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Spectrum and Thermodynamics of the 1D Supersymmetric t-J Model with $1/r^2$ Exchange and Hopping

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We derive the spectrum and thermodynamics of the 1D supersymmetric t-J model with long range hopping and spin exchange using a set of maximal spin eigenstates. This spectrum confirms the recent conjecture that the asymptotic Bethe-ansatz spectrum is exact. By empirically determining the spinon degeneracies of each state, we are able to explicitly construct the free energy.

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The explicit construction of low dimensional models with Jastrow ground-state wavefunctions has attracted considerable recent interest [1–6]. In one dimension, Shastry and Haldane [5,6] have demonstrated that the ground-state of the 1D Heisenberg model with a $1/r^2$ exchange interaction is a Gutzwiller state for the half filled infinite-U Hubbard model. Haldane has shown how the spectrum of this model can be written in terms of a generalized type of Jastrow wavefunction with excitations of novel statistics [7].

Kuramoto and Yokoyama [8] have recently extended these results to include holes, demonstrating that the corresponding 1D supersymmetric t-J model is also characterized by a Gutzwiller ground-state. Most recently, Kawakami has obtained an asymptotic Bethe-ansatz (ABA) solution for the model, based on the observation that the ground-state wavefunction is a product of two-body functions [9]. Assuming factorizability, he derived the spectrum of the system, which was conjectured to be exact. The low-energy critical behavior of the model has been identified as a Luttinger liquid [10,11]; the spin and charge excitations are described independently by $c = 1$ conformal field theories.

In the case of the $1/r^2$ Bose gas [12], and the Shastry-Haldane $1/r^2$ Heisenberg chain [7], the ABA has been shown to furnish the correct spectrum, despite the long-range nature of the interactions. A remarkable feature of these models is that excited states are obtained from the ground-state by introducing zeros into the Jastrow wavefunction, in a manner reminiscent of Laughlin's description of quasiparticles in the fractional quantum Hall effect. This motivates us to examine the $1/r^2$ supersymmetric t-J model in a similar vein. Here, we show how this philosophy can be used to construct the excited state Jastrow wavefunctions of the $1/r^2$ supersymmetric t-J model and indeed, the spectrum confirms Kawakami's conjecture. In addition to the spectrum, we are able to obtain the spin degeneracies of each state, permitting us to write the the free energy in closed form.

The Hamiltonian for the one-dimensional t-J model is given by

$$H = \sum_{i\neq j,\sigma} \left[-t_{ij} c^{\dagger}_{i\sigma} c_{j\sigma}\right] + \sum_{i\neq j} \left[J_{ij}(\mathbf{S}_i \cdot \mathbf{S}_j - \tfrac{1}{4} n_i n_j)\right], \tag{1}$$

where we implicitly project out any double occupancies. We take $t_{ij} = J_{ij} = t/d^2(i-j)$ where

$d(n) = \frac{N}{\pi}\sin(n\pi/N)$ is the chord distance consistent with periodic boundary conditions on N lattice sites [13].

States in the Hilbert space can be represented by spin and hole excitations from the fully-polarized up-spin state $|P\rangle$ [14]. If we let Q denote the number of holes and M denote the number of down-spins, then S_z is given by $S_z = (N-Q)/2 - M$. The wavefunctions are given by

$$|\psi\rangle = \sum_{x,y} \psi(x,y) \prod_{\alpha} S^{-}_{x_\alpha} \prod_i h^{\dagger}_{y_i} |P\rangle, \tag{2}$$

where the amplitude $\psi(x,y)$ is symmetric in $x \equiv (x_1, x_2, \ldots, x_M)$, the positions of the down-spins, and antisymmetric in $y \equiv (y_1, y_2, \ldots, y_Q)$, the positions of the holes. $S^{-}_{x_\alpha} = c^{\dagger}_{x_\alpha\downarrow} c_{x_\alpha\uparrow}$ is the spin-lowering operator at site x_α and $h^{\dagger}_{y_i} = c_{y_i\uparrow}$ creates a hole at site y_i.

We can construct a general class of states corresponding to states of uniform motion and spin polarization. To describe these states, we generalize Kuramoto and Yokoyama's Jastrow ground-state [8] as follows

$$\begin{aligned} \psi_G(x,y;J_s,J_h) &= \exp\left[\frac{2\pi i}{N}\left(J_s \sum_\alpha x_\alpha + J_h \sum_i y_i\right)\right] \mathbf{\Psi}_0(x,y) \\ \mathbf{\Psi}_0(x,y) &= \prod_{\alpha<\beta} d^2(x_\alpha - x_\beta) \prod_{i<j} d(y_i - y_j) \prod_{\alpha,i} d(x_\alpha - y_i). \end{aligned} \tag{3}$$

Here, J_s and J_h govern the (uniform) momenta of down-spins and holes respectively. J_s and J_h take on either integral or half-integral values as appropriate to insure that ψ_G has the correct periodicities under $x_\alpha \to x_\alpha + N$ and $y_i \to y_i + N$.

The Hamiltonian can be broken up into four parts, $H = T^{\uparrow} + T^{\downarrow} + H^0 + H^{\rm int}$, where $T^{\uparrow}$ ($T^{\downarrow}$) is the up (down) spin transfer operator, H^0 is the spin exchange operator and $H^{\rm int}$ is the diagonal interaction term. When H acts on ψ_G, $T^{\uparrow}$ only affects the y variables and H^0 only affects the x variables. As a result, these operators are easy to treat and yield only two and three body terms when appropriate conditions on J_s and J_h are met [5,6,8].

However, because $T^{\downarrow}$ exchanges pairs of x_α and y_i, this term must be treated differently. In general, it is *not* true that $T^{\uparrow}|\psi_G\rangle = T^{\downarrow}|\psi_G\rangle$ because $T^{\uparrow}$ does not commute with the spin raising operator. This difficulty was overlooked in earlier work [8]. To deal with $T^{\downarrow}$, we use an alternate representation for $|\psi_G\rangle$ in terms of up-spins and holes. Let us introduce the $N-M-Q$ coordinates $u \equiv (u_1, u_2, \ldots, u_{N-M-Q})$ which give the location of the up-spins. Wavefunctions in this representation are given by the spin rotated version of Eq. (2) where the x are replaced by u and M is replaced by $N-M-Q$. Making this transformation, we find

$$\psi_G(x, y; J_s, J_h) = A\psi_G(u, y; N - J_s, J_h - J_s + \frac{N}{2}), \tag{4}$$

where the set of N coordinates (x, y, u) exhausts the entire lattice. A is a constant independent of the spin and hole coordinates. Using this identity, the down-spin transfer operator gives

$$\frac{T^{\downarrow}\psi_G(x, y)}{\psi_G(x, y)} = \frac{T^{\downarrow}\psi_G(u, y)}{\psi_G(u, y)}, \tag{5}$$

and can thus be treated in a similar manner as $T^{\uparrow}$. The result gives two and three body terms in the variables u and y. These terms can then be converted into sums over the x and y variables by making use of the fact that (x, y, u) runs over the entire lattice.

When the separate terms that contribute to the Hamiltonian are combined, we find that the two body terms drop out and the three body terms combine to give constants. As a result, ψ_G with total momentum $P = \frac{2\pi}{N}(J_s M + J_h Q)$ is an exact eigenstate of H with energy

$$\begin{aligned}\frac{N^2}{\pi^2 t}E = &\frac{2}{3}M(M^2-1) - 2MJ_s(N-J_s)\\ &+Q\Big[\frac{1}{3}(N^2-1) + \frac{2}{3}(Q^2-1) + \frac{1}{2}(M+Q)(2M-Q)\\ &\quad -2J_h(N-J_h) + 2(J_s-J_h)^2\Big].\end{aligned} \tag{6}$$

The cancellation of the many body terms, and thus this result, is only valid under the conditions $|J_s - N/2| \leq N/2 - (M - 1 + Q/2)$, $|J_h - N/2| \leq N/2 - (M + Q - 1)/2$ and $|J_h - J_s| \leq (M+1)/2$. For a given S_z, the minimum energy is given when J_s and J_h are as close to $N/2$ as possible. The ground state is given when S_z is either 0 or 1/2 and is a singlet whenever possible [8]. When $Q = 0$ this reduces to the result for the Heisenberg chain [5,6]. These energy levels have also been found by Ha and Haldane [15], where $J_\uparrow = J_h - N + (M+Q+1)/2$ and $J_\downarrow = J_h - J_s - (M-1)/2$. From these energy levels we find the spin and charge velocities to be identical to the previous results [8,15].

To investigate the other excited states of the system, we introduce zeros into the wavefunction by premultiplying it with polynomials of $X_\alpha = \exp(2\pi i x_\alpha/N)$ and $Y_i = \exp(2\pi i y_i/N)$. The wavefunctions thus take the following modified Kalmeyer-Laughlin form [16]:

$$\psi(x,y) = \Phi_s(X,Y)\Phi_h(Y)\mathbf{\Psi_0}, \tag{7}$$

where Φ_s and Φ_h are completely symmetric under pairwise interchange of their arguments. These states will be termed "fully-polarized spinon states". Loosely speaking, the polynomials Φ_s and Φ_h can be regarded as spin and charge quasiparticle wavefunctions respectively.

When the Hamiltonian acts on this wavefunction, once again all three-body terms combine to give constants. However, in this case, some two-body terms remain and we are left with the eigenvalue equation

$$\frac{N^2}{\pi^2 t} E\Phi_s\Phi_h = E_0\Phi_s\Phi_h + H_1 + H_2 + H_3, \tag{8}$$

where

$$\begin{aligned} H_1 &= 2\Phi_h\left[\sum_\mu \partial_\mu^2 + \sum_{\mu<\nu}\frac{W_\mu+W_\nu}{W_\mu-W_\nu}(\partial_\mu-\partial_\nu)\right]\Phi_s \\ &\quad +4\Phi_s\left[\sum_i \partial_i^2 + \frac{1}{2}\sum_{i<j}\frac{Y_i+Y_j}{Y_i-Y_j}(\partial_i-\partial_j)\right]\Phi_h \\ H_2 &= 4\sum_i \partial_i\Phi_s\partial_i\Phi_h \\ H_3 &= 2\Phi_h\sum_{\alpha<\beta}\frac{X_\alpha+X_\beta}{X_\alpha-X_\beta}(\partial_\alpha-\partial_\beta)\Phi_s. \end{aligned} \tag{9}$$

Here we denote $W \equiv (X,Y) \equiv (X_1\ldots X_M, Y_1\ldots Y_{Q+M})$ and $\partial_\mu \equiv W_\mu\partial/\partial W_\mu$. In deriving this, we have shifted Φ_s by the ground state configuration, $\prod_\mu W_\mu^{N/2}$. As a result, E_0 is given by Eq. (6) with $J_s = J_h = N/2$. We require that $|\text{degree } \Phi_s| \le N/2 - (M-1+Q/2)$, $|\text{degree } \Phi_s\Phi_h| \le N/2 - (M+Q-1)/2$ and $|\text{degree } \Phi_h| \le (M+1)/2$, which is to hold for each variable X_α or Y_i independently.

The first term, H_1, does not mix Φ_s and Φ_h and has been solved by Sutherland [12]. However, H_2 mixes Φ_s and Φ_h and H_3 does not act symmetrically on Φ_s. As a result, they are harder to deal with. We follow Sutherland and start by choosing the following symmetric basis functions:

$$\begin{aligned} \Phi_s(W;\{n\}) &= \sum_{\{P_\mu\}} \prod_\mu W_\mu^{n_{P_\mu}} \\ \Phi_h(Y;\{m\}) &= \sum_{\{P_i\}} \prod_i Y_i^{m_{P_i}}, \end{aligned} \tag{10}$$

where the quantum numbers $\{n_1, \ldots, n_{M+Q}\}$ and $\{m_1, \ldots, m_Q\}$ are taken to be in increasing order and $\{P_\mu\}$ and $\{P_i\}$ denote permutations of the indices. These quantum numbers are integral or half-integral as required by periodic boundary conditions.

In this basis, labeled by the two sets of quantum numbers $\{n_\mu\}$ and $\{m_i\}$, the Hamiltonian, considered as a matrix, can be shown to be upper-triangular. Eigenvalues are found by reading the diagonal-elements labeled in terms of the quantum numbers $\{n_\mu\}$ and $\{m_i\}$ [17]. The result simplifies when written in terms of a conjugate set of quantum numbers $\{J_1, J_2, \ldots, J_{M+Q}\}$ and $\{I_1, I_2, \ldots, I_Q\}$ defined by

$$\begin{aligned} J_\mu &= n_\mu + n_\mu^0 \qquad & n_\mu^0 &= \frac{1}{2}(2\mu - (M+Q) - 1) \\ I_i &= m_i + m_i^0 \qquad & m_i^0 &= \frac{1}{2}(2i - Q - 1), \end{aligned} \tag{11}$$

where $\{n_\mu\}$ and $\{m_i\}$ must satisfy the conditions specified before. This translates into the conditions $|J_\mu| \leq (N - M + 1)/2$ and $|I_i| \leq (M+Q)/2$. The energy is

$$\frac{E}{t} = \frac{\pi^2}{3} Q\left(1 - \frac{1}{N^2}\right) + \frac{1}{2} \sum_{\mu=1}^{M+Q} (p_\mu^2 - \pi^2), \tag{12}$$

where the pseudomomenta, p_μ, are given by the following equations:

$$\begin{aligned} p_\mu N &= 2\pi J_\mu - \pi \sum_{i=1}^{Q} \operatorname{sgn}(p_\mu - q_i) + \pi \sum_{\nu=1}^{M+Q} \operatorname{sgn}(p_\mu - p_\nu) \\ 2\pi I_i &= \pi \sum_{\mu=1}^{M+Q} \operatorname{sgn}(q_i - p_\mu). \end{aligned} \tag{13}$$

The above equations correspond to the asymptotic Bethe-ansatz equations obtained by Kawakami [9]. Our result thus confirms that the ABA spectrum is exact.

Here the resulting $\{p_\mu\}$ and $\{q_i\}$ must lie between $-\pi$ and π. The set of $M+Q$ distinct quantum numbers J_μ is in ascending order and governs the spin excitations. We restrict them to take values in the range $[-(N-M-1)/2, (N-M-1)/2]$ to guarantee that they generate fully spin-polarized states. There are $N-M$ values in this range, of which $M+Q$ are occupied and $2S_z$ are empty. A spin configuration can be represented by a sequence of $N-M$ digits such as $\{S\} = (0111001011)_s$, where 1 represents an occupied quantum number and 0 an unoccupied quantum number. These empty values are identified as spinons [7]; a sequence of $2j_r$ consecutive zeros corresponds to a symmetric bound-state of $2j_r$ spinons, thereby creating an excitation of spin j_r with spin degeneracy $2j_r+1$. On these physical grounds, we anticipate a spin degeneracy in the thermodynamic limit given by

$$w_S = \prod_j (2j+1)^{n(j)} \tag{14}$$

where $n(j)$ is the number of sequences of zeros of length $2j$. The set of Q distinct quantum numbers I_i, in ascending order and taking values in the range $[-(M+Q)/2, (M+Q)/2]$, governs charge excitations.

To complete the study of the model and confirm our interpretation of the quasiparticle degeneracies, we looked at exact diagonalization of small systems ($N \leq 10$ with holes). As an example, the low-lying states of the $N=10$, $Q=2$ model is shown in Fig. 1. We summarize the numerical result as follows:

1. The spectrum described in terms of the real pseudomomenta $\{p_\mu\}$ and $\{q_i\}$ span the *full* set of energy levels of the system.

2. The real pseudomomentum states are all highest weight states when $\{p_\mu\} \neq \pm\pi$.

3. The spin degeneracy rule is obeyed for all internal sequences of zeros.

Certain small corrections to the spin degeneracy rule apply when there are zeros at either end of $\{S\}$ which we shall not enumerate here, and which are not important in the thermodynamic limit [17].

Finally, we may use the spectrum generated by the "fully-polarized spinon states" and the supermultiplicity rule to obtain the free energy of the model in the thermodynamic limit. Besides the "particle-state" solutions of equations (12) and (13), we have to take into account the "hole-state" solutions [18]. At thermal equilibrium, the distribution functions of these solutions are determined by minimizing the free energy functional [19,20], $F = E - TS - \mu(N - Q)$, with the constraint that each "fully-polarized spinon state" described by quantum numbers $\{J_\mu\}$ and $\{I_i\}$ is associated with a spin degeneracy w_S as given in (14), where μ is the chemical potential. Following the standard methods of Takahashi [19], minimizing the free energy for a given quantum number distribution gives the following free energy

$$F(T)/N = -\mu - \frac{T}{2\pi} \int_{-\pi}^{\pi} dp \, \ln \left[1 + e^{-\beta \epsilon_s(p)} \right], \tag{15}$$

where ϵ_s is determined by the coupled equations

$$\begin{aligned} 2\epsilon_s(p) &= \epsilon_0(p) - 2a - T \ln \left[1 + e^{-\beta \epsilon_c(p)} \right] \\ \epsilon_c(q) &= 2a - T \ln \left[1 + e^{-\beta \epsilon_s(q)} \right]. \end{aligned} \tag{16}$$

Here $\epsilon_0(p) = \frac{1}{2} t(p^2 - \frac{\pi^2}{3}) + \mu$ and $a = \frac{1}{6} t\pi^2 + \frac{1}{2}\mu$. In the limit of half filling, $\mu \to \infty$, $\epsilon_s \to t(p^2 - \pi^2)/4$, and the free energy reverts to the form obtained by Haldane for the corresponding Heisenberg model [7,21]. For general μ, elimination of $\epsilon_c(p)$ yields the result

$$\epsilon_s(p) = \epsilon_o(p) - T \ln \left[\frac{1}{2} + \left(\frac{1}{4} + 2e^{\beta(\epsilon_0(p)+a)} \cosh(\beta a) \right)^{1/2} \right], \tag{17}$$

We have verified that high temperature expansion of this free energy in powers of β correctly reproduces the first two non-trivial terms in the high temperature perturbation theory.

From the free energy, it is not clear whether we may make a unique identification of the statistics of the spin and charge excitations. We note that the $S = 1/2$ spinon excitations always combine into a state with a symmetric spin wavefunction, thus $2S$ spinons form a state with total spin S. In the limit of zero doping, the free energy is that of spinless fermions [7,21]. We can equally well regard the spinon excitations as $S = 1/2$ fermions in a state with a fully antisymmetric spatial wavefunction; or alternatively, as hardcore $S = 1/2$ bosons in a fully symmetric spatial state.

In summary, we have derived the spectrum of the 1D t-J model with $1/r^2$ long-range exchange and hopping by the introduction of zeros into Jastrow ground-state wavefunctions. Our solution confirms Kawakami's conjecture that the ABA provides the exact spectrum, suggesting that despite the long-range nature of the interactions, two-body scattering dominates the long-wavelength physics. By interpreting multiple occupancy of momentum states in the spinon wavefunction as symmetric bound-complexes of spinons, we have been able to determine the degeneracies of the states needed to construct the free energy. Further work is required to determine the integrability conditions of this model. There are also several possible generalizations: most notably, SU(N) generalizations and the appealing possibility of Jastrow-integrable impurity models.

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FIGURES

FIG. 1. Low-lying energy levels of the 10 site 2 hole system from exact diagonalization. The numbers associated with each state list the spin degeneracies starting with spin 0 on the left. For example, the number “331” indicates that we have 3 states with $S = 0$, 3 states with $S = 1$ and 1 state with $S = 2$.