) for all classical simple Lie algebras.

Another class of problems in which the Bethe ansatz has been applied successfully is the class of Calogero–Moser–Sutherland quantum many body problems on the line with elliptic potentials, see [FV1], [FV2]. In the case of two bodies, the Bethe ansatz goes actually back to Hermite, who solved in this way the generalized Lamé equation, cf. [WW]. These integrable Schrödinger operators admit a q -deformation due to Ruijsenaars [R]. In Sect. 7 we present a q -deformation of Hermite's result, i.e., we give eigenfunctions for the two-body Ruijsenaars operator. These eigenfunctions are parametrized by a *spectral curve*, similarly to the differential case. It is a double covering of a hyperelliptic curve. This result follows from our result and the observation that the transfer matrix associated to an evaluation representation is equal to the Ruijsenaars operator up to a scalar factor depending on the spectral parameter. A similar observation, relating transfer matrices to Ruijsenaars operators, has been made recently by Hasegawa [H] in the context of Sklyanin's algebra, indicating that our construction should be extendable to the N -body case.

2. MODULES OVER $E_{\tau,\eta}(sl_2)$ AND TRANSFER MATRICES

In this paper, we construct eigenvectors of the transfer matrix of the elliptic quantum group $E_{\tau,\eta}(sl_2)$, [F, FV3], associated to certain highest weight modules.

Let us recall the definitions: we fix two complex parameters τ, η , such that $\text{Im}(\tau) > 0$. The definition of $E_{\tau,\eta}(sl_2)$ is based on an R -matrix $R(z, \lambda)$ which we now introduce. Let

$$\theta(t) = - \sum_{j \in \mathbb{Z}} e^{\pi i(j + \frac{1}{2})^2 \tau + 2\pi i(j + \frac{1}{2})(t + \frac{1}{2})}, \quad (1)$$

be Jacobi's theta function and

$$\alpha(z, \lambda) = \frac{\theta(\lambda + 2\eta)\theta(z)}{\theta(\lambda)\theta(z - 2\eta)}, \quad \beta(z, \lambda) = -\frac{\theta(\lambda + z)\theta(2\eta)}{\theta(\lambda)\theta(z - 2\eta)},$$

Let V be a two dimensional complex vector space with basis $e[1], e[-1]$, and let $E_{ij}e[k] = \delta_{jk}e[i]$, $h = E_{11} - E_{-1,-1}$. Then, for $z, \lambda \in \mathbb{C}$, $R(z, \lambda) \in$

$\text{End}(V \otimes V)$ is the matrix

$$\begin{aligned} R(z, \lambda) &= E_{11} \otimes E_{11} + E_{-1,-1} \otimes E_{-1,-1} + \alpha(z, \lambda) E_{11} \otimes E_{-1,-1} \\ &\quad + \alpha(z, -\lambda) E_{-1,-1} \otimes E_{11} + \beta(z, \lambda) E_{1,-1} \otimes E_{-1,1} + \beta(z, -\lambda) E_{-1,1} \otimes E_{1,-1}. \end{aligned}$$

This R -matrix obeys the dynamical (or modified) quantum Yang–Baxter equation

$$\begin{aligned} R^{(12)}(z-w, \lambda-2\eta h^{(3)}) R^{(13)}(z, \lambda) R^{(23)}(w, \lambda-2\eta h^{(1)}) = \\ R^{(23)}(w, \lambda) R^{(13)}(z, \lambda-2\eta h^{(2)}) R^{(12)}(z-w, \lambda) \end{aligned}$$

in $\text{End}(V \otimes V \otimes V)$, $z, w, \lambda \in \mathbb{C}$. The meaning of this notation is the following: $R^{(12)}(\lambda-2\eta h^{(3)})v_1 \otimes v_2 \otimes v_3$ is defined as

$$(R(z, \lambda-2\eta\mu_3)v_1 \otimes v_2) \otimes v_3,$$

if $hv_3 = \mu_3v_3$. The other terms are defined similarly: in general, let V_1, \dots, V_n be modules over the one dimensional Lie algebra $\mathfrak{h} = \mathbb{C}h$ with one generator h , such that, for all i , V_i is the direct sum of finite dimensional eigenspaces $V_i[\mu]$ of h , labeled by the eigenvalue μ . We call such modules diagonalizable \mathfrak{h} -modules. If $X \in \text{End}(V_i)$ we denote by $X^{(i)} \in \text{End}(V_1 \otimes \dots \otimes V_n)$ the operator $\dots \otimes \text{Id} \otimes X \otimes \text{Id} \otimes \dots$ acting non-trivially on the i th factor, and if $X = \sum X_k \otimes Y_k \in \text{End}(V_i \otimes V_j)$ we set $X^{(ij)} = \sum X_k^{(i)} Y_k^{(j)}$. If $X(\mu_1, \dots, \mu_n)$ is a function with values in $\text{End}(V_1 \otimes \dots \otimes V_n)$, then $X(h^{(1)}, \dots, h^{(n)})v = X(\mu_1, \dots, \mu_n)v$ if $h^{(i)}v = \mu_i v$, for all $i = 1, \dots, n$.

Now, by definition, a module over $E_{\tau, \eta}(sl_2)$ is a diagonalizable \mathfrak{h} -module $W = \bigoplus_{\mu \in \mathbb{C}} W[\mu]$, together with an L -operator $L(z, \lambda) \in \text{End}_{\mathfrak{h}}(V \otimes W)$ (a linear map commuting with $h^{(1)} + h^{(2)}$) depending meromorphically on $z, \lambda \in \mathbb{C}$ and obeying the relations

$$\begin{aligned} R^{(12)}(z-w, \lambda-2\eta h^{(3)}) L^{(13)}(z, \lambda) L^{(23)}(w, \lambda-2\eta h^{(1)}) = \\ L^{(23)}(w, \lambda) L^{(13)}(z, \lambda-2\eta h^{(2)}) R^{(12)}(z-w, \lambda) \end{aligned}$$

For example, $W = V$, $L(w, \lambda) = R(w - z_0, \lambda)$ is a module over $E_{\tau, \eta}(sl_2)$, called the fundamental representation, with evaluation point z_0 . In [FV3] more general examples of such modules were constructed: in particular, for any pair of complex numbers Λ, z we have an *evaluation Verma module* $V_{\Lambda}(z)$. Also, we have a notion of tensor products of modules over $E_{\tau, \eta}(sl_2)$. The main examples considered in this paper will be tensor products $V_{\Lambda_1}(z_1) \otimes \dots \otimes V_{\Lambda_n}(z_n)$ of evaluation Verma modules and some of their subquotients.

For any module W over $E_{\tau, \eta}(sl_2)$, we define the associated operator algebra, an algebra of operators on the space $\text{Fun}(W)$ of meromorphic functions of $\lambda \in \mathbb{C}$ with values in W . It is generated by h , acting on the values, and operators $a(z), b(z), c(z), d(z)$. Namely, let $\tilde{L}(z) \in \text{End}(V \otimes \text{Fun}(W))$ be the operator defined by $(\tilde{L}(z)(v \otimes f))(\lambda) = L(z, \lambda)(v \otimes f(\lambda - 2\eta\mu))$ if $hv = \mu v$.

View $\tilde{L}(z)$ as a 2 by 2 matrix with entries in $\text{End}(\text{Fun}(W))$:

$$\begin{aligned}\tilde{L}(z)(e[1] \otimes f) &= e[1] \otimes a(z)f + e[-1] \otimes c(z)f, \\ \tilde{L}(z)(e[-1] \otimes f) &= e[1] \otimes b(z)f + e[-1] \otimes d(z)f.\end{aligned}$$

The relations obeyed by these operators are described in detail in [FV3] (in [FV3] these operators are denoted by $\tilde{a}(z), \tilde{b}(z)$ and so on).

Theorem 1. [FV3] *For any module W over $E_{\tau, \eta}(sl_2)$, the transfer matrices $T(z) = a(z) + d(z)$ preserve the space $H = \text{Fun}(W)[0]$ of functions with values in the zero weight space $W[0]$, and commute pairwise on H : $T(z)T(w) = T(w)T(z)$ on H .*

Theorem 2. [FV3] *Let $W = V_{\Lambda_1}(z_1) \otimes \cdots \otimes V_{\Lambda_n}(z_n)$ be a tensor product of evaluation modules, and let $\Lambda = \Lambda_1 + \cdots + \Lambda_n$. Then $W[\Lambda] = \mathbb{C}v_0$, where v_0 , viewed as a constant function in $\text{Fun}(W)$ obeys the following highest weight condition: for every z , $c(z)v_0 = 0, a(z)v_0 = A(z, \lambda)v_0, d(z)v_0 = D(z, \lambda)v_0$, with highest weight functions*

$$A(z, \lambda) = 1, \quad D(z, \lambda) = \frac{\theta(\lambda - 2\eta\Lambda)}{\theta(\lambda)} \prod_{j=1}^n \frac{\theta(z - p_j)}{\theta(z - q_j)}, \quad (2)$$

where we set $p_j = z_j + \eta(-\Lambda_j + 1), q_j = z_j + \eta(\Lambda_j + 1)$

A highest weight module W of highest weight (Λ, A, D) , is a module with a highest weight vector $v_0 \in \text{Fun}(W)$ such that $c(z)v_0 = 0, a(z)v_0 = A(z, \lambda)v_0, d(z)v_0 = D(z, \lambda)v_0$, for all z, λ , and so that $\text{Fun}(W)$ is spanned by the vectors of the form $b(t_1) \cdots b(t_j)v_0$, as a vector space over the field of meromorphic functions of λ . It is shown in [FV3] that if A, D are of the form (2), for some p_k, q_k , then every irreducible highest weight module of weight (Λ, A, D) is isomorphic to a subquotient of $\otimes V_{\Lambda_i}(z_i)$, where the parameters z_j, Λ_j are related to p_k, q_k as in the theorem. If p_k, q_k are generic under the condition that $\sum(p_i - q_i) = 2\eta\Lambda$, then all highest weight modules of weight (Λ, A, D) are isomorphic to $\otimes V_{\Lambda_i}(z_i)$ itself.

3. BETHE ANSATZ

In this section we fix a highest weight module W of weight (Λ, A, D) of the form (2), with highest weight vector v_0 . We assume that Λ is an even integer $2m \geq 0$, so that the zero-weight space $W[0]$ can be nontrivial.

We follow the strategy of [TF]: we seek common eigenvectors of $T(w)$ in the form $b(t_1) \cdots b(t_m)v$, where $v \in \text{Fun}(W)[\Lambda]$. The problem is to find conditions for t_1, \dots, t_m, v so that we have an eigenvector. The question of completeness, i.e., whether “all” eigenvectors can be obtained in this way, will not be addressed here, except in the example of the q -analogue of the Lamé equation discussed below.

Any non-zero vector $v \in \text{Fun}(W)[\Lambda]$ is of the form $v = g(\lambda)v_0$, for some meromorphic function $g \neq 0$. To find for which values of t_1, \dots, t_m and which

choice of v we get an eigenvector we commute a and d with the b 's using the relations of the quantum group. The relevant commutation relations are (see [FV3])

$$\begin{aligned} a(w)b(t) &= r(t-w, \lambda)b(t)a(w) + s(t-w, \lambda)b(w)a(t), \\ d(w)b(t) &= r(w-t, \lambda-2\eta h)b(t)d(w) - s(t-w, \lambda-2\eta h)b(w)d(t). \end{aligned}$$

The functions r, s are defined by the formulae

$$r(t, \lambda) = \frac{\theta(t-2\eta)\theta(\lambda)}{\theta(t)\theta(\lambda-2\eta)}, \quad s(t, \lambda) = \frac{\theta(t+\lambda)\theta(2\eta)}{\theta(t)\theta(\lambda-2\eta)}.$$

Note that these coefficients have poles for $t \in \mathbb{Z} + \tau\mathbb{Z}$, so we have to be careful to avoid infinities in the calculations: we assume that w, t_1, \dots, t_m are all distinct modulo $\mathbb{Z} + \tau\mathbb{Z}$. Using repeatedly the commutation relations to bring a to the right of the b 's, we get

$$\begin{aligned} a(w)b(t_1) \cdots b(t_m) &= A_0 b(t_1) \cdots b(t_m) a(w) \\ &\quad + \sum_{j=1}^n A_j b(t_1) \cdots b(t_{j-1}) b(w) b(t_{j+1}) \cdots b(t_m) a(t_j) \end{aligned}$$

for some complex coefficients $A_j = A_j(w, t_1, \dots, t_m, \lambda)$. The first term is called the “wanted” term, and the others are “unwanted” terms. Similarly, we have

$$\begin{aligned} d(w)b(t_1) \cdots b(t_m) &= D_0 b(t_1) \cdots b(t_m) d(w) \\ &\quad + \sum_{j=1}^n D_j b(t_1) \cdots b(t_{j-1}) b(w) b(t_{j+1}) \cdots b(t_m) d(t_j) \end{aligned}$$

for some complex coefficients $D_j = D_j(w, t_1, \dots, t_m, \lambda)$. The coefficients A_0 and A_1 are easy to compute, since they are products of coefficients appearing in the commutation relations:

$$\begin{aligned} A_0 &= \prod_{j=1}^m r(t_j - w, \lambda + 2\eta(j-1)), \\ A_1 &= s(t_1 - w, \lambda) \prod_{j=2}^m r(t_j - t_1, \lambda + 2\eta(j-1)). \end{aligned}$$

A similar calculation gives

$$\begin{aligned} D_0 &= \prod_{j=1}^m r(w - t_j, \lambda - 2\eta(j-1)), \\ D_1 &= -s(t_1 - w, \lambda) \prod_{j=2}^m r(t_1 - t_j, \lambda - 2\eta(j-1)). \end{aligned}$$

A direct calculation of the other coefficients is more complicated. However the answer is simple, thanks to the

Lemma 3. *For any permutation σ of m letters, and all $j = 1, \dots, m$,*

$$\begin{aligned} A_j(w, t_{\sigma(1)}, \dots, t_{\sigma(m)}, \lambda) &= A_{\sigma(j)}(w, t_1, \dots, t_m, \lambda), \\ D_j(w, t_{\sigma(1)}, \dots, t_{\sigma(m)}, \lambda) &= D_{\sigma(j)}(w, t_1, \dots, t_m, \lambda), \end{aligned}$$

The proof of this Lemma is deferred to Sect. 8.

The next step is to find conditions for cancellation of unwanted terms. We have, with $v(\lambda) = g(\lambda)v_0$,

$$\begin{aligned} T(w)b(t_1) \cdots b(t_m)v &= C_0 b(t_1) \cdots b(t_m)v \\ &+ \sum_{j=1}^n C_j b(t_1) \cdots b(t_{j-1})b(w)b(t_{j+1}) \cdots b(t_m)v \end{aligned} \quad (3)$$

for some coefficients C_j . The condition of cancellation is $C_j = 0$, $j \geq 1$. Let us first consider C_1 . It is given by

$$\begin{aligned} C_1(w, t, \lambda) &= A_1(w, t, \lambda) \frac{g(\lambda + 2\eta(m-1))}{g(\lambda + 2\eta m)} \\ &+ D_1(w, t, \lambda) D(t_1, \lambda + 2\eta m) \frac{g(\lambda + 2\eta(m+1))}{g(\lambda + 2\eta m)}. \end{aligned}$$

The condition $C_1 = 0$ is then equivalent to

$$\begin{aligned} &r(t_2 - t_1, \lambda + 2\eta) \cdots r(t_m - t_1, \lambda + 2\eta(m-1)) g(\lambda + 2\eta(m-1)) \\ &= r(t_1 - t_2, \lambda - 2\eta) \cdots r(t_1 - t_m, \lambda - 2\eta(m-1)) D(t_1, \lambda + 2\eta m) g(\lambda + 2\eta(m+1)) \end{aligned}$$

which can be written as

$$\prod_{j=2}^m \frac{\theta(t_j - t_1 - 2\eta)}{\theta(t_j - t_1 + 2\eta)} \prod_{k=1}^n \frac{\theta(t_1 - q_k)}{\theta(t_1 - p_k)} \frac{\theta(\lambda + 2\eta(m-1))\theta(\lambda + 2\eta m)g(\lambda + 2\eta(m-1))}{\theta(\lambda)\theta(\lambda - 2\eta)g(\lambda + 2\eta(m+1))} = 1$$

In particular, the left-hand side of this equation should be independent of λ . This holds if g is taken in the form

$$g(\lambda) = e^{c\lambda} \prod_{j=1}^m \frac{\theta(\lambda - 2\eta j)}{\theta(2\eta)}.$$

Thus C_1 vanishes if g has this form and the t_i obey the equation

$$\prod_{j=2}^n \frac{\theta(t_j - t_1 - 2\eta)}{\theta(t_j - t_1 + 2\eta)} \prod_{k=1}^n \frac{\theta(t_1 - q_k)}{\theta(t_1 - p_k)} = e^{4\eta c}$$

Using Lemma 3 we find the conditions for C_j to vanish for $j = 1, \dots, m$. The result is:

Theorem 4. *Let W be a highest weight module over $E_{\tau, \eta}(sl_2)$ of highest weight (Λ, A, D) with A, D of the form (2) and $\Lambda = 2m \in 2\mathbb{Z}_{\geq 0}$. Let $T(w) \in \text{End}(H)$, $w \in \mathbb{C}$, be the corresponding transfer matrices. Let*

$v(\lambda) = e^{c\lambda} \prod_{j=1}^m \theta(\lambda - 2\eta j) v_0$. Then, for any solution (t_1, \dots, t_m) of the Bethe ansatz equations

$$\prod_{j:j \neq i} \frac{\theta(t_j - t_i - 2\eta)}{\theta(t_j - t_i + 2\eta)} \prod_{k=1}^n \frac{\theta(t_i - q_k)}{\theta(t_i - p_k)} = e^{4\eta c}, \quad i = 1, \dots, m, \quad (4)$$

such that, for all $i < j$, $t_i \neq t_j \pmod{\mathbb{Z} + \tau\mathbb{Z}}$, the vector $b(t_1) \cdots b(t_m)v$ is a common eigenvector of all transfer matrices $T(w)$ with eigenvalues

$$\epsilon(w) = e^{-2\eta c} \prod_{j=1}^m \frac{\theta(t_j - w - 2\eta)}{\theta(t_j - w)} + e^{2\eta c} \prod_{j=1}^m \frac{\theta(t_j - w + 2\eta)}{\theta(t_j - w)} \prod_{k=1}^n \frac{\theta(w - p_k)}{\theta(w - q_k)}$$

What is left to prove is the formula for the eigenvalue, which is given by C_0 in (3). By definition

$$C_0 = A_0 \frac{g(\lambda + 2\eta(m-1))}{g(\lambda + 2\eta m)} + D_0 \frac{g(\lambda + 2\eta(m+1))}{g(\lambda + 2\eta m)} D(w, \lambda + 2\eta m).$$

By inserting the formulas for A_0, D_0, g, D , we see that the λ dependence disappears for the same reason as before, and we are left with the formula for ϵ given in the theorem.

We conclude this section by giving an explicit formula for $b(t_1) \cdots b(t_m)v$. It is sufficient to consider the case where $W = V_{\Lambda_1}(z_1) \otimes \cdots \otimes V_{\Lambda_n}(z_n)$, for any other highest weight module considered in the previous Theorem is a quotient of the submodule of a module of this form generated by the product of highest weight vectors. Recall [FV3] that $V_{\Lambda}(z)$ is defined to be the infinite dimensional vector space with basis e_0, e_1, e_2, \dots , such that $he_j = (\Lambda - 2j)e_j$, and with the action of the generators of $E_{\tau, \eta}(sl_2)$ given by explicit formulae.

Theorem 5. *With the notations as in the previous theorem,*

$$\begin{aligned} b(t_1) \cdots b(t_m)v &= (-1)^m e^{c(\lambda + 2\eta m)} \sum_{I_1, \dots, I_n} \prod_{l=1}^n \prod_{i \in I_l} \prod_{k=l+1}^n \frac{\theta(t_i - p_k)}{\theta(t_i - q_k)} \\ &\times \prod_{k < l} \prod_{i \in I_k, j \in I_l} \frac{\theta(t_i - t_j - 2\eta)}{\theta(t_i - t_j)} \\ &\times \prod_{k=1}^n \prod_{j \in I_k} \frac{\theta(\lambda + t_j - q_k + 2\eta m_k - 2\eta \sum_{l=k+1}^n (\Lambda_l - 2m_l))}{\theta(t_j - q_k)} e_{m_1} \otimes \cdots \otimes e_{m_n}. \end{aligned}$$

The summation is over all partitions of $\{1, \dots, m\}$ into n disjoint subsets I_1, \dots, I_n . The cardinality of I_j is denoted by m_j .

In particular, we see that our eigenvector $\psi(\lambda)$ is an entire function of λ obeying $\psi(\lambda + 1) = (-1)^m e^c \psi(\lambda)$. The proof of this Theorem is contained in Section 8.

4. DISCRETE MODELS

The construction of the transfer matrix admits the following variation. The difference operators $a(w), \dots, d(w)$ shift the argument of functions by $\pm 2\eta$. Therefore we may replace $\text{Fun}(W)$ by the space $\text{Fun}_\mu(W)$ of all functions from the set $C_\mu = \{\mu + 2\eta j, j \in \mathbb{Z}\}$ to W . The operators are then well-defined on $\text{Fun}_\mu(W)$ if μ is generic. Also, it follows from Theorem 5 that the restriction to C_μ of the Bethe ansatz eigenfunctions is well defined for all μ . We thus have:

Corollary 6. *Suppose t_1, \dots, t_m is a solution to the Bethe ansatz equations (4). Then, for generic μ , the restriction to C_μ of $b(t_1) \cdots b(t_m)v$ is a common eigenfunction of the operators $T(w) \in \text{End}(\text{Fun}_\mu(W)[0])$.*

5. INTERACTION-ROUND-A-FACE MODELS

In this section, we consider a special case of the above construction, and relate $T(w)$ to the transfer matrix of the interaction-round-a-face (IRF) (also called solid-on-solid) models of [B], [ABF]. Therefore, our formulae give in particular eigenvectors for transfer matrices of IRF models.

Notice first (see [F]) that if we define¹ a ‘‘Boltzmann weight’’ $w(a, b, c, d; z)$, depending on complex parameters a, b, c, d, z , such that $a - b, b - c, c - d, a - d \in \{1, -1\}$, by

$$R(z, -2\eta d)e[c - d] \otimes e[b - c] = \sum_a w(a, b, c, d; z)e[b - a] \otimes e[a - d],$$

(the sum is over one or two allowed values of a) then the dynamical quantum Yang–Baxter equation translates into the star-triangle relation

$$\begin{aligned} & \sum_g w(a, b, g, f; z - w)w(f, g, d, e; z)w(g, b, c, d; w) \\ &= \sum_g w(f, a, g, e; w)w(a, b, c, g; z)w(g, c, d, e; z - w). \end{aligned}$$

We let $W = V^{\otimes n}$ be the tensor product of fundamental representations with evaluation points z_1, \dots, z_n . Then

$$L(z, \lambda) = R^{(01)}(z - z_1, \lambda - 2\eta \sum_{j=2}^n h^{(j)}) \cdots R^{(0, n-1)}(z - z_{n-1}, \lambda - 2\eta h^{(n)}) R^{(0n)}(z - z_n, \lambda).$$

In this formula the factors of V in $V \otimes W = V^{\otimes n+1}$ are numbered from 0 to n . The module W is a highest weight module with highest weight functions of the form (2) with $p_j = z_j, q_j = z_j + 2\eta$.

Let us now introduce a basis $|a_1, \dots, a_n\rangle$ of $\text{Fun}_\mu(W[0])$, labeled by $a_i \in \mu + \mathbb{Z}$ with $a_i - a_{i+1} \in \{1, -1\}, i = 1, \dots, n - 1$, and $a_n - a_1 \in \{1, -1\}$. We let $\delta(\lambda) = 1$ if $\lambda = 0$ and 0 otherwise. Then we define

$$|a_1, \dots, a_n\rangle(\lambda) = \delta(\lambda + 2\eta a_1)e[a_1 - a_2] \otimes e[a_2 - a_3] \otimes \cdots \otimes e[a_n - a_1].$$

¹We adopt here a convention for the definition of the Boltzmann weights which is slightly different than in [F] and in better agreement with the literature.

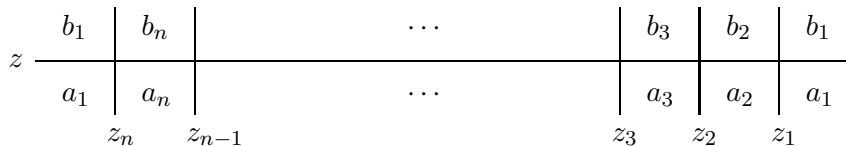


FIGURE 1. Graphical representation of the row-to-row transfer matrix of an IRF model. Each crossing represents a Boltzmann weight w whose arguments are the labels of the adjoining regions and the difference of the parameters associated to the lines.

If Γ is the shift operator $\Gamma f(\lambda) = f(\lambda - 2\eta)$, then $\Gamma|a_1, \dots, a_n\rangle = |a_1 - 1, \dots, a_n - 1\rangle$. Using this, and the fact that $h^{(j)}|a_1, \dots, a_n\rangle = (a_{j+1} - a_j)|a_1, \dots, a_n\rangle$, we get

$$T(z)|a_1, \dots, a_n\rangle = \sum_{b_1, \dots, b_n} \prod_{j=1}^n w(b_j, a_j, a_{j+1}, b_{j+1}; z - z_j)|b_1, \dots, b_n\rangle,$$

with the understanding that $b_{n+1} = b_1$, $a_{n+1} = a_1$. The (finite) sum is over the values of the indices b_i for which the Boltzmann weights are defined. Comparing with [B], we see that $T(z)$, in this basis, is the row-to-row transfer matrix of the (inhomogeneous) interaction-round-a-face model associated to the solution $w(a, b, c, d; z)$ of the star-triangle equation (see [B]). The situation is best visualized by looking at the graphical representation of Fig. 1.

This construction can be in principle extended to higher representations, and we obtain in this way eigenvectors of transfer matrices of the IRF models of [DJMO].

6. THE EIGHT-VERTEX MODEL

We show in this section how to obtain from our result eigenvectors for the transfer matrix of the eight-vertex model.

The eight-vertex model is based on Baxter's solution

$$\begin{aligned} R_{8V}(z) = & a_{8V}(z)(E_{1,1} \otimes E_{1,1} + E_{-1,-1} \otimes E_{-1,-1}) \\ & + b_{8V}(z)(E_{1,-1} \otimes E_{1,-1} + E_{-1,1} \otimes E_{-1,1}) \\ & + c_{8V}(z)(E_{1,-1} \otimes E_{-1,1} + E_{-1,1} \otimes E_{1,-1}) \\ & + d_{8V}(z)(E_{1,1} \otimes E_{-1,-1} + E_{-1,-1} \otimes E_{1,1}), \end{aligned}$$

of the Yang–Baxter equation. The coefficients are

$$\begin{aligned} a_{8V}(z) &= \frac{\theta_0(z)\theta_0(2\eta)}{\theta_0(z-2\eta)\theta_0(0)}, & b_{8V}(z) &= \frac{\theta_1(z)\theta_0(2\eta)}{\theta_1(z-2\eta)\theta_0(0)}, \\ c_{8V}(z) &= -\frac{\theta_0(z)\theta_1(2\eta)}{\theta_1(z-2\eta)\theta_0(0)}, & d_{8V}(z) &= -\frac{\theta_1(z)\theta_1(2\eta)}{\theta_0(z-2\eta)\theta_0(0)}, \end{aligned}$$

in terms of the theta functions with characteristics

$$\theta_1(z) = -\sum_{j \in \mathbb{Z}} e^{2\pi i(j+\frac{1}{2})^2\tau + 2\pi i(j+\frac{1}{2})(z+\frac{1}{2})}, \quad \theta_0(z) = ie^{-i\pi i(z+\tau/2)}\theta_1(z).$$

Remarks. Baxter uses $-z$ instead of z . The classical Jacobi notation, used by Baxter, is $\theta_1(z) = \mathbb{H}(2Kz)$, $\theta_0(z) = \Theta(2Kz)$.

For any (generic) z_1, \dots, z_n , one then defines commuting transfer matrices

$$T_{8V}(z) = \text{tr}_0 R_{8V}(z - z_1)^{(01)} \dots R_{8V}(z - z_n)^{(0n)}. \quad (5)$$

acting on $(\mathbb{C}^2)^{\otimes n}$. The relation with our transfer matrices is based on the following identity, which is a version of Baxter’s vertex-IRF transformation:

Proposition 7. *Let $S(z, \lambda)$ be the matrix*

$$\frac{1}{\theta(\lambda)} \begin{pmatrix} \theta_0(z - \lambda + 1/2) & -\theta_0(-z - \lambda + 1/2) \\ -\theta_1(z - \lambda + 1/2) & \theta_1(-z - \lambda + 1/2) \end{pmatrix}.$$

Then

$$S(w, \lambda)^{(2)} S(z, \lambda - 2\eta h^{(2)})^{(1)} R(z - w, \lambda) = R_{8V}(z - w) S(z, \lambda)^{(1)} S(w, \lambda - 2\eta h^{(1)})^{(2)}$$

Proof: We need to recall some well-known facts about the functions θ_α and Baxter’s R -matrix R_{8V} .

The functions θ_α , $\alpha = 0, 1$ are entire, and uniquely determined up to normalization by the properties $\theta_\alpha(z+1) = (-1)^\alpha \theta_\alpha(z)$, $\theta_\alpha(z+\tau) = ie^{-\pi i(z+\tau/2)} \theta_{1-\alpha}(z)$. Also, $\theta_\alpha(-z) = (-1)^\alpha \theta_\alpha(z)$, and the zeros of θ_α are simple and of the form $r + 2s\tau$ if $\alpha = 1$ and $r + (2s+1)\tau$ if $\alpha = 0$, ($r, s \in \mathbb{Z}$). We have $\theta_0(z)\theta_1(z) = C(\tau)\theta(z)$ for some constant $C(\tau)$, as one can see comparing transformation properties under translations by $\mathbb{Z} + \tau\mathbb{Z}$.

Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $R_{8V}(z)$ commutes with $A \otimes A$ and $B \otimes B$. Moreover $R_{8V}(z+1) = A^{(1)} R_{8V}(z) A^{(1)}$, $R_{8V}(z+\tau) = e^{-2\pi i\eta} B^{(1)} R_{8V}(z) B^{(1)}$, and $R_{8V}(0) = P$, the flip $u \otimes v \mapsto v \otimes u$. As a function of z , R_{8V} is meromorphic. Its poles are simple and are at 2η modulo $\mathbb{Z} + \tau\mathbb{Z}$. The residue of $R_{8V}(z)$ at $z = 2\eta$ is $\theta(2\eta)/\theta'(0)$ times the antisymmetrization operator $\Pi : u \otimes v \mapsto u \otimes v - v \otimes u$.

Let $\phi(z) = (\theta_1(z), \theta_0(z))$. This vector is the (up to normalization) unique vector with entire holomorphic components such that $\phi(z+1) = A\phi(z)$, $\phi(z+\tau) = ie^{-i\pi(z+\tau/2)} B\phi(z)$.

We claim that the statement of the Proposition is equivalent to the following set of identities, which, incidentally, is essentially the standard form of the vertex-IRF transformation.

Lemma 8. *Let $\phi^\pm(z, \lambda) = \phi(\mp z - \lambda + \frac{1}{2})$. Then*

$$\begin{aligned} R_{8V}(z-w)\phi^\pm(z, \lambda \mp 2\eta) \otimes \phi^\pm(w, \lambda) &= \phi^\pm(z, \lambda) \otimes \phi^\pm(w, \lambda \mp 2\eta) \\ R_{8V}(z-w)\phi^\pm(z, \lambda \pm 2\eta) \otimes \phi^\mp(w, \lambda) &= \alpha(z-w, \pm\lambda)\phi^\pm(z, \lambda) \otimes \phi^\mp(w, \lambda \mp 2\eta) \\ &\quad + \beta(z-w, \pm\lambda)\phi^\mp(z, \lambda) \otimes \phi^\pm(w, \lambda \pm 2\eta). \end{aligned}$$

The (known) proof of this lemma consists in comparing transformation properties under lattice translations and poles of both sides of the equations as functions of z , w , and using the uniqueness of ϕ .

It remains to show that these identities are equivalent to the statement of the Proposition. Let us write them in matrix form: Let $\hat{S}(z, \lambda)$ be the 2 by 2 matrix whose columns are ϕ^+ and ϕ^- then the previous lemma reads

$$R_{8V}(z-w)\hat{S}(w, \lambda)^{(2)}\hat{S}(z, \lambda-2\eta h^{(2)})^{(1)} = \hat{S}(z, \lambda)^{(1)}\hat{S}(w, \lambda-2\eta h^{(1)})^{(2)}R(z-w, \lambda)^t.$$

The transposed of a matrix in $\text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is defined by the rule $(X \otimes Y)^t = X^t \otimes Y^t$, $X, Y \in \text{End}(\mathbb{C}^2)$. The statement of the Proposition is proved by transposing both sides of the equation, and inverting the matrices \hat{S} . The identity we need is $(\hat{S}^t)^{-1} = \text{const } S$, which follows from the identity $\det \hat{S}(z, \lambda) = \text{const } \theta(z)\theta(\lambda)$. The latter formula can be proved by comparing the transformation properties under lattice translations of z and λ , and using the fact that θ is the unique function (up to normalization) such that $\theta(z+1) = -\theta(z)$ and $\theta(z+\tau) = -e^{-\pi i(\tau+2z)}\theta(z)$. \square

The transfer matrix $T(z)$ of the previous section has the form $T(z)\psi(\lambda) = a(z, \lambda)\psi(\lambda - 2\eta) + d(z, \lambda)\psi(\lambda + 2\eta)$, where $a(z, \lambda)$, $d(z, \lambda) \in \text{End}((\mathbb{C}^2)^{\otimes n}[0])$ are the diagonal matrix elements of $L(z, \lambda)$.

Corollary 9. *Fix generic $z_1, \dots, z_n \in \mathbb{C}$. Let*

$$S_n(\lambda) = S(z_n, \lambda)^{(n)}S(z_{n-1}, \lambda - 2\eta h^{(n)})^{(n-1)} \dots S(z_1, \lambda - 2\eta \sum_{j \geq 2} h^{(j)})^{(1)}.$$

Then, for all generic complex z, λ ,

$$T_{8V}(z)S_n(\lambda) = S_n(\lambda + 2\eta)a(z, \lambda + 2\eta) + S_n(\lambda - 2\eta)d(z, \lambda - 2\eta),$$

on the zero weight space $(\mathbb{C}^2)^{\otimes n}[0]$.

Proof: Let us write $T_{8V}(z) = \text{tr}_0 L_{8V}(z)$, see (5). The matrix $L_{8V}(z)$ acts on the tensor product of $n+1$ copies of \mathbb{C}^2 , numbered from 0 to n . By iterating Proposition 7, we obtain

$$S_n(\lambda)S(z, \lambda - 2\eta h)^{(0)}L(z, \lambda) = L_{8V}(z)S(z, \lambda)^{(0)}S_n(\lambda - 2\eta h)^{(0)}. \quad (6)$$

In this formula $S_n(\lambda)$ acts on the factors numbered from 1 to n , and $h = h^{(1)} + \dots + h^{(n)}$. Let $L_\alpha^\beta(z, \lambda)$ be defined by

$$L(z, \lambda)e[\alpha] = \sum_{\beta=\pm 1} e[\beta] \otimes L_\alpha^\beta(z, \lambda),$$

and set $\psi_\alpha = S(z, \lambda + 2\eta\alpha)e[\alpha]$. Replacing λ by $\lambda + 2\eta\alpha$ in (6) and acting on a vector of the form $e[\alpha] \otimes u$, $\alpha \in \{1, -1\}$, where $hu = 0$, yields

$$\sum_{\beta} \psi_\beta \otimes S_n(\lambda + 2\eta\alpha)L_\alpha^\beta(z, \lambda + 2\eta\alpha)u = L_{8V}(z)(\psi_\alpha \otimes S_n(\lambda)u),$$

where we used the fact that $L(z, \lambda)$ commutes with $h^{(0)} + h$. Since, for generic parameters, ψ_1, ψ_{-1} form a basis of \mathbb{C}^2 , and $L_1^1(z, \lambda) = a(z, \lambda)$, $L_{-1}^{-1}(z, \lambda) = d(z, \lambda)$, the proof is complete. \square

Theorem 10. *Let $f \mapsto \int f(\lambda)$ be a linear function on a space \mathcal{F} of functions of $\lambda \in \mathbb{C}$, such that $\int f(\lambda + 2\eta) = \int f(\lambda)$ for all $f \in \mathcal{F}$. Extend \int to vector-valued functions by acting componentwise. Then for each eigenfunction $\psi(\lambda)$ of $T(z)$, the vector $\int S_n(\lambda)\psi(\lambda)$, if defined, is an eigenvector of $T_{8V}(z)$ with the same eigenvalue.*

This theorem is an easy consequence of the previous corollary. What is left to do is to find linear forms \int defined on the components of $S(\lambda)\psi(\lambda)$ for the Bethe ansatz eigenfunctions ψ of Theorems 4, 5.

This can be done easily in two situations: notice that both $S(\lambda)$ and $\psi(\lambda)$ are periodic in λ with period 2, if the parameter c belongs to $\pi i\mathbb{Z}$. 1. If $\eta = p/q$ is rational, we may choose

$$\int f(\lambda) = \sum_{j=0}^{q-1} f(\mu + 2\eta j),$$

for any generic μ .

2. If η is real, we set

$$\int f(\lambda) = \int_0^2 f(\mu + \lambda)d\lambda,$$

for generic μ .

7. q -DEFORMED LAMÉ EQUATION

We consider here the special case where W is the evaluation module $V_{2m}(0)$ with positive integer m . Then the zero weight space is one-dimensional. From the expression for a and d given in [FV3], we see that the transfer matrix has the form (cf. [H], where a similar observation is made in the context of the Sklyanin algebra)

$$T(z) = \frac{\theta(z - \eta)}{\theta(z - (2m + 1)\eta)}L, \quad (7)$$

where the difference operator L is independent of z and is given by

$$L\psi(\lambda) = \frac{\theta(\lambda + 2\eta m)}{\theta(\lambda)}\psi(\lambda - 2\eta) + \frac{\theta(\lambda - 2\eta m)}{\theta(\lambda)}\psi(\lambda + 2\eta).$$

This operator is in fact conjugated to the two-body Ruijsenaars operator [R]. It obeys $(L\phi, \psi) = (\phi, L\psi)$ with respect to the symmetric bilinear form

$$(\phi, \psi) = \int \frac{\phi(\lambda)\psi(-\lambda)}{\prod_{j=1}^m \theta(\lambda - 2\eta j)\theta(\lambda + 2\eta j)},$$

where \int is a linear function invariant under shift by 2η , and change of sign of the argument, defined on a suitable space of functions. If η is real, we may take \int to be the integral over $\gamma = \gamma_+ + \gamma_-$, where the straight path γ_+ goes from $\tau/2$ to $1 + \tau/2$ and γ_- goes from $-1 - \tau/2$ to $-\tau/2$. This linear form is defined, say, on the space of meromorphic functions whose poles are contained in $\{2\eta j + k + l\tau, j, k, l \in \mathbb{Z}\}$, which is preserved by the action of L .

Note that L has periodic coefficients, and therefore preserves the space of Bloch functions ψ such that $\psi(\lambda + 1) = \mu\psi(\lambda)$. The bilinear form is well-defined on this space of functions.

We consider the eigenvalue problem

$$L\psi = \epsilon\psi. \quad (8)$$

It is a q -deformation of the generalized Lamé equation: we have the expansion of L in powers of η :

$$L_\eta = 2\text{Id} + 4\eta^2\left(\frac{d^2}{d\lambda^2} - 2m\frac{\theta'(\lambda)}{\theta(\lambda)} + m^2\frac{\theta''(\lambda)}{\theta(\lambda)}\right) + O(\eta^4).$$

The differential operator appearing in the second order coefficient is the generalized Lamé operator (up to conjugation by $\theta(\lambda)^m$). Our Bethe ansatz solution is a q -deformation of Hermite's solution of the (generalized) Lamé equation (see the last pages of [WW]).

Theorem 11. *Let (t_1, \dots, t_m, c) be a solution of the Bethe ansatz equations:*

$$\frac{\theta(t_i - \eta(1 + 2m))}{\theta(t_i - \eta(1 - 2m))} \prod_{j:j \neq i} \frac{\theta(t_j - t_i - 2\eta)}{\theta(t_j - t_i + 2\eta)} = e^{4\eta c}, \quad i = 1, \dots, m, \quad (9)$$

such that $t_i \neq t_j \pmod{\mathbb{Z} + \tau\mathbb{Z}}$ if $i \neq j$. Then

$$\psi(\lambda) = e^{c\lambda} \prod_{j=1}^m \theta(\lambda + t_j - \eta), \quad (10)$$

is a solution of the q -deformed Lamé equation $L\psi = \epsilon\psi$ with eigenvalue

$$\epsilon = e^{-2\eta c} \frac{\theta(4\eta m)}{\theta(2\eta m)} \prod_{j=1}^m \frac{\theta(t_j + (2m - 3)\eta)}{\theta(t_j + (2m - 1)\eta)}.$$

Proof: This theorem is a special case of Theorem 4. It follows from (7) that the formula for the eigenvalue is $\epsilon = \frac{\theta(z - (2m+1)\eta)}{\theta(z - \eta)} \epsilon(z)$, where $\epsilon(z)$ is the function in Theorem 4. Since this expression is independent of z , we can evaluate it at any z . The formula given in this theorem is obtained by taking $z = (1 - 2m)\eta$, so that the second term in $\epsilon(z)$ vanishes. \square

The Bethe ansatz equations have the form $b_i(t) = e^{4\eta c}$, $i = 1, \dots, m$. We may eliminate c and consider the set of (t_1, \dots, t_m) obeying the equations $b_i(t)b_j(t)^{-1} = 1$. The functions $b_i(t)b_j(t)^{-1}$ are doubly periodic meromorphic functions with periods 1 and τ in each of the variables t_j . Also, eigenfunctions corresponding to solutions related by shifts of the variables t_i by 1 or τ are proportional. Therefore the set

$$X = \{t \in (\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z})^m / S_m \mid b_i(t) = b_j(t) \neq \infty, i = 1, \dots, m, t_i \neq t_j (i \neq j)\}$$

is an algebraic subvariety of the symmetric power of our elliptic curve, which parametrizes our eigenfunctions. We call it the Hermite–Bethe variety. On this variety, we have a single-valued function ϵ^2 , the square of the eigenvalue, which is the restriction to X of a doubly periodic function. Therefore it is an algebraic function $\epsilon^2 : X \rightarrow \mathbb{C}$. The eigenvalue, which is the square root of this function, is two-valued on X . This is connected to the fact that generically there are two eigenfunctions corresponding to a single point of the Hermite–Bethe variety: a point of X determines c only modulo $(4\eta)^{-1}2\pi i\mathbb{Z}$. If ψ is an eigenfunction corresponding to a point of X , then $e^{\pi i \lambda / 2\eta} \psi(\lambda)$ is an eigenfunction with the opposite eigenvalue corresponding to the same point but with c translated by $2\pi i / 4\eta$.

Note that the space of meromorphic solutions of the difference equation (8) is a vector space over the field K of meromorphic 2η -periodic functions of λ . If (t_1, \dots, t_m, c) is a solution of the Bethe ansatz equations, then, for all $k \in \mathbb{Z}$, $(t_1, \dots, t_m, c + 2\pi i k / 2\eta)$ is also a solution with the same value of ϵ . The corresponding eigenfunction is proportional (with a 2η -periodic coefficient) to the original eigenfunction. Therefore we should consider c modulo $\pi i \eta^{-1} \mathbb{Z}$.

Note also that we have an involutive automorphism σ of X , sending (t_1, \dots, t_m) to $(2\eta - t_1, \dots, 2\eta - t_m)$. If (t, c) is a solution of the Bethe ansatz equations, then also $(\sigma(t), -c)$. The corresponding eigenfunctions are related by the ‘‘Weyl reflection’’ $\psi(\lambda) \mapsto \psi(-\lambda)$ and have the same eigenvalue.

We now turn to the question of completeness. Let $S^m E_\tau = (\mathbb{Z} + \tau\mathbb{Z})^m / S_m$ be the symmetric power of the elliptic curve.

Theorem 12. *Suppose that $\eta \in \mathbb{C}$ is generic. For generic $\epsilon \in \mathbb{C}$, there are precisely two solutions*

$$(t_1, \dots, t_m, c) \quad \text{and} \quad (2\eta - t_1, \dots, 2\eta - t_m, -c),$$

in $S^m E_\tau \times (\mathbb{C}/\pi i \eta^{-1} \mathbb{Z})$ of the Bethe ansatz equations (9) with given ϵ . The corresponding eigenfunctions ψ_\pm are linearly independent over the field K of 2η -periodic meromorphic functions of λ , and all solutions of the q -Lamé equation (8) are linear combinations of ψ_+ , ψ_- with coefficients in K .

Proof: It is straightforward to check that the reflection $t_i \mapsto 2\eta - t_i$, $c \mapsto -c$, maps solutions to solutions preserving the value of the eigenvalue ϵ .

Let \bar{X} be the closure of X in the symmetric power of the elliptic curve. Then \bar{X} contains the point $P = ((1-2m)\eta, \dots, -3\eta, -\eta)$. In a neighborhood

of P we can introduce local coordinates $u_1 = t_1 - (1 - 2m)\eta$, $u_j = t_j - t_{j-1} - 2\eta$, $2 \leq j \leq m$. In these coordinates, \bar{X} is described by the equations $u_j = c_j u_m^{m-j+1} (1 + O(u_m))$, $j = 1, \dots, m-1$, for some constants c_j . Therefore, in this neighborhood, \bar{X} is a non-singular curve, and u_m is a local parameter at P . The eigenvalue has the form

$$\epsilon = \text{const } u_m^{-1/2} (1 + O(u_m))$$

in a neighborhood of P . Therefore ϵ is a non-constant function on X . Similarly one shows that c appearing in the Bethe ansatz equation is a non-constant function on X (c diverges at P). Since ϵ^2 is algebraic, we have, for any generic value of ϵ , a solution (t_1, \dots, t_m, c) of the Bethe ansatz equations, and thus an eigenfunction ψ_+ . Let ψ_- be the eigenfunction with the same eigenvalue associated to the reflected solution $(2\eta - t_1, \dots, 2\eta - t_m, -c)$. Since generically ψ_+ and ψ_- have different multipliers, they are linearly independent over K : suppose $\psi_+ = C\psi_-$ for some $C \in K$. Then C is 2η -periodic and obeys $C(\lambda + 1) = e^{2c}C(\lambda)$. If η is real and irrational and $e^{2c} \neq 1$, we get a contradiction, so ψ_+ and ψ_- are linearly independent. The case of generic η is treated by analytic continuation.

Next we show that every solution is a linear combination of ψ_+ and ψ_- . The (difference) *Wronskian* $W(f, g)$ of two functions f, g is the function $f(\lambda + 2\eta)g(\lambda) - f(\lambda)g(\lambda + 2\eta)$. Two meromorphic functions are linearly dependent over K if and only if their Wronskian vanishes. If f, g are solutions of (8) then their Wronskian obeys the difference equation $W(\lambda + 2\eta) = u(\lambda)W(\lambda)$, where the function u is a combination of the coefficients of the (8). Thus for any solution f , the functions $A_{\pm} = \pm W(f, \psi_{\mp})/W(\psi_+, \psi_-)$ are 2η -periodic. On the other hand, $A_{\pm}(\lambda)$ are (by Cramer's rule) the coefficients in the expression of $(f(\lambda), f(\lambda + 2\eta))$ as a linear combination of the linearly independent vectors $(\psi_{\pm}(\lambda), \psi_{\pm}(\lambda + 2\eta))$. In particular,

$$f(\lambda) = A_+(\lambda)\psi_+(\lambda) + A_-(\lambda)\psi_-(\lambda).$$

Therefore f is a linear combination of ψ_{\pm} with 2η -periodic coefficients.

It remains to prove that for each generic ϵ there are not more than two solutions of the Bethe ansatz equation with given ϵ . Suppose that there were a third solution (t'_1, \dots, t'_m, c') distinct from the two we have constructed. In particular (t'_1, \dots, t'_m) represents a point in the symmetric power of the elliptic curve which is distinct from the two points represented by (t_1, \dots, t_m) and $(2\eta - t_1, \dots, 2\eta - t_m)$. Let ψ' be the corresponding eigenfunction. Thus $\psi' = a\psi_+ + b\psi_-$ with 2η -periodic coefficients a, b . We have that $a = W(\psi', \psi_-)/W(\psi_+, \psi_-)$. The three functions ψ', ψ_+, ψ_- have all the form (10). Therefore the meromorphic function a obeys the equations $a(\lambda + 2\eta) = a(\lambda)$, $a(\lambda + 1) = C_1 a(\lambda)$, $a(\lambda + \tau) = C_2 a(\lambda)$ for some constants C_1, C_2 . If η is generic, this implies that a is of the form $a_0 e^{s\pi i/\eta}$ for some constant a_0 and some integer s (a cannot have poles since it would have a dense set of singularities). Similarly, b has the same form. We may moreover assume that e^c is generic, so comparing multipliers we see that either $a = 0$ or $b = 0$.

But the zeros of ψ' are not equal to the zeros of ψ_+ or to the zeros of ψ_- , a contradiction. \square

We see from the proof of this result that the irreducible component(s) of \bar{X} containing P and its reflected point $\sigma(P)$ form a curve $Y \subset \bar{X}$. On this part of \bar{X} , ϵ^2 takes every generic value precisely twice.

Let us summarize our results.

Theorem 13. *The closure \bar{X} in $S^m E_\tau$ of the Hermite–Bethe variety contains an algebraic curve Y . It is a two-fold ramified covering $\epsilon^2 : Y \rightarrow \mathbb{P}^1$ of the Riemann sphere. It has an involutive automorphism $\sigma : Y \rightarrow Y$ permuting the sheets and preserving ϵ^2 . For each generic point $t \in Y$, there are two solutions (t, c) , $(t, c + i\pi/2\eta)$ of the Bethe ansatz equation in $Y \times \mathbb{C}/\pi i\eta^{-1}\mathbb{Z}$. The corresponding eigenfunctions are related by $\psi(\lambda) \rightarrow e^{\pi i\lambda/\eta}\psi(\lambda)$, and the eigenvalues are the two square roots of $\epsilon^2(t)$.*

One way to formulate this result is that eigenfunctions are parametrized by the spectral curve, the double covering of Y on which ϵ is single-valued.

8. PROOFS

This section contains the proofs of Lemma 3 and Theorem 5. These proofs are based on the following technical result.

Lemma 14. *Let $V(z)$ with basis $e[1], e[-1]$ denote the fundamental representation of $E_{\tau, \eta}(sl_2)$ with evaluation point z (see Sect. 2). Let $t_1, \dots, t_m, z_1, \dots, z_m, \tau, \eta$ be generic complex numbers, such that $\text{Im}(\tau) > 0$, and let $W = V(z_1) \otimes \dots \otimes V(z_m)$. Then the 2^m vectors*

$$\left(\prod_{j \in J} b(t_j) \right) (e[1] \otimes \dots \otimes e[1]) \in \text{Fun}(W),$$

where J runs over all subsets of $\{1, \dots, m\}$, are linearly independent over the field of meromorphic functions of λ .

Proof: Let us denote these vector by w_J . It is sufficient to show that the values $w_J(\lambda)$ are linearly independent for some value of λ , and some value of the parameters. This will be shown by considering the matrix relating $w_J(\lambda)$ to the basis $e_J = e[\sigma_1] \otimes \dots \otimes e[\sigma_m]$ with $\sigma_j = 1$ iff $j \in J$, and showing that, in some limit of the parameters, this matrix is upper triangular with respect to the lexicographical ordering of binary numbers: $J \geq K$ iff $\sum_{j \in J} 2^j \geq \sum_{j \in K} 2^j$. The limit is obtained by first taking $\tau \rightarrow i\infty$ which amounts to replacing $\theta(x)$ by $\sin(\pi x)$, then $\lambda \rightarrow \infty$. In this limit, $b(t)$ acts on $V(z)$ as $b(t)e[-1] = 0$, $b(t)e[1] = B(\eta)(1 - \exp(-2\pi i(z - t + 2\eta)))^{-1}e[-1]$ for some constant $B(\eta) \neq 0$. Now let us choose $t_j = z_j + \epsilon$ for some fixed generic ϵ , and set $z_j = \sqrt{-1}Nj$, $j = 1, \dots, m$. Let $e[\sigma]_k$ be basis vectors of $V(z_k)$. Then, as $N \rightarrow \infty$, the vector $b(t_j)e[1]_k$ tends to zero if $j < k$ and tends to a nonzero multiple of $e[-1]_k$ if $j = k$. On the other hand, the

matrix elements of a and d tend to non-zero finite values in this limit. But the action of $b(t)$ on a tensor product $u_1 \otimes \cdots \otimes u_m$ consists of m terms, the j 'th one being given by the action of b on the j th factor and the action of a or d on the other $m - 1$ factors (see [FV3]).

It follows that $b(t_{j_1}) \cdots b(t_{j_r})$ is a linear combination of vectors e_K where $K = \{k_1, \dots, k_r\}$ with $k_1 \geq j_1, \dots, k_r \geq j_r$, and that the diagonal matrix elements (such that $K = \{j_1, \dots, j_r\}$) are non-zero. \square

Proof of Lemma 3. The coefficients A_j, D_j can be in principle computed using the commutation relations repeatedly, giving them as some universal polynomials in the values of the functions α, β , independent of the highest weight modules we are considering. Since the b 's commute, we obtain $m!$ ways of representing $a(w)b(t_1) \cdots b(t_m)$ as a linear combination of $b(t_1) \cdots b(t_{j-1})b(w)b(t_{j+1}) \cdots b(t_m)a(t_j)$ (and similarly for d). Since by the previous Lemma there exists a vector in a module so that these operators applied to this vector yield linearly independent vectors, it follows that all these representations must coincide. \square

Proof of Theorem 5. We have to write an explicit formula for $b(t_1) \cdots b(t_m)v_0$, where v_0 is a tensor product of highest weight vectors of some highest weight modules. Let us first consider the case where $v_0 = u \otimes v$ is the tensor product of two highest weight vectors. Then, using the rules for the action on tensor products (see [FV3]) and the commutation relations, we obtain a linear combination of terms of the form

$$\Gamma(-2\eta h^{(2)})[b(t_{i_1}) \cdots b(t_{i_s})a(t_{j_1}) \cdots a(t_{j_{m-s}})]u \otimes b(t_{k_1}) \cdots b(t_{k_{m-s}})d(t_{l_1}) \cdots d(t_{l_s})v,$$

where $I = \{i_1 < \dots < i_s\}$, $J = \{j_1 < \dots < j_{m-s}\}$, $I \cup J = \{1, 2, \dots, m\}$, $K = \{k_1 < \dots < k_{m-s}\}$, $L = \{l_1 < \dots < l_s\}$, $K \cup L = \{1, 2, \dots, m\}$. The notation $\Gamma(-2\eta h^{(2)})$ indicates that the argument of the matrix valued function of λ in the square bracket must be shifted by -2η times the weight of the vector in the second factor.

The coefficients can be computed using only the commutation relations and are therefore independent of the choice of highest weight modules. It is convenient to take u and v to be vectors of the form of the previous Lemma, since it then follows from the linear independence that the coefficients are *uniquely* determined.

Suppose that we compute the coefficients by first applying $b(t_m)$ to the tensor product, then $b(t_{m-1})$ and so on, and then use the commutation relations to shift d 's and a 's to the right of the b 's. Then it is clear that the coefficients of the terms with $I \ni 1$ and $K \ni 1$ vanishes. Similarly, if we use the fact that the b 's commute to act with $b(t_j)$ at the end, we see that the coefficients of the terms with $I \cap K \ni j$ must vanish. We conclude that the only terms appearing with non-vanishing coefficients must have $I \cap K = \emptyset$.

Therefore

$$b(t_1) \cdots b(t_m)(u \otimes v) = \sum_I C_I \Gamma(-2\eta h^{(2)}) \left[\prod_{i \in I} b(t_i) \prod_{j \notin I} a(t_j) \right] u \otimes \prod_{j \notin I} b(t_j) \prod_{i \in I} d(t_i)v,$$

for some universal, uniquely defined, coefficients $C_I(t_1, \dots, t_m)$, $I \subset \{1, \dots, m\}$. By the commutativity of the b 's, these coefficients can be computed in different ways with the result that

$$C_{\sigma(I)}(t_1, \dots, t_m) = C_I(t_{\sigma(1)}, \dots, t_{\sigma(m)}),$$

for all permutations $\sigma \in S_m$. Therefore, it is sufficient to calculate C_I for $I = \{1, \dots, s\}$, for all $s \in \{1, \dots, m\}$, which can be done straightforwardly using the tensor product and commutation rules. Taking u equal to the highest weight vector of an evaluation representation $V_\Lambda(z)$ and v a tensor product of such highest weight vectors gives a recursive procedure to compute all coefficients given in Theorem 5. \square

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