The FZZ duality with boundary

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ABSTRACT: The Fateev-Zamolodchikov-Zamolodchikov (FZZ) duality relates Witten's cigar model to sine-Liouville theory. This duality was proven in the path integral formulation and extended to the case of higher genus closed Riemann surfaces by Schomerus and one of the authors. In this note we further extend the duality to the case with boundary. Specifically, we relate D1-branes in the cigar model to D2-branes in the sine-Liouville theory. In particular, the boundary action for D2-branes in the sine-Liouville theory is constructed. We also consider the fermionic version of the FZZ duality. This duality was proven as a mirror symmetry by Hori and Kapustin, but we give an alternative proof in the path integral formulation which directly relates correlation functions. Also here the case with boundary is investigated and the results are consistent with those for branes in $\mathcal{N} = 2$ super Liouville field theory obtained by Hosomichi.

Contents

1 Introduction

In this paper we derive the Fateev-Zamolodchikov-Zamolodchikov (FZZ) duality [1] in the case where the world-sheet has a boundary. The duality is between Witten's sigma model [2] for the Euclidean 2d black hole and sine-Liouville field theory. We also consider the fermionic version of the black hole sigma model and its duality with the $\mathcal{N} = (2, 2)$ super-Liouville field theory. The fermionic FZZ duality was proven as a mirror symmetry in [3]. The bosonic two-dimensional black hole is described by the $H_3^+/\mathbb{R}$ coset sigma model with world-sheet coupling given by the level k. Here $H_3^+ = \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2, \mathbb{R})$ is the Euclidean version of AdS$_3$. The geometry describes a semi-infinite cigar with asymptotic radius $\sqrt{k}$.

In this paper we work in units where the string length is set to $l_s \equiv \sqrt{\alpha'} = 1$. The dual sine-Liouville theory has interaction terms which are defined on a cylinder with the inverse radius $1/\sqrt{k}$. The potential gives exponential suppression in one direction of the cylinder, but with a coupling constant which is the inverse as on the cigar side. For large k the cigar model is at weak curvature and thus has weak world-sheet coupling, while the dual sine-Liouville potentials give strong coupling, and vice versa.

The main motivation for studying these target space dualities is their strong/weak coupling nature. Such dualities allow us to do calculations in the strongly coupled regions which are normally very hard to access. Calculations at strong coupling are important in many settings with a prominent example being the AdS/CFT duality [4]. Here the string side with strong world-sheet coupling should be compared with a weakly coupled gauge theory dual, and being able to calculate on both sides will allow non-trivial checks of the conjecture. The FZZ duality involves Euclidean AdS$_3$ and has been used to get quantum corrections to the thermodynamics of the two-dimensional black hole via holography [5]. The fermionic version would be useful to study holography involving NS5-branes as, e.g., in [6].

The bosonic FZZ duality on closed world-sheets has recently been proven by Schomerus and one of the authors [7] using a path-integral method. Let us remind ourselves of the key steps in the proof. First the coset $H_3^+/\mathbb{R}$ is written as a gauged WZNW model and embedded in the $H_3^+ \times U(1)$ model [8]. One then uses the fact that the correlators of H_3^+ model can be written in terms of Liouville field theory correlators with extra insertions of degenerate fields [9, 10]. The essential non-perturbative step is here to use Liouville theory's strong/weak self-duality. Then, some treatments of fields and operators yield the correlators of sine-Liouville theory. The supersymmetric version of the FZZ duality has been derived by Hori and Kapustin [3] as mirror duality using the standard, but rather indirect method of gauged linear sigma models. In this paper we give an alternative proof using the method of [7] and thus establish a direct path-integral derivation of mirror duality. This also has the advantage that we do not only get a duality for the actions, but we have a precise relation of correlation functions including the coupling dependent normalization. Further, it extends readily to higher genus closed world-sheets [7].

The main part of this note is devoted to extend the FZZ and mirror duality to the case with open strings. It is certainly an important problem to understand boundary conformal field theories with non-compact target spaces. In eventually solving these models, knowledge of the boundary Lagrangian complying with the boundary conditions is important.[1] In our case such a Lagrangian description is essential since we need a path integral formulation of correlation functions to derive the dualities. Such a boundary action can luckily be found for AdS$_2$-type branes in the H_3^+ model [19]. In our coset $H_3^+/\mathbb{R}$ such branes descend to 1-dimensional branes on the cigar. Using the boundary action we can then derive the duality in a similar manner as the bulk case. As with T-duality we expect Dirichlet and Neumann boundary conditions to be exchanged in the circular direction. In-

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[1]Examples of such models studied recently are deformations of Liouville field theory and H_3^+ model [11], as well as supergroup WZNW models [12–17]. Finding boundary actions is in general a difficult task and has not been solved in generality, see however [18].

deed, the corresponding dual branes are of the two-dimensional FZZT-type [20, 21]. As a result correlators of sine-Liouville and the cigar coincide, and thus the known disc one-point functions of the cigar [22] can be used to write down those of sine-Liouville. Moreover, we obtain a Lagrangian for the boundary sine-Liouville theory which also allows to compute correlators directly in this model.

The article is organized as follows: In section two we consider the bosonic FZZ duality for open strings. In section 2.1 we give the action for the disk including the boundary part when we consider D1-branes.[2] In section 2.2 the vertex operators are considered and the correlation functions are written in the path integral formalism. The correspondence to Liouville field theory is shown in section 2.3, and using the Liouville self-duality, the duality with sine-Liouville theory is derived in section 2.4. In section 3 we consider the supersymmetric FZZ duality, starting with the path integral formulation in section 3.1. Finally, the duality is derived for respectively closed and open world-sheets in sections 3.2 and 3.3. Conclusions are given in section 4. The paper is closed with five appendices. In appendix A we recall the geometry of the branes that we consider, and their possible boundary actions. The gauged sigma model describing the cigar is considered in appendix B where we also explain the Chan-Paton factors appearing in our calculations. Some Jacobians from the change of measure are calculated in appendix C. In appendix D we derive the reflection coefficient of vertex operators that we need in the proofs of the dualities. Finally, in appendix E we discuss branes in sine-Liouville theory.

2 Boundary FZZ duality

The FZZ duality conjecture was proven in [7] and generalized to correlation functions on closed Riemann surfaces of arbitrary genus. In this section we would like to extend the duality to disk amplitudes. Before proceeding let us briefly state the main result. For the derivation of the FZZ duality with boundary, the main ingredient is to show the identity between correlators in the cigar model and sine-Liouville theory on the disk or equivalently upper half-plane

$$\left\langle \prod_{a=1}^{N} \Psi_{m_a,\bar{m}_a}^{j_a}(z_a) \prod_{c=1}^{M} \Psi_{m_c}^{l_c,i_c}(u_c) \right\rangle^{\text{cig}} \tag{2.1}$$

$$= \mathcal{N} \left\langle \prod_{a=1}^{N} e^{2b(j_a+1)\phi+i\frac{2}{\sqrt{k}}(m_a X_L - \bar{m}_a X_R)}(z_a) \prod_{c=1}^{M} \frac{\sigma_{i_c}}{\sqrt{2}} e^{b(l_c+1)\phi+i\frac{1}{\sqrt{k}}m_c(X_L - X_R)}(u_c) \right\rangle^{\text{sL}} .$$

On the cigar side we have N bulk primary operators $\Psi_{m_a,\bar{m}_a}^{j_a}$ in the positions z_a. These have representation labels $m, \bar{m}, j$ where the discrete labels $m, \bar{m}$ are related to momentum and winding along the circular direction, and the continuous label j is related to momentum in the semi-infinite direction. The boundary conditions in the cigar corresponds to a single D1-brane. There are M boundary operators $\Psi_{m_c}^{l_c,i_c}$ located at the points u_c. These have labels m, l, i. Here m describes the winding in the circular direction which can be half-integer (see figure 1 below), and l is again momentum in the non-compact direction. Further

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[2]Since we are in Euclidean space, Dp-branes denote p-dimensional branes.

$i = 0, 1, 2, 3$ labels an SU(2) Chan-Paton factor (2.9) which will be important for us. It is related to which branches the string ends are attached to. There is a single coupling constant k in the cigar model which is the square of the cigar radius at infinity. The precise definitions of the above bulk and boundary operators will be given in eqs. (2.10)–(2.18).

On the sine-Liouville side the theory is described by two scalars ϕ, X which are non-compact and compact respectively. There are also N bulk fields and M boundary fields which will depend on k under the mapping. The boundary conditions correspond to a D2-brane (after T-dualizing the X direction), and the boundary operators will also have Chan-Paton factors which are traced over in the evaluation of the correlator. The action will depend on the coupling $1/b = \sqrt{k-2}$, and the exact form of the boundary action will be derived, see eq. (2.54). Finally, there is a constant, $\mathcal{N}$, relating the two correlators, which only depends on N, M, the coupling k and the total winding number.

2.1 D1-branes in the 2d black hole

The derivation of the duality follows the method used in [7] which is based on path integral techniques. We thus need the boundary action for D1-branes in the 2d black hole. This is not known yet, so we first have to find it. The starting point is the H_3^+ WZNW model, where $H_3^+ = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2,\mathbb{R})$ is the Euclidean version of AdS$_3$. The sigma model of the 2d black hole is the H_3^+ WZNW model gauged by $\mathbb{R}$, and it can be embedded in the $H_3^+ \times U(1)$ WZNW model [8]. Branes in the cigar then descend from branes in the WZNW model [22], and it was found that there are D0-, D1- and D2-branes in the 2d black hole. The D0-branes descend from fuzzy spherical branes in H_3^+, the D1-branes from AdS$_2$ branes, and the D2-branes from H_2^+ branes. We consider D1-branes, since AdS$_2$ branes have a nice Lagrangian description [19]. This allows us to find an action for D1-branes as we will now explain. In particular, we will find that a Chan-Paton factor should be included in the boundary action.

Let us start with the bulk theory before going into the details of the boundary theory. By the standard technique the coset model $H_3^+/\mathbb{R}$ can be described by the product of the H_3^+ model, a U(1) free boson X, and a (b, c)-ghost system. The action of the bulk 2d black hole is then given by

$$S = S^H + \frac{1}{\pi} \int d^2w \partial X \bar{\partial} X \ (+S_{b,c}) \ , \tag{2.2}$$

where we use the action for the H_3^+ model in the free field realization as[3]

$$S^H = \frac{1}{\pi} \int d^2w \left(\partial\phi\bar{\partial}\phi - \beta\bar{\partial}\gamma - \bar{\beta}\partial\bar{\gamma} + \frac{Q_\phi}{4}\sqrt{g}\mathcal{R}\phi - \pi\lambda\beta\bar{\beta}e^{2b\phi} \right) \ . \tag{2.3}$$

The theory consists of a (β, γ)-system with conformal dimension $(1, 0)$ and a free boson ϕ with background charge $Q_\phi = b = 1/\sqrt{k-2}$. As mentioned in the introduction we

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[3]Here the measure is $d^2w = dxdy$ with $w = x + iy$. Thus there is a factor 2 difference from the one in [7]. The world-sheet metric and its curvature are given by $ds^2 = |\rho(w)|^2 dw^2$ and $\sqrt{g}\mathcal{R} = -4\partial\bar{\partial}\ln|\rho|$. The regularization at the same position is done as $\lim_{w\to z}|w-z|^2 = -\ln|\rho(z)|^2$. In this note we set $\rho = 1$ and suppress the curvature terms.

set $\alpha' = 1$. Namely, the operator products are given by $\phi(z, \bar{z})\phi(0, 0) \sim X(z, \bar{z})X(0, 0) \sim -\ln|z|$. Since the (b, c)-ghost system does not appear in the following discussion, we neglect the part associated with the action $S_{b,c}$.

D-branes in the cigar model were investigated in [22]. The semi-classical analysis was done by using DBI actions as world-volume theories. Moreover, exact solutions were obtained by making use of the fact that correlation functions on a disk are given by the product of those in the H_3^+ model and the U(1) free boson. In this paper we consider D1-branes, and their classical geometry can be examined as follows. For D1-branes, it was argued that we should assign boundary conditions corresponding to AdS$_2$ branes in the H_3^+ model, and Dirichlet boundary condition in the U(1) direction. It is well known that branes in WZNW models of group manifolds are described by twisted conjugacy classes [23]. The situation is similar in a coset, as $H_3^+ = \text{SL}(2, \mathbb{C})/\text{SU}(2, \mathbb{R})$, as well as in a gauged WZNW model, at least if we gauge by an abelian group. In appendix A.1 we explain the geometric meaning of the branes in the product theory that descend to D1-branes in the coset. The brane of the U(1) part is just a point $e^{i\theta_0}$, i.e. it has to satisfy Dirichlet conditions. The H_3^+ part is a left-translate of a twisted conjugacy class, translated by $e^{i\theta_0}$. The metric of the 2d black hole may be given by

$$ds^2 = k(d\rho^2 + \tanh^2 \rho d\theta^2) \ , \tag{2.4}$$

and the D1-branes are characterized by the equation

$$\sinh \rho \sin(\theta - \theta_0) = \sinh r \tag{2.5}$$

with two parameters r, θ_0 (see figure 1). The parameter r corresponds to the distance from the tip of cigar ($\rho = 0$) to the D1-brane. The other parameter θ_0 represents the position of D1-branes in the θ-direction at $\rho \to \infty$. In particular, the D1-brane reaches to the infinity $\rho \to \infty$ at the two point $\theta_0, \theta_0 + \pi$.

One of the important steps to generalize the FZZ duality for a disk amplitude is to obtain the boundary action for D1-branes in the cigar. This is because we closely follow the method used for closed strings in [7], where the path integral formulation is essential. The D1-branes in the cigar model descend from AdS$_2$ branes in the H_3^+ model. In equation (2.3) we have already written the action for the H_3^+ model, and the boundary action for AdS$_2$ branes is proposed in [19] as

$$S_{\text{bint}} = i\lambda_B \int du \beta e^{b\phi} \ , \tag{2.6}$$

where the parameter λ_B is related to r as

$$\lambda_B = \sqrt{\frac{\lambda}{\sin \pi b^2}} \sinh r \ . \tag{2.7}$$

We may treat the interaction terms perturbatively, then the boundary conditions for free fields are Neumann boundary condition for ϕ, and moreover $\gamma + \bar{\gamma} = 0$, $\beta + \bar{\beta} = 0$. However, it turns out that this boundary action cannot be used directly for the D1-branes in the cigar model.
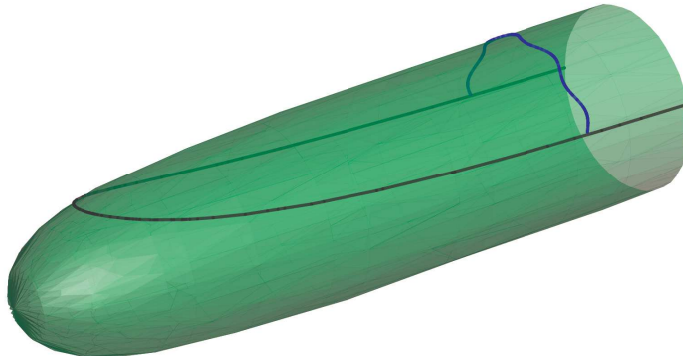
Figure 1. *The cigar with a D1-brane (drawn in black) having $r \neq 0$ and a 1/2 winding string (drawn in blue) stretched between its two branches. ρ is the coordinate along the cigar axis, and θ the angular coordinate. The tip of the cigar is located at $\rho = 0$. The brane parameter r describes the minimal ρ-value for the brane, and the parameter θ_0 its angular orientation.*

We would like to propose the following modified version as

$$S_{\text{bint}} = i\lambda_B e^{i\theta_0} \sigma_3 \int du \beta e^{b\phi} \qquad (2.8)$$

with Dirichlet boundary condition for X additionally. There are two differences from the one in (2.6). One is the factor $e^{i\theta_0}$, which represents the position of the D1-brane in θ-direction. This factor is not so important, and in fact we can and will remove it by the shift of the coordinate θ. The important one is the Chan-Paton factor σ_3, where we use the notation for the Pauli matrices as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (2.9)$$

and $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. The Chan-Paton factor may be realized using boundary fermions, and the classical consistency of (2.8) is explained in appendix A.2.

We can find a reason for the Chan-Paton factor when we follow the boundary conditions from the cigar model via the gauged WZNW model to the product theory, see appendix B. In the cigar model we can have boundary operators corresponding to half-winding open strings going between the two branches of the same D1-brane as in figure 1. In fact, the spectrum of open strings between a D1-brane was obtained in [22] with the help of modular transformation of the annulus amplitude. It was found, for instance in eq. (4.20) of that paper, that the spectrum includes both integer and half-integer winding modes. In the product theory such half-winding strings will not go between the same brane, but between two different branes located oppositely on the U(1) circle and having parameters r and $-r$
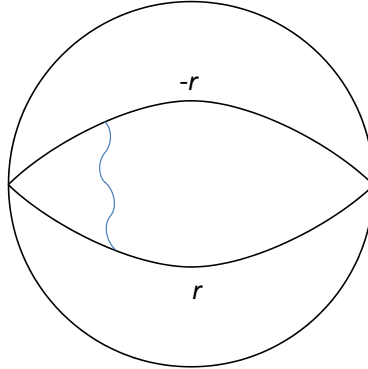
Figure 2. H_3^+ *seen from above with two AdS_2 branes labelled r and $-r$. The half integer windings strings – sketched in blue as in figure 1 – now go between the two different branes.*

for the AdS_2-part in H_3^+ as in figure 2. The Chan-Paton factor appearing in the action corresponds to having these two branes, and one can explicitly see how it appears in the action when going to the product theory. See also the discussions at the end of section 4.3 of [22].

Let us remark that there are other ways to see the necessity of the Chan-Paton factors. First consider the geometry of a D1-brane in the cigar which is characterized by (2.5). This equation is invariant under the exchange of parameters (r, θ_0) and $(-r, \theta_0 + \pi)$. In other words, the branes with parameters (r, θ_0) and $(-r, \theta_0 + \pi)$ should not be two different branes, but identified. In order to realize this, we take a sum of two "fractional" branes to construct one "bulk" brane in the cigar model, just as for bulk branes in an orbifold model. This treatment is consistent with the presence of the Chan-Paton factor σ_3. Secondly, when we study the action, we are treating it in a perturbative way. In the asymptotic region $\rho \to \infty$, we can neglect the interaction term and the geometry is just a cylinder. In this sense, the interaction terms deform the geometry in the small ρ regions. From the geometry of D1-brane (2.5), we can see that there are two D1-branes with opposite orientations in the $\rho \to \infty$ region. In a flat background, it is known how to describe systems with two branes, that is, we just need to include a 2×2 Chan-Paton factor for each open string operator.

2.2 Correlation functions on a disk

In order to prove that the two theories are equivalent, we have to show that all correlation functions in the two theories match. As in [7] we first show that correlation functions of tachyon operators yield the same quantities. After this we just need to check that the symmetry of the theories is the same since descendants can be generated with the help of currents of underlying symmetry. First we need to develop the method to compute correlation functions on a disk with D1-brane boundary conditions in the cigar model.

According to the general formalism, the correlators are given by products of those of H_3^+ and $U(1)$ models roughly speaking, but details should be fixed. For instance, the boundary operators have to be associated with Chan-Paton factors, since the boundary action includes σ_3 as we have already observed. In the later subsections, we relate the amplitudes to those of sine-Liouville theory.

We would like to compute correlation functions on a disk with insertions of bulk and boundary operators. We calculate in the product theory where the cigar model is described by the $H_3^+ \times U(1)$ model along with the ghost. Gauge invariant bulk operators can be constructed from the following products of operators in $H_3^+ \times U(1)$ (see, for instance, [7])

$$\Psi^j_{m,\bar{m}}(z) = \Phi^j_{m,\bar{m}}(z) V^X_{m,\bar{m}}(z) . \tag{2.10}$$

Here $\Phi^j_{m,\bar{m}}$ is a primary operator of H_3^+ model, and it is defined as

$$\Phi^j_{m,\bar{m}}(z) = N^j_{m,\bar{m}} \int \frac{d^2\mu}{|\mu|^2} \mu^m \bar{\mu}^{\bar{m}} V_j(\mu|z) , \qquad V_j(\mu|z) = |\mu|^{2j+2} e^{\mu\gamma - \bar{\mu}\bar{\gamma}} e^{2b(j+1)\phi} \tag{2.11}$$

with

$$N^j_{m,\bar{m}} = \frac{\Gamma(-j-m)}{\Gamma(j+1+\bar{m})} , \qquad m = \frac{kw+n}{2} , \qquad \bar{m} = \frac{kw-n}{2} . \tag{2.12}$$

Here n and w take integer values and correspond to momentum and winding number, respectively. The $U(1)$ vertex operator

$$V^X_{m,\bar{m}}(z) = e^{i\frac{2}{\sqrt{k}}(mX_L - \bar{m}X_R)} \tag{2.13}$$

corresponds to a state with non-trivial winding w.

Along the boundary of world-sheet, we can insert boundary operators, which are going to be constructed. As in the bulk case, the operators should be given by certain products of those in the H_3^+ and $U(1)$ models. The boundary operators for AdS$_2$ branes in the H_3^+ model were constructed in [24] as[4]

$$\Phi^l_{m,\eta}(u) = N^l_{m,\eta} \int \frac{d\nu}{|\nu|} |\nu|^m \mathrm{sgn}^\eta(\nu) V_l(\nu|u) , \qquad V_l(\nu|u) = |\nu|^{l+1} e^{(\nu\gamma - \nu\bar{\gamma})/2} e^{b(l+1)\phi} . \tag{2.14}$$

Here u is the coordinate of the boundary of the world-sheet and the coefficient is

$$N^l_{m,\eta} = 2i^\eta \Gamma(-l-m) \sin\frac{\pi}{2}(-l-1-m-\eta) . \tag{2.15}$$

For D1-branes in the cigar model, we would like to propose that the boundary operators are expressed as

$$\Psi^{l,i}_{m,\eta}(u) = \frac{\sigma_i}{\sqrt{2}} \Phi^l_{m,\eta} V^X_m(u) , \tag{2.16}$$

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[4] The parameter η enters in the boundary case. The integration over ν in (2.14) has a singularity at $\nu = 0$, so we have to separate the integration domain as $\nu < 0$ and $\nu > 0$. Alternatively, we assign the weight $\mathrm{sgn}^\eta(\nu)$ as in (2.14). The label η is related to the behavior of operator under the parity transformation $\gamma \to -\gamma$. This can be easily seen from the fact that the transformation $\gamma \to -\gamma$ can be absorbed by the change as $\nu \to -\nu$ and hence the operator (2.14) has a factor $(-1)^\eta$ under $\gamma \to -\gamma$.

where the $U(1)$ boundary vertex operators with Dirichlet boundary condition are

$$V_m^X(u) = e^{i\frac{1}{\sqrt{k}}m(X_L - X_R)} \ . \tag{2.17}$$

The non-trivial point here is the Chan-Paton factor. We have associated σ_i to each boundary operator, since the boundary action in (2.8) includes a Chan-Paton factor σ_3.

The presence of the Chan-Paton factor affects the values of parameters for the boundary operator. Inserting a boundary operator changes the boundary conditions, so it also has labels $[\Phi_{m,\eta}^l]_{L,L'}$ with the parameters of boundary conditions $L = (r, M)$. The parameter r appears in (2.8) through (2.7), so the coefficient of the boundary action changes across the boundary operator. The parameter M is related to the position of D1-brane as $M = \frac{k}{2\pi}\theta_0$. Since X satisfies the Dirichlet boundary condition, the label m represents the length of open string $\Delta\theta = 2\pi m/\sqrt{k}$ stretched between two D1-branes at the infinity. Thus, m takes value

$$m = M - M' + \frac{kw}{2} \ . \tag{2.18}$$

In order to understand the meaning of w, let us think of open strings on the same brane, namely with $M = M'$. As illustrated in figure 1, one D1-brane reaches two points at the infinity $\rho \to \infty$. If we associate σ_i with $i = 0, 3$ in (2.16), then the corresponding open string is stretching between the same side of the D1-brane. Therefore, the length of open string should be $\Delta\theta \in 2\pi\sqrt{k}\mathbb{Z}$, which implies $w \in 2\mathbb{Z}$ in (2.18). In the same way, if we associate σ_i with $i = 1, 2$ in (2.16), then the corresponding open string is between the opposite sides of the D1-brane. Therefore, the length of open string should be $\Delta\theta \in 2\pi\sqrt{k}(\mathbb{Z} + 1/2)$, which implies $w \in 2\mathbb{Z} + 1$ in (2.18). As argued below, we have to use $\eta = 0$ for $i = 0, 1$ and $\eta = 1$ for $i = 2, 3$, so we may suppress the label η from now on.

Let us see how the relation between the labels η and i arises. The boundary action is given by (2.8), and the Chan-Paton factor σ_3 corresponds to having two branes. The first and the second brane have respectively the labels r and $-r$ without the Chan-Paton factor in (2.8). Notice that the sign change can be absorbed by $\beta \to -\beta$ and $\gamma \to -\gamma$. First we consider the open strings stretched between the same branes. Then the Chan-Paton factor should be $\hat{\sigma}_+$ for an open string on the first brane and $\hat{\sigma}_-$ for an open string on the second brane. Here we have defined $\hat{\sigma}_\pm = \frac{1}{2}(\sigma_0 \pm \sigma_3)$. In other words, if we use $\sigma_0 = \hat{\sigma}_+ + \hat{\sigma}_-$, then the open string is invariant under the exchange of the first and the second brane. Since the exchange of branes can be reproduced by the parity transformation $\gamma \to -\gamma$, the open string associated with σ_0 should be invariant under the parity transformation. Thus we should choose $\eta = 0$ in (2.16). In the same way, the Chan-Paton factor σ_3 implies a minus sign under the parity transformation, and hence $\eta = 1$. Open strings between different branes can be analyzed in the same way. The Chan-Paton factor σ_1 is invariant under the exchange of branes by definition, and thus the corresponding open string should be invariant under the parity transformation. On the other hand, the Chan-Paton factor σ_2 yields a minus sign under the exchange, and we should choose $\eta = 1$ in (2.16).

Now that bulk and boundary operators are constructed, we can write down the correlation functions. We would like to compute a disk amplitude with D1-brane boundary

conditions and with the insertions of N bulk and M boundary operators

$$\left\langle \prod_{a=1}^{N} \Psi_{m_a,\bar{m}_a}^{j_a}(z_a) \prod_{b=1}^{M} \Psi_{m_b}^{l_b,i_b}(u_b) \right\rangle . \tag{2.19}$$

The total winding number

$$\sum_a (m_a + \bar{m}_a) + \sum_b m_b = \frac{kS}{2} \tag{2.20}$$

is not necessarily zero due to the cigar-shape background and U-shape brane geometry. As we will see below, the violation number is limited as $|S| \leq 2N + M - 2$. If it is non-zero, then the spectral flow operation in the H_3^+-sector is important as emphasized in [7].

Let us first recall the operator inducing the spectral flow action with S' for the bulk case, which is represented by $v^{S'}(\zeta)$. Here we assume $S' \geq 0$, but $S' < 0$ case can be analyzed in a similar way. The insertion of $v^{S'}(\zeta)$ has two effects in our free field realization of the H_3^+ model with the action (2.3). One is the insertion of a vertex operator $\exp(S'\phi/b)$. The other is the restriction of the integration domain for $\beta, \bar{\beta}$ such that they have a zero at $z = \zeta$ of order S'. See [7] for the relation to the usual definition of spectral flow action. For the boundary case, we would like to introduce a boundary operator which induces the spectral flow action with S units. Here we again assume that $S \geq 0$. We propose that it is given by $\sigma_1^S v^S(\xi)$. The action of $v^S(\xi)$ is almost the same as the bulk case. Namely, the boundary operator $\exp(S\phi/2b)$ is inserted, and the integration domain for $\beta = -\bar{\beta}$ is restricted such that it has a zero at $z = \xi$ of order S. The boundary operator should be associated with a Chan-Paton factor σ_i^S. From the parity property, we should choose $i = 0$ or $i = 1$. Among them we adopt $i = 1$ since the spectral flow with odd S maps open strings between the same brane to the branes with r and $-r$, and vise versa as mentioned in [22].

With the preparation of spectral flow operator, the correlation function (2.19) is now written in terms of H_3^+ and $U(1)$ models as

$$\left\langle \prod_{a=1}^{N} \Psi_{m_a,\bar{m}_a}^{j_a}(z_a) \prod_{b=1}^{M} \Psi_{m_b}^{l_b,i_b}(u_b) \right\rangle \tag{2.21}$$

$$= \prod_{a=1}^{N} \left[N_{m_a,\bar{m}_a}^{j_a} \int \frac{d^2\mu_\nu}{|\mu_\nu|^2} \mu_\nu^{m_\nu} \bar{\mu}_\nu^{\bar{m}_\mu} \right] \prod_{b=1}^{M} \left[N_{m_b,\eta_b}^{l_b} \int \frac{d\nu_b}{|\nu_b|} |\nu_b|^{m_b} \mathrm{sgn}^{\eta_b}(\nu_b) \right]$$

$$\times \mathrm{tr}\, P \left\langle \sigma_1^S v^S(\xi) V_{-\frac{kS}{2}}^X(\xi) \prod_{a=1}^{N} V_{j_a}(\mu_a|z_a) V_{m_a,\bar{m}_a}^X(z_a) \prod_{b=1}^{M} \frac{\sigma_{i_b}}{\sqrt{2}} V_{l_b}(\nu_b|u_b) V_{m_b}^X(u_b) \right\rangle ,$$

where P represents the path ordering of boundary operators. The spectral flow action $\sigma_1^S v^S(\xi)$ is inserted by using the fact that the identity operator in the cigar model can be represented by

$$\mathbf{1} = \sigma_1^S v^S(\xi) V_{-\frac{kS}{2}}^X(\xi) . \tag{2.22}$$

This is actually the very definition of the spectral flow operator. Total winding number in the $U(1)$ model is now conserved due to the insertion of $V_{-kS/2}^X$. Notice that the end result would not depend on the position of the insertion ξ since we have just inserted the identity operator. For more details, see [7].

2.3 Relation to Liouville field theory

The task is now to express the $N + M$ point correlation function (2.19) of the boundary cigar in terms of a $N + M$ point correlator of sine-Liouville theory. As in the bulk case [7], this can be achieved in two steps; The first one is to rewrite the correlator in terms of a correlation function in Liouville field theory plus a free boson with additional degenerate field insertions. This is the subject of this subsection. For this purpose we use the relation between correlators of H_3^+ model and Liouville field theory [9, 10], which was extended to the case with boundary in [19, 24]. The second step is to show that after applying the self-duality of Liouville field theory, rotations of the fields and reflection relations of operator lead to the correlator in sine-Liouville theory. This will be done in the next subsection.

As in [7], we use the path integral form of the $N + M$ point correlation function (2.19) as

$$\left\langle \prod_{a=1}^{N} \Psi_{m_a, \bar{m}_a}^{j_a}(z_a) \prod_{b=1}^{M} \Psi_{m_b}^{l_b, i_b}(u_b) \right\rangle$$
$$= \operatorname{tr} P \int \mathcal{D}\phi \mathcal{D}^2 \beta \mathcal{D}^2 \gamma \mathcal{D}X e^{-S} \prod_{a=1}^{N} \Psi_{m_a, \bar{m}_a}^{j_a}(z_a) \prod_{b=1}^{M} \Psi_{m_b}^{l_b, i_b}(u_b) . \qquad (2.23)$$

Here the action S is given by (2.2) with the boundary interaction term (2.8). Now we work on the upper half plane $\operatorname{Im} z \geq 0$ and the boundary is at the line $\operatorname{Im} z = 0$. According to the standard doubling trick, the anti-holomorphic part of fields are mapped to the region of $\operatorname{Im} z < 0$ such that the boundary conditions are satisfied along the boundary $\operatorname{Im} z = 0$.

Following [10, 19], we integrate over β, γ to reduce the theory to the one with two remaining fields ϕ, X. Notice that the field γ appears only linearly in the exponent of the path integral (2.23). This is because the action includes γ only in the kinetic term of (2.2) and the vertex operators are expressed in terms of (2.11) and (2.14). Therefore, by integrating over γ, we would have a delta functional for $\bar{\partial}\beta$, which implies that the field β can be replaced by a function $\mathcal{B}(z)$. Integrating over the world-sheet coordinate, the function is obtained as

$$\mathcal{B}(w) = \sum_{a=1}^{N} \frac{\mu_a}{w - z_a} + \sum_{a=1}^{N} \frac{\bar{\mu}_a}{w - \bar{z}_a} + \sum_{b=1}^{M} \frac{\nu_b}{w - u_b} . \qquad (2.24)$$

In the same way, $\bar{\beta}(\bar{z})$ is replaced by $-\bar{\mathcal{B}}(\bar{z})$. This form might be understood from the operator product between β in the interaction terms and $\gamma, \bar{\gamma}$ in the vertex operators. See also [19].

One essential ingredient of the H_3^+-Liouville relation is the change of variables corresponding to Sklyanin's separation of variables as in [9, 10]. With the boundary, the formula for the change of variables is given by (see [19, 24] and also appendix C)

$$\mathcal{B}(w) = u \frac{(w - \xi)^S \prod_{a'=1}^{N'}(w - y_{a'})(w - \bar{y}_{a'}) \prod_{b'=1}^{M'}(w - t_{b'})}{\prod_{a=1}^{N}(w - z_a)(w - \bar{z}_a) \prod_{b=1}^{M}(w - u_b)} . \qquad (2.25)$$

Here $t_{b'}$ and $y_{a'}$ denote zeros respectively on the boundary and in the bulk. The equality gives a map of parameters from $(\mu_a, \bar{\mu}_a, \nu_b)$ to $(u, y_{a'}, \bar{y}_{a'}, t_{b'})$, where the numbers of $y_{a'}$

and $t_{b'}$ satisfy the equation $2N' + M' + S + 2 = 2N + M$. This follows from $\mathcal{B}(z)$ being a meromorphic one-form on the full plane hence having two more poles than zeros. Notice that the numbers N', M' would vary when we change the values of $(\mu_a, \bar{\mu}_a, \nu_b)$. Moreover, since the numbers of insertions should be non-negative, we have a restriction for the total winding number $S \le 2N + M - 2$. The presence of $v^S(\xi)$ forces β to have a zero of order S, and this is possible only if

$$\ell_n(\xi) = \sum_{a=1}^{N} \frac{\mu_a}{(\xi - z_a)^n} + \sum_{a=1}^{N} \frac{\bar{\mu}_a}{(\xi - \bar{z}_a)^n} + \sum_{b=1}^{M} \frac{\nu_b}{(\xi - u_b)^n} = 0 \qquad (2.26)$$

for $n = 0, 1, \cdots, S$. Due to the $S+1$ constraints, the number of parameters matches. Since the correlation function is given by the integration over $(\mu_a, \bar{\mu}_a, \nu_b)$ in (2.21), we need the formula for the Jacobian due to the change of variables. It is given by

$$\int \prod_{a=1}^{N} \frac{d^2\mu_a}{|\mu_a|^2} \prod_{b=1}^{b} \frac{d\nu_b}{|\nu_b|} \prod_{n=0}^{S} \delta(\ell_n(\xi)) = \sum_{N',M'} \frac{1}{N'!M'!} \int \frac{du}{|u|^{2+S}} \prod_{a'=1}^{N'} d^2 y_{a'} \prod_{b'=1}^{M'} dt_{b'} |\Xi| \qquad (2.27)$$

with

$$\Xi = \prod_{i<j} |z_{ij}|^2 \prod_{i,j} (z_i - \bar{z}_j) \prod_{i,a} |z_i - u_a|^2 \prod_{a<b} u_{ab} \prod_{i'<j'} |y_{i'j'}|^2 \prod_{i',j'} (y_{i'} - \bar{y}_{j'}) \prod_{i',a'} |y_{i'} - t_{a'}|^2 \prod_{a'<b'} t_{a'b'}$$
$$\times \prod_{i,i'} |z_i - y_{i'}|^{-2} \prod_{i,i'} |z_i - \bar{y}_{i'}|^{-2} \prod_{i,a'} |z_i - t_{a'}|^{-2} \prod_{a,i'} |u_a - y_{i'}|^{-2} \prod_{a,a'} (u_a - t_{a'})^{-1} . \qquad (2.28)$$

Here we should sum over N', M' with the condition $2N' + M' + S + 2 = 2N + M$ since the numbers N', M' depend on the explicit values of $(\mu_a, \bar{\mu}_a, \nu_b)$. For details, see appendix C.

The action of H_3^+ model is given by (2.3), and now $\gamma(w)$ are integrated over and $\beta(w)$ is replaced by a function $\mathcal{B}(w)$. Now the theory is like the Liouville field theory, but the interaction term includes coordinate dependent coefficients as $|\mathcal{B}(w)|^2 \exp(2b\phi(w))$. As in [7, 10], we change the field ϕ as

$$\varphi(w, \bar{w}) = \phi(w, \bar{w}) + \frac{1}{2b} \ln |\mathcal{B}(w)|^2 , \qquad (2.29)$$

then the interaction term becomes $\exp(2b\varphi(w))$ as desired. The boundary interaction term (2.8) is now

$$S_{\text{bint}} = i\lambda_B \sigma_3 \int dt \, \text{sgn}\, \mathcal{B}(t) e^{b\varphi} , \qquad (2.30)$$

which has a sign function $\text{sgn}\, \mathcal{B}(t)$ as noticed in [19, 24]. This implies that the parameter of boundary interaction changes when it crosses the boundary positions u_b and $t_{b'}$. As noticed in [7], it is also necessary to shift $X(w, \bar{w}) = X_L(w) + X_R(\bar{w})$ as

$$\chi_L(w) = X_L(w) - i\frac{\sqrt{k}}{2} \ln \mathcal{B}(w) , \qquad \chi_R(\bar{w}) = X_R(\bar{w}) + i\frac{\sqrt{k}}{2} \ln \bar{\mathcal{B}}(\bar{w}) , \qquad (2.31)$$

where $\chi_R(\bar{w})$ is the complex conjugate of $\chi_L(w)$. This means that $X = X_L + X_R$ is changed with $-i\sqrt{k}/2 \ln \mathcal{B}/\bar{\mathcal{B}}$ whereas the dual field $\tilde{X} = X_L - X_R$ is changed with $-i\sqrt{k}/2 \ln |\mathcal{B}|^2$. The action is now

$$S = \frac{1}{\pi} \int d^2w \left(\partial\varphi\bar{\partial}\varphi + \partial\chi\bar{\partial}\chi + \frac{\sqrt{g}}{4} \mathcal{R}(Q_\varphi \varphi + Q_{\tilde{\chi}} \tilde{\chi}) + \pi\lambda e^{2b\varphi} \right) \tag{2.32}$$

with boundary interaction term (2.30). The dual field is denoted by $\tilde{\chi} = \chi_L - \chi_R$ and background charges are found to be shifted as $Q_\varphi = b + b^{-1}$ and $Q_{\tilde{\chi}} = -i\sqrt{k}$ as shown in [7, 10].

The change of fields (2.29) and (2.31) also affects the kinetic terms. We write the action as $-\int d^2w \phi \partial\bar{\partial}\phi$ and insert the expression in (2.29). Then, from the term $\partial\bar{\partial} \ln |\mathcal{B}(w)|^2$, we obtain delta functions localized at $z_a, u_b, y_{a'}, t_{b'}$. Integrating over the world-sheet coordinate, we find shifts of momenta in the existing vertex operators at z_a, u_b and insertions of new operators at $y_{a'}, t_{b'}$, since the action is in the exponent of the path integral (2.23). The similar things happen for X due to the shift (2.31). Closely following the analysis in [7, 10], the correlation function (2.19) is now written as

$$\left\langle \prod_{a=1}^{N} \Psi_{m_a,\bar{m}_a}^{j_a}(z_a) \prod_{b=1}^{M} \Psi_{m_b}^{l_b,i_b}(u_b) \right\rangle = \sum_{M',N'} \frac{1}{N'!M'!} \prod_{a'=1}^{N'} \int d^2 y_{a'} \prod_{b'=1}^{M'} \int dt_{b'} \tag{2.33}$$

$$\times \prod_{a=1}^{N} N_{m_a,\bar{m}_a}^{j_a} \prod_{b=1}^{M} N_{m_b,\eta_b}^{l_b} \mathrm{sgn}^{\eta_b}(\mathcal{B}(u_b)) \, \mathrm{tr}\, P \left\langle \sigma_1^S \prod_{a=1}^{N} V_{\alpha_a}(z_a) V_{m_a-\frac{k}{2},\bar{m}_a-\frac{k}{2}}^{\chi}(z_a) \right.$$

$$\left. \times \prod_{b=1}^{M} B_{\beta_b}^{i_b}(u_b) V_{m_b-\frac{k}{2}}^{\chi}(u_b) \prod_{a'=1}^{N'} V_{-\frac{1}{2b}}(y_{a'}) V_{\frac{k}{2},\frac{k}{2}}^{\chi}(y_{a'}) \prod_{b'=1}^{M'} B_{-\frac{1}{2b}}^{0}(t_{b'}) V_{\frac{k}{2}}^{\chi}(t_{b'}) \right\rangle .$$

The right hand side is computed using the action (2.32) with the boundary interaction term (2.30). The bulk and boundary operators in the boundary Liouville field theory are defined as

$$V_\alpha(z) = e^{2\alpha\varphi(z)} , \qquad B_\beta^i(t) = \frac{\sigma_i}{\sqrt{2}} e^{\beta\varphi(t)} , \tag{2.34}$$

respectively. The shifts of parameters for bulk and boundary operators at z_a, u_b are

$$\alpha_\nu = b(j_\nu + 1) + \frac{1}{2b} , \qquad \beta_\nu = b(l_\nu + 1) + \frac{1}{2b} , \tag{2.35}$$

and new operators are inserted at $y_{a'}, t_{b'}$. In the above expression $\mathcal{B}(u_b)$ is actually not well-defined since $\mathcal{B}(z)$ has a pole at $z = u_b$. We just represent $\mathrm{Res}_{z \to u_b} \mathcal{B}(z) (= \nu_b)$ by $\mathcal{B}(u_b)$ for simplicity. From the terms like $\ln|\mathcal{B}(w)| \partial\bar{\partial} \ln|\mathcal{B}(w)|$, we would obtain a pre-factor in the correlation function, but as found in [7] we can see that it cancels the Jacobian factor (2.28). Moreover, the insertion $\exp(S\phi/2b)$ coming form $v^S(\xi)$ disappears due to the shift of momenta at ξ.

The method in [7] can be applied almost straightforwardly, one exception is, however, the insertion of the new boundary operator

$$B_{-\frac{1}{2b}}^{0}(t_{b'}) = \frac{\sigma_0}{\sqrt{2}} e^{-\frac{1}{2b}\varphi(t_{b'})} \tag{2.36}$$

with a non-trivial factor $\sigma_0/\sqrt{2}$. If we follow the analysis in [10, 19], then we obtain the insertion of $e^{-\varphi(t)/2b}$ along the boundary. However, it is required to associate a Chan-Paton factor to the insertion at the boundary. The factor should be proportional to the identity operator σ_0, but the overall normalization may differ from one. Let us first examine the operator corresponding to the identity state. The identity state $|1\rangle$ has the norm $\langle 1|1\rangle = 1$. On the other hand, the identity operator $\mathcal{O}_1$ is defined as $\mathcal{O}_1 \cdot \mathcal{O} = \mathcal{O}$ for all operators $\mathcal{O}$. Due to the state-operator correspondence, we may define $|1\rangle := C\mathcal{O}_1|0\rangle$ up to an overall factor C. For our case $\mathcal{O}_1 = \sigma_0$, and $\langle 1|1\rangle = C^2 \operatorname{tr}\sigma_0^2\langle 0|0\rangle = 2C^2$. Therefore, we should define as $|1\rangle := \sigma_0/\sqrt{2}|0\rangle$, and the operator corresponding to the identity state is $\sigma_0/\sqrt{2}$. This reasoning also explains the factor $\sigma_0/\sqrt{2}$ in (2.36).

The aim of this subsection was to rewrite the correlation function of the cigar model in terms of Liouville field theory and free boson with the action (2.32), (2.30) as we have already done in (2.33). However, in order to proceed furthermore, it is convenient to remove the sign factors $\operatorname{sgn}\mathcal{B}(t)$ in the boundary action (2.30) and $\operatorname{sgn}^{\eta_b}(\mathcal{B}(u_b))$ in front of the correlator (2.33). This is possible by making use of the anti-commutativity of the Pauli matrices. Since the boundary action (2.30) includes σ_3, the boundary operator $B_\beta^i(t)$ commutes with it for $i = 0, 3$ and anti-commutes with it for $i = 1, 2$. As mentioned below equation (2.30), the role of $\operatorname{sgn}\mathcal{B}(t)$ is to multiply (-1) when it crosses the positions of boundary operators. Therefore, we can remove the sign function by replacing $i = 0, 3$ and $i = 1, 2$. Moreover, the other sign functions $\operatorname{sgn}^{\eta_b}(\mathcal{B}(u_b))$ implies that we receive extra minus sign when boundary operators with $\eta_b = 0$ and $\eta_b = 1$ are exchanged. Recalling that $i = 0, 1$ for $\eta = 0$ and $i = 2, 3$ for $\eta = 1$, the effect of $\operatorname{sgn}^{\eta_b}(\mathcal{B}(u_b))$ can be reproduced by replacing $i = 0$ and $i = 1$ regardless of the replacement of $i = 2$ and $i = 3$. Combining the both, the rule may be summarized such that σ_i in front of the boundary operator is replaced by $\sigma_1\sigma_i$. Effectively we may insert $1 = \sigma_1^{M+M'+S}$ with even $M + M' + S$ to the correlation function. Now the expression becomes a bit simpler[5]

$$
\left\langle \prod_{a=1}^{N} \Psi_{m_a, \bar{m}_a}^{j_a}(z_a) \prod_{b=1}^{M} \Psi_{m_b}^{l_b, i_b}(u_b) \right\rangle = \sum_{M', N'} \frac{1}{N'!M'!} \prod_{a'=1}^{N'} \int \frac{d^2 y_{a'}}{N'!} \prod_{b'=1}^{M'} \int \frac{dt_{b'}}{M'!} \tag{2.37}
$$

$$
\times \prod_{a=1}^{N} N_{m_a, \bar{m}_a}^{j_a} \prod_{b=1}^{M} N_{m_b, \eta_b}^{l_b} \operatorname{tr} P \left\langle \prod_{a=1}^{N} V_{\alpha_a}(z_a) V_{m_a - \frac{k}{2}, \bar{m}_a - \frac{k}{2}}^{\chi}(z_a) \right.
$$

$$
\left. \times \prod_{b=1}^{M} \sigma_1 B_{\beta_b}^{i_b}(u_b) V_{m_b - \frac{k}{2}}^{\chi}(u_b) \prod_{a'=1}^{N'} V_{-\frac{1}{2b}}(y_{a'}) V_{\frac{k}{2}, \frac{k}{2}}^{\chi}(y_{a'}) \prod_{b'=1}^{M'} B_{-\frac{1}{2b}}^{1}(t_{b'}) V_{\frac{k}{2}}^{\chi}(t_{b'}) \right\rangle
$$

with boundary interaction term

$$
S_{\mathrm{bint}} = i\lambda_B \sigma_3 \int dt\, e^{b\varphi} . \tag{2.38}
$$

The function $\mathcal{B}(w)$ is proportional to u as defined in (2.24), but the dependence of u disappears in the last expression. For instance, the dependence of $\operatorname{sgn}(u)$ can be absorbed by the rotation of Chan-Paton factors as $\sigma_2, \sigma_3 \to -\sigma_2, -\sigma_3$.

---

[5]The equality is up to a trivial factor.

2.4 Duality with boundary sine-Liouville theory

In the previous subsection, we obtained a relation between the cigar model and a Liouville-like theory, but it is not what we wanted to have. In order to relate the cigar model to the sine-Liouville theory, we take three steps as in [7]. First step is to perform the self-duality of the Liouville theory exchanging $b \leftrightarrow b^{-1}$. With this duality the relation becomes a strong/weak duality for k. Next step is to realize that the extra insertions at $y_{a'}, t_{b'}$ can be seen as an expansion of an interaction term in the action. Thus we obtain a relation between $N + M$ point correlation functions. The final step is to perform a rotation of fields and utilize the reflection relation such as to arrive at the boundary sine-Liouville theory.

First step is the Liouville self-duality. It is known that the Liouville field theory is self-dual under the exchange of b by b^{-1} followed by the replacement of λ by $\tilde{\lambda}$ as [25]

$$\mathcal{L} = \tilde{\lambda} e^{2b^{-1}\varphi} , \qquad \pi\tilde{\lambda}\gamma(1/b^2) = (\pi\lambda\gamma(b^2))^{1/b^2} , \tag{2.39}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1 - x)$. The self-duality is extended to the case with boundary, and the dual interaction term is given by [20]

$$\mathcal{L}_B = \tilde{\lambda}_B e^{b^{-1}\varphi} , \qquad \lambda_B = \sqrt{\frac{\lambda}{\sin \pi b^2}} \cosh bs , \qquad \tilde{\lambda}_B = \sqrt{\frac{\tilde{\lambda}}{\sin \pi b^{-2}}} \cosh b^{-1}s \tag{2.40}$$

for the case without Chan-Paton factor. In our case, it is convenient to rewrite the Pauli-matrix as $\sigma_3 = \sigma_+\sigma_- - \sigma_-\sigma_+$, such that the interaction term can be treated as two single branes. Applying the formula (2.40) for each term with $\sigma_+\sigma_-$ and $\sigma_-\sigma_+$, the dual interaction term for the boundary is obtained as

$$\mathcal{L}_B = f(\sigma_i)e^{b^{-1}\varphi} , \tag{2.41}$$

with

$$f(\sigma_i) = \sqrt{\frac{\tilde{\lambda}}{\sin \pi b^{-2}}} \left(\cosh \left(b^{-2} \left(r + \frac{\pi i}{2} \right) \right) \sigma_+\sigma_- + \cosh \left(b^{-2} \left(r - \frac{\pi i}{2} \right) \right) \sigma_-\sigma_+ \right) \tag{2.42}$$

$$= \sqrt{\frac{\tilde{\lambda}}{\sin \pi b^{-2}}} \left(\cosh \frac{r}{b^2} \cos \frac{\pi}{2b^2} \sigma_0 + i \sinh \frac{r}{b^2} \sin \frac{\pi}{2b^2} \sigma_3 \right) .$$

The second equality comes from $\sigma_\pm\sigma_\mp = \frac{1}{2}(\sigma_0 \pm \sigma_3)$. The same result may be obtained with the method in appendix D. With the help of the self-duality, we would obtain a strong/week duality.

Next, we focus on the extra insertions of bulk and boundary operators

$$V(y_{a'}) = V_{-\frac{1}{2b}} V^{\chi}_{\frac{k}{2}, \frac{k}{2}} = e^{-b^{-1}\varphi + i\sqrt{k}\tilde{\chi}} , \qquad V_B(t_{b'}) = B^1_{-\frac{1}{2b}} V^{\chi}_{\frac{k}{2}} = \frac{\sigma_1}{\sqrt{2}} e^{-\frac{1}{2}b^{-1}\varphi + i\frac{\sqrt{k}}{2}\tilde{\chi}} . \tag{2.43}$$

The H_3^+-Liouville relation maps the parameters μ_a, ν_b to the positions of extra insertions $y_{a'}, t_{b'}$. However, in the coset model, we should use the m-basis expressions as in (2.11) and (2.14) by performing Fourier transforms. These lead to the integration over μ_a, ν_b and

hence $y_{a'}, t_{b'}$ after the map as in (2.37). Since the positions of the extra insertions are integrated over the whole world-sheet $y_{a'}$ and the whole boundary $t_{b'}$, we may deal with them as a part of the interaction terms in the action. Notice that only the terms in (2.37) contribute after expanding the interaction terms due to the momentum conservation along the χ direction. The momentum conservation still allows different numbers of bulk and boundary insertions N', M' if they satisfy the condition $2N' + M' = 2N + M - S - 2$. After the above two steps, the interaction terms for the bulk are now the sum of

$$\mathcal{L}_1 = \tilde{\lambda} e^{2b^{-1}\varphi} \ , \qquad \mathcal{L}_2 = -e^{-b^{-1}\varphi + i\sqrt{k}\tilde{\chi}} \ , \tag{2.44}$$

and for the boundary

$$\mathcal{L}_{B,1} = f(\sigma_i) e^{b^{-1}\varphi} \ , \qquad \mathcal{L}_{B,2} = -\frac{\sigma_1}{\sqrt{2}} e^{-\frac{1}{2}b^{-1}\varphi + i\frac{\sqrt{k}}{2}\tilde{\chi}} \ . \tag{2.45}$$

Since the extra insertions are treated as the interaction terms, we now have a relation between $N + M$ point correlation functions of the cigar model and the theory with the above interaction terms. However, the new theory is not yet the sine-Liouville field theory.

Fortunately, the new theory can be mapped to sine-Liouville theory as in [7]. Since the background charges for φ and χ are different from those for sine-Liouville theory, we redefine as

$$\phi = (k - 1)\varphi - i\sqrt{k}b^{-1}\tilde{\chi} \ , \qquad \tilde{X} = -i\sqrt{k}b^{-1}\varphi - (k - 1)\tilde{\chi} \ . \tag{2.46}$$

The background charges then become the desired ones as $Q_\phi = b = 1/\sqrt{k-2}$ and $Q_{\tilde{X}} = 0$. After this rotation the interaction terms are

$$\mathcal{L}_1 = \tilde{\lambda} e^{2b^{-1}(k-1)\phi - 2i\sqrt{k}b^{-2}\tilde{X}} \ , \qquad \mathcal{L}_2 = -e^{b^{-1}\phi - i\sqrt{k}\tilde{X}} \ , \tag{2.47}$$

and for the boundary

$$\mathcal{L}_{B,1} = f(\sigma_i) e^{b^{-1}(k-1)\phi - i\sqrt{k}b^{-2}\tilde{X}} \ , \qquad \mathcal{L}_{B,2} = -\frac{\sigma_1}{\sqrt{2}} e^{\frac{1}{2}b^{-1}\phi - i\frac{\sqrt{k}}{2}\tilde{X}} \ . \tag{2.48}$$

Here we should note that the rotation is consistent with the boundary conditions since ϕ and $\tilde{X}$ satisfy Neumann boundary condition.

We also need to apply the reflection relations of Liouville theory to the interaction terms in order to arrive at the sine-Liouville theory [7]. Since Liouville theory has interaction term of the exponential type, in-coming and out-going modes are related by reflection relations. One trick here is to treat $\mathcal{L}_2$ as the Liouville term instead of $\mathcal{L}_1$. More precisely speaking, we introduce a new field $\hat{\phi} = -i/\sqrt{2}(b^{-1}\phi - i\sqrt{k}\tilde{X})$, then $\mathcal{L}_2 = -e^{i\sqrt{2}\hat{\phi}}$ becomes a Liouville interaction term with $\hat{b} = i/\sqrt{2}$. We apply the reflection relation with (D.2) for the $\hat{b} = i/\sqrt{2}$ theory to the bulk operator in $\mathcal{L}_1$ keeping intact the part orthogonal to the $\hat{\phi}$ direction. Rotating back to the original fields, we have

$$\mathcal{L}_1 = -\tilde{\lambda} \pi^{-1-2b^{-2}} \gamma(1 + b^{-2}) e^{b^{-1}\phi + i\sqrt{k}\tilde{X}} \ , \qquad \mathcal{L}_2 = -e^{b^{-1}\phi - i\sqrt{k}\tilde{X}} \tag{2.49}$$

for the bulk interaction terms. They are the interaction terms of bulk sine-Liouville theory as desired. More detailed explanations are given in [7]. For the boundary case we need

the reflection amplitude with boundary interaction term $\mathcal{L}_{B,2} = -2^{-1/2}\sigma_1 e^{i\hat{\phi}/\sqrt{2}}$, which is studied in appendix D. With the result from there, we have

$$\mathcal{L}_{B,1} = \frac{\tilde{c}}{\sqrt{2}}(e^{\frac{r}{b^2}}\sigma_+ + e^{-\frac{r}{b^2}}\sigma_-)e^{\frac{1}{2b}\phi+i\frac{\sqrt{k}}{2}\tilde{X}} \ , \qquad \mathcal{L}_{B,2} = -\frac{1}{\sqrt{2}}(\sigma_+ + \sigma_-)e^{\frac{1}{2b}\phi-i\frac{\sqrt{k}}{2}\tilde{X}} \quad (2.50)$$

with

$$\tilde{c} = \sqrt{\tilde{\lambda}\pi^{-1-2b^{-2}}\gamma(1+b^{-2})} \ . \tag{2.51}$$

In the end of the computation we have performed a SU(2) rotation of Chan-Paton factors as

$$(\sigma_1, \sigma_2, \sigma_3) \rightarrow (\sigma_1, -\sigma_3, \sigma_2) \tag{2.52}$$

just to make the expression simpler. We would like to claim that they are the interaction terms for D2-branes in the sine-Liouville theory.

If we want to obtain a symmetric expression, then we would shift the zero mode of $\tilde{X}$ appropriately. For the bulk interaction we have

$$\mathcal{L}_1 = \tilde{c}e^{b^{-1}\phi+i\sqrt{k}\tilde{X}} \ , \qquad \mathcal{L}_2 = \tilde{c}e^{b^{-1}\phi-i\sqrt{k}\tilde{X}}, \tag{2.53}$$

and for the boundary interaction

$$\mathcal{L}_{B,1} = i\sqrt{\frac{\tilde{c}}{2}}\left(e^{\frac{r}{2b^2}}\sigma_+ + e^{-\frac{r}{2b^2}}\sigma_-\right)e^{\frac{1}{2b}\phi+i\frac{\sqrt{k}}{2}\tilde{X}} \ , \tag{2.54}$$

$$\mathcal{L}_{B,2} = i\sqrt{\frac{\tilde{c}}{2}}\left(e^{-\frac{r}{2b^2}}\sigma_+ + e^{\frac{r}{2b^2}}\sigma_-\right)e^{\frac{1}{2b}\phi-i\frac{\sqrt{k}}{2}\tilde{X}} \ .$$

This form of the boundary action is quite analogous to the one for B-branes in $\mathcal{N} = 2$ super Liouville field theory in (5.8) with (5.27) of [26]. From this fact, we are confident in the correctness of the boundary action for D2-branes in sine-Liouville theory. In the next section we study the fermionic version of FZZ duality, and indeed we obtain the boundary action for a B-brane in $\mathcal{N} = 2$ super Liouville field theory from the one for a D1-brane in the fermionic cigar model.

We should also apply the same rotation of fields and the reflection relations to the vertex operators in order to establish the relation between correlation functions. For the bulk operator it was done in [7] as

$$N_{m,\bar{m}}^j V_\alpha(z)V_{m-\frac{k}{2},\bar{m}-\frac{k}{2}}^X(z) = -\pi^{-1-2j-m-\bar{m}}e^{2b(j+1)\phi+i\frac{2}{\sqrt{k}}(mX_L-\bar{m}X_R)} \ . \tag{2.55}$$

For the boundary operator we have in the same way

$$N_m^l \sigma_1 B_\beta^i(u)V_{m-\frac{k}{2}}^X(u) = -(-1)^\eta \pi^{-l-m}\sigma_i e^{b(l+1)\phi+i\frac{1}{\sqrt{k}}m(X_L-X_R)} \ , \tag{2.56}$$

where the pre-factor is computed by using the result in appendix D. Here $\eta = 0$ for $i = 0, 1$ and $\eta = 1$ for $i = 2, 3$ as mentioned in subsection 2.2. Combining all the results obtained

so far, the equation (2.37) leads to the final result mentioned already in equation (2.1)

$$
\left\langle \prod_{a=1}^{N} \Psi_{m_a,\bar{m}_a}^{j_a}(z_a) \prod_{c=1}^{M} \Psi_{m_c}^{l_c,i_c}(u_c) \right\rangle^{\text{cig}}
$$

$$
= \mathcal{N} \left\langle \prod_{a=1}^{N} e^{2b(j_a+1)\phi + i\frac{2}{\sqrt{k}}(m_a X_L - \bar{m}_a X_R)}(z_a) \prod_{c=1}^{M} \frac{\sigma_{i_c}}{\sqrt{2}} e^{b(l_c+1)\phi + i\frac{1}{\sqrt{k}}m_c(X_L - X_R)}(u_c) \right\rangle^{\text{sL}} ,
$$

where the left hand side and the right hand side are computed in the cigar model and sine-Liouville theory, respectively.

Up to now we have shown that the correlation functions for the cigar model and sine-Liouville theory among tachyon vertex operators are the same. For the equivalence of theory we have to establish the relation among descendants as well. As mentioned in subsection 2.2, since the descendants can be constructed by the action of symmetry current generators, we just need to show the both boundary theories preserve the same symmetry. In the cigar model, we are considering branes preserving one pair of parafermionic currents. As shown in [27], we can construct the generators of so-called $\hat{W}_\infty(k)$ algebra from the parafermionic currents. In fact, it was already shown in [28] that the boundary actions (2.54) in sine-Liouville theory preserve the $\hat{W}_\infty(k)$ symmetry.

3   Fermionic FZZ duality

In this section we study the fermionic version of the FZZ duality which relates the fermionic 2d black hole and $\mathcal{N} = 2$ super Liouville field theory. In [3] the authors show that they are related by a mirror symmetry [29]. Here we would like to give another proof by utilizing the method developed in [7]. The motivation to consider the fermionic FZZ duality is twofold. Firstly the $\mathcal{N} = 2$ SL(2)/U(1) coset model or dual $\mathcal{N} = 2$ super Liouville theory appears frequently in the context of superstring theory. For instance, string compactification on a singular Calabi-Yau 3-fold can be described with these models (see, e.g., [6]). Second one is to check that the boundary action (2.54) for D2-brane in sine-Liouville theory is the correct one. This is possible since the counterpart in the fermionic version is B-brane in $\mathcal{N} = 2$ super Liouville field theory and its boundary action has been obtained in [26]. D-branes in $\mathcal{N} = 2$ SL(2)/U(1) coset and $\mathcal{N} = 2$ super Liouville theory have been also studied in, e.g., [30–33].

3.1   Fermionic 2d black hole

The fermionic 2d black hole is the $\mathcal{N} = 2$ supersymmetric model based on the coset SL(2)/U(1) which is given by the Kazama-Suzuki construction [34]. As in the bosonic case, we start from the fermionic 2d black hole and show that it is equivalent to $\mathcal{N} = 2$ super Liouville field theory. In this and the next subsections, we restrict ourselves to the bulk case for the simplicity of expressions, and in the last subsection, the analysis is extended to the case with boundary.

The super coset is defined by the $\mathcal{N} = 1$ supersymmetric SL(2) WZNW model gauged by a $U(1)$ direction, where the supersymmetry is enhanced to $\mathcal{N} = 2$. In addition to the

bosonic SL(2) currents j^a ($a = 1, 2, 3$) with level $k_B = k + 2$, we have three fermions ψ^a with OPEs $\psi^a(z)\psi^b(0) \sim \delta^{ab}/z$. It is convenient to bosonize as

$$\psi^\pm = \frac{1}{\sqrt{2}}(\psi^1 \pm i\psi^2) , \qquad \psi^\pm = e^{\pm i\sqrt{2}H_L} , \tag{3.1}$$

where $H_L(z)H_L(0) \sim -1/2 \ln z$. The $U(1)$ direction we are gauging is generated by

$$J^3 = j^3 + \psi^+\psi^- = j^3 + i\sqrt{2}\partial H_L = i\sqrt{\frac{k}{2}}\partial X_L , \tag{3.2}$$

where the last equality defines one free boson X_L with $X_L(z)X_L(0) \sim -1/2 \ln z$. One of the fermions ψ^3 is also decoupled due to the gauging procedure. Above we have discussed the holomorphic part, but the anti-holomorphic part can be defined in the same way.

In total, the action is given by

$$S = S^H + \frac{1}{\pi}\int d^2w \partial H \bar\partial H + \frac{1}{\pi}\int d^2w \partial X \bar\partial X \ (+S_{b,c}) , \tag{3.3}$$

where we use the action S^H for the H_3^+ model as in (2.3). Here we combined holomorphic and anti-holomorphic parts as $H = H_L + H_R$ and $X = X_L + X_R$. This action should be obtained through the standard procedure of [8] as well. The ghost system with b, c enters through the gauge fixing but it will decouple from the other parts as before. The parameter is now set as $Q_\phi = b = 1/\sqrt{k_B - 2} = 1/\sqrt{k}$. Vertex operators invariant under the gauge transformation are given by (see e.g. [35])

$$\Psi^{j,s}_{m,\bar m}(z) = \Phi^j_{m,\bar m}(z)V^H_{s,\bar s}(z)V^X_{m+s,\bar m+\bar s}(z) . \tag{3.4}$$

Here $\Phi^j_{m,\bar m}$ is a primary operator of the H_3^+ model, which is defined in (2.11). The other vertex operators are

$$V^H_{s,\bar s}(z) = e^{i\sqrt{2}(sH_L - \bar s H_R)} , \qquad V^X_{m+s,\bar m+\bar s}(z) = e^{i\frac{2}{\sqrt{k}}((m+s)X_L - (\bar m+\bar s)X_R)} . \tag{3.5}$$

The correlation function is now written as

$$\left\langle \prod_{a=1}^N \Psi^{j_a,s_a}_{m_a,\bar m_a}(z_a) \right\rangle = \prod_{a=1}^N \left[N^{j_a}_{m_a,\bar m_a} \int \frac{d^2\mu_a}{|\mu_a|^2} \mu_a^{m_a}\bar\mu_a^{\bar m_a} \right] \times \tag{3.6}$$

$$\times \left\langle V^X_{-\frac{kS}{2},-\frac{kS}{2}}(\zeta)V^H_{S,S}(\zeta)v^S(\zeta)\prod_{a=1}^N V_{j_a}(\mu_a|z_a)V^H_{s_a,\bar s_a}(z_a)V^X_{m_a+s_a,\bar m_a+\bar s_a}(z_a) \right\rangle .$$

Here S represents the violation of total winding number as $\sum_a(m_a + s_a) = \sum_a(\bar m_a + \bar s_a) = kS/2$ and $v^S(\zeta)$ denotes the spectral flow operator as before. The operator $v^S(\zeta)$ again means that β has a zero of order S at ζ and the vertex operator $\exp(S\phi/b)$ is inserted.

3.2 Duality with $\mathcal{N} = 2$ super Liouville field theory

Since the fermionic cigar model and $\mathcal{N} = 2$ super Liouville theory both preserve $\mathcal{N} = 2$ super conformal symmetry, we just need to show the correlators of tachyon vertex operators

for the two theories agree. Moreover, in the correlation function (3.6), the fermion sector with H enters only through the direct products. It is thus natural to expect that we can apply the same method as in [7, 10] at least to the SL(2) sub-sector. With the rotation of fields involving H, we can show that the correlator in (3.6) is mapped to the one of the $\mathcal{N} = 2$ super Liouville theory.

We follow the strategy in [7, 10], which was reviewed in the previous section. We integrate out first $\gamma, \bar{\gamma}$ and then $\beta, \bar{\beta}$. Then β and $\bar{\beta}$ are replaced by $\mathcal{B}$ and $-\bar{\mathcal{B}}$ with

$$\mathcal{B}(w) = \sum_{a=1}^{N} \frac{\mu_\nu}{w - z_a} = u \frac{(w - \xi)^S \prod_{a'=1}^{N-2-S}(w - y_{a'})}{\prod_{a=1}^{N}(w - z_a)} . \tag{3.7}$$

In order to remove $\mathcal{B}$ from the action, we perform a shift of ϕ as

$$\varphi(w, \bar{w}) = \phi(w, \bar{w}) + \frac{1}{2b} \ln |\mathcal{B}(w)|^2 . \tag{3.8}$$

We furthermore perform a shift of H_L along with X_L as

$$\chi_L(w) = X_L(w) - i \frac{\sqrt{k}}{2} \ln \mathcal{B}(w) , \qquad h_L(w) = H_L(w) + i \frac{1}{\sqrt{2}} \ln \mathcal{B}(w) . \tag{3.9}$$

The anti-holomorphic parts χ_R and h_R are defined by the complex conjugates. With the above new fields, the correlation function becomes

$$\left\langle \prod_{a=1}^{N} \Psi_{m_a, \bar{m}_a}^{j_a, s_a}(z_a) \right\rangle = \prod_{i=1}^{N-2-S} \int \frac{d^2 y_i}{(N - 2 - S)!} \prod_{a=1}^{N} N_{m_a, \bar{m}_a}^{j_a} \times \tag{3.10}$$

$$\times \left\langle \prod_{a=1}^{N} V_{\alpha_a}(z_a) V_{s_a+1, \bar{s}_a+1}^{h}(z_a) V_{m_a+s_a-\frac{k}{2}, \bar{m}_a+\bar{s}_a-\frac{k}{2}}^{\chi}(z_a) \prod_{a'=1}^{N-2-S} V_{-\frac{1}{2b}}(y_{a'}) V_{-1,-1}^{h}(y_{a'}) V_{\frac{k}{2}, \frac{k}{2}}^{\chi}(y_{a'}) \right\rangle ,$$

where the right hand side is computed with the action

$$S = \frac{1}{\pi} \int d^2 w \left(\partial \varphi \bar{\partial} \varphi + \partial h \bar{\partial} h + \partial \chi \bar{\partial} \chi + \frac{\sqrt{g}}{4} \mathcal{R}(Q_\varphi \varphi + Q_{\tilde{h}} \tilde{h} + Q_{\tilde{\chi}} \tilde{\chi}) + \pi \lambda e^{2b\varphi} \right) . \tag{3.11}$$

Here $\tilde{\chi}$ and $\tilde{h}$ are the dual fields and background charges are $Q_\varphi = b + b^{-1}$, $Q_{\tilde{\chi}} = -i\sqrt{k}$ and $Q_{\tilde{h}} = i\sqrt{2}$. The vertex operator is $V_\alpha = \exp(2\alpha\varphi)$ with $\alpha = b(j + 1) + 1/2b$. In this way we rewrite the correlation function (3.6) in terms of bosonic Liouville field theory with φ and two additional free bosons with χ, h.

As in the bosonic case we first apply the self-duality of the Liouville field theory under $b \leftrightarrow b^{-1}$, and we then treat the vertex operators inserted at $y_{a'}$ as a perturbation operator. Now we have the interactions as

$$\mathcal{L}_1 = \tilde{\lambda} e^{2b^{-1}\varphi} , \qquad \mathcal{L}_2 = -e^{-b^{-1}\varphi + i\sqrt{k}\tilde{\chi} - i\sqrt{2}\tilde{h}} . \tag{3.12}$$

The dual parameter $\tilde{\lambda}$ is defined in (2.39). Next we look for suitable field redefinitions. First we take a linear combination of χ and h such that a new field has no background charge. An orthogonal basis is given by

$$\tilde{\chi}^+ = \frac{1}{\sqrt{k_B}} \left(\sqrt{2}\tilde{\chi} + \sqrt{k}\tilde{h} \right) , \qquad \tilde{\chi}^- = \frac{1}{\sqrt{k_B}} \left(\sqrt{k}\tilde{\chi} - \sqrt{2}\tilde{h} \right) , \tag{3.13}$$

whose background charges are $Q_{\tilde{\chi}^+} = 0$ and $Q_{\tilde{\chi}^-} = -i\sqrt{k_B}$. With these new fields, the interaction terms become

$$\mathcal{L}_1 = \tilde{\lambda}e^{2b^{-1}\varphi} , \qquad \mathcal{L}_2 = -e^{-b^{-1}\varphi + i\sqrt{k_B}\tilde{\chi}^-} . \tag{3.14}$$

Fortunately, these interactions are exactly the same as (2.43) in the bosonic case. Therefore, the rest is almost the same as before. We perform field redefinitions as

$$\phi = (k_B - 1)\varphi - i\sqrt{k_B}b^{-1}\tilde{\chi}^- , \qquad \tilde{X}^- = -i\sqrt{k_B}b^{-1}\varphi - (k_B - 1)\tilde{\chi}^- , \tag{3.15}$$

giving the background charges as $Q_\phi = b = 1/\sqrt{k}$ and $Q_{\tilde{X}} = 0$. Then we have

$$\mathcal{L}_1 = -\tilde{\lambda}\pi^{-1-2b^{-2}}\gamma(1 + b^{-2})e^{b^{-1}\phi + i\sqrt{k_B}\tilde{X}^-} , \qquad \mathcal{L}_2 = -e^{b^{-1}\phi - i\sqrt{k_B}\tilde{X}^-} , \tag{3.16}$$

where we have utilized the reflection relation of Liouville theory with $\hat{b} = i/\sqrt{2}$.

In order to obtain $\mathcal{N} = 2$ super Liouville field theory, we need to rotate the fields furthermore. We consider

$$\tilde{X} = \frac{1}{\sqrt{k_B}}\left(\sqrt{k}\tilde{X}^- + \sqrt{2}\tilde{\chi}^+\right) , \qquad \tilde{H} = \frac{1}{\sqrt{k_B}}\left(-\sqrt{2}\tilde{X}^- + \sqrt{k}\tilde{\chi}^+\right) , \tag{3.17}$$

such that the field content is the same as that for $\mathcal{N} = 2$ super Liouville field theory. Namely, we have ϕ with background charge $Q_\phi = b = 1/\sqrt{k}$, a free boson X and a bosonized fermion H. See also appendix C of [36]. The interaction terms are changed to

$$\mathcal{L}_1 = -\tilde{\lambda}\pi^{-1-2b^{-2}}\gamma(1 + b^{-2})e^{\sqrt{k}(\phi + i\tilde{X}) - i\sqrt{2}\tilde{H}} , \qquad \mathcal{L}_2 = -e^{\sqrt{k}(\phi - i\tilde{X}) + i\sqrt{2}\tilde{H}} , \tag{3.18}$$

which are those for $\mathcal{N} = 2$ super Liouville field theory. The coefficients of the interaction terms can be changed by the shift of zero mode of $\tilde{X}$ as

$$\mathcal{L}_1 = \tilde{c}e^{\sqrt{k}(\phi + i\tilde{X}) - i\sqrt{2}\tilde{H}} , \qquad \mathcal{L}_2 = \tilde{c}e^{\sqrt{k}(\phi - i\tilde{X}) + i\sqrt{2}\tilde{H}} , \tag{3.19}$$

where $\tilde{c}$ is as in (2.51).

Moving to the vertex operators, we rewrite them in a suitable form as

$$V_\alpha(z)V^H_{s+1,\bar{s}+1}(z)V^X_{m+s-\frac{k}{2},\bar{m}+\bar{s}-\frac{k}{2}}(z) \tag{3.20}$$

$$= e^{2b(j+1+\frac{1}{2b^2})\phi + i\frac{2}{\sqrt{k_B}}\left((m-\frac{k_B}{2})\chi^-_L - (\bar{m}-\frac{k_B}{2})\chi^-_R\right)} \cdot e^{2i\sqrt{\frac{2}{kk_B}}\left((\frac{k_B}{2}s+m)\chi^+_L - (\frac{k_B}{2}\bar{s}+\bar{m})\chi^+_R\right)}$$

in terms of the new fields (3.13). Then we observe that the first factor on the right hand side is of the same form as in the bosonic case. Therefore, we can perform the reflection relation in the same way. The result is

$$N^j_{m,\bar{m}}V_\alpha(z)V^H_{s+1,\bar{s}+1}(z)V^X_{m+s-\frac{k}{2},\bar{m}+\bar{s}-\frac{k}{2}}(z) \tag{3.21}$$

$$= -\pi^{-1-2j-m-\bar{m}}e^{\frac{2}{\sqrt{k}}((j+1)\phi + i(m+s)X_L - i(\bar{m}+\bar{s})X_R) + i\sqrt{2}(sH_L - \bar{s}H_R)} ,$$

which are the vertex operators of $\mathcal{N} = 2$ super Liouville field theory. In the last equation we have used the field redefinition (3.17). Thus we have established the relation between correlation functions as

$$\left\langle \prod_{a=1}^{N} \Psi_{m_a,\bar{m}_a}^{j_a,s_a}(z_a) \right\rangle^{\text{fcig}} = \mathcal{N} \left\langle \prod_{a=1}^{N} e^{\frac{2}{\sqrt{k}}((j_a+1)\phi+i(m_a+s_a)X_L-i(\bar{m}_a+\bar{s}_a)X_R)+i\sqrt{2}(s_aH_L-\bar{s}_aH_R)}(z_a) \right\rangle^{\text{N=2L}} ,$$

where the left hand side and the right hand side are respectively computed in the fermionic cigar model and in the $\mathcal{N} = 2$ super Liouville field theory.

3.3 Fermionic FZZ duality with boundary

The above analysis shows that the fermionic FZZ duality naturally comes from the bosonic FZZ duality through the proper change of fields. In particular, it is easy to extend it to the case with closed Riemann surfaces of arbitrary genus. In this subsection the fermionic FZZ duality is generalized to disk amplitudes with proper boundary conditions. We consider D1-branes in the fermionic 2d black hole, which should be mirror to B-branes in the $\mathcal{N} = 2$ super Liouville field theory studied in [26]. Since the analysis is almost the same as before, we explain it only briefly.

The boundary action for D1-branes is the same as the one for the bosonic case in (2.8)

$$S_{\text{bint}} = i\lambda_B\sigma_3 \int du \beta e^{b\phi} , \qquad \lambda_B = \sqrt{\frac{\lambda}{\sin \pi b^2}} \sinh r , \qquad (3.22)$$

but with the bulk action in (3.11). We assign Dirichlet boundary conditions as $\beta + \bar{\beta} = 0$, $\gamma + \bar{\gamma} = 0$, and Neumann boundary conditions for ϕ. We also assign $X_L + X_R = \sqrt{k}\theta_0$ and $\psi^{\pm} = e^{\pm 2i\alpha}\bar{\psi}^{\mp}$ for the fermions. Boundary operators are defined as

$$\Psi_{m,i}^{l,s}(z) = \frac{\sigma_i}{\sqrt{2}} \Phi_{m,\eta}^l V_s^H V_{m+s}^X(z) , \qquad (3.23)$$

where the operator for H_3^+ is given in (2.14) and the other operators are

$$V_s^H(z) = e^{i\sqrt{2}sH_L} , \qquad V_{m+s}^X(z) = e^{i\frac{2}{\sqrt{k}}(m+s)X_L} . \qquad (3.24)$$

Since the boundary operator changes the boundary conditions generically, we may label as $[\Psi_{m,i}^{l,s}]_{L,L'}$ with $L = (r, M, \alpha)$. The labels take values $s = \alpha - \alpha' + S$ and $m + s = M - M' + \alpha - \alpha' + \frac{k}{2}w$ with $S, w \in \mathbb{Z}$. We should use $w \in 2\mathbb{Z}$ for $i = 0, 3$ and $w \in 2\mathbb{Z} + 1$ for $i = 1, 2$ as before. Similarly, $\eta = 0$ for $i = 0, 1$ and $\eta = 1$ for $i = 2, 3$.

Once we have the boundary operators, the rest is straightforward. However, to compare with known results, let us first discuss the Chan-Paton factors. It is possible to rewrite the non-Abelian action by an Abelian action with the introduction of boundary fermions, see [37–40]. One good review is in section 3 of [39]. Basically, when we compute correlation functions with an action including matrix coefficients, we have to perform a trace with keeping path ordering. The same effect can be made with the introduction of boundary fermions $\Theta, \bar{\Theta}$. The kinetic term may be given by

$$S = \int du \Theta \partial_u \bar{\Theta} , \qquad (3.25)$$

which leads to

$$\langle \Theta(u_1)\bar{\Theta}(u_2) \rangle = \langle \bar{\Theta}(u_1)\Theta(u_2) \rangle = \frac{1}{2}\mathrm{sgn}(u_1 - u_2) \ . \tag{3.26}$$

Then the algebra of $\sigma_+, \sigma_-, \sigma_3$ can be reproduced by $\sqrt{2}\Theta, \sqrt{2}\bar{\Theta}, 2(\Theta\bar{\Theta} - \bar{\Theta}\Theta)$. However, the off-diagonal terms become Grassmann odd in this formulation.

With the above argument in mind, we proceed further. Following the previous analysis we find the boundary interaction terms as

$$\mathcal{L}_{B,1} = i\sqrt{\frac{\tilde{c}}{2}}\left(e^{\frac{r}{2b^2}}\sigma_+ + e^{-\frac{r}{2b^2}}\sigma_-\right)e^{\frac{1}{2b}\phi + i\frac{\sqrt{k_B}}{2}\tilde{X}^-} \ , \tag{3.27}$$

$$\mathcal{L}_{B,2} = i\sqrt{\frac{\tilde{c}}{2}}\left(e^{-\frac{r}{2b^2}}\sigma_+ + e^{\frac{r}{2b^2}}\sigma_-\right)e^{\frac{1}{2b}\phi - i\frac{\sqrt{k_B}}{2}\tilde{X}^-}$$

in terms of the fields (3.15). Applying the rotation in (3.17) and rewriting the Chan-Paton factors, we now have[6]

$$\mathcal{L}_{B,1} = i\sqrt{\tilde{c}}\left(e^{\frac{r}{2b^2}}\Theta + e^{-\frac{r}{2b^2}}\bar{\Theta}\right)\psi^- e^{\frac{\sqrt{k}}{2}(\phi + i\tilde{X})} \ , \tag{3.28}$$

$$\mathcal{L}_{B,2} = i\sqrt{\tilde{c}}\left(e^{-\frac{r}{2b^2}}\Theta + e^{\frac{r}{2b^2}}\bar{\Theta}\right)\psi^+ e^{\frac{\sqrt{k}}{2}(\phi - i\tilde{X})} \ .$$

One advantage to introduce the boundary fermions is on the co-cycle factor. Since we have fermionized as $\psi^\pm = e^{\pm i\sqrt{2}H_L}$, we should take care of its Grassmann parity. Fortunately, we replaced the Pauli matrices $\sigma^\pm$ by boundary fermions at the same time, thus the boundary interaction terms remain bosonic. Notice the above interaction terms are the same as (5.8) with (5.27) of [26]. In particular, matrix factorization suggests that the bulk cosmological constant is the square of boundary cosmological constant.

We should work out the vertex operators as well. With the same method as before, we find that the vertex operators are mapped to

$$B_{m,i}^{l,s} = \frac{\sigma_i}{\sqrt{2}}e^{\frac{1}{\sqrt{k}}((l+1)\phi + i(m+s)(X_L - X_R)) + i\sqrt{2}sH_L} \ . \tag{3.29}$$

We should again replace the Chan-Paton factor by boundary fermions. Then, these boundary operators coincide with those in [26] (see eq.(5.18) of the paper and arguments given below). Therefore, we have reproduced the results for B-branes in $\mathcal{N} = 2$ super Liouville field theory from D1-branes in the fermionic 2d black hole.

4 Outlook

In this article, we derive the FZZ duality [1] for disc correlators, which is a strong/weak duality between the cigar model and sine-Liouville field theory. For the purpose, we have used perturbative path integral methods of [7] in combination with the strong/weak self-duality of Liouville field theory. This was extended to the duality between the supersymmetric

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[6]This form of boundary interactions suggests that we are dealing with a $D\bar{D}$ system. Moreover, open string tachyons between brane and anti-brane are condensed in this system. See [39, 40] in more details.

cigar and $\mathcal{N} = 2$ super-Liouville theory as in [3] both for correlators on the sphere and the disc. There are several possible applications, generalizations and future directions.

For D1- or A-branes in $\mathcal{N} = 2$ super Liouville theory, a Lagrangian description is known [26]. We would expect these branes to be dual to branes in the cigar which wind the circular direction. Thus for D2-branes in the (super)cigar model, it might be possible to find a boundary Lagrangian and derive the duality with the branes in sine-Liouville (super Liouville) theory. D0-branes in sine-Liouville theory are not covered by our derivation, so one should study these branes directly in the sine-Liouville theory. They are of ZZ-type [41] and thus can be studied by looking at Cardy conditions.

One should be able to apply our methods to other models. One example is the coset $OSP(1|2)/\mathbb{R}$ studied in [42]. The $OSP(1|2)$ WZNW model is in correspondence with $\mathcal{N} = 1$ super Liouville field theory, and the derivation is analogous to the one between H_3^+ model and Liouville field theory [12, 43]. It is likely that also the $OSP(1|2)/\mathbb{R}$ coset possesses a strong/weak dual which is yet to be determined. As our derivation is quite constructive, it might be possible to use it to find and derive the duality in one step. The H_3^+-Liouville correspondence extends to e.g. correspondences between $OSP(N|2)$ and $SU(M|2)$ supergroup WZNW models and superconformal field theories respectively with $SO(N)$- and $U(M)$-extended supersymmetry [44]. These extended supersymmetry algebras were introduced in [45, 46]. The derivation of these correspondence is again via the path integral and these superconformal field theories seem to be strong/weak self-dual.[7] Again one can investigate cosets thereof [47] and look for possible weakly coupled dual theories. This should be particularly interesting for cosets of the $PSU(2|2)$ and $OSP(4|2)$ WZNW models as these supergroups are important in the AdS/CFT correspondence.

Acknowledgments

A   Branes

In this appendix we recall geometry and action of branes in the coset. Geometry of branes in cosets has been considered in [48, 49] and the boundary action of AdS$_2$ branes was given in [19].

A.1   Geometry of Branes

An element in H_3^+ satisfies $g^\dagger = g$. AdS_2 branes are described by the restriction of twisted conjugacy classes of $SL(2,\mathbb{C})$ to H_3^+, i.e.

$$C_a^\omega = \{hah^t \mid h \in SL(2,\mathbb{R}) \}. \tag{A.1}$$

---

[7]Except for $\mathcal{N} = 2$ super Liouville theory, which is the case in this article.

Here, $a = a^\dagger$ is in H_3^+ and hence the same is true for every element of C_a^ω. The coset $H_3^+/\mathbb{R}$ is the twisted adjoint one, consisting of elements g in H_3^+ modulo the twisted adjoint action

$$g \sim ugu \tag{A.2}$$

where $u = u^t = u^\dagger$ in the t^3 direction. Branes in cosets are now described by elements of the form gv^{-1} where g is in the twisted conjugacy class describing the H_3^+ brane and v in the conjugacy class for the subgroup we mod out, i.e. $\mathbb{R}$ [49]. In addition, this translate of a twisted conjugacy class has to be invariant under the twisted adjoint action defining the coset (A.2). For us this is true for either Dirichlet (v constant) or Neumann boundary conditions in the $\mathbb{R}$ directions, i.e.

$$uhah^t v^{-1}u = uhah^t uv^{-1} = uhah^t u^t v^{-1} = uha(uh)^t v^{-1}. \tag{A.3}$$

We choose to consider Dirichlet boundary conditions, i.e. $v = e^{i\theta_0} = $ constant. The embedding of the gauged WZNW model in the product theory of $H_3^+ \times U(1)$ is realized by parameterizing the gauge fields as $A = i\partial X$, $\bar{A} = -i\bar{\partial}X$ and changing variables $g' = e^{iXt^3}ge^{-iXt^3}$.

A.2 Boundary actions and Chan-Paton factors

In this subsection, we argue for consistent boundary actions including boundary fermions, which can be seen as Chan-Paton factors. The bulk action is

$$S_{\text{bulk}} = \frac{1}{2\pi}\int d^2z\left(\partial\phi\bar{\partial}\phi - \gamma\bar{\partial}\beta - \bar{\gamma}\partial\bar{\beta} - \mu b^2\beta\bar{\beta}e^{2b\phi}\right). \tag{A.4}$$

As a boundary term, we choose

$$S_{\text{bdy}} = \frac{i\mu_B}{2\pi}\int du\left(\lambda\partial_u\bar{\lambda} + f(\lambda,\bar{\lambda})\beta e^{b\phi}\right), \tag{A.5}$$

and we vary under the Dirichlet constraint $\beta = -e^{i\theta_0}\bar{\beta}$. We want to determine functions $f(\lambda,\bar{\lambda})$ that are consistent with preserving current symmetry at the boundary. Then the variation of the action has the following boundary contribution

$$\delta S|_{\text{bdy}} = \frac{i}{2\pi}\int du\left(\delta\phi((\bar{\partial}-\partial)\phi + f(\lambda,\bar{\lambda})bc\beta e^{b\phi}) + \delta\beta(\gamma + e^{-i\theta_0}\bar{\gamma}+\right.$$
$$+ f(\lambda,\bar{\lambda})\mu_B e^{b\phi}) + \delta\lambda(\partial_u\bar{\lambda} + \frac{d}{d\lambda}f(\lambda,\bar{\lambda})\mu_B\beta e^{b\phi})+ \tag{A.6}$$
$$\left.+ \delta\bar{\lambda}(\partial_u\lambda + \frac{d}{d\bar{\lambda}}f(\lambda,\bar{\lambda})\mu_B\beta e^{b\phi})\right).$$

Moreover, the bulk equations of motion imply

$$\mu b^2 e^{2b\phi}\bar{\beta} = \bar{\partial}\gamma \quad \text{and} \quad \mu b^2 e^{2b\phi}\beta = \partial\bar{\gamma}. \tag{A.7}$$

The currents take the following form

$$\begin{aligned} J^- &= \beta \quad,\quad J^3 = \beta\gamma + b^{-1}\partial\phi \quad,\quad J^+ = \beta\gamma^2 + 2b^{-1}\gamma\partial\phi - (k-2)\partial\gamma, \\ \bar{J}^- &= \bar{\beta} \quad,\quad \bar{J}^3 = \bar{\beta}\bar{\gamma} + b^{-1}\bar{\partial}\phi \quad,\quad \bar{J}^+ = \bar{\beta}\bar{\gamma}^2 + 2b^{-1}\bar{\gamma}\bar{\partial}\phi - (k-2)\bar{\partial}\bar{\gamma}. \end{aligned} \tag{A.8}$$

The gluing conditions for AdS_2 branes are

$$J^- = - e^{i\theta_0}\bar{J}^- \quad , \quad J^3 = \bar{J}^3 \quad \text{and} \quad J^+ = - e^{-i\theta_0}\bar{J}^+ . \tag{A.9}$$

The vanishing of the variation of above boundary part of the action implies that classically $J^- = -e^{i\theta_0}\bar{J}^-$ and $J^3 = \bar{J}^3$. It also implies that $J^+ = -e^{-i\theta_0}\bar{J}^+$ as follows. Using $J^- = -e^{i\theta_0}\bar{J}^-$ and $J^3 = \bar{J}^3$, we see that the conditions

$$
\begin{aligned}
J^+ &= - e^{-i\theta_0}\bar{J}^+ \quad \Leftrightarrow \\
0 &= -b^{-1}(e^{-i\theta_0}\bar{\gamma} + \gamma)(\partial + \bar{\partial})\phi + (k-2)e^{-i\theta_0}\bar{\partial}\bar{\gamma} + (k-2)\partial\gamma
\end{aligned}
\tag{A.10}
$$

are equivalent. Using the bulk equations of motion (A.7), $\beta = -e^{i\theta_0}\bar{\beta}$ and $\delta S = 0$ (A.6), we rewrite

$$
\begin{aligned}
\partial\gamma + e^{-i\theta_0}\bar{\partial}\bar{\gamma} &= (\partial + \bar{\partial})(\gamma + e^{-i\theta_0}\bar{\gamma}) + e^{-i\theta_0}\partial\bar{\gamma} - \bar{\partial}\gamma \\
&= (\partial + \bar{\partial})(\gamma + e^{-i\theta_0}\bar{\gamma}) \quad \text{because of (A.7) and } \beta = -e^{i\theta_0}\bar{\beta} \quad \text{(A.11)} \\
&= -(\partial + \bar{\partial})\big(f(\lambda, \bar{\lambda})\mu_B e^{b\phi}\big) \quad \text{because of (A.6)} .
\end{aligned}
$$

Inserting this in (A.10) and using $b = 1/\sqrt{k-2}$, we see that classically

$$J^+ = -e^{-i\theta_0}\bar{J}^+ \quad \Leftrightarrow \quad 0 = (\partial + \bar{\partial})f(\lambda, \bar{\lambda}) . \tag{A.12}$$

Recall the equations of motion of λ and $\bar{\lambda}$ from (A.6)

$$0 = \partial_u\bar{\lambda} + \frac{d}{d\lambda}f(\lambda, \bar{\lambda})\mu_B\beta e^{b\phi} = \partial_u\lambda + \frac{d}{d\bar{\lambda}}f(\lambda, \bar{\lambda})\mu_B\beta e^{b\phi} . \tag{A.13}$$

Hence, $(\partial + \bar{\partial})f(\lambda, \bar{\lambda})$ vanishes if it is one of the following

$$
\begin{aligned}
f(\lambda, \bar{\lambda}) &= c , \\
f(\lambda, \bar{\lambda}) &= c\lambda , \\
f(\lambda, \bar{\lambda}) &= c\bar{\lambda} , \\
f(\lambda, \bar{\lambda}) &= c\lambda\bar{\lambda}
\end{aligned}
\tag{A.14}
$$

for some constant c. This can be related to the u(2) algebra and means that the Chan-Paton factors σ_0, σ_3 and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ are consistent with preserving current symmetry.

B The gauged sigma model

B.1 The bulk theory

To warm up we consider the coset $H_3^+/\mathbb{R}$ on the sphere.

Coset as gauged sigma model

As in [7] we write elements in H_3^+ as

$$h = e^{\gamma \sigma^+} e^{\phi \sigma^3} e^{\bar{\gamma} \sigma^-} \ . \tag{B.1}$$

The symmetries are $g \mapsto BhB^\dagger$ generated by $aJ - a^* \bar{J}$ where we use[8]

$$J = -k \partial h h^{-1}, \qquad \bar{J} = k h^{-1} \bar{\partial} h. \tag{B.2}$$

We gauge the direction corresponding to $J_3 - \bar{J}_3 = \text{tr}(J - \bar{J}) \sigma^3$, the gauged model is

$$S = \frac{k}{2\pi} \int d^2 z \left((\bar{\partial}\phi + \bar{A})(\partial\phi + A) + e^{-2\phi}(\bar{\partial} + \bar{A})\gamma(\partial + A)\bar{\gamma} \right) \ . \tag{B.3}$$

This is invariant under the gauge symmetry

$$h \mapsto e^{\lambda(z,\bar{z})\sigma^3/2} h e^{\lambda(z,\bar{z})\sigma^3/2}, \qquad A \mapsto A - d\lambda(z,\bar{z}), \tag{B.4}$$

where we have used the symbol A for the whole one-form as well as its holomorphic component. On the fields this acts as

$$\phi \mapsto \phi + \lambda(z,\bar{z}), \qquad \gamma \mapsto e^{\lambda(z,\bar{z})}\gamma, \qquad \bar{\gamma} \mapsto e^{\lambda(z,\bar{z})}\bar{\gamma} \ . \tag{B.5}$$

To get the cigar model we note that the equations of motion for $A, \bar{A}$ are

$$A = -\frac{\partial\phi + e^{-2\phi}\gamma\partial\bar{\gamma}}{1 + e^{-2\phi}\gamma\bar{\gamma}} = -\partial\phi - \frac{v\partial\bar{v}}{1 + v\bar{v}} \ , \tag{B.6}$$

$$\bar{A} = -\frac{\bar{\partial}\phi + e^{-2\phi}\bar{\gamma}\bar{\partial}\gamma}{1 + e^{-2\phi}\gamma\bar{\gamma}} = -\bar{\partial}\phi - \frac{\bar{v}\bar{\partial}v}{1 + v\bar{v}} \ , \tag{B.7}$$

where we have introduced the gauge invariant coordinates

$$v = e^{-\phi}\gamma \ , \qquad \bar{v} = e^{-\phi}\bar{\gamma} \ . \tag{B.8}$$

The action then takes the cigar form

$$S = \frac{k}{2\pi} \int d^2 z \left(\frac{1}{1 + v\bar{v}} \bar{\partial}v\partial\bar{v} \right) \ . \tag{B.9}$$

On the other hand we can fix the gauge field using a complex valued gauge $U = \exp(\alpha + ix)$, note that x is here 2π periodic,

$$A = U^{-1}\partial U = \partial\alpha + i\partial x, \qquad \bar{A} = (A)^* = U^{*-1}\bar{\partial}U^* = \bar{\partial}\alpha - i\bar{\partial}x. \tag{B.10}$$

The real part, α, is pure gauge and the integration over x gives the volume of the gauge group, but the imaginary part cannot be gauged away, but can be decoupled by making a transformation conjugate to the gauge symmetry i.e. along $iJ_3 + i\bar{J}_3$:

$$h \mapsto e^{ix(z,\bar{z})\sigma^3/2} h e^{-ix(z,\bar{z})\sigma^3/2} \tag{B.11}$$

---

[8]Note that the components for the anti-chiral currents in last appendix are related to this definition with an extra minus transposed automorphism.

which simply takes

$$\gamma \mapsto e^{ix(z,\bar{z})}\gamma, \qquad \bar{\gamma} \mapsto e^{-ix(z,\bar{z})}\bar{\gamma} \ . \tag{B.12}$$

This gives the action of the product theory $H_3^+ \times U(1)$

$$S = \frac{k}{2\pi} \int d^2z \left(\bar{\partial}\phi\partial\phi + \bar{\partial}x\partial x + e^{-2\phi}\bar{\partial}\gamma\partial\bar{\gamma} \right) \ . \tag{B.13}$$

Strings without winding

Let us now consider what happens to the vertex operators. The primary fields of H_3^+ in the m-basis may be written (after a rescaling of ϕ)

$$\Phi_{m\bar{m}}^j \propto \gamma^{m-(j+1)}\bar{\gamma}^{\bar{m}-(j+1)}e^{2b(j+1)\phi}, \tag{B.14}$$

where

$$m = \frac{n+ip}{2}, \qquad \bar{m} = \frac{-n+ip}{2}, \qquad n \in \mathbb{Z}, p \in \mathbb{R} \ . \tag{B.15}$$

Note that these are operators with no winding around the boundary of H_3^+. We must require gauge invariance of our operators. The basis is such that $J_3 = -2m, \bar{J}_3 = 2\bar{m}$. Thus we require $m + \bar{m} = ip = 0$. Under the decoupling of the imaginary gauge field we then get

$$\Phi_{m\bar{m}}^j \mapsto \Phi_{m\bar{m}}^j e^{(m-\bar{m})ix} = \Phi_{m\bar{m}}^j e^{nix}, \tag{B.16}$$

i.e. also no winding in the U(1) direction. If we introduce currents for the U(1) part

$$J_0 = -ik\partial x, \qquad \bar{J}_0 = ik\bar{\partial}x, \tag{B.17}$$

we see that the total currents

$$J_3^{\text{Total}} = J_3 - 2J_0, \qquad \bar{J}_3^{\text{Total}} = (J_3^{\text{Total}})^* = \bar{J}_3 + 2\bar{J}_0 \tag{B.18}$$

are zero when acting on the primary fields.

Winding strings

We now consider states that wind asymptotically in H_3^+ i.e. spectrally flowed states. To this end we define the scalar field H by $J^3 = 2\partial H$. If we use the first order formalism for H_3^+ and bosonize the β, γ system such that $\gamma = \exp(Y_L - Z_L)$, $\beta = -\exp(-Y_L + Z_L)\partial Z$ we have

$$H = -\phi/b - \beta\gamma = -\phi/b - Y. \tag{B.19}$$

This field has OPE $HH \sim -k/2\ln|z-w|^2$ and background charge $1/2$. The w times spectrally flowed state (i.e. w times winding) is then

$$\Phi_{m\bar{m}}^{jw} = : e^{wH}\Phi_{m\bar{m}}^j : \tag{B.20}$$

Under gauge transformations (B.4) we have $Y_L \mapsto Y_L + \lambda(z,\bar{z})$, $Y_R \mapsto Y_R + \lambda(z,\bar{z})$ and hence

$$H \mapsto H - k\lambda(z,\bar{z}) \ . \tag{B.21}$$

Thus demanding gauge invariance of these operators means

$$m = \frac{n + kw}{2}, \qquad \bar{m} = \frac{-n + kw}{2}, \qquad n, w \in \mathbb{Z}, \tag{B.22}$$

i.e. $ip \mapsto w$. H is invariant under the transformation decoupling the U(1) part so

$$\Phi_{m\bar{m}}^{jw} \mapsto \Phi_{m\bar{m}}^{jw} e^{inx} \equiv \Phi_{m\bar{m}}^{jw} V_n. \tag{B.23}$$

So now we have winding in the H_3^+ part, but none in the U(1) part. Again the total currents (B.18) are zero. We can use the spectral flow covariance of the correlators to move all spectral flow to the identity operator at infinity [50]. There will however be a prefactor on the correlator depending on the positions of the vertex operators. To avoid this we do a similar, but chiral spectral flow in the U(1) theory such that

$$\Phi_{m\bar{m}}^{jw} V_n \mapsto \Phi_{m\bar{m}}^{j} V_{nw} \propto \gamma^{m-(j+1)} \bar{\gamma}^{\bar{m}-(j+1)} e^{2b(j+1)\phi} e^{inx+iwk\tilde{x}}, \tag{B.24}$$

with $\tilde{x} = x_L - x_R$. Doing the spectral flow in both H_3^+ and U(1) also means that the total currents (B.18) are kept zero. Further, the dimension of the operators on the left and right hand side can be shown to be the same, and this is the reason that we get an equality of the correlators. For the identity operator at infinity in the case of violated winding number we have

$$1 \mapsto e^{-\sum_i w_i H} e^{-i \sum_i w_i k \tilde{x}}. \tag{B.25}$$

Identifying $X = x/\sqrt{k}$, we have now reached the operators used in the main text which have no winding in H_3^+ direction, but in U(1).

Winding by Wilson lines

As shown in [8] (see eq. (50) in section 4) there is an alternative way to make H_3^+ vertex operators with $m + \bar{m} \neq 0$ gauge invariant – simply add Wilson lines:

$$\prod_i \Phi_{m_i \bar{m}_i}^{j_i}(\xi_i) e^{\int_C A}, \tag{B.26}$$

where C is a chain with $\delta C = \sum_i (m_i + \bar{m}_i)\xi_i$. This is nicely gauge invariant. The demand that $m + \bar{m}$ is real in this case comes from locality.

When one decouples the gauge field, we exactly get the states (B.24) which wind in the U(1) direction. To show that (B.26) and (B.23) gives the same state when going to the cigar model, one could try to construct the first order formalism for the gauged sigma model, but we will refrain from doing that here.

B.2 The boundary theory

We now consider the gauged model where the world-sheet is a disk. We are going to argue that the boundary action

$$S_{\text{bint}} = i\lambda_B e^{i\theta_0} \sigma_3 \int du \beta e^{b\phi} \tag{B.27}$$

arises from a single brane in the cigar. The boundary conditions used for the decoupled H_3^+ model are (c and λ_B are proportional)

$$e^{i\theta_0}\partial\bar\gamma + e^{-i\theta_0}\bar\partial\gamma = 0, \qquad (e^{i\theta_0}\beta + e^{-i\theta_0}\bar\beta = 0),$$
$$e^{-i\theta_0}\gamma + e^{i\theta_0}\bar\gamma = \pm ce^\phi, \qquad \bar\partial\phi - \partial\phi = \pm cbe^{i\theta_0}\beta e^{b\phi}, \qquad z = \bar z . \tag{B.28}$$

The boundary action, seen as a perturbative interaction term, will geometrically deform the brane with $c = 0$ to the brane with non-zero c.

Let us start with a single brane in the cigar given by the boundary conditions

$$e^{-i\theta_c}v + e^{i\theta_c}\bar v = 2Re(e^{-i\theta_c}v) = c, \tag{B.29}$$

where the coordinates $v, \bar v$ were defined in (B.8). Of course, there is rotational invariance, so we could simply set $\theta_c = 0$. Note that for $v = \sinh\rho\, e^{i(\theta-\pi/2)}$ we get

$$\sinh\rho\sin(\theta - \theta_c) = c/2 \propto \sinh r , \tag{B.30}$$

as in (2.5).

The boundary conditions in the gauged WZNW model would have to be a gauge invariant version of the ones in H_3^+ model. To descend to (B.29) they will be

$$e^{i\theta_c}(\partial+A)\bar\gamma + e^{-i\theta_c}(\bar\partial+\bar A)\gamma = 0, \qquad e^{-i\theta_c}\gamma + e^{i\theta_c}\bar\gamma = ce^\phi, \qquad \partial\phi + A - \bar\partial\phi - \bar A = cbe^{i\theta_c}\beta e^{b\phi},$$
$$z = \bar z. \tag{B.31}$$

Further, one needs to assign boundary conditions to A. There are two obvious choices $A = \pm\bar A$. The boundary conditions above are gauge invariant by construction, so by gauge-fixing $A = d\alpha + i * dx$ we can remove α. However, we cannot separate x out of the condition unless we choose Dirichlet boundary condition on x i.e.

$$\partial x + \bar\partial x = 0, \qquad A - \bar A = 0 , \tag{B.32}$$

which will be our choice of boundary conditions.

Going to the product theory by decoupling x we get

$$e^{-i(x_0-\theta_c)}\partial\bar\gamma + e^{i(x_0-\theta_c)}\bar\partial\gamma = 0 , \qquad (e^{-i(x_0-\theta_c)}\beta + e^{i(x_0-\theta_c)}\bar\beta = 0) ,$$
$$e^{i(x_0-\theta_c)}\gamma + e^{-i(x_0-\theta_c)}\bar\gamma = ce^\phi, \qquad \partial\phi - \bar\partial\phi = cbe^{-i(x_0-\theta_c)}\beta e^{b\phi}, \qquad z = \bar z . \tag{B.33}$$

This is just the boundary condition for the rotated brane. Here x_0 is the boundary value of x

$$x_L + x_R = x_0, \qquad z = \bar z . \tag{B.34}$$

Before continuing let us consider the boundary states. First, we should remove all momentum in the circular direction, i.e. set $n = 0$ since our strings are attached to the D-branes. This also easily follows using the boundary conditions on the bulk states. With winding we expect two types of strings in the cigar. Strings with integer winding and strings with half-integer winding of course ending on the same brane since we only have one brane in the cigar, see figure 1 (see also figures in [22]).

A boundary operator analogous to the bulk state (B.24) is

$$\Phi_w^j V_w \propto \gamma^{kw/4-(j+1)/2}\bar{\gamma}^{kw/4-(j+1)/2}e^{b(j+1)\phi}e^{ikw/2\,\tilde{x}}. \tag{B.35}$$

Now the strings in the cigar with integer/half-integer winding will correspond to strings in the U(1) direction with integer/half-integer winding. This means that we have two branes in the product model. One located in the U(1) space at $x_0 = \theta_c - \theta_0$, in the H_3^+ directions this will be a brane with labels (θ_0, r) as we see from (B.33). Secondly there must be a brane at $x_0 = \theta_c - \theta_0 + \pi$ and in the H_3^+ directions it will have label $(\theta_0, -r)$ or equivalently $(\theta_0 + \pi, r)$. We can label the two branes $(\theta_c - \theta_0, (\theta_0, r))$ and $(\theta_c - \theta_0 + \pi, (\theta_0, -r))$.

If we start with a string with zero winding in the cigar it will map to a string with zero winding in the U(1) direction and which starts and ends on the same brane say $(\theta_c - \theta_0, (\theta_0, r))$. If we do a half-integer spectral flow we get a string in the cigar going between two branches of the same brane, but in the product theory it will now be half winding in the U(1) direction and going between the two branes in the product theory. This is sketched in figure 2, see also figure 6 in [22].

We can also relate the above discussion to the boundary action. We know what the boundary action looks like in the H_3^+ model for a single brane. The action in a first order formalism for the gauged theory must be exactly the same i.e.

$$S_{\text{bint}} = ie^{i\theta_c}\lambda_B \int du\beta e^{b\phi} \ , \tag{B.36}$$

since this is nicely gauge invariant. However when we decouple x in the rest of the action it will still appear in the boundary term:

$$S_{\text{bint}} = i\lambda_B e^{-i(x_0-\theta_c)} \int du\beta e^{b\phi}. \tag{B.37}$$

This corresponds to eq. (B.33).

Chan-Paton factors

In order to calculate in the path integral we introduce Chan-Paton factors corresponding to the two branes. The new feature is that the boundary action will depend on which brane we are on, i.e. it will depend on the Chan-Paton factor. From (B.37) we see that the form is exactly what we wanted, i.e.

$$S_{\text{bint}} = i\lambda_B e^{i\theta_0}\sigma_3 \int du\beta e^{b\phi}. \tag{B.38}$$

Here have taken the brane $(\theta_c - \theta_0, (\theta_0, r))$ as the first brane and $(\theta_c - \theta_0 + \pi, (\theta_0, -r))$ as the second. Boundary operators with half-integer winding in the U(1) direction should have a Chan-Paton factor that anti-commutes with this i.e. be σ_1 or σ_2. To discriminate between these two we note that the whole theory including branes is invariant under parity transformation in the $\beta\gamma$-system together with a π rotation in the U(1) direction. This will map our two branes onto each other. The map of the fields to the cigar will be exactly the same under this conjugation. The conjugation is used in the main text to fix the Chan-Paton factors.

C Change of variables

When we relate the correlation functions of the cigar model to those of Liouville theory plus a free boson, we utilize a change of variables

$$\sum_{a=1}^{N} \frac{\mu_a}{w - z_a} + \sum_{a=1}^{N} \frac{\bar{\mu}_a}{w - \bar{z}_a} + \sum_{b=1}^{M} \frac{\nu_b}{w - u_b} \tag{C.1}$$
$$= u \frac{(w - \xi)^S \prod_{a'=1}^{N'}(w - y_{a'})(w - \bar{y}_{a'}) \prod_{b'=1}^{M'}(w - t_{b'})}{\prod_{a=1}^{N}(w - z_a)(w - \bar{z}_a) \prod_{b=1}^{M}(w - u_b)}$$

as in (2.24) and (2.25) subject to constraints (2.26). The Jacobian due to the change of variables is used as in (2.27) with (2.28), and we would like to derive it in this appendix. For the bulk case, the Jacobian due to the change of variables was obtained in [9, 50] and extended to the case with closed Riemann surfaces of arbitrary genus in [7]. For the boundary case, the variables are changed from $(\mu_a, \bar{\mu}_a, \nu_b)$ to $(u, y_{a'}, \bar{y}_{a'}, t_{b'})$, where there are N' y's and M' t's with $M' + 2N' + S + 2 = M + 2N$. One non-trivial point here is that the number of $y_{a'}$ and $t_{b'}$ changes when we vary μ_a, ν_b. Thus we first show that the map (C.1) gives a one-to-one map modulo permutations among $y_{a'}$ and $t_{b'}$, and hence the integral regions of the both side match. Then we obtain the weight of the integral as given in the Jacobian (2.27). For $S = 0$ it was already given in [24].

In order to show that (C.1) defines a one-to-one map, we start with the map from $(u, y_{a'}, \bar{y}_{a'}, t_{b'})$ to $(\mu_a, \bar{\mu}_a, \nu_b)$. From the residues of (C.1) at $z = z_a$ and $z = u_b$, we have

$$\mu_a = u \frac{(z_a - \xi)^S \prod_{a'=1}^{N'}(z_a - y_{a'})(z_a - \bar{y}_{a'}) \prod_{b'=1}^{M'}(z_a - t_{b'})}{(z_a - \bar{z}_a) \prod_{c=1, c \neq a}^{N}(z_a - z_c)(z_a - \bar{z}_c) \prod_{b=1}^{M}(z_a - u_b)} , \tag{C.2}$$

$$\nu_b = u \frac{(u_b - \xi)^S \prod_{a'=1}^{N'}(u_b - y_{a'})(u_b - \bar{y}_{a'}) \prod_{b'=1}^{M'}(u_b - t_{b'})}{\prod_{a=1}^{N}|u_b - z_a|^2 \prod_{d=1, d \neq b}^{M}(u_b - u_d)} , \tag{C.3}$$

and similarly for $\bar{\mu}_a$. Therefore, if we choose the values of $(u, y_{a'}, \bar{y}_{a'}, t_{b'})$, then we can obtain $(\mu_a, \bar{\mu}_a, \nu_b)$ uniquely from the above equations. A point here is that we can use arbitrary numbers of $y_{a'}$ and $t_{b'}$ if they satisfy $2N' + M' = 2N + M - S - 2$.

To show that the map is onto the whole region of $(\mu_a, \bar{\mu}_a, \nu_b)$ we consider a given value of these and rewrite

$$\sum_{a=1}^{N} \frac{\mu_a}{z - z_a} + \sum_{a=1}^{N} \frac{\bar{\mu}_a}{z - \bar{z}_a} + \sum_{b=1}^{M} \frac{\nu_b}{z - u_b} \tag{C.4}$$
$$= u \frac{(z - \xi)^S (z^{2N'+M'} + a_1 z^{2N'+M'-1} + \cdots + a_{2N'+M'})}{\prod_{a=1}^{N}(z - z_a)(z - \bar{z}_a) \prod_{b=1}^{M}(z - u_b)}$$

with $u = \sum_a 2 \operatorname{Re} \mu_a z_a + \sum_b \nu_b u_b$. The term proportional to $z^{2N+M-1} = z^{2N'+M'+S+1}$ vanishes due to the delta function $\delta(\ell_0(\xi))$, where $\ell_n(\xi)$ is given by (2.26). With the other delta functions $\delta(\ell_n(\xi))$ with $n = 1, 2, \cdots, S$, the left hand side can be factorized by $(z-\xi)^S$. Since the left hand side is always real when we set $z = t$ with $t \in \mathbb{R}$, the coefficients a_i have to be real. Here we use the theorem that an algebraic equation of order P with real valued

coefficients have N' complex roots along with their complex conjugates and M' real roots with $2N' + M' = P$. Therefore, once we choose some values of $(\mu_a, \bar{\mu}_a, \nu_b)$, then we can find $(u, y_{a'}, \bar{y}_{a'}, t_{b'})$ from the roots of the above algebraic equation of order $2N' + M'$ with real coefficients up to permutations among $y_{a'}$ and $t_{b'}$.

In this way, we have shown that the map (C.1) is one-to-one between the two regions of parameters. Therefore, in order to establish the relation (2.27) we just need to obtain the weight of measure. In the case with $S = 0$, the Jacobian for the change of parameters was already given in [24] as (2.27) with $S = 0$. It can be also obtained by utilizing the bosonization formula for (b, c)-ghosts as in appendix C of [7]. The correlation function we should use is

$$\left\langle \prod_{a=1}^{N} c(z_a)\bar{c}(\bar{z}_a) \prod_{i=1}^{M} c(u_i) \prod_{a'=1}^{N'} b(y_{a'})\bar{b}(\bar{y}_{a'}) \prod_{i'=1}^{M'} b(t_{i'}) \right\rangle . \tag{C.5}$$

Notice that the point $t_{i'} = t_{j'}$ is really a singularity since we cannot across the point continuously contrary to the point $y_{a'} = y_{b'}$. Indeed, this is the point where the number of $t_{i'}$'s would change. The same is true for the point $y_{a'} = \bar{y}_{a'}$ where the number of $y_{a'}$'s would change. The generalization to the case with $S \neq 0$ is actually quite straightforward if we utilize the method in [7]. In that paper, the Jacobian is found by induction in S, and with the same trick the Jacobian is found as (2.27) with (2.28).

D   Reflection relations of boundary operators

In the derivation of the FZZ duality, we utilize reflection relations of bulk and boundary operators in the Liouville field theory with the action

$$S = \frac{1}{\pi} \int d^2 w \left(\partial\phi\bar{\partial}\phi + \frac{\sqrt{g}}{4}\mathcal{R}Q\phi + \mu\pi e^{2b\phi} \right) + \mu_B \sigma_1 \int du\, e^{b\phi} . \tag{D.1}$$

Here the parameters are $Q = b + b^{-1}$, $b = i/\sqrt{2}$, $\mu = -1$ and $\mu_B = -1/\sqrt{2}$. Notice that this Liouville field theory is different from the one obtained from the relation to H_3^+ model. Rather, it is obtained by treating the extra insertions as the Liouville interaction terms. The conformal dimension of the bulk operator $V_\alpha = \exp(2\alpha\phi)$ is given by $\Delta_\alpha = \alpha(Q - \alpha)$, which implies that V_α is related to $V_{Q-\alpha}$ since they have the same conformal dimension. In fact, we have the reflection relation as $V_\alpha = D(\alpha)V_{Q-\alpha}$, where $D(\alpha)$ can be obtained from the two point function. With $b = i/\sqrt{2}$ the reflection coefficient is simplified as [7]

$$D(\alpha) = -(-\mu\pi)^{-1+2\sqrt{2}i\alpha}\frac{\Gamma(1 - \sqrt{2}i\alpha)}{\Gamma(\sqrt{2}i\alpha)} \tag{D.2}$$

with $\mu = -1$. In the same way, we should have reflection relations for the boundary operators as

$$[B_\beta^i(z)]_{s_1,s_2} = d(\beta|s_1, s_2)^i{}_j [B_{Q-\beta}^j(z)]_{s_1,s_2} , \qquad B_\beta(z) = \frac{\sigma_i}{\sqrt{2}} e^{\beta\phi} . \tag{D.3}$$

The aim of this appendix is to compute the coefficient $d(\beta|s_1, s_2)^i{}_j$ with our specific values of parameters. Since the boundary action includes the Chan-Paton factor, the relations are different from those obtained for the case without Chan-Paton factor [20, 21]. Here we derive the reflection relations from those without Chan-Paton factor. The idea is to change the basis for the boundary operators such that the Chan-Paton factor in the boundary action is diagonal and we can apply the formula for the FZZT-branes.

Before introducing the Chan-Paton factor, we summarize some useful formulas for the FZZT-branes in [20, 21]. The action of the boundary Liouville field theory is given as

$$S = \frac{1}{\pi} \int d^2 w \left(\partial\phi\bar\partial\phi + \frac{\sqrt{g}}{4} \mathcal{R}Q\phi + \pi\mu e^{2b\phi} \right) + \mu_B \int du\, e^{b\phi} \ . \tag{D.4}$$

The boundary parameter μ_B may be expressed by s as

$$\cosh \pi b s = \mu_B \sqrt{\frac{\sin \pi b^2}{\mu}} \ . \tag{D.5}$$

The reflection relation for boundary action is then written as

$$[B_\beta(z)]_{s_1, s_2} = d(\beta|s_1, s_2)[B_{Q-\beta}(z)]_{s_1, s_2} \ , \qquad B_\beta(z) = e^{\beta\phi} \ . \tag{D.6}$$

The reflection coefficient is found as

$$d(\beta|s_1, s_2) = \frac{(\pi\mu\gamma(b^2)b^{2-2b^2})^{(Q-2\beta)/2b} \mathbf{G}(Q - 2\beta)\mathbf{G}^{-1}(2\beta - Q)}{\mathbf{S}(\beta + \frac{i}{2}(s_1 + s_2))\mathbf{S}(\beta - \frac{i}{2}(s_1 + s_2))\mathbf{S}(\beta + \frac{i}{2}(s_1 - s_2))\mathbf{S}(\beta - \frac{i}{2}(s_1 - s_2))} \ , \tag{D.7}$$

where the functions $\mathbf{G}(x)$ and $\mathbf{S}(x)$ are defined as

$$\log \mathbf{S}(x) = \int_0^\infty \frac{dt}{t} \left[\frac{\sinh(Q - 2x)t}{2\sinh(bt)\sinh(t/b)} - \frac{(Q/2 - x)}{t} \right] \ , \tag{D.8}$$

$$\log \mathbf{G}(x) = \int_0^\infty \frac{dt}{t} \left[\frac{e^{-Qt/2} - e^{-xt}}{(1 - e^{-bt})(1 - e^{-t/b})} + \frac{(Q/2 - x)^2}{2} e^{-t} + \frac{(Q/2 - x)}{t} \right] \ . \tag{D.9}$$

We will use the shift relations

$$\mathbf{S}(x + b) = 2\sin(\pi b x)\mathbf{S}(x) \ , \qquad \mathbf{S}(x + 1/b) = 2\sin(\pi x/b)\mathbf{S}(x) \ . \tag{D.10}$$

For more details, see [20, 21].

Now we include the effects of Chan-Paton factor. The action we consider is (D.1) and the boundary operator is in (D.3). For the boundary operators, it is convenient to take linear combinations as

$$B_\beta^\pm(u) = \frac{1}{2}(\sigma_0 \pm \sigma_1)e^{\beta\phi} \ , \qquad B_\beta'^{\pm}(u) = \frac{1}{2}(\sigma_3 \mp i\sigma_2)e^{\beta\phi} \ . \tag{D.11}$$

Notice that they behave as

$$\sigma_1 B_\beta^\pm = \pm B_\beta^\pm \ , \qquad B_\beta^\pm \sigma_1 = \pm B_\beta^\pm \ , \qquad \sigma_1 B_\beta'^\pm = \pm B_\beta'^\pm \ , \qquad B_\beta'^\pm \sigma_1 = \mp B_\beta'^\pm, \tag{D.12}$$

when the Chan-Paton factor in the boundary action acts on them. Let us start with B_β^+. It is easy to see that the reflection relation is the same as for the FZZT-branes. Therefore, we obtain $B_\beta^+ = d(\beta|s,s)B_{Q-\beta}^+$, where $s = i5/4b^{-1}$ from (D.5) and the explicit values of μ, μ_B. Here, and in the following, we restrict the domain of s as $0 \leq -ibs < 2$ to avoid ambiguity. For B_β^-, we may shift $s \to s - i/b$ since $\sigma_1 B_\beta^- = B_\beta^- \sigma_1 = -B_\beta^-$ with (D.5). Thus the reflection coefficient is

$$d(\beta|s - \tfrac{i}{b}, s - \tfrac{i}{b}) = \frac{\sin(\tfrac{\pi}{b}(\beta + is - \tfrac{1}{b}))}{\sin(\tfrac{\pi}{b}(\beta - is))} d(\beta|s,s) = -d(\beta|s,s) \tag{D.13}$$

where we have used the shift relations (D.10). Moreover, we have set $1/b^2 = -2$ and $s = i5/4b^{-1}$. The reflection relations are thus

$$B_\beta^0 = d(\beta|s,s)B_{Q-\beta}^1 \,, \qquad B_\beta^1 = d(\beta|s,s)B_{Q-\beta}^0 \tag{D.14}$$

in the original basis. We now move to the other cases with $B_\beta'^\pm$. From (D.12), we propose that the reflection relations are given as

$$B_\beta'^+ = d(\beta|s - \tfrac{i}{2b}, s + \tfrac{i}{2b})B_{Q-\beta}'^+ \,, \qquad B_\beta'^- = d(\beta|s + \tfrac{i}{2b}, s - \tfrac{i}{2b})B_{Q-\beta}'^- \,, \tag{D.15}$$

where we have used $s = i5/4b^{-1}$. There might be other choices, but it turns out that they do not work well. Noticing that $b^{-1} = -2b$, we can show

$$d(\beta|s - \tfrac{i}{2b}, s + \tfrac{i}{2b}) = \cot(\pi b\beta)d(\beta|s,s) = d(\beta|s + \tfrac{i}{2b}, s - \tfrac{i}{2b}) \,. \tag{D.16}$$

Therefore, we have

$$B_\beta^2 = \cot(\pi b\beta)d(\beta|s,s)B_{Q-\beta}^2 \,, \qquad B_\beta^3 = \cot(\pi b\beta)d(\beta|s,s)B_{Q-\beta}^3 \tag{D.17}$$

in the original basis.

Above, we have seen that the reflection coefficient in (D.3) can be written in terms of $d(\beta|s,s)$ for the FZZT-branes defined in (D.6). In fact, we can show that the function $d(\beta|s,s)$ with $b = i/\sqrt{2}$ and $s = i5/4b^{-1}$ has a simple expression. Since it is quite difficult to directly compute the functions (D.8) and (D.9), we take a different route. We choose to utilize the relation shown in [20]

$$d(\beta|s,s) = c_-(\beta)d(\beta + b|s,s) \,, \tag{D.18}$$

which the reflection coefficient has to satisfy. Here $c_-(\beta)$ is simplified as

$$\begin{aligned} c_-(\beta) &= (-4\mu/\pi)\sin^2(\pi\beta b)\sin(\pi\beta b + \pi/4)\sin(\pi\beta b - \pi/4) \\ &\quad \times \Gamma(1 - 2\beta b)\Gamma(2\beta b - 1)\Gamma(3/2 - 2\beta b)\Gamma(2\beta b - 1/2) \\ &= -\pi\mu\frac{\sin(\pi\beta b)}{\cos(\pi\beta b)}\frac{\Gamma(2(\beta + b)b)}{\Gamma(2\beta b)} \end{aligned} \tag{D.19}$$

for our choice of s. The relation (D.18) can be solved quite easily. Assigning the unitarity relation

$$d(\beta)d(Q - \beta) = 1 \,, \tag{D.20}$$

we find for the reflection coefficient

$$d(\beta|s,s) = (-\mu\pi)^{(Q-2\beta)/(2b)}\sqrt{\frac{\pi}{2}}\frac{1}{\Gamma(2\beta b)\sin(\pi b\beta + \pi/2)} \ . \tag{D.21}$$

We also need

$$\cot(\pi b\beta)d(\beta|s,s) = (-\mu\pi)^{(Q-2\beta)/(2b)}\sqrt{\frac{\pi}{2}}\frac{1}{\Gamma(2\beta b)\sin(\pi b\beta)} \ . \tag{D.22}$$

E   Branes in sine-Liouville theory

Branes in sine-Liouville theory have not been investigated yet. However, it is easy to guess what they are by two means. Firstly, one can utilize the boundary FZZ duality, but of course we cannot use this to prove the duality itself. Secondly, one can mimic the arguments in [26] for branes in $\mathcal{N} = 2$ Liouville theory. The details may be changed, but qualitatively they should be the same as branes in sine-Liouville theory. In particular, we should be able to obtain the branes in sine-Liouville theory by repeating the analysis in [26].

Following the terminology in [26], we call D1-branes for A-branes and D2-branes for B-branes. A-branes are relatively easy since we can construct boundary states for them only from modular transformations of annulus amplitudes. According to [26], we cannot construct B-branes from modular invariance of annulus amplitudes, and we have to study the factorization constraint from the two-point function on a disk. Fortunately, we have shown that B-branes in sine-Liouville theory can be obtained from D1-branes in the cigar model, so we just need to study A-branes.

Since the branes are labeled by the representation of sl$(2, \mathbb{R})$, A-branes may be classified according the representation as in section 3.1 of [26]. Following the paper, we have six types as

1. Identity representation

2. Non-chiral non-degenerate representations

3. Non-chiral degenerate representations

4. Anti-chiral representations

5. Chiral representations

6. Degenerate representations

Types 2,4,5 and types 1,6 would correspond to D2-branes and D1-branes in the cigar model, respectively. Type 3 requires some consideration. We just need to repeat the analysis in [26] to construct boundary states for the above branes, but replacing the characters for the $\mathcal{N} = 2$ superconformal algebra by those for the $\hat{W}_\infty(k)$-algebra (or equivalently parafermions), see section 4 of [27].

In order to go further, we need to compute two-point functions of bulk operators on a disk with one degenerate operator and two-point functions of boundary operators on a disk. Even though it is a very hard task to compute them directly, it is easy to guess the boundary actions. For B-branes we have already obtained this as

$$\mathcal{L}_{\mathrm{bdy}} = (\mu_B \sigma_+ + \mu_{\bar{B}} \sigma_-) e^{\frac{1}{2b}\phi + i\frac{\sqrt{k}}{2}\tilde{X}} + (\bar{\mu}_B \sigma_+ + \bar{\mu}_{\bar{B}} \sigma_-) e^{\frac{1}{2b}\phi - i\frac{\sqrt{k}}{2}\tilde{X}} \ ,$$

with

$$(\mu_B, \mu_{\bar{B}}, \bar{\mu}_B, \bar{\mu}_{\bar{B}}) = i\sqrt{\frac{\tilde{c}}{2}}(e^{i\pi(J-M)}, e^{-i\pi(J-M)}, e^{-i\pi(J+M)}, e^{i\pi(J+M)}) \ . \tag{E.1}$$

This form of the action implies that B-branes consist of two D2-branes and a non-trivial Wilson loop is included. For A-branes corresponding to D2-branes in the cigar model, we propose the following boundary action (see (5.28) in [26])

$$\mathcal{L}_{\mathrm{bdy}} = \mu_A \sigma_+ e^{\frac{1}{2b}\phi + i\frac{\sqrt{k}}{2}\tilde{X}} + \bar{\mu}_A \sigma_- e^{\frac{1}{2b}\phi - i\frac{\sqrt{k}}{2}\tilde{X}} \tag{E.2}$$

with

$$\mu_A = \sqrt{2\tilde{c}} \sin \pi(J - M) \ , \qquad \bar{\mu}_A = \sqrt{2\tilde{c}} \sin \pi(J + M) \ . \tag{E.3}$$

This form of the action implies that A-branes consist of two D1-branes and tachyonic states from open strings between two branes are condensed. Following the analysis in [28] we can show that the boundary action preserves the $\hat{W}_\infty(k)$ symmetry of sine-Liouville theory, just like the boundary action for A-branes in the $\mathcal{N} = 2$ super Liouville theory preserves superconformal symmetry. Therefore, the problem left is to fix the constant coefficients including the Chan-Paton factors. However, in order to show that our proposal is correct, we would need to study correlation functions.

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