EFI-92-67
IFP-434-UNC
hep-th/9211056

Magnetic Monopoles in String Theory

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Abstract

Magnetic monopole solutions to heterotic string theory are discussed in toroidal compactifications to four spacetime dimensions. Particular emphasis is placed on the relation to previously studied fivebrane solutions in ten dimensions and on the possibility of constructing exact monopole solutions related to symmetric fivebranes.

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1. Introduction

Although a wide variety of approximate and exact soliton solutions to string theory are now known, many of the most important questions involving solitons in string theory are still open. These include the proper treatment of collective coordinates, and the possibility of a strong-weak coupling duality in string theory [1] modeled after the conjectured electric-magnetic duality in $N = 4$ gauge theory [2,3]. In this regard magnetic monopoles provide a particularly important subset of possible soliton solutions to string theory. A number of approximate monopole solutions have already been studied. Recently it has been claimed that there are also monopole solutions which provide exact solutions of string theory [4]. This work was motivated by the desire to better understand these solutions. We will begin in section 2 with a quick summary of previous work on monopole solutions to string theory. In section 3 we will discuss the relation between fivebrane solutions and magnetic monopoles. We show that two previously known monopole solutions can be constructed from a periodic array of "gauge" and "neutral" fivebranes, respectively. We then to turn to the construction of solutions corresponding to a periodic array of "symmetric" fivebranes. In section 4 we develop the properties of these solutions and compare our results with previous work. We end with our conclusions in section 5.

2. Summary of Previous Work

There are a number of possible magnetic monopole solutions to string theory compactified from ten to four dimensions. First consider standard compactifications of heterotic string theory on a Calabi-Yau space K with $H^1(K) = 0$ and gauge symmetry breaking by Wilson lines. If the unbroken gauge group has a $U(1)$ factor one might expect magnetic monopole solutions to exist. Demanding that the asymptotic form of the $U(1)$ gauge field is that of a magnetic monopole, one asks if the configuration can be extended smoothly over the whole of space. It turns out that this condition places some rather subtle topological restrictions on the allowed magnetic charges [5]. Within each topological class satisfying these restrictions we would expect that there is a magnetic monopole solution to string theory. However, not much is known about the detailed form of the solution since it involves massive Kaluza-Klein states in a non-trivial way.

If we consider less realistic compactifications there are several new possibilities for monopole solutions. In particular suppose that the compact space has the form $S^1 \times K'$ where K' is arbitrary as long as it provides a solution to the string equations of motion.

Then the low-energy four-dimensional theory will have (at least) two $U(1)$ gauge fields coming from the components $g_{\mu 4}$ of the metric and $B_{\mu 4}$ of the antisymmetric tensor field. Here μ is a spacetime index and x^4 is the coordinate along the S^1. At the level of low-energy field theory there are magnetic monopole solutions carrying magnetic charge under both of these $U(1)$ factors. At lowest order the "metric" magnetic monopole is just that constructed by Sorkin [6] and by Gross and Perry [7]. In [8] it was argued that starting from this solution one can construct a solution to string theory to all orders in perturbation theory in the parameter α'/R^2 with R the radius of the S^1. By utilizing the string duality $R \to \alpha'/R$ a solution involving $B_{\mu 4}$ was also constructed in [8]. Monopole solutions of heterotic string theory involving $B_{\mu 4}$ were discussed in a general context in [9] where it was emphasized that the gauge invariant field strength is given by the antisymmetric three-form $H_{(4)\mu\nu} \equiv H_{4\mu\nu}$ and that the Yang-Mills Chern-Simons contributions can play an important role. We will elaborate on this point later.

Finally there exist a class of magnetic monopole solutions which are perhaps most closely related to the usual 't Hooft-Polyakov monopole. These occur when a non-abelian gauge symmetry of the four-dimensional field theory is spontaneously broken by a light Higgs field. Two closely related examples have been studied. In the first example we consider the theory close to the self-dual radius of the S^1. At the self-dual point there is a well-known $SU(2)$ gauge symmetry and a set of massless scalars in the triplet representation of $SU(2)$. As one moves away from this point the $SU(2)$ is broken to $U(1)$ and we expect monopole solutions. This solution was constructed in [8] in a perturbative expansion away from the self-dual point. A very similar situation occurs in a toroidal compactification of arbitrary radius down to four dimensions ($K' = T^5$) in which case one has $N = 4$ supersymmetry and the gauge supermultiplets contain scalars in the adjoint representation of the gauge group. Since a potential for these scalars is forbidden by supersymmetry, one can assign arbitrary vacuum expectation values for these fields again leading to symmetry breaking with $U(1)$ factors in the unbroken gauge group. The corresponding monopoles were constructed in [10] where it was shown that the monopoles preserve half of the supersymmetries and saturate a Bogomol'nyi bound.

Although all of these monopoles can presumably be extended to solutions to string theory, they all receive corrections to higher orders in α'. This makes their description in terms of conformal field theory problematic. It would be very nice to have solutions which are exact without any higher order corrections. Such solutions have been proposed in [4] based on "symmetric" fivebrane solutions [11]. We will argue that the construction of [4] does in fact lead to exact monopole solutions, although there are some crucial differences in interpretation between this work and [4]. In addition, similar constructions allow us to relate aspects of the solutions of [8] and [10] to the "neutral" and "gauge" fivebranes.

3. Fivebranes and Monopoles

A fivebrane is an extended soliton solution to 10 dimensional string theory with 5+1 dimensional translational symmetry. Explicit fivebrane solutions have been constructed from a generalization of Yang-Mills instantons in which the four-dimensional instanton sits in the directions transverse to the fivebrane. When such objects are compactified to four dimensions, they can be classified by the embedding of the core instanton in space-time and internal space [1]. In particular, if the instanton lies in 3 space directions and 1 internal direction, it appears as a particle from the four dimensional point of view. In the following, we show that these pointlike solitons can be further identified as magnetic monopoles.

3.1. Gauge, neutral and symmetric fivebranes

The fivebrane solutions are constructed from the low-energy effective action for the massless fields of the heterotic string. At lowest order in α', the effective action is given by $N = 1$ super Yang-Mills coupled to supergravity theory. In "sigma model" variables, the bosonic part is

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{3}H^2 - \frac{\alpha'}{30}\mathrm{Tr}F^2 \right] \tag{3.1}$$

where the Yang-Mills gauge fields are in the adjoint representation of $E_8 \times E_8$ or $SO(32)$ with the trace conventionally normalized so that $\mathrm{tr}(t^a t^b) = \delta^{ab}$ in the fundamental representation.

The H field is given at lowest order by $H = dB + \alpha'(\omega_3^L - \frac{1}{30}\omega_3^{YM})$ with ω_3^L and ω_3^{YM} being the Lorentz and Yang-Mills Chern-Simons three-forms respectively. Without the Lorentz Chern-Simons form this action has a well known supersymmetric completion. When this term is included one must keep including additional terms in a power series in α' in order to maintain supersymmetry. In carrying out this procedure it is found that the generalized spin connections $\Omega_{\pm M}^{AB} \equiv {\omega_M}^{AB} \pm {H_M}^{AB}$ play a central role [12]. In particular, one finds that the three-form H is recursively defined by evaluating the Lorentz Chern-Simons term in the definition of H above using the generalized connection Ω_+. Thus the Bianchi identity that supplements (3.1) is most informatively written as

$$dH = \alpha'(\mathrm{tr}R(\Omega_+) \wedge R(\Omega_+) - \frac{1}{30}\mathrm{Tr}F \wedge F) + O(\alpha'^2). \tag{3.2}$$

The role of the generalized connections can also be understood from the sigma-model point of view. For a discussion of this in the context of fivebrane solutions see [13].

The supersymmetry transformations for the fermionic fields are, to lowest order, given by

$$\begin{aligned} \delta\chi &= F_{MN}\gamma^{MN}\epsilon \\ \delta\lambda &= \left(\gamma^M\partial_M\phi - \frac{1}{6}H_{MNP}\gamma^{MNP}\right)\epsilon \\ \delta\psi_M &= \left(\partial_M + \frac{1}{4}\Omega_{-M}^{AB}\gamma_{AB}\right)\epsilon. \end{aligned} \tag{3.3}$$

The fivebrane ansatz preserves a chiral half of the supersymmetries and is given by [1,11]

$$\begin{aligned} F_{\mu\nu} &= \pm\frac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\sigma}F_{\lambda\sigma} \\ H_{\mu\nu\lambda} &= \mp\epsilon_{\mu\nu\lambda}{}^{\sigma}\partial_\sigma\phi \\ g_{\mu\nu} &= e^{2\phi}\delta_{\mu\nu}, \qquad g_{ab} = \eta_{ab} \end{aligned} \tag{3.4}$$

where $\mu, \nu, \ldots$ denote transverse space, and $a, b, \ldots$ orthogonal space indices. For the gauge part of the solution, the first line indicates that $F_{\mu\nu}$ is an (anti-)self-dual field strength, and in particular can be solved by an instanton configuration in an $SU(2)$ subgroup of the gauge group.

From this starting point, there are two approaches to constructing the rest of the fivebrane solution. The first is to solve the Bianchi identity (3.2) perturbatively in α'. Starting with a $F_{\mu\nu} = O(1)$ instanton solution, the Bianchi identity tells us that the dilaton and hence the curvature is $O(\alpha')$, so to lowest order, we can drop the $R \wedge R$ term to give

$$\nabla_\rho\nabla^\rho\phi = \mp\frac{\alpha'}{120}\epsilon^{\mu\nu\lambda\sigma}\mathrm{Tr}F_{\mu\nu}F_{\lambda\sigma} \tag{3.5}$$

which can be solved for a given multi-instanton configuration. For the charge one self-dual instanton of scale size ρ one obtains

$$e^{2\phi} = e^{2\phi_0} + 8\alpha'\frac{(x^2 + 2\rho^2)}{(x^2 + \rho^2)^2}. \tag{3.6}$$

These fivebrane solutions are referred to as gauge fivebranes [1,11] and receive higher order corrections in α'. Nevertheless, it is possible to maintain supersymmetry and construct a solution order by order in α' using non-renormalization arguments based on the six-dimensional symmetry of the low-energy fivebrane effective action [14].

The neutral fivebrane solution is obtained from the gauge fivebrane (3.6) by taking the limit $\rho \to 0$ to get

$$e^{2\phi} = e^{2\phi_0} + \frac{n\alpha'}{x^2}. \tag{3.7}$$

Although only the solution with $n = 8$ is obtained in this limit the solution exists for all positive integers n[1]. The most general neutral multi-fivebrane configuration is obtained by solving $\Box e^{2\phi} = 0$ assuming S^3 symmetry, giving

$$e^{2\phi} = e^{2\phi_0} + \sum_I \frac{n_I \alpha'}{(x - x_I)^2} \tag{3.8}$$

for some positive integers n_I and where x_I denote the locations of the fivebranes. Like the gauge fivebranes, the neutral solutions will also receive higher order corrections in α'.

The second approach to completing the fivebrane solution is, in analogy with Calabi-Yau compactifications, to embed the generalized spin connection in the gauge group, $\Omega_+ = A$, so that dH vanishes to all orders in α'. The condition for this to hold is simply $\Box e^{2\phi} = 0$, just as in the neutral solution (3.8). In this case Ω_+, using the above metric ansatz, is given by

$$\Omega^{mn}_{+\mu} = \sigma_{\mu\nu}{}^{mn} \partial^\nu 2\phi \tag{3.9}$$

with

$$\sigma_{\mu\nu}{}^{mn} = \delta_{\mu\nu}{}^{mn} \mp \frac{1}{2}\epsilon_{\mu\nu}{}^{mn} \tag{3.10}$$

being anti-self-dual (self-dual) in both pairs of indices. Using the condition $\Box e^{2\phi} = 0$, this implies that Ω_+ is an (anti-)self-dual $SU(2)$ connection (embedded in $SO(4)$), ensuring the consistency of equating it with an (anti-)self-dual Yang-Mills connection (restricted to be in the form of the 't Hooft ansatz, discussed below). The solution with the dilaton given by (3.7) and the instanton size given by $\rho = e^{-\phi_0}\sqrt{n\alpha'}$ is known as the symmetric fivebrane [11]. It is an exact solution of string theory without higher order corrections as can be seen from various points of view, including construction of the explicit underlying

[1] If n is negative the dilaton becomes imaginary when $x^2 < -n\alpha' e^{-2\phi_0}$ and the physical interpretation of the solution is unclear.

superconformal field theory [13]. When the dilaton is given by (3.8) and the instanton sizes are given by $\rho_I = e^{-\phi_0}\sqrt{n_I\alpha'}$ we have a multi symmetric fivebrane configuration.

To conclude this review of the fivebrane solutions we note that we have taken into account the quantization condition on the three-form H that is required for the consistent propagation of strings in this background [9]. Specifically, H must satisfy

$$Q = -\frac{1}{2\pi^2\alpha'}\int_M H, \qquad Q \in Z \tag{3.11}$$

where the integral is over an arbitrary closed three manifold M. All of the fivebrane solutions satisfy this condition with $Q = 8$ for the gauge fivebrane and $Q = \sum_I n_I$ for the neutral and symmetric multi-fivebrane solutions. The "anti-fivebrane" solutions are obtained using the lower sign in (3.4) and although the form of the dilaton is the same as for the fivebranes (upper sign), they have opposite H-charge Q.

3.2. *Periodic instantons and monopoles*

In constructing magnetic monopole solutions from fivebranes we will exploit a small generalization of the relation between Yang-Mills instantons and monopoles which we will now review. For an $SU(2)$ connection, a general N instanton configuration is described by $8N-3$ parameters: the positions, sizes and relative $SU(2)$ angles of the N instantons. The 't Hooft ansatz gives an explicit $5N$ parameter multi-instanton solution in which all instantons have identical gauge orientations (see, for example, [15]). The ansatz involves writing the $SU(2)$ gauge field as

$$A_\mu(x) = \overline{\Sigma}_{\mu\nu}\nabla^\nu \ln f(x) \tag{3.12}$$

where the matrix valued 't Hooft tensor, $\overline{\Sigma}_{\mu\nu}$, is antisymmetric and anti-self-dual[2].

The self-duality condition then becomes $f^{-1}\Box f = 0$, which can be solved (assuming S^3 symmetry) to give

$$f(x) = \sum_{I=1}^{N+1} \frac{\rho_I^2}{(x-x_I)^2}. \tag{3.13}$$

[2] Note that $\sigma_{\mu\nu}$ defined in (3.10) is a possible choice for $\overline{\Sigma}_{\mu\nu}$ if one interprets it as the doublet representation of $SU(2)$ having been embedded in the fundamental representation of $SO(4)$.

This form of the solution most directly exhibits the conformal symmetry of the solution. If we take the limit $\rho_{N+1} \to \infty$ and $x_{N+1} \to \infty$ with $\rho_{N+1}/x_{N+1} = 1$ we obtain the perhaps more familiar form of the solution

$$f(x) = 1 + \sum_{I=1}^{N} \frac{\rho_I^2}{(x - x_I)^2} \tag{3.14}$$

where x_I and ρ_I can now be interpreted as the position and size of the $I^{\rm th}$ instanton.

To see how monopoles arise from periodic instantons, we consider making one of the four transverse coordinates (*e.g.* x^4) periodic with period $2\pi R$ and look for solutions to the self-dual equation on the space $R^3 \times S^1$.

We are thus interested in instanton solutions which are periodic in x^4. In general, a single periodic instanton can be constructed by taking an infinite string of identical (up to a gauge transformation) instantons lined up in the compact direction with spacing $2\pi R$. Starting with the 't Hooft ansatz (3.14) for instantons with identical gauge orientation and performing the sum to enforce the periodicity $x^4 \equiv x^4 + 2\pi R$ gives a single periodic instanton [16]:

$$\begin{aligned} f^{(1)}(\vec{x}, x^4) &= 1 + \sum_{k=-\infty}^{\infty} \frac{\rho^2}{r^2 + (x^4 - x_0^4 + 2\pi k R)^2} \\ &= 1 + \frac{\rho^2}{2Rr} \sinh \frac{r}{R} \Big/ \left(\cosh \frac{r}{R} - \cos \frac{x^4 - x_0^4}{R} \right) \end{aligned} \tag{3.15}$$

with $r = |\vec{x} - \vec{x}_0|$ and $(\vec{x}_0, x_0^4)$ the location of the instanton.

A periodic multi-instanton can be constructed in the same way by starting with n such strings. This gives

$$f^{(n)}(\vec{x}, x^4) = 1 + \sum_{I=1}^{n} \frac{\rho_I^2}{2Rr_I} \sinh \frac{r_I}{R} \Big/ \left(\cosh \frac{r_I}{R} - \cos \frac{x^4 - x_I^4}{R} \right) \tag{3.16}$$

where $r_I = |\vec{x} - \vec{x}_I|$ is the 3-radius. The n periodic instantons are located at $(\vec{x}_I, x_I^4)$.

While an uncompactified instanton has a single scale, ρ, a single periodic instanton has two relevant scale parameters, the original instanton size and the compactification radius of the S^1. The behavior of the non-abelian field strength of the periodic instanton depends on the ratio of these scales. For a single periodic instanton (centered at the origin) in the asymptotic limit $r \gg R$, (3.16) reduces to

$$f = 1 + \frac{\rho^2}{2Rr} + O(e^{-r/R}). \tag{3.17}$$

From (3.12), we see that $A_\mu \sim 1/r^2$ provided $r \gg \rho^2/2R$, so the field strength falls off as $1/r^3$ as $r \to \infty$. In fact, a more careful study shows that asymptotically the gauge field has the characteristic of a three-dimensional dipole [17]. In the region $R \ll r \ll \rho^2/2R$ (provided it exists), we find instead that $A_\mu \sim 1/r$ and the space-time components of the field strength looks like that of a magnetic monopole.

This correspondence with a magnetic monopole can be made exact when we set $\rho = \infty$ so that the monopole region above extends to infinity. In this limit, the single periodic instanton ansatz, (3.15), becomes conformally invariant and is in fact gauge equivalent to the BPS magnetic monopole solution when we identify A_4 (the component of the gauge field in the periodic direction) with the Higgs field in the BPS limit and $1/R$ with the vacuum expectation value of the Higgs [18]. A remarkable aspect of this solution is that it is actually independent of the periodic coordinate. This identification between the periodic instanton and the BPS monopole only holds for the single monopole case. Anti-monopole solutions are obtained in this framework by starting with a periodic array of anti-instantons instead of instantons.

3.3. Gauge fivebranes and BPS gauge monopoles

Given the above identification of a single conformal periodic instanton with a BPS monopole and the close connection between fivebranes and instantons, it is natural to try to construct monopole solutions in string theory starting from the fivebrane solutions. Specifically, we trivially compactify five of the spatial dimensions tangent to the fivebrane in (3.4) on a five-torus and then look for solutions where in addition the fourth transverse direction is compactified on an S^1 of radius R.

For the gauge fivebrane the relation is very simple. To construct the lowest-order single monopole solution of [10] we take an array of gauge fivebranes periodic in the x^4 direction. In particular, we take an instanton string with spacing $2\pi R = 2\pi/C$ and with size $\rho = \infty$. Because the fivebrane energy is independent of ρ, no singularities arise in this limit. Up to an x^4 dependent gauge transformation, this gauge field configuration is equivalent to the BPS solution used in [10], so through (3.5), it yields the identical solution for the gravity fields given by the dilaton configuration

$$e^{2\phi} = e^{2\phi_0} + 2\alpha' \frac{1}{r^2}[1 - K^2 + 2H] \tag{3.18}$$

where $H = Cr \coth Cr - 1$ and $K = Cr/\sinh Cr$ are the BPS functions. This BPS gauge monopole is independent of the internal coordinates and is thus not only a solution of the compactified ten-dimensional theory but also of the $N=4$ super Yang-Mills supergravity theory constructed by dimensional reduction to four dimensions.

Using the ansatz (3.4) for the self-dual solution, we deduce that the non-zero components of the resulting three-form field strength are given by

$$\begin{aligned} H_{ij4} &= -2\alpha' \epsilon_{ijk} \frac{x^k}{r^4} H(1-K^2) \\ &\approx -2\alpha' C \epsilon_{ijk} \frac{x^k}{r^3} \qquad r \to \infty. \end{aligned} \tag{3.19}$$

Since $H_{\mu\nu 4}$ is the gauge invariant field strength of the $U(1)$ field coming from $B_{\mu 4}$ in the compactification, we see that the BPS gauge monopole is also an $H_{(4)}$ monopole with magnetic charge $-8\pi\alpha'/R$. We will discuss the quantization of the magnetic $H_{(4)}$ charge in the next section.

When the instanton size is finite, the periodic gauge fivebrane is no longer independent of the internal direction. This finite size instanton solution is equally valid as a solution of the compactified theory, but it differs from the previous solution in that it cannot be viewed as a purely four-dimensional solution and it does not have an interpretation as a non-abelian monopole since the Yang-Mills gauge field strength falls off like a dipole sufficiently far from the core. However, it still has the interpretation of a magnetic $H_{(4)}$ monopole with the same magnetic charge. We will calculate the magnetic charge in the next section.

3.4. Neutral fivebranes and neutral H monopoles

In a similar fashion, neutral monopole solutions can also be constructed out of periodic configurations of neutral fivebranes. Starting with n periodic stacks of neutral fivebranes spaced at a distance $2\pi R$ apart in the compact dimension, (3.8) can be summed to give

$$e^{2\phi}(\vec{x}, x^4) = e^{2\phi_0} + \sum_{I=1}^{n} \frac{n_I \alpha'}{2R r_I} \sinh \frac{r_I}{R} \Big/ \left(\cosh \frac{r_I}{R} - \cos \frac{x^4 - x^4_I}{R} \right). \tag{3.20}$$

The asymptotic form of the H field can be calculated giving

$$H_{ij4} = -\sum_{I=1}^{n} \frac{n_I \alpha'}{4R} \epsilon_{ijk} \frac{(x - x_I)^k}{r_I^3} + O(e^{-r_I/R}) \tag{3.21}$$

for $r_I \gg R$. Thus, these solutions are multi $H_{(4)}$ monopoles with total magnetic charge $-\sum_I n_I \pi\alpha'/R$. Since this solution is dependent on the internal coordinate x^4, it must be viewed as a compactification of the original 10 dimensional theory.

An x^4 independent solution is indicated by considering the formal limit $R \to 0$. Specifically we want to take this limit while keeping $n_I\alpha'/2R$ fixed to give

$$e^{2\phi} = e^{2\phi_0} + \sum_{I=1}^{n} \frac{n_I\alpha'}{2R}\frac{1}{r_I}. \tag{3.22}$$

There are many reasons why we should be concerned about this limit. Firstly, because of the quantization of H and hence the discreteness of n_I, the limit isn't properly defined. In addition the limit involves topology change. However, (3.22) is indeed a well defined solution for arbitrary R as can be seen by returning to the derivation of the neutral fivebrane (3.8). Here we want to solve $\Box e^{2\phi} = 0$ assuming $S^2 \times S^1$ symmetry and no dependence on the S^1 which has radius R. The solution (3.22) satisfies these conditions and now the H field given by (3.21) is valid everywhere for this solution. The quantization of the coefficients in (3.22) comes from the quantization condition on $H_{(4)}$ that we will discuss in the next section.

The solution (3.22) was first constructed to lowest order in [8]. The solution was obtained by a duality transformation of a solution based on the Sorkin-Gross-Perry monopole. In showing that this solution can be obtained using the fivebrane ansatz (3.4), we have also shown that this solution is a *supersymmetric* solution to string theory.

3.5. Symmetric fivebranes and symmetric monopoles

It should now be obvious that one can construct symmetric monopole solutions from an array of symmetric fivebranes. Starting with the periodic instanton configuration (3.16), the symmetric monopole solution is then given by the dilaton field

$$e^{2\phi} = e^{2\phi_0} f(\vec{x}, x^4) \tag{3.23}$$

and obviously satisfies the consistency condition $\Box e^{2\phi} = 0$. In fact, recalling the equivalence of the dilaton fields for the neutral and symmetric fivebrane solutions, it should be no surprise that this is the same as for the neutral monopole solution (3.20) after relating the gauge and gravitational instanton sizes by defining $\rho_I = e^{-\phi_0}\sqrt{n_I\alpha'}$.

Similarly, an x^4 independent solution is in fact obtained by taking the limit $R \to 0$ and $\rho_I \to 0$ with $m_I = \rho_I^2/2R$ fixed. In this limit, we find

$$f(x) = e^{-2\phi_0}e^{2\phi} = 1 + \sum_{I=1}^{n} \frac{m_I}{r_I} \tag{3.24}$$

which reproduces the solution of [4]. The quantization condition on $H_{(4)}$ requires that $m_I = e^{-2\phi_0} n_I \alpha'/2R$ and we see that the dilaton for this symmetric solution is the same as for the neutral solution (3.22). We will explain in the next section that since the Yang-Mills field has a dipole structure at spatial infinity it cannot be interpreted as a Yang-Mills monopole.

In order to have a true BPS symmetric monopole solution, we instead take the limit $\rho \to \infty$ for a single periodic symmetric fivebrane. Up to an overall rescaling of the metric, this is equivalent to dropping the 1 in the ansatz (3.16). Thus for a finite rescaled ρ, the BPS symmetric monopole is given by

$$f(x) = e^{-2\phi_0} e^{2\phi} = \frac{\rho^2}{2Rr} \sinh \frac{r}{R} \bigg/ \left(\cosh \frac{r}{R} - \cos \frac{x^4}{R} \right) . \tag{3.25}$$

Although the x^4 dependence of the Yang-Mills field can be gauge transformed away, the dilaton and gravity fields remain x^4 dependent. In particular, the x^4 dependence of the generalized connection cannot be transformed away since the coordinate is compact. Nevertheless, the Bianchi identity is still satisfied since both $\mathrm{Tr} F \wedge F$ and $\mathrm{tr} R \wedge R$ are x^4 independent.

A comparison with the neutral solutions shows that all of these symmetric solutions are $H_{(4)}$ (multi-)monopoles with unit of magnetic charge given by $-\pi\alpha'/R$.

4. Properties of the Solutions

4.1. H vs Yang-Mills monopoles

The compactification from ten to four dimensions introduces several $U(1)$ gauge fields, six from the $g_{\mu a}$ components of the metric and six from $B_{\mu a}$ where $a = 4, \ldots, 9$. As was discussed in [9] the appropriate gauge invariant field strengths for the $U(1)$ fields coming from the antisymmetric tensor are given by $H_{(a)\mu\nu} \equiv H_{a\mu\nu}$. Of these $U(1)$ gauge fields, only $H_{(4)}$ is excited in the solutions we have been discussing.

All of the solutions presented in the last section are $H_{(4)}$ monopole solutions. In [9] such monopoles were discussed in a general context and the construction of some solutions was sketched by postulating the existence of an asymptotic monopole field strength and then demanding that the field could be smoothly continued to all of space. Assuming that space had the topology of R^3, it was argued that to avoid Dirac singularities, gauge field instantons played an important role in the solution via the Bianchi identity $dH =$

$-\frac{1}{30}\alpha'\mathrm{Tr}F\wedge F+\ldots$. The gauge monopole solution, (3.18) and (3.19), is an explicit (and supersymmetric) realization of this kind of solution. It is interesting to note that for this solution the B field is not excited and the contributions to $H_{(4)}$ come entirely from the gauge Chern-Simons term.

In the last section we showed that the BPS gauge monopole solution (3.18) had $H_{(4)}$ magnetic charge of $-8\pi\alpha'/R$ by an explicit calculation of $H_{(4)}$. An alternative way to calculate this and also to calculate the $H_{(4)}$ charge of the gauge solution with finite instanton size is to follow an argument presented in [9]. Using the fact that the topology of space is R^3 and that $H_{(4)}$ is asymptotically independent of x^4, we can relate the $H_{(4)}$ charge to the instanton number through the use of the Bianchi identity, (3.2), giving

$$\begin{aligned} g_{(4)} &\equiv \int_{S^2} H_{(4)} = \frac{1}{2\pi R}\int_{S^2\times S^1} H = \frac{1}{2\pi R}\int_{R^3\times S^1} dH \\ &= -\frac{\alpha'}{2\pi R}\int_{R^3\times S^1}\mathrm{tr}F\wedge F + O({\alpha'}^2) = -8\pi\alpha' q/R \end{aligned} \tag{4.1}$$

where

$$q = \frac{1}{16\pi^2}\int \mathrm{tr}F\wedge F \in Z \tag{4.2}$$

gives the instanton number. For the BPS gauge monopole, $q=1$, so that $g_{(4)}=-8\pi\alpha'/R$.

Since the gauge fields are not excited at all for the neutral monopole solutions (3.20), they do not fit into the scheme discussed in [9]. The reason that the asymptotic monopole field strength can be continued into the interior of space is that now the topology is not R^3 but $R^3-\{0\}$ as will be discussed in the next subsection. Although an instanton gauge field is excited for the symmetric monopole solutions, (3.23) and (3.24), the considerations of the H field is more like that of the neutral monopoles than that discussed in [9]. Note also that because of the topology of these solutions the $H_{(4)}$ magnetic charge cannot be calculated using (4.1).

We now discuss the Dirac quantization condition for $H_{(4)}$ which comes from the quantization of the three-form H. Specifically, choosing the manifold M in (3.11) to be an asymptotic $S^2\times S^1$ with S^2 at spatial infinity and S^1 the compact dimension and assuming that $H_{(4)}$ is asymptotically independent of the coordinate on S^1 we obtain

$$\int_{S^2} H_{(4)} = ng \qquad g = -\frac{\pi\alpha'}{R} \tag{4.3}$$

where n is an integer (positive for the solutions we have been considering) and g is the unit of magnetic charge. The states electrically charged with respect to $H_{(4)}$ come from strings that wind around the S^1. To calculate the unit of electric charge we note that the coupling of the string to the antisymmetric tensor contains a term

$$S = \frac{1}{\pi\alpha'} \int d^2\sigma \left[\dot{X}^i \partial_\sigma X^4 - \dot{X}^4 \partial_\sigma X^i \right] B_{i4}. \tag{4.4}$$

Looking at a configuration that winds around the S^1 once, in the center of mass frame we have a term

$$S = \frac{2R}{\alpha'} \int dt \dot{X}^i B_{i4}. \tag{4.5}$$

Thus the winding state couples to the $U(1)$ gauge field like a charged particle with unit of electric charge given by $e = 2R/\alpha'$. From (4.3) we see that the charges satisfy the Dirac quantization condition $g_{(4)}e = 2\pi n$.

We now turn to a discussion of the Yang-Mills fields. Now we are interested in whether the solutions can be thought of as monopoles of the $U(1)$ arising from the spontaneous symmetry breaking of $SU(2)$. The BPS gauge monopole of [10], constructed out of a periodic gauge fivebrane, and the BPS symmetric monopole (3.25), constructed out of a periodic symmetric fivebrane, are obviously of this type. Asymptotically, the gauge field strength constructed from (3.25) is that of a non-abelian magnetic monopole, $F^a_{ij} \sim -\epsilon_{ijk} x^a x^k / r^4$. Before gauge transforming, the "Higgs field" behaves as $\varphi^a \equiv A^a_4 \sim x^a/r^2$ which has vanishing expectation value. However, with an x^4 dependent gauge transformation[18], $\varphi \to U(\varphi + \partial_4)U^{-1}$ such that $U\partial_4 U^{-1} = O(1)$, we recover the BPS solution, $\varphi^a \sim Cx^a/r$ with the asymptotic F^a_{ij} unchanged.

On the other hand, for the symmetric monopole composed of finite sized periodic instantons, the non-abelian magnetic field strength, F^a_{ij}, has the characteristics of a dipole and falls off as $1/r^3$ as $r \to \infty$. The "Higgs field" in this case falls off as $1/r^2$. We can again change the asymptotics of φ^a by an x^4 dependent gauge transformation, but since F^a_{ij} is essentially unchanged, it still falls off as a dipole. Another way to understand why these solutions are not non-abelian monopoles is to think about quantization of the collective coordinates for these solutions. Because of the falloff of the Higgs and gauge fields at infinity there will be normalizable zero modes corresponding to global $SU(2)$ gauge rotations of the solution (in contrast to true non-abelian monopoles for which these zero modes are not normalizable). Quantization of the corresponding collective coordinates will give a spectrum of states in definite representations of the unbroken gauge group, much as in the

Skyrme model of hadrons. As a result we conclude that both the symmetric monopole solutions (3.23) and (3.24) carry no non-abelian magnetic charge[3].

4.2. Spacetime properties of the monopoles

We now turn to the spacetime properties of the various monopole solutions. When investigating these properties, it is important to keep track of the various possible metrics. In 10 dimensions, the above solutions are constructed with the "sigma model" metric, g_{MN}. This is related by a Weyl rescaling, $g_{MN} = e^{\phi/2}\hat{g}_{MN}$, to the "Einstein" metric $\hat{g}_{MN}$ where the action takes the canonical Einstein-Hilbert form. When compactified to four dimensions, the contribution of the internal space volume to the four-dimensional gravitational coupling can be scaled out to give the four-dimensional canonical metric $\tilde{g}_{\mu\nu} = \Delta^{1/2}\hat{g}_{\mu\nu}$ where $\Delta = \det \hat{g}_{ij}$ (with μ, ν being space time and i, j internal space indices). In particular, for the fivebrane ansatz, the 10 dimensional sigma model metric is

$$ds^2 = -dt^2 + e^{2\phi}(dr^2 + r^2 d\Omega_{(2)}^2 + (dx^4)^2) + dx_I^2 \tag{4.6}$$

where x_I stands for the remaining 5 internal directions orthogonal to the fivebrane that can be thought of as being compactified on a five torus. The four-dimensional canonical line element has the isotropic form

$$d\tilde{s}^2 = -e^{-\phi}dt^2 + e^{\phi}(dr^2 + r^2 d\Omega_{(2)}^2) \tag{4.7}$$

where the isotropic coordinate r can be related to the Schwarzschild radial coordinate $\overline{r}$, by $\overline{r} = re^{\phi(r)/2}$.

Examining the four-dimensional metric, (4.7), for the BPS gauge monopole (3.18), we find that it has the interesting property that there are no event horizons or singularities regardless of the Higgs vacuum expectation value $C = 1/R$. The mass density of the monopole is given by the 00 component of the stress energy tensor in an orthonormal basis, $\rho(\overline{r}) = T_{\hat{0}\hat{0}}(\overline{r})$, and the Schwarzschild mass is $M(\overline{r}) = 4\pi \int_0^{\overline{r}} \rho(\overline{r})\overline{r}^2 d\overline{r}$. Working in isotropic coordinates, we calculate for the BPS gauge monopole

$$\rho(r) = T_{\hat{0}\hat{0}}(r) = \frac{\alpha' e^{-3\phi}}{\kappa_4^2 r^4}\left[2((1-K^2)^2 + 2H^2K^2) + \frac{7\alpha' e^{-2\phi}}{r^2}H^2(1-K^2)^2\right] \tag{4.8}$$

[3] In this respect, our interpretation differs from that of [4]. Without a Higgs vacuum expectation value, the unbroken $U(1)$ electromagnetic field strength tensor of 't Hooft used in [4] loses its meaning.

where the first term is due to the gauge and Higgs field and the second term is due to the gravity fields. Here, κ_4 is the four-dimensional gravitational coupling. The mass as a function of the isotropic coordinate r is most easily calculated from the metric and is given by

$$M(r) = \frac{8\pi\alpha' e^{-3\phi/2}}{\kappa_4^2 r} H(1-K^2)\left[1 - \frac{\alpha' e^{-2\phi}}{2r^2} H(1-K^2)\right]. \tag{4.9}$$

In the limit $r \to \infty$, we find the ADM mass of the monopole to be $M = 8\pi\alpha' C e^{-3\phi_0/2}/\kappa_4^2 = -g_{(4)} e^{-3\phi_0/2}/\kappa_4^2$. The behavior of this monopole is governed by the dimensionless scale parameter $\lambda^2 = \alpha' C^2 e^{-2\phi_0}$. For $\lambda \ll 1$, the gauge field dominates, and the monopole has a core size $r_{\rm core} \approx 1.5 e^{-\phi_0}\sqrt{\alpha'}/\lambda$. The core has a constant density $\rho_{\rm core} = \frac{2e^{\phi_0}}{3\alpha'\kappa_4^2}\lambda^4$ which falls off as $1/r^4$ outside the core.

For a BPS monopole coupled to gravity, since the mass and inverse size of the monopole is proportional to C, we expect the monopole to become a black hole when C becomes sufficiently large [19]. This fate is avoided in the present situation because of the dilaton field. For $\lambda \gg 1$, the core mass density still comes from the gauge field. However, because of the dilaton coupling, we now find $\rho_{\rm core} = \frac{e^{\phi_0}}{3\sqrt{2}\alpha'\kappa_4^2}\lambda$, and $r_{\rm core} \approx 2.17 e^{-\phi_0}\sqrt{\alpha'}/\lambda$. The fraction of the mass concentrated at the core approaches 0 as $\lambda \to \infty$. In this limit, the monopole develops an intermediate region between $r_{\rm core}$ and $r_{\rm max} \approx 10 e^{-\phi_0}\sqrt{\alpha'}\lambda$ in which the mass density falls off only as $1/r^2$. The energy in this region arises from the second term in (4.8), and accounts for the majority of the total mass.

One caveat of this above analysis is that the BPS gauge monopole given by (3.18) is only valid to lowest order in α'. The higher order corrections will presumably be important in the case when $\lambda \gg 1$. Nevertheless, we expect the qualitative behavior to hold for all values of λ. An alternate way to see that the BPS gauge solution is always well behaved is to note that the underlying gauge fivebrane solutions are everywhere regular, regardless of the instanton sizes.

Since the dilaton field has the same form for the neutral and symmetric solutions they have identical spacetime properties. For the neutral and symmetric monopole solutions, (3.20) and (3.23), and at distances larger than the compactification radius R, the four-dimensional metric approaches (4.7) with

$$e^{2\phi} = e^{2\phi_0}\left[1 + \sum_{I=1}^{n} \frac{m_I}{r_I}\right]. \tag{4.10}$$

In this limit, the metric is identical to the 4-metric of the Sorkin-Gross-Perry Kaluza-Klein monopole [6,7]. Since these monopoles are equivalent to a periodic array of neutral or symmetric fivebranes, their geometry has the structure of the underlying fivebrane [11]. For distances $r^2 + (x^4)^2 \ll R^2$, each monopole core has the geometry of a semi-infinite wormhole in the original 10 dimensional "sigma model" metric (4.6). As mentioned earlier this "cylindrical" topology evades the relation between the H charge and gauge field instanton charge discussed in [9], essentially by pushing the instanton charge off to the end of the infinite wormhole.

For the neutral and symmetric solutions, (3.22) and (3.24), which describe the limit $R \to 0$, the asymptotic 4-metric with (4.10) is valid everywhere. In this case, there is a singularity at the location of the core of each monopole shielded by a horizon located at $r_i = -m_I$. Since m_I is positive this corresponds to a naked singularity. It is important to remember here that the anti-monopole solution is obtained not simply by taking $m_I \to -m_I$ but requires going back and starting from an anti-self-dual fivebrane solution. The relation between the H charge and m_I (or equivalently n_I), (3.21), will then have an additional minus sign so that the anti-monopole solution will also have a naked singularity. Near each monopole core, the metric approaches

$$ds^2 = -\sqrt{r/me^{2\phi_0}}dt^2 + \sqrt{me^{2\phi_0}/r}(dr^2 + r^2 d\Omega^2_{(2)}). \tag{4.11}$$

The spatial part of this metric can be converted to Schwarzschild coordinates according to

$$ds^2 = (r/r_0)^s(dr^2 + r^2 d\Omega^2_{(d-1)}) = \frac{1}{A^2}(d\overline{r}^2 + A^2\overline{r}^2 d\Omega^2_{(d-1)}) \tag{4.12}$$

where $\overline{r} = r_0(r/r_0)^A$ and $A = 1 + s/2$. This metric has a conical singularity at the origin and in d dimensions has a curvature $R_{(d)} = (d-1)(d-2)(1-A^2)/\overline{r}^2$ which is non-vanishing for $d > 2$. In this case, $s = -1/2$ and the 3-geometry at the origin is that of a conical space with deficit solid angle $\Omega \equiv 4\pi(1 - A^2) = 7\pi/4$.

What are we to make of these singular solutions? There seem to be two possibilities. If we truly consider the limit $R \to 0$ then at least in the context of string theory we are ignoring the effects of the light string winding modes which will certainly alter the structure of the solution. Put another way, as $R \to 0$ we should be using a different low-energy effective field theory for the winding modes rather than the low-energy theory for the momentum modes. On the other hand, although we obtained the solutions (3.22) and (3.24) by considering the limit $R \to 0$ we can consider this to be just a formal trick for obtaining an x_4 independent solution and once we have it, reinstate a non-zero compactification radius R. This is essentially what is done in [4].

It thus seems that the symmetric monopole solution (3.24) is a singular but exact solution to string theory. It was suggested in [4] that the divergence in the curvature cancels against the gauge field singularity, at least in the calculation of the action. It would be interesting to see whether string theory expanded about this solution really is non-singular.

The ADM masses of the neutral and symmetric solutions can be read off from the asymptotic behavior of the metric where the dilaton is given by (4.10). The result is $M = (2\pi e^{\phi_0/2}/\kappa_4^2)\sum_I m_I$ which can be rewritten as $M = (\pi\alpha' e^{-3\phi_0/2}/\kappa_4^2 R)\sum_I n_I = -g_{(4)} e^{-3\phi_0/2}/\kappa_4^2$. This relation between the mass and the $H_{(4)}$ charge is related to the saturation of a Bogomol'nyi bound for these solitons and holds for *all* the above monopole solutions that have an asymptotically flat metric.

In the case of the BPS symmetric monopole, (3.25), we find $e^{2\phi} \sim me^{2\phi_0}/r$ in the $r \to \infty$ limit. In this case, neither the original 10 dimensional sigma model metric nor the 4-metric is asymptotically flat. The resulting conical metric at infinity is an indication that the monopole mass is divergent. This diverging action is the result of identifying the metric instanton connection with the gauge connection in the limit of an infinitely large instanton. Since the gravity fields do not obey the Yang-Mills equations, they give a divergent instead of a zero contribution to the (gravitational) energy density. In particular, $T_{\hat{0}\hat{0}} \sim 1/\overline{r}^2$ which is reminiscent of non-gauge monopoles coupled to gravity[20]. The reason this is not a problem for the BPS gauge monopole is that in that case the instanton scale is independent of the scale of the gravity fields.

5. Conclusions

In this paper we have tried to give a unified description of the known magnetic monopole solutions to string theory (or its low-energy limit) by relating them to the three known fivebrane solutions. We have clarified the supersymmetry of the solutions which is particularly significant for the solutions that receive higher order corrections in α' since non-renormalization theorems can be used to show that the corrections will not destabilize the solutions. We have explained how the symmetric solutions discussed in [4] are actually limiting cases of a more general class of symmetric monopole solutions. We have also emphasized that these symmetric solutions are *not* non-abelian monopoles but are rather monopoles of a $U(1)$ group resulting from compactification of the antisymmetric tensor field. It would be interesting to study the analogous dyon solutions, the structure of the $N = 4$ superconformal field theory which underlies the monopole solutions, and the implications of these solutions for both $R \to 1/R$ duality and the more general $SL(2, R)$ duality recently discussed by Sen in the context of monopole solutions [21].

Acknowledgements

We would like to thank Ramzi Khuri, Rafael Sorkin and Andy Strominger for helpful discussions. This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG05-85ER-40219 and by NSF Grant No. PHY90-00386. J.P.G. is supported by a grant from the Mathematical Discipline Center of the Department of Mathematics, University of Chicago. J.H. also acknowledges the support of NSF PYI Grant No. PHY-9196117.

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