

Severe constraints on Loop-Quantum-Gravity energy-momentum dispersion relation from black-hole area-entropy law

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We explore a possible connection between two aspects of Loop Quantum Gravity which have been extensively studied in the recent literature: the black-hole area-entropy law and the energy-momentum dispersion relation. We observe that the original Bekenstein argument for the area-entropy law implicitly requires information on the energy-momentum dispersion relation. Recent results show that in first approximation black-hole entropy in Loop Quantum Gravity depends linearly on the area, with small correction terms which have logarithmic or inverse-power dependence on the area. Preliminary studies of the Loop-Quantum-Gravity dispersion relation reported some evidence of the presence of terms that depend linearly on the Planck length, but we observe that this possibility is excluded since it would require, for consistency, a contribution to black-hole entropy going like the square root of the area.

I. INTRODUCTION

The intuition that the entropy of a black hole should be proportional to its (horizon-surface) area, up to corrections that can be neglected when the area A is much larger than the square of the Planck length L_p , has provided an important element of guidance for quantum-gravity research. It is noteworthy that, as shown by Bekenstein [1], this contribution to black hole entropy can be obtained from very simple ingredients. One starts from the general-relativity result [2] that the minimum increase of area when the black hole absorbs a classical particle of energy E and size s is $\Delta A \simeq 8\pi L_p^2 E s$ (in “natural units” with $\hbar = c = 1$). Taking into account the quantum properties of particles one can estimate s as roughly given by the position uncertainty δx , and, since a particle with position uncertainty δx should at least [3] have energy $E \sim 1/\delta x$, this leads to the conclusion [1,4] that the minimum change in the black-hole area must be of order L_p^2 , independently of the size of the area. Then using the fact that, also independently of the size of the area, this minimum increase of area should correspond to the minimum (“one bit”) change of entropy one easily obtains [1] the proportionality between black-hole entropy and area.

It is remarkable that, in spite of the humble ingredients of this Bekenstein analysis, the entropy-area relation introduced such a valuable constraint for quantum-gravity research. And a rather challenging constraint, since attempts to reproduce the entropy-area-linearity result using directly some quantum properties of black holes were unsuccessful for nearly three decades. But over the last few years both in String Theory and in Loop Quantum Gravity the needed techniques for the analysis of entropy on the basis of quantum properties of black holes were developed. These results [5–8] now go even beyond the entropy-area-proportionality contribution: they establish that the leading correction should be of log-area type, so that one expects (for $A \gg L_p^2$) an entropy-area relation for black holes of the type

$$S = \frac{A}{4L_p^2} + \rho \ln \frac{A}{L_p^2} + O\left(\frac{L_p^2}{A}\right). \quad (\text{I.1})$$

For the case of Loop Quantum Gravity, which is here of interest, there is still no consensus on the coefficient of the logarithmic correction, ρ , but it is established [6–8] that there are no correction terms with stronger-than-logarithmic dependence on the area.

We observe that the availability of results on the log-area correction might provide motivation for reversing the Bekenstein argument: the knowledge of black-hole entropy up to the leading log correction can be used to establish the Planck-scale modifications of the ingredients of the Bekenstein analysis.

In particular, the mentioned role of the relation $E \geq 1/\delta x$ in the Bekenstein analysis appears to provide an opportunity to put under scrutiny some scenarios for the energy-momentum dispersion relation in Loop Quantum Gravity. Several recent studies have tentatively argued that the Loop-Quantum-Gravity dispersion relation might involve a term with a linear dependence on the Planck length, and, as we observe in Section II, this in turn requires a Planck-length modification of the relation $E \geq 1/\delta x$ between the energy and position uncertainty of a particle. However, as we show in Section III, the resulting modification of the $E \geq 1/\delta x$ relation would in turn lead, following the Bekenstein argument, to a contribution to black-hole entropy that goes like the square root of the area. Since such a square-root contribution is, as mentioned, excluded by direct analysis of black-hole entropy in Loop Quantum Gravity, we conclude that the presence in the energy-momentum dispersion relation of a term with linear dependence on the Planck length is also excluded.

II. LOOP-QUANTUM-GRAVITY DISPERSION RELATION AND ITS IMPLICATIONS FOR THE $E \geq 1/\delta X$ RELATION

The possibility of Planck-scale modifications of the dispersion relation has been considered extensively in the recent quantum-gravity literature [9–11] and in particular in Loop Quantum Gravity [12–15].

Some calculations in Loop Quantum Gravity [12,13] provide support for the idea of an energy-momentum dispersion relation that for a particle of high energy would take the approximate form

$$E \simeq p + \frac{m^2}{2p} + \alpha L_p E^2, \quad (\text{II.1})$$

where α is a coefficient of order 1. However, these results must be viewed as preliminary [14,15] since they essentially consider perturbations of “weave states” [12,13], rather than perturbations of the ground state of the theory. It is not surprising (and therefore not necessarily insightful) that there would be some states of the theory whose excitations have a modified spectrum. If instead a relation of the type (II.1) was applicable to excitations of the ground state of the theory this would provide a striking characteristic of the Loop-Quantum-Gravity approach.

Several papers have been devoted to the derivation of tighter and tighter experimental limits on coefficients of the α type for Loop Quantum Gravity (see, *e.g.*, Ref. [16] and references therein). As announced we intend to show here that the linear-in- L_p term can be excluded already on theoretical grounds, because of an inconsistency with the black-hole-entropy results.

In this section we start by observing that a modified dispersion relation implies a modification of the relation $E \geq 1/\delta x$ between the energy of a particle and its position uncertainty. We can see this by simply following the familiar derivation [3] of the relation $E \geq 1/\delta x$, substituting, where applicable, the standard special-relativistic dispersion relation with the Planck-scale modified dispersion relation. It is convenient to focus first [3] on the case of a particle of mass M at rest, whose position is being measured by a procedure involving a collision with a photon of energy E_γ and momentum p_γ . In order to measure the particle position with precision δx one should use a photon with momentum uncertainty $\delta p_\gamma \geq 1/\delta x$. Following the standard argument [3], one takes this $\delta p_\gamma \geq 1/\delta x$ relation and converts it into the relation $\delta E_\gamma \geq 1/\delta x$, using the special-relativistic dispersion relation, and then the relation $\delta E_\gamma \geq 1/\delta x$ is converted into the relation $M \geq 1/\delta x$ because the measurement procedure requires¹ $M \geq \delta E_\gamma$. If indeed Loop Quantum Gravity hosts a Planck-scale-modified dispersion relation of the form (II.1), it is easy to see that, following the same reasoning, one would obtain from $\delta p_\gamma \geq 1/\delta x$ the requirement $M \geq (1/\delta x)[1 + 2\alpha(L_p/\delta x)]$.

These results strictly apply only to the measurement of the position of a particle at rest, but they can be straightforwardly generalized [3] (simply using a boost) to the case of measurement of the position of a particle of energy E . In the case of the standard dispersion relation (without Planck-scale modification) one obtains the familiar $E \geq 1/\delta x$. In the case of (II.1) one instead easily finds that

$$E \geq \frac{1}{\delta x} \left(1 + 2\alpha \frac{L_p}{\delta x} \right). \quad (\text{II.2})$$

¹One must take into account the fact [3] that the measurement procedure should ensure that the relevant energy uncertainties are not large enough to possibly produce extra copies of the particle whose position one intends to measure.

III. A REQUIREMENT OF CONSISTENCY WITH THE BLACK-HOLE ENTROPY ANALYSIS

We now intend to show that the linear-in- L_p modification of the relation between the energy of a particle and its position uncertainty, which follows from the corresponding modification of the energy-momentum dispersion relation, should be disallowed in Loop Quantum Gravity since it leads to a contribution to the black-hole entropy-area relation which has already been excluded in direct black-hole-entropy analyses.

We do this by following the original Bekenstein argument [1]. As done in Ref. [1] we take as starting point the general-relativistic result which establishes that the area of a black hole changes according to $\Delta A \geq 8\pi E s$ when a classical particle of energy E and size s is absorbed. In order to describe the absorption of a quantum particle one must describe the size of the particle in terms of the uncertainty in its position [1,4], $s \sim \delta x$, and take into account a ‘‘calibration² factor’’ [17–19] $(\ln 2)/2\pi$ that connects the $\Delta A \geq 8\pi E s$ classical-particle result with the quantum-particle estimate $\Delta A \geq 4(\ln 2)L_p^2 E \delta x$. Following the original Bekenstein argument [1] one then enforces the relation $E \geq 1/\delta x$ (and this leads to $\Delta A \geq 4(\ln 2)L_p^2$), but we must take into account the Planck-length modification in (II.2), obtaining

$$\Delta A \geq 4(\ln 2) \left[L_p^2 + 2 \frac{\alpha L_p^3}{\delta x} \right] \simeq 4(\ln 2) \left[L_p^2 + 2 \frac{\alpha L_p^3}{R_S} \right] \simeq 4(\ln 2) \left[L_p^2 + \frac{\alpha 4\sqrt{\pi} L_p^3}{\sqrt{A}} \right],$$

where we also used the fact that in falling in the black hole the particle acquires [18,21,22] position uncertainty $\delta x \sim R_S$, where R_S is the Schwarzschild radius (and of course $A = 4\pi R_S^2$).

Next, following again Bekenstein [1], one assumes that the entropy depends only on the area of the black hole, and one uses the fact that according to information theory the minimum increase of entropy should be $\ln 2$, independently of the value of the area:

$$\frac{dS}{dA} \simeq \frac{\min(\Delta S)}{\min(\Delta A)} \simeq \frac{\ln 2}{4(\ln 2)L_p^2 \left[1 + \alpha 4\sqrt{\pi} \frac{L_p}{\sqrt{A}} \right]} \simeq \left(\frac{1}{4L_p^2} - \frac{\alpha\sqrt{\pi}}{L_p\sqrt{A}} \right). \quad (\text{III.1})$$

From this one easily obtains (up to an irrelevant constant contribution to entropy):

$$S \simeq \frac{A}{4L_p^2} - 2\alpha\sqrt{\pi} \frac{\sqrt{A}}{L_p}. \quad (\text{III.2})$$

We therefore conclude that when a quantum-gravity theory predicts the presence of a linear-in- L_p contribution to the energy-momentum dispersion relation it should correspondingly predict the presence of \sqrt{A} contribution to black-hole entropy. Since in Loop Quantum Gravity such a \sqrt{A} contribution to black-hole entropy has already been excluded [6–8] in direct black-hole entropy studies, we conclude that in Loop Quantum Gravity the presence of linear-in- L_p contributions to the energy-momentum dispersion relation is excluded.

It is instead plausible that Loop Quantum Gravity might host a dispersion relation of the type

$$E \simeq p + \frac{m^2}{2p} + \tilde{\alpha} L_p^2 E^3, \quad (\text{III.3})$$

with a quadratic-in- L_p contribution. In fact, the careful reader can easily adapt our analysis to the case of the dispersion relation (III.3), finding that the quadratic-in- L_p contribution to the dispersion relation ultimately leads to a leading correction to the black-hole-entropy formula which is of log-area type, consistently with the indications obtained in direct black-hole entropy studies [6–8].

²Clearly some calibration is needed in order to adapt the classical-gravity result for absorption of a classical particle to the case of a quantum black hole absorbing a quantum particle. In particular, a calibration should arise in the description of a quantum particle with position uncertainty δx in terms of a classical particle of size s . A direct evaluation of the calibration coefficient within quantum gravity is presently beyond reach; however, several authors (see, *e.g.*, Refs. [17–19]) have used the independent analysis of black-hole entropy by Hawking [20] to infer indirectly this calibration needed in the Bekenstein argument. We adopt this calibration for consistency with previous literature, but the careful reader will notice that this calibration does not affect our line of analysis (the calibration could be reabsorbed in the free parameter α).

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