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Weak disorder in the stochastic mean-field model of distance II

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In this paper, we study the complete graph K_n with n vertices, where we attach an independent and identically distributed (i.i.d.) weight to each of the $n(n-1)/2$ edges. We focus on the weight W_n and the number of edges H_n of the minimal weight path between vertex 1 and vertex n .

It is shown in (*Ann. Appl. Probab.* **22** (2012) 29–69) that when the weights on the edges are i.i.d. with distribution equal to that of E^s , where $s > 0$ is some parameter, and E has an exponential distribution with mean 1, then H_n is asymptotically normal with asymptotic mean $s \log n$ and asymptotic variance $s^2 \log n$. In this paper, we analyze the situation when the weights have distribution E^{-s} , $s > 0$, in which case the behavior of H_n is markedly different as H_n is a tight sequence of random variables. More precisely, we use the method of Stein–Chen for Poisson approximations to show that, for almost all $s > 0$, the hopcount H_n converges in probability to the nearest integer of $s + 1$ greater than or equal to 2, and identify the limiting distribution of the recentered and rescaled minimal weight. For a countable set of special s values denoted by $\mathcal{S} = \{s_j\}_{j \geq 2}$, the hopcount H_n takes on the values j and $j + 1$ each with *positive* probability.

Keywords: complete graph; extreme value theory; first passage percolation; hopcount; minimal path weight; Poisson approximation; Stein–Chen method; stochastic mean-field model; weak disorder

1. Introduction

One of the central themes of modern discrete probability is the study of the effect of random edge disorder on various properties of the underlying network. The base network itself could be *deterministic*, for example, a large finite box in the lattice or the complete graph on n vertices, or *random*, for example, the giant component of the Erdős–Rényi random graph or the configuration model. Each edge is assigned a random edge weight,

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whose interpretation varies depending on the context. One can think of this weight as the *cost* in traversing the edge, yielding first passage percolation-type models. Alternatively, one could think of the underlying graph as an electrical network and the assigned weights as resistances, yielding a random resistor network, or as capacities on edges and the underlying graph as a flow carrying network, entrusted with carrying flow (commodities, information, etc.) between various parts of the network.

One graph model that has resulted in many problems of fundamental interest is the complete graph K_n on n vertices with random edge weights. In the various contexts mentioned above, this model both gives rise to very interesting conjectures as well as generates new techniques and insights in probability theory that can then be applied in a number of other contexts. While providing a complete list of references of the various models that have been studied in this context would be impractical, we direct the interested reader to [14] for one of the first refined results in first passage percolation in this context, [10] for an analysis of the cost of the minimal spanning tree, [11] for an analysis of random electrical networks on the complete graph, [2] for a study of the random assignment problem, [4] for an analysis of the multicommodity flow problem and the survey paper [1], where a number of other examples are analyzed via the powerful *local weak convergence* method.

Let us now focus on the particular problem dealt with in this study and the motivations behind it. Suppose we start with a connected graph \mathcal{G}_n (deterministic such as K_n or random) on n vertices. Suppose each edge e is assigned a random positive edge weight E_e . We shall assume that the weights are i.i.d. over the edges with some distribution F , with density f . Fix two vertices (say chosen uniformly at random from \mathcal{G}_n), and let us denote them by 1 and n . For any path \mathcal{P} between the two vertices, let the weight of the path $w(\mathcal{P})$ be defined by

$$w(\mathcal{P}) := \sum_{e \in \mathcal{P}} E_e,$$

that is, the sum of weights of the edges in the path. The *optimal* or *minimal weight* path (which is unique since the edge weights have a density) is the path that minimizes the above weight function. In the study of random systems, this regime is often called the *weak disorder* regime, while probabilists know this problem as “first passage percolation.” The mental picture one can have is that the network is entrusted with carrying flow between various nodes of the network, and the way it performs this duty is via routing flow through optimal paths. We shall defer a more extensive discussion of the relevant literature to Section 3.

Another regime which is of tremendous interest is the *strong disorder regime*. Here the weight of a path is either the *maximum* or the *minimum* weight of all edges in the path. We denote the weight functions as

$$w_{\max}(\mathcal{P}) := \max_{e \in \mathcal{P}} E_e, \tag{1.1}$$

and

$$w_{\min}(\mathcal{P}) := \min_{e \in \mathcal{P}} E_e. \tag{1.2}$$

In both situations, one is interested in properties of the path which minimizes the above weight function. One is also interested in formulating a model, depending on a real-valued parameter, the “inverse parameter,” which interpolates between these two models. One can then study questions such as phase transitions, where there is a change in the behavior of the system from the weak disorder regime to the strong disorder regime. Given the set of edge weights E_e , one method of doing this is as follows: assign each edge a cost E_e^β where $\beta \in \mathbb{R}$ is a real-valued parameter. With these edge weights, suppose that, as before, we consider the weak disorder regime, so that now the weight of a path \mathcal{P} is

$$w_\beta(\mathcal{P}) := \sum_{e \in \mathcal{P}} E_e^\beta.$$

Then, we can identify the following special cases:

- (a) *Original model*: $\beta = 1$ is our original model.
- (b) *Graph distance*: $\beta = 0$ gives us the graph distance between the chosen vertices in the graph \mathcal{G}_n .
- (c) *Strong disorder, max edge weight*: The case $\beta \rightarrow +\infty$ gives us the strong disorder regime where the weight of a path is given by (1.1). This is also called the *minimal spanning tree* regime as the optimal path between the two vertices is the same as the path in the minimal spanning tree on \mathcal{G}_n with edge weights E_e .
- (d) *Strong disorder, min edge weight*: $\beta \rightarrow -\infty$ gives us the strong disorder model where the weight of a path is given by (1.2).

Thus this model allows us to interpolate between various regimes of interest. We shall denote the optimal path by $\mathcal{P}_{\text{opt}}(\beta)$. Given a particular base network \mathcal{G}_n and edge weight distribution E_e , two statistics are of paramount interest:

- (i) *Minimal weight*: This is the actual weight of the optimal path, namely $W_n = \sum_{e \in \mathcal{P}_{\text{opt}}(\beta)} E_e^\beta$.
- (ii) *Hopcount*: This is defined as the number of edges in the optimal path $\mathcal{P}_{\text{opt}}(\beta)$. We shall denote this random variable by $H_n(\beta)$.

Aim of this paper: In this paper, we shall specialize to the case where the graph \mathcal{G}_n is the complete graph K_n and each edge originally has edge weight E_e^β , where E_e is exponentially distributed with rate 1 ($E_e \stackrel{d}{=} \text{Exp}(1)$). We shall study the case where $\beta < 0$. The case where $\beta > 0$ has been solved in [6], where it was proved that, for $\beta > 0$,

$$\frac{H_n(\beta) - \beta \log n}{\sqrt{\beta^2 \log n}} \xrightarrow{d} Z, \quad (1.3)$$

where Z denotes a standard normal random variable, and \xrightarrow{d} denotes convergence in distribution. In the same paper it was proved that, for the optimal weight $W_n = W_n(\beta)$, there exists a constant $\lambda = \lambda(\beta) > 0$ and a non-degenerate real-valued random variable $\Xi(\beta)$ such that

$$W_n(\beta) - \frac{1}{\lambda} \log n \xrightarrow{d} \Xi(\beta). \quad (1.4)$$

In this study, we shall derive asymptotics for the two random variables of interest $W_n(\beta)$ and $H_n(\beta)$ as $n \rightarrow \infty$, and see that the behavior in the case when $\beta < 0$ is markedly different.

Throughout the paper, we make use of the following standard notation. We let \xrightarrow{d} denote convergence in distribution, and $\xrightarrow{\mathbb{P}}$ convergence in probability. For a sequence of random variables $(X_n)_{n \geq 1}$, we write $X_n = O_{\mathbb{P}}(1)$ when $|X_n|$ is a tight sequence of random variables as $n \rightarrow \infty$, and $X_n = o_{\mathbb{P}}(1)$ when $|X_n| \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$. For a non-negative function $n \mapsto g(n)$, we write $f(n) = O(g(n))$ when $|f(n)|/g(n)$ is uniformly bounded, and $f(n) = o(g(n))$ when $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$. We let $\text{Exp}(\lambda)$ denote an exponential random variable with rate λ and $\text{Poi}(\lambda)$ a Poisson random variable with mean λ . We write that a sequence of events $(\mathcal{E}_n)_{n \geq 1}$ occurs *with high probability (w.h.p.)* when $\mathbb{P}(\mathcal{E}_n) \rightarrow 1$. Finally, for $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the largest integer smaller than or equal to x and by $\lceil x \rceil$ the smallest integer larger than or equal to x .

We now state our main results and defer a further discussion to Section 3.

2. Results

Before stating the main result, we need some further notation. We study the complete graph K_n with i.i.d. edge weights $E_{\{i,j\}}^{-s}$, $1 \leq i < j \leq n$, on the edges of K_n . Thus, compared to the discussion in the previous section, we have taken $s = -\beta$, and we shall study the $s > 0$ regime. For fixed $s > 0$, define the function

$$g_s(x) = \frac{x^{s+1}}{(x-1)^s}, \quad x \geq 2. \quad (2.1)$$

Observe that, for $0 < s \leq 1$, the function $g_s(x)$, $x \geq 2$, is increasing, while for $s > 1$, the function is strictly convex with unique minimum at $x = s + 1$. We shall be interested in minimizing this function only on the set \mathbb{Z}_+ of positive integers. Then there is a sequence of values $s = s_j$, $j \geq 2$, for which the minimum integer of g_s is *not* unique. From the equation $g_s(j) = g_s(j+1)$, and the bounds $j-1 < s < j$, it is not hard to verify that

$$s_j = \frac{\log(1+j^{-1})}{\log(1+(j^2-1)^{-1})} \in (j-1, j), \quad j = 2, 3, \dots \quad (2.2)$$

We will need to deal with these special points separately. When $s \notin \mathcal{S} = \{s_2, s_3, \dots\}$, then there is a unique integer which minimizes the function $g_s(x)$ on \mathbb{Z}_+ .

Below and in the remainder of the paper, for notational simplicity, we take $p = 1/s$, for $s > 0$. Let us now state the main theorems:

Theorem 2.1 (Hopcount and weight asymptotics). *For any fixed $s > 0$ with $s \notin \mathcal{S}$, let $k^*(s) \in \{\lfloor s+1 \rfloor, \lceil s+1 \rceil\}$ denote the unique integer that minimizes the function defined in (2.1). Then:*

(a) the hopcount $H_n = H_n(s)$ converges in probability to $k^*(s)$ as $n \rightarrow \infty$:

$$\mathbb{P}(H_n = k^*(s)) \rightarrow 1;$$

(b) the optimal weight $W_n = W_n(s)$, properly normalized converges in distribution as $n \rightarrow \infty$,

$$\begin{aligned} & \mathbb{P}\left(\frac{k-1}{sg_s(k)}(\log n)^{s+1}\left(W_n - \frac{g_s(k)}{(\log n)^s}\right) + \frac{k-1}{2}\log\log n - \frac{p(k-1)\log g_s(k)}{2} > t\right) \\ & \rightarrow \exp(-a_k e^t), \quad t \in \mathbb{R}, \end{aligned}$$

where $k = k^*(s)$, and the sequence of constants $(a_k)_{k \geq 1}$ is defined by

$$a_k = \left(\frac{2\pi p}{1+p}\right)^{(k-1)/2} k^{((k-1)p-1)/2}. \quad (2.3)$$

Theorem 2.1 states that the hopcount H_n converges to the optimal value of the function $x \mapsto g_s(x)$ defined in (2.1), while the rescaled and recentered minimal weight W_n converges in distribution to a Gumbel distribution. We can intuitively understand this as follows. For fixed k , the minimal path of length k is similar to an *independent* minimum of copies of sums of k random variables E^{-s} . The number of independent copies is equal to the number of disjoint paths between vertices 1 and n , which is of order n^{k-1} . While on K_n , the appearing paths do *not* have independent weights, the paths that are particularly short are almost independent. Now, the independent problem can be handled in two steps. First, we analyze the behavior of the random variable $Z_k = E_1^{-s} + \dots + E_k^{-s}$. In this analysis, the function g_s appears in the lower tail of the distribution. Second, we study the asymptotics of the minimum of n^{k-1} of such random variables, which can be seen to be of order $g_s(k)/(\log n)^s$. This explains why the minimal integer value of g_s is the crucial value for the hopcount, while the minimum of a large number of independent random variables with distribution Z_k , properly rescaled and recentered, converges to a Gumbel distribution by standard extreme value arguments. This intuitively explains Theorem 2.1. The main difficulty in the proof is to handle the fact that the weights of paths in the complete graph are actually *not* independent, and we use the method of Stein–Chen for the Poisson approximation to deal with the available dependence.

Let us now deal with the case where $s \in \mathcal{S}$.

Theorem 2.2 (The special set \mathcal{S}). *Suppose $s \in \mathcal{S}$, so that both $\lfloor s+1 \rfloor$ and $\lceil s+1 \rceil$ minimize $g_s(\cdot)$ over \mathbb{Z}_+ . Define a sequence of independent random variables $(\Xi_k)_{k \geq 2}$, where, for any $k \geq 2$, Ξ_k has the Gumbel survival function*

$$\mathbb{P}(\Xi_k > t) = \exp(-a_k e^t), \quad t \in \mathbb{R}, \quad (2.4)$$

with $(a_k)_{k \geq 2}$ defined in (2.3). Then:

(a) the optimal weight $W_n = W_n(s)$, properly normalized, converges in distribution as $n \rightarrow \infty$. More precisely

$$\frac{(\log n)^{s+1}}{sg^*} \left(W_n - \frac{g^*}{(\log n)^s} \right) + \frac{1}{2} \log \log n - \frac{p \log g^*}{2} \xrightarrow{d} \min \left(\frac{\Xi_{\lfloor s+1 \rfloor}}{\lfloor s+1 \rfloor - 1}, \frac{\Xi_{\lceil s+1 \rceil}}{\lceil s+1 \rceil - 1} \right),$$

where $g^* = g_s(\lfloor s+1 \rfloor) = g_s(\lceil s+1 \rceil)$;

(b) the hopcount $H_n = H_n(s)$ converges in distribution as $n \rightarrow \infty$, that is

$$H_n(s) \xrightarrow{d} H^*,$$

where

$$H^* = \arg \min \{ \Xi_k / (k-1) : k \in \{ \lfloor s+1 \rfloor, \lceil s+1 \rceil \} \}.$$

Another quantity of interest is the distribution of optimal paths between vertex 1 and a *set* of vertices. In telecom, this is called multicast, since one source sends to a multiple number of users. This also follows from the analysis in the paper. We shall give a brief idea of the proof in Section 4.6. The result is stated for $s \notin \mathcal{S}$, but one could state an equivalent result for $s \in \mathcal{S}$ as well. Before we state the result we need some further notation. Recall that we used $k^*(s)$ to denote the unique minimizer of $g_s(\cdot)$ over \mathbb{Z}_+ . For fixed $m \geq 1$, let $\{\eta_i\}_{1 \leq i \leq m}$ denote independent copies of the Gumbel random variable defined in (2.4) with $k = k^*(s)$.

Corollary 2.3 (Multipoint distances). Fix $m \geq 1$ distinct vertices say $2, 3, \dots, m+1$ in K_n . Suppose $s \notin \mathcal{S}$, and let $\{W_n^{(j)}\}_{2 \leq j \leq m+1}$ denote the weight of the optimal path from 1 to these vertices. Write

$$\tilde{W}_n^{(j)} = (\log n)^{s+1} \left(W_n^{(j)} - \frac{g_s(k)}{(\log n)^s} \right) + \frac{k-1}{2} \log \log n - \frac{p(k-1) \log g_s(k)}{2},$$

where $k = k^*(s)$. Then, as $n \rightarrow \infty$,

$$(\tilde{W}_n^{(j)})_{2 \leq j \leq m+1} \xrightarrow{d} (\eta_j)_{1 \leq j \leq m}.$$

Organization of the paper: The paper is organized as follows. We first discuss the relevance of our results and techniques in Section 3. We shall then continue to prove the main results in Section 4.

3. Discussion

We now provide a discussion of the various concepts used in this paper and the relevance of the results.

(a) *Stochastic mean-field model of distance:* This notion refers to the complete graph with exponentially distributed edge weights having unit mean. The model gives a simpler but mathematically more tractable model of distances between random points in high

dimensions. While one can consider other edge distributions, the memoryless property of exponential random variable allows one to give clean proofs in a number of different contexts, including first passage percolation; see [14] where this property is used to great effect to derive refined asymptotics. We also refer to [1], where many other computations are derived in this context with the help of a powerful infinite random structure called the *weighted infinite tree*.

(b) *Weak and strong disorder*: The last few years, with the availability of an enormous amount of data on real-world networks, has witnessed an explosion in network models for these real-world networks as well as dynamics on them. Physicists have been highly interested in understanding the effect of random disorder on the various flow carrying properties of these network models. Via simulations, they have predicted a number of fascinating phenomena in these networks. Regarding the notions of weak and strong disorder mentioned in Section 1, we refer the interested reader to [8, 9, 12] and [17] and the references therein.

(c) *First passage percolation*: First passage percolation problems have been of great interest to probabilists for quite a while now, not just because of their origin from physical motivations of modeling disordered random flow systems, but also because this process and its variants (e.g., oriented first passage percolation and last passage percolation) arise as basic constructing blocks for more complicated problems, such as the contact process. There has been an intensive study of this model on the d -dimensional lattice (see, e.g., [15, 16] and [13]). The case of the complete graph with exponential edge weights was analyzed in [14], where, in particular, it was proved that the weight and hopcount of the optimal path satisfy

$$nW_n - \log n \xrightarrow{d} \Xi,$$

and

$$\frac{H_n}{\log n} \xrightarrow{\mathbb{P}} 1,$$

as $n \rightarrow \infty$, where $\Xi \in \mathbb{R}$ is a non-degenerate random variable.

In the last few years, due to the connections to real-world networks described above, these questions have taken on an added significance, and a number of studies both at the non-rigorous level [8] and rigorous level (see, e.g., [7]) have been undertaken to study such questions in many other random graph models.

(d) *Proof techniques*: A number of different techniques have been used in the analysis of first passage percolation asymptotics in various contexts, ranging from subadditivity methods in the context of the lattice, to continuous-time branching process embeddings and renewal theory in the context of various random graph models. The paper [6] used embeddings into a particular continuous-time branching process to derive the results in (1.3) and (1.4). As far as we know, the present paper is the first paper that uses the method of Stein–Chen for Poisson approximation to derive refined asymptotics in the first passage percolation context. In particular the results here complete the program started in [6] and show that for $\beta \leq 0$, $H_n(\beta)$ is a tight sequence of random variables, while for $\beta > 0$, $H_n(\beta)/(\beta \log n) \xrightarrow{\mathbb{P}} 1$. Clearly, this further shows that there are at least

two universality classes for first-passage percolation on the complete graph in terms of the edge weight distribution. When $\beta \leq 0$, the weights E^β are in the same universality class as the weight 1, in the sense that the hopcount remains bounded, while for $\beta > 0$, they are in the same universality class as the exponential distribution arising for $\beta = 1$. As discussed in more detail in [6], this raises the question of what the universality classes for first passage percolation on K_n are. In particular, does H_n always satisfy a central limit theorem whenever $H_n \rightarrow \infty$? Or are there classes of edge weight distributions where the behavior is even different? For example, is there a class of random edge weights where the behavior is similar as for the minimal spanning tree, where H_n is of the order $n^{1/3}$.

(e) *Multi-point distances and exchangeability*: The classical probability theory of exchangeability has been used in the last few years to analyze various complex random structures; see [3] for a nice, modern survey. In the context of Corollary 2.3, one can analyze such questions in a number of different contexts (such as the stochastic mean-field model of distance or your favorite random graph model with your favorite random edge weights). For the stochastic mean-field model, the multi-point optimal path weights converge (after proper rescaling and recentering) to an *exchangeable* sequence of random variables. In the present model, we can once again show convergence but to an independent sequence of random variables.

4. Proofs

This section contains the proofs of the main results. We start with an outline of the proof. In Section 4.1, we shall show that the hopcount $H_n(s), s > 0$, is a tight sequence of random variables as $n \rightarrow \infty$. In Section 4.2, we shall derive the asymptotic behavior, for $z \downarrow 0$, of the distribution function $F_k(z)$, where

$$F_k(z) = \mathbb{P}(E_1^{-s} + E_2^{-s} + \dots + E_k^{-s} \leq z), \quad (4.1)$$

and where E_1, E_2, \dots is an i.i.d. sequence of $\text{Exp}(1)$ random variables. Denoting the weight of the minimal path with exactly k edges between vertex 1 and vertex n by $W_k(n)$, we then show that for each K , and each $0 < \varepsilon < 1$, uniformly in $2 \leq k \leq K$, w.h.p.,

$$(\log n)^s W_k(n) \geq (1 - \varepsilon) g_s(k), \quad (4.2)$$

where $g_s(x)$ is defined in (2.1).

Inequality (4.2) yields a first-order lower bound for all values of $k \geq 2$. We will show in the paper that the function $g_s(x)$ determines the first-order asymptotics of the weights $W_k(n)$. The behavior of g_s near its minimum value determines the asymptotic behavior of the hopcount H_n . Roughly speaking, the hopcount H_n will converge in probability to the integer $k = k^*(s)$ that minimizes the function $g_s(x), x \geq 2$, over the set \mathbb{Z}_+ . The above statement about the convergence of H_n is true for every $s > 0$ for which the minimizing integer of $g_s(x)$ for $x \geq 2$ is unique, that is, $s \notin \mathcal{S}$. In Section 4.3, we study the minimum of an independent number of n^{k-1} random variables. Each of these n^{k-1} variables is the sum of k i.i.d. random variables with distribution E^{-s} . The result is used to complete

the proof of Theorem 2.1 when $0 < s \leq 1$. In Section 4.4, we extend the analysis to $s > 1$ and complete the proof of Theorem 2.1 by studying the second order asymptotics of the minimal weight of paths of length k in the complete graph K_n .

For $s_j \in \mathcal{S}$, to decide whether the hopcount H_n converges in probability either to $\lfloor s_j + 1 \rfloor$ or to $\lceil s_j + 1 \rceil$, we need the second order asymptotics of $W_k(n)$, which is carried out in detail in Section 4.5. In Section 4.6, we sketch the proof of Corollary 2.3.

4.1. Tightness of the hopcount

Note that the minimal weight W_n satisfies the following inequality:

$$W_n \geq H_n \cdot \min_{1 \leq j \leq n(n-1)/2} E_j^{-s}, \quad (4.3)$$

where $E_j \sim \text{Exp}(1)$ are independent. Since the maximum of n independent exponentials scales like $(1 + o_{\mathbb{P}}(1)) \log n$, we obtain from (4.3) that w.h.p.

$$W_n \geq H_n \log^{-s}(n(n-1)/2)(1 + o_{\mathbb{P}}(1)). \quad (4.4)$$

On the other hand, W_n is, at most, equal to the minimal weight of all two-edge paths between 1 and n . Here, a two-edge path is a path of the form $1 \rightarrow j \rightarrow n, j = 2, 3, \dots, n-1$, so that

$$W_n \leq \min_{2 \leq j \leq n-1} ((E'_j)^{-s} + (E''_j)^{-s}), \quad (4.5)$$

where $E'_j, 2 \leq j \leq n-1$, and $E''_j, 2 \leq j \leq n-1$, are independent $\text{Exp}(1)$ random variables. It is not hard to verify (see Lemma 4.1 in Section 4.2) that (4.5) implies that w.h.p.,

$$W_n \leq \frac{C}{(\log n)^s}. \quad (4.6)$$

Inequalities (4.4) and (4.6) together imply that w.h.p.,

$$H_n \leq C(\log 2)^s. \quad (4.7)$$

We conclude that H_n is a tight sequence of random variables. The remainder of the proof will reveal that, in fact, H_n converges in distribution, either to a constant $k^*(s)$ when $s \notin \mathcal{S}$, or to a random variable giving positive mass to two values when $s \in \mathcal{S}$.

4.2. The first-order lower bound

We start with an investigation of the distribution function F_k of an *independent* sum of k inverse powers of exponentials, that is,

$$Z_k = E_1^{-s} + \dots + E_k^{-s}, \quad s > 0. \quad (4.8)$$

Lemma 4.1 (Sums of inverse powers of exponentials). Fix $s > 0$, and put $p = 1/s$. Then, for $z \downarrow 0$,

$$F_k(z) \sim a_k z^{-(k-1)p/2} e^{-k^{p+1} z^{-p}}, \quad (4.9)$$

where $(a_k)_{k \geq 1}$ is defined in (2.3), and where, for arbitrary real functions g and h , $g(z) \sim h(z), z \downarrow 0$, means that $\lim_{z \downarrow 0} g(z)/h(z) = 1$.

Proof. The result for $k = 1$ is immediate from $F_1(z) = e^{-z^{-p}}, z > 0$. We proceed by induction. Suppose that (4.9) holds for some integer $k \geq 1$, then

$$\begin{aligned} F_{k+1}(z) &\sim \int_0^z a_k (z-y)^{-(k-1)p/2} e^{-k^{p+1}(z-y)^{-p}} d(e^{-y^{-p}}) \\ &= p a_k z^{-(k-1)p/2} z^{-p} \int_0^1 x^{-p-1} (1-x)^{-(k-1)p/2} e^{-z^{-p} h_k(x)} dx, \end{aligned}$$

where $h_k(x) = x^{-p} + k^{p+1}(1-x)^{-p}$. The function h_k has a minimum at $x = 1/(k+1)$, since $h_k(1/(k+1)) = (k+1)^{p+1}$, $h'_k(1/(k+1)) = 0$, and

$$h''_k(1/(k+1)) = p(p+1)(k+1)^{p+2} \left(1 + \frac{1}{k}\right).$$

Hence, from a standard Laplace-method argument, we obtain

$$\begin{aligned} F_{k+1}(z) &\sim p a_k z^{-(k-1)p/2} z^{-p} \int_0^1 x^{-p-1} (1-x)^{-(k-1)p/2} e^{-z^{-p} h_k(x)} dx \\ &\sim p a_k z^{-(k-1)p/2} z^{-p} (k+1)^{p+1} (k/(k+1))^{-(k-1)p/2} \\ &\quad \times e^{-z^{-p} h_k(1/(k+1))} \sqrt{\frac{2\pi}{z^{-p} h''_k(1/(k+1))}}. \end{aligned}$$

From the latter expression, we obtain

$$a_{k+1} = \left(\frac{2\pi p}{1+p}\right)^{1/2} a_k (k+1)^{(kp-1)/2} k^{-((k-1)p-1)/2}.$$

This recursion is telescoping in k . Defining $c = \left(\frac{2\pi p}{1+p}\right)^{1/2}$ and $b_k = k^{((k-1)p-1)/2}$, we find

$$a_{k+1} = c a_k \frac{b_{k+1}}{b_k} \quad \Rightarrow \quad a_{k+1} = c^k \frac{b_{k+1}}{b_1} a_1 = c^k b_{k+1},$$

which yields (2.3). \square

Using the above lemma, we obtain the following first-order lower bound for $W_k(n)$:

Theorem 4.2 (First-order lower bound). Fix $s > 0$ and an arbitrary large integer K . For each $0 < \varepsilon < 1$, with the function g_s defined in (2.1), w.h.p. and uniformly in $k \in \{2, 3, \dots, K\}$,

$$(\log n)^s W_k(n) \geq (1 - \varepsilon) g_s(k).$$

Proof. Fix $s > 0$ and $2 \leq k < n$, and define, for $0 < \varepsilon < 1$,

$$x_{k,n} = x_{k,n}(\varepsilon) = (1 - \varepsilon) \frac{g_s(k)}{(\log n)^s}. \quad (4.10)$$

Let $N_k^{(n)}(x), x > 0$, be the number of paths between 1 and n with exactly k edges and weight at most x . Note that the total number of paths with exactly k edges between 1 and n is $\prod_{j=2}^k (n - j)$. Thus, according to Lemma 4.1, for $x \downarrow 0$,

$$\mathbb{E}[N_k^{(n)}(x)] = \left[\prod_{j=2}^k (n - j) \right] F_k(x) \sim \left[\prod_{j=2}^k (n - j) \right] a_k x^{-(k-1)p/2} e^{-k^{p+1} x^{-p}}. \quad (4.11)$$

For $n \rightarrow \infty$ the expression $x_{k,n} \downarrow 0$, and the term $x_{k,n}^{-(k-1)p/2}$ blows up only polynomially fast, while $\exp\{-k^{p+1} x_{k,n}^{-p}\}$ tends to 0 exponentially fast. Using that $\prod_{j=2}^k (n - j) < n^{k-1}$ and abbreviating $N_k^{(n)} = N_k^{(n)}(x_{k,n})$, we reach to the conclusion that

$$\mathbb{E}[N_k^{(n)}] \leq n^{k-1} \exp\{-k^{p+1} x_{k,n}^{-p}\} = \exp\left\{-\left(\frac{1}{(1 - \varepsilon)^p} - 1\right)(k - 1) \log n\right\}.$$

Boole's inequality and the Markov inequality together yield

$$\begin{aligned} & \mathbb{P}\left(\bigcup_{k=2}^K \{(\log n)^s W_k(n) < (1 - \varepsilon) g_s(k)\}\right) \\ & \leq \sum_{k=2}^K \mathbb{P}((\log n)^s W_k(n) < (1 - \varepsilon) g_s(k)) \leq \sum_{k=2}^K \mathbb{P}(N_k^{(n)} \geq 1) \\ & \leq \sum_{k=2}^K \mathbb{E}[N_k^{(n)}] \leq \sum_{k=1}^{\infty} \exp\left\{-\left(\frac{1}{(1 - \varepsilon)^p} - 1\right) k \log n\right\}. \end{aligned}$$

Since the summand on the right-hand side is of order $n^{-p\varepsilon k}$, we may conclude that the probability that $(\log n)^s W_k(n) < (1 - \varepsilon) g_s(k)$, for some $2 \leq k \leq K$, tends to 0 as $n \rightarrow \infty$. \square

4.3. Second-order asymptotics

In this section we identify the second-order asymptotics for the *independent* minimum of n^{k-1} random variables, where each of these random variables has distribution function

$F_k(z)$, $z > 0$. The proof of Theorem 2.1 for $0 < s \leq 1$ follows quite easily from this and the lower bound (4.2). The proof of Theorem 2.1 for $s > 1$ is postponed to the next section.

We write

$$W_k^{(\text{ind})} = \min_{1 \leq j \leq n^{k-1}} Y_{k,j}, \quad (4.12)$$

where $Y_{k,1}, \dots, Y_{k,n^{k-1}}$ are i.i.d. with distribution function F_k . The following theorem derives the asymptotics of $W_k^{(\text{ind})}$:

Theorem 4.3 (Minimum for independent r.v.s). *For $k \geq 2$ fixed, the minimal weight $W_k^{(\text{ind})}$ defined in (4.12), satisfies*

$$\begin{aligned} & \mathbb{P}\left(\frac{k-1}{sg_s(k)}(\log n)^{s+1}\left(W_k^{(\text{ind})} - \frac{g_s(k)}{(\log n)^s}\right) + \frac{k-1}{2}\log\log n - \frac{p(k-1)\log g_s(k)}{2} > t\right) \\ & \rightarrow e^{-a_k e^t}, \end{aligned} \quad (4.13)$$

where $(a_k)_{k \geq 1}$ is defined in (2.3).

Proof. We compute $z_n = z_n(t)$, such that

$$(1 - F_k(z_n))^{n^{k-1}} \rightarrow \exp\{-a_k e^t\}.$$

Taking logarithms on both sides and using that $\log\{1 - F_k(z_n)\} \sim -F_k(z_n)$ for $z_n \rightarrow 0$, this is equivalent to

$$n^{k-1}F_k(z_n) \rightarrow a_k e^t, \quad (4.14)$$

or

$$(k-1)\log n + \log\{F_k(z_n)\} \rightarrow t + \log a_k. \quad (4.15)$$

Put $z_n = \kappa(\log n)^{-s} + \zeta_n(t)$, where $\kappa = g_s(k)$. From Lemma 4.1, we find that (4.15) is equivalent to

$$\begin{aligned} & (k-1)\log n - (k-1)p/2\log(\kappa(\log n)^{-s} + \zeta_n(t)) \\ & - k^{p+1}(\kappa(\log n)^{-s} + \zeta_n(t))^{-p} \rightarrow t. \end{aligned} \quad (4.16)$$

Writing

$$\kappa(\log n)^{-s} + \zeta_n(t) = \kappa(\log n)^{-s}(1 + \zeta_n(t)(\log n)^s/\kappa),$$

yields

$$\begin{aligned} & (k-1)\log n - (k-1)p/2\log(\kappa(\log n)^{-s}(1 + \zeta_n(t)(\log n)^s/\kappa)) \\ & - k^{p+1}\kappa^{-p}\log n \cdot (1 + \zeta_n(t)(\log n)^s/\kappa)^{-p} \rightarrow t. \end{aligned}$$

Using that $k^{p+1}\kappa^{-p} = k - 1$ and $ps = 1$, we arrive at

$$\begin{aligned} & (k-1)\log n + (k-1)/2\log(\log n) - (k-1)p/2\log(1 + \zeta_n(t)(\log n)^s/\kappa) \\ & - (k-1)\log n \cdot (1 + \zeta_n(t)(\log n)^s/\kappa)^{-p} \rightarrow t + (k-1)p/2\log \kappa. \end{aligned}$$

Now we choose

$$\zeta_n(t) = (\log n)^{-s-1} \cdot (\zeta \log \log n + h(t)) \quad \text{or} \quad \zeta_n(t)(\log n)^s = \frac{(\zeta \log \log n + h(t))}{\log n}. \quad (4.17)$$

Then

$$(k-1)p/2\log(1 + \zeta_n(t)(\log n)^s/\kappa) = O\left(\frac{\log \log n}{\log n}\right) \rightarrow 0,$$

and

$$\begin{aligned} -(k-1)\log n \cdot (1 + \zeta_n(t)(\log n)^s/\kappa)^{-p} & \sim -(k-1)\log n \cdot (1 - p\zeta_n(t)(\log n)^s/\kappa) \\ & = -(k-1)\log n + \frac{(k-1)p}{\kappa}(\zeta \log \log n + h(t)), \end{aligned}$$

resulting in

$$\zeta = -\kappa/2p \quad \text{and} \quad \frac{(k-1)ph(t)}{\kappa} = t + \frac{1}{2}(k-1)p\log \kappa. \quad (4.18)$$

Hence, we can choose

$$\begin{aligned} z_n(t) & = g_s(k)(\log n)^{-s} + \zeta_n(t) \\ & = g_s(k)(\log n)^{-s} + (\log n)^{-s-1} \cdot (\zeta \log \log n + h(t)) \\ & = \frac{g_s(k)}{(\log n)^s} + \frac{g_s(k)}{(\log n)^{s+1}} \left[-\frac{\log \log n}{2p} + \frac{t}{(k-1)p} + \frac{\log g_s(k)}{2} \right]. \quad \square \end{aligned} \quad (4.19)$$

We now turn to the proof of Theorem 2.1 in the case where $0 < s \leq 1$:

Proof of Theorem 2.1 in case $0 < s \leq 1$. Observe from Theorem 4.2 that for any K , w.h.p. and uniform in $k \in \{2, 3, \dots, K\}$,

$$(\log n)^s W_k(n) \geq (1 - \varepsilon)g_s(3) > g_s(2), \quad (4.20)$$

where the latter inequality follows since for the indicated values of s , the function g_s is increasing on $[2, \infty)$ and where we can take $\varepsilon < \min_{0 < s \leq 1} [1 - g_s(2)/g_s(3)] = 1/9$. On the complete graph with n vertices the paths of length 2 have *independent* total weight, since they are disjoint. The number of paths of length 2 is equal to $n - 2 \sim n$, so that we can conclude from the Theorem 4.3 that, for any $\varepsilon > 0$ and w.h.p.,

$$(\log n)^s W_2(n) \in (g_s(2) - \varepsilon, g_s(2) + \varepsilon). \quad (4.21)$$

From (4.20) and (4.21) it is immediate that, w.h.p., the minimal-weight path is either a path of length 2 or has a length exceeding K . Since K can be taken arbitrary large and the hopcount H_n is tight (see 4.7), we conclude that

$$H_n(s) \xrightarrow{\mathbb{P}} 2$$

for $0 < s \leq 1$. Consequently, $W_n = W_n(2)$, w.h.p., and statement (b) of Theorem 2.1 follows from (4.13) for $0 < s \leq 1$ and $k = 2$. \square

4.4. The case $s > 1$

In this section we treat the case $s > 1$. The number of paths with $k \geq 2$ edges between the vertices 1 and n is equal to $\prod_{j=2}^k (n-j) \sim n^{k-1}$. Let $\mathcal{S}_k(n)$ denote the set of all such paths. As before, we let F_k denote the distribution function of the sum of k independent random variables each with distribution equal to the distribution of E^{-s} , and by $N_k^{(n)}(z)$, $z > 0$, the number of paths with k edges which have total weight $w_s(\mathcal{P}) = \sum_{e \in \mathcal{P}} E_e^{-s}$ less than z . Recall the definition of $z_n(t)$ in (4.19). From Theorem 4.3 and its proof (compare (4.14)), we conclude that, as $n \rightarrow \infty$,

$$\lambda_k^{(n)}(t) := \mathbb{E}[N_k^{(n)}(z_n(t))] \sim n^{k-1} F_k(z_n(t)) \rightarrow a_k e^t. \quad (4.22)$$

We shall prove the following proposition:

Proposition 4.4 (Poisson approximation for small weight paths). *Fix $s > 1$ and let $\text{Poi}_k^{(n)}(t)$ be a Poisson random variable with mean $\lambda_k^{(n)}(t)$. Then, both for $k = \lfloor s + 1 \rfloor$ and $k = \lceil s + 1 \rceil$, as $n \rightarrow \infty$,*

$$d_{\text{TV}}(N_k^{(n)}(z_n(t)), \text{Poi}_k^{(n)}(t)) \rightarrow 0,$$

where d_{TV} denotes the total variation distance.

Assuming the proposition let us first show how to complete the proof of Theorem 2.1.

Proof of Theorem 2.1 in case $s > 1$ and $s \notin \mathcal{S}$. Observe that

$$\mathbb{P}(W_k(n) > z_n(t)) = \mathbb{P}(N_k^{(n)}(z_n(t)) = 0). \quad (4.23)$$

Now Proposition 4.4 together with (4.22) imply that, for $k = \lfloor s + 1 \rfloor$ and $k = \lceil s + 1 \rceil$, as $n \rightarrow \infty$,

$$\mathbb{P}(W_k(n) > z_n(t)) \rightarrow \exp(-a_k e^t). \quad (4.24)$$

Note that the weak convergence shows in particular that $(\log n)^s W_k(n)$ converges in probability to $g_s(k)$ for the two indicated values of k . This together with the lower bound proven in Theorem 4.2, and an argument similar to the case $0 < s \leq 1$ then completes

the proof of Theorem 2.1, in case the integer that minimizes $g_s(x)$ is unique, that is, in case $s \notin \mathcal{S}$. \square

Proof of Proposition 4.4. We shall use [5], Theorem 1.A. Before quoting this result, we shall need to setup some notation. Let \mathcal{I} be a finite index set, and let $\{I_\alpha : \alpha \in \mathcal{I}\}$ be a family of indicator random variables and write $p_\alpha = \mathbb{E}[I_\alpha]$. Let

$$W = \sum_{\alpha \in \mathcal{I}} I_\alpha, \quad \lambda = \mathbb{E}[W] = \sum_{\alpha \in \mathcal{I}} p_\alpha.$$

Now suppose for each α we can decompose the index set \mathcal{I} as $\mathcal{I} = \{\alpha\} \cup \mathcal{I}^*(\alpha) \cup \mathcal{S}^*(\alpha)$, where we shall think of $\{I_\beta : \beta \in \mathcal{I}^*(\alpha)\}$ to be the set of random variables which “strongly depend” on I_α while $\{I_{\beta'} : \beta' \in \mathcal{S}^*(\alpha)\}$ consists of the set of random variables which only “weakly depend” on I_α . Now let $Z_\alpha = \sum_{\beta \in \mathcal{I}^*(\alpha)} I_\beta$, while

$$Y_\alpha = W - I_\alpha - Z_\alpha = \sum_{\beta' \in \mathcal{S}^*(\alpha)} I_{\beta'}.$$

Then with this notation, the following is just one example of the power of the Stein–Chen machinery for Poisson approximation for weakly dependent indicator random variables:

Theorem 4.5 (Stein–Chen Poisson approximation ([5], Theorem 1.A)). *With the above notation,*

$$d_{\text{TV}}(W, \text{Poi}(\lambda)) \leq \min(1, \lambda^{-1}) \sum_{\alpha \in \mathcal{I}} (p_\alpha^2 + p_\alpha \mathbb{E}[Z_\alpha] + \mathbb{E}[I_\alpha Z_\alpha]) + \min(1, \lambda^{-1/2}) \sum_{\alpha \in \mathcal{I}} \eta_\alpha,$$

where η_α is such that

$$|\mathbb{E}[I_\alpha g(Y_\alpha + 1)] - p_\alpha \mathbb{E}[g(Y_\alpha + 1)]| \leq \eta_\alpha \|g\|, \quad \alpha \in \mathcal{I}$$

for all bounded functions g on \mathbb{Z}_+ , and where $\|\cdot\|$ is the supremum norm.

To apply Theorem 4.5 to the situation at hand, we take $\mathcal{I} = \mathcal{S}_k(n)$, the set of paths between 1 and n having precisely k edges. For $\alpha \in \mathcal{S}_k(n)$, we denote by

$$I_\alpha = I_\alpha(z_n(t)) = \mathbb{1}_{\{w_s(\alpha) \leq z_n(t)\}}, \quad (4.25)$$

where, as before, $w_s(\alpha) = \sum_{e \in \alpha} E_e^{-s}$ denotes the weight of the path α , and where $\mathbb{1}_A$ denotes the indicator of event A . Furthermore, let $p_k^{(n)}(t)$ denote the expectation of $I_\alpha(z_n(t))$, that is,

$$p_k^{(n)}(t) = \mathbb{P}(w_s(\alpha) \leq z_n(t)) = F_k(z_n(t)). \quad (4.26)$$

Let $\mathcal{I}^*(\alpha) \subseteq \mathcal{S}_k(n)$ denote the set of paths (not including α) which have at least one edge in common with α (i.e., $\mathcal{I}^*(\alpha)$ is the set of paths β for which I_β is “strongly” dependent on I_α), and let $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}_k(n)$ denote the set of paths that do not overlap on any edge with α . Note that the random variable $w_s(\alpha)$ is independent of $\{w_s(\beta) : \beta \in \mathcal{S}^*(\alpha)\}$.

Finally, in the above notation, note that

$$Z_\alpha = \sum_{\beta \in \mathcal{I}^*(\alpha)} \mathbb{1}_{\{w_s(\beta) \leq z_n(t)\}}.$$

The independence of $w_s(\alpha)$ and $\{w_s(\beta): \beta \in \mathcal{S}^*(\alpha)\}$ implies that we can take $\eta_\alpha = 0$ in Theorem 4.5. Thus applying Theorem 4.5, we get

$$\begin{aligned} d_{\text{TV}}(N_k^{(n)}(z_n(t)), \text{Poi}_k^{(n)}(t)) &\leq \frac{\sum_{\alpha \in \mathcal{S}_k(n)} [(p_k^{(n)}(t))^2 + p_k^{(n)}(t) \mathbb{E}[Z_\alpha] + \mathbb{E}[I_\alpha Z_\alpha]]}{\lambda_k^{(n)}(t)} \\ &= p_k^{(n)}(t) + \mathbb{E}[Z_\alpha] + \frac{\mathbb{E}[I_\alpha Z_\alpha]}{p_k^{(n)}(t)}, \end{aligned} \quad (4.27)$$

where the last equality follows since $\lambda_k^{(n)}(t) = |\mathcal{S}_k(n)| p_k^{(n)}(t)$ and since $\mathbb{E}[Z_\alpha]$ and $\mathbb{E}[I_\alpha Z_\alpha]$ are independent of α . As before, by the choice of $z_n(t)$,

$$n^{k-1} p_k^{(n)}(t) \rightarrow a_k e^t.$$

Thus, in particular, $p_k^{(n)}(t) \rightarrow 0$, as $n \rightarrow \infty$. Further, there exists a constant C_k such that, as $n \rightarrow \infty$,

$$\mathbb{E}[Z_\alpha] = |\mathcal{I}^*(\alpha)| p_k^{(n)}(t) \leq C_k n^{k-2} p_k^{(n)}(t) \rightarrow 0.$$

Thus, the first two terms in (4.27) vanish as $n \rightarrow \infty$. The last term requires some more analysis. We note that

$$\mathbb{E}[I_\alpha Z_\alpha] = \sum_{j=1}^{k-2} |\mathcal{I}_{k,j}^*(\alpha)| p_{k,j}^{(n)}(t). \quad (4.28)$$

Here $\mathcal{I}_{k,j}^*(\alpha) \subseteq \mathcal{S}_k(n)$ consists of the set of paths of length k which overlap with α in exactly j edges, while

$$p_{k,j}^{(n)}(t) = \mathbb{P}(X_{k,k} \leq z_n(t), X_{k,j} \leq z_n(t)),$$

where $X_{k,k} = \sum_{r=1}^k E_r^{-s}$, while $X_{k,j} = \sum_{r=1}^j E_r^{-s} + \sum_{r=j+1}^k \tilde{E}_r^{-s}$, $1 \leq j \leq k-2$ and $(E_i)_{i=1}^k$ and $(\tilde{E}_r)_{r=1}^k$ are two independent vectors of i.i.d. $\text{Exp}(1)$ random variables. We bound the probability $p_{k,j}^{(n)}(t)$ in the same way as before, using the standard Laplace's method:

Lemma 4.6 (Correlated sums of inverse powers of exponentials). *Fix $k \geq 3$, and let $1 \leq i \leq k-2$. Then, for $z \downarrow 0$,*

$$p_{k,j}^{(n)}(t) = \mathbb{P}(X_{k,k} \leq z, X_{k,i} \leq z) \sim C_{k,i} \frac{1}{z^{(k-i-1)p+ip/2}} \exp(-z^{-p}[(k-i)\nu + i]^{p+1}),$$

where $\nu = 2^{1/(p+1)}$ and $C_{k,i} > 0$ is a constant.

Proof. The proof is given by straightforward computation using Laplace's method:

$$\begin{aligned}
& \mathbb{P}(X_{k,k} \leq z, X_{k,i} \leq z) \\
&= \mathbb{P}\left(\sum_{r=1}^k E_r^{-s} \leq z, \sum_{r=1}^i E_r^{-s} + \sum_{r=i+1}^k (\tilde{E}_r)^{-s} \leq z\right) = \int_0^z F_{k-i}^2(z-y) dF_i(y) \\
&\sim a_i a_{k-i}^2 \int_0^z (z-y)^{-(k-i-1)p} e^{-2(k-i)^{p+1}(z-y)^{-p}} dy^{-(i-1)p/2} e^{-i^{p+1}y^{-p}} \\
&= a_i a_{k-i}^2 \int_0^z y^{-(i-1)p/2-1} e^{-i^{p+1}y^{-p}} (pi^{p+1}y^{-p} - (i-1)p/2) \\
&\quad \times (z-y)^{-(k-i-1)p} e^{-2(k-i)^{p+1}(z-y)^{-p}} dy \\
&= a_i a_{k-i}^2 z^{-(k-i-1)p-(i-1)p/2} \int_0^1 x^{-(i-1)p/2-1} (pi^{p+1}x^{-p}z^{-p} - (i-1)p/2) \\
&\quad \times (1-x)^{-(k-i-1)p} \exp\{-z^{-p}h_{k,i}(x)\} dx,
\end{aligned}$$

where we abbreviate

$$h_{k,i}(x) = i^{p+1}x^{-p} + 2(k-i)^{p+1}(1-x)^{-p}.$$

Put $\nu = 2^{1/(p+1)}$. Then the minimum arises in the point $x_{k,i}$, satisfying $h'_{k,i}(x_{k,i}) = 0$, which yields

$$x_{k,i} = \frac{i}{(k-i)\nu + i}.$$

Furthermore, $h_{k,i}(x_{k,i}) = ((k-i)\nu + i)^{p+1}$, while

$$h''_{k,i}(x_{k,i}) = \frac{p(p+1)}{i(k-i)\nu} ((k-i)\nu + i)^{p+3}.$$

Applying Laplace's method then yields

$$\begin{aligned}
& \mathbb{P}(X_{k,k} \leq z, X_{k,i} \leq z) \\
&\sim a_i a_{k-i}^2 z^{-(k-i-1)p-(i-1)p/2} (x_{k,i})^{-(i-1)p/2-1} (pi^{p+1}(zx_{k,i})^{-p} - (i-1)p/2) \quad (4.29) \\
&\quad \times (1-x_{k,i})^{-(k-i-1)p} \exp\{-z^{-p}h_{k,i}(x_{k,i})\} \sqrt{\frac{2\pi}{z^{-p}h''_{k,i}(x_{k,i})}}. \quad \square
\end{aligned}$$

Recall (4.27). The first two terms on the right-hand side vanish, as $n \rightarrow \infty$, hence it suffices to show that

$$\frac{\mathbb{E}[I_\alpha Z_\alpha]}{p_k^{(n)}(t)} \rightarrow 0.$$

Using that $p_k^{(n)}(t) = O(n^{-(k-1)})$ and that $|\mathcal{I}_{k,j}^*(\alpha)| \sim n^{k-j-1}$ it follows from (4.28) that we now need to show for $1 \leq j \leq k-2$,

$$n^{2k-j-2} p_{k,j}^{(n)}(t) \rightarrow 0,$$

as $n \rightarrow \infty$. Now the polynomial terms (z^k type terms) in the approximation of $p_{k,j}^{(n)}(t)$ should not play a role. Thus, using the fact that up to the first-order

$$(z_n(t))^{-p} \sim \left(\frac{g_s(k)}{\log^s n} \right)^{-p} = \frac{(k-1) \log n}{k^{1+p}}, \quad (4.30)$$

we need to show that for $1 \leq j \leq k-2$ and with $\nu = 2^{1/(p+1)}$,

$$\left[\left(\left(1 - \frac{j}{k} \right) \nu + \frac{j}{k} \right)^{p+1} - \left(2 - \frac{j}{k-1} \right) \right] > 0. \quad (4.31)$$

The above inequality is not true for s close to 0 and larger values of k . However, it is true for $s > 1$ and $k \in \{\lfloor s+1 \rfloor, \lceil s+1 \rceil\}$ as we will now show. Indeed, define, for $x \in [0, 1]$,

$$u_k(x) = [(1-x)2^{s/(s+1)} + x]^{(1+1/s)} - \left(2 - \frac{k}{k-1}x \right),$$

and note that $u_k(j/k)$ is equal to the left-hand side of (4.31). Hence, if we show that when $s > 1$ for both $k = \lfloor s+1 \rfloor$ and $k = \lceil s+1 \rceil$, the function $u_k(x) > 0$ for all $x \in (0, 1)$, then we are done. Differentiating $x \mapsto u_k(x)$ with respect to x yields

$$u'_k(x) = -a[(1-x)2^{1/a} + x]^{a-1}(2^{1/a} - 1) + \frac{k}{k-1},$$

where $a = (s+1)/s > 1$. The function u'_k is increasing as can easily be seen from the second derivative

$$u''_k(x) = a(a-1)(1-2^{1/a})^2[(1-x)2^{1/a} + x]^{a-2} > 0.$$

Hence, since $u_k(0) = 0$, it suffices to show that $u'_k(0) > 0$, for the two indicated values of k .

Claim. Fix $s > 1$, then the statement $u'_k(0) > 0$ is true for both $k = \lfloor s+1 \rfloor$ and $k = \lceil s+1 \rceil$.

Proof. Since $a = (s+1)/s$, the condition $u'_k(0) > 0$ is equivalent to

$$sk(2^{1/(s+1)} - 1) > (k - (s+1))(2 - 2^{1/(s+1)}).$$

This inequality is trivially true for $k = \lfloor s+1 \rfloor$, since then the right-hand side is smaller than or equal to 0, whereas the left-hand side is positive. We now turn to the case where

$k = \lceil s + 1 \rceil$. Since $2^{1/(s+1)} \geq 1$ and $\lceil s + 1 \rceil - (s + 1) \leq 1$, the right-hand side is bounded by 1, that is,

$$(\lceil s + 1 \rceil - (s + 1))(2 - 2^{1/(s+1)}) \leq 1, \quad s > 1.$$

A lower bound for the left-hand side on the domain $s > 1$, is attained in the limit as $s \downarrow 1$ and equals $3(\sqrt{2} - 1) = 1.2426\dots$, that is,

$$sk(2^{1/(s+1)} - 1) = s\lceil s + 1 \rceil(2^{1/(s+1)} - 1) \geq 3(\sqrt{2} - 1), \quad s > 1.$$

This shows that the above claim holds and hence that the Poisson approximation holds both for $k = \lfloor s + 1 \rfloor$ and $k = \lceil s + 1 \rceil$. \square

With the verification of the above claim the proof of Proposition 4.4 is complete. \square

4.5. The case $s \in \mathcal{S}$, the special set

In this section, we will prove Theorem 2.2. To this end, we fix $s_j \in \mathcal{S}$ and write $k = \lfloor s_j + 1 \rfloor$, so that $k + 1 = \lceil s_j + 1 \rceil$. Let $N_k^{(n)} = N_k^{(n)}(z_n(x))$ denote the number of paths from 1 to n of length k and with weight at most $z_n(x)$, with $z_n(\cdot)$ given by (4.19), and similarly we denote by $M_k^{(n)} = M_k^{(n)}(z_n(y))$ the number of paths from 1 to n of length $k + 1$ and with weight at most $z_n(y)$, where $z_n(y)$ is given by the right-hand side of (4.19), with t replaced by y and k by $k + 1$. Note that the change from k to $k + 1$ is for many aspects irrelevant, because for $s = s_j$, we have $g_s(k) = g_s(k + 1)$. We are therefore, in particular, allowed to use the same quantity $z_n(y)$ in the definition of $M_k^{(n)}$. We show below that the total variation distance between $N_k^{(n)} + M_k^{(n)}$ and a Poisson variable with mean $\mu_k^{(n)}(x, y) = \mathbb{E}[N_k^{(n)} + M_k^{(n)}]$ converges to 0 as $n \rightarrow \infty$; that is, we show that

$$d_{\text{TV}}(N_k^{(n)} + M_k^{(n)}, \text{Poi}(\mu_k^{(n)}(x, y))) \rightarrow 0. \quad (4.32)$$

Let us first prove that (4.32) implies Theorem 2.2.

Proof of Theorem 2.2 assuming (4.32). The convergence in total variation in (4.32) implies that

$$\begin{aligned} \mathbb{P}(W_k(n) > z_n(x), W_{k+1}(n) > z_n(y)) &= \mathbb{P}(N_k^{(n)} = 0, M_k^{(n)} = 0) = \mathbb{P}(N_k^{(n)} + M_k^{(n)} = 0) \\ &\rightarrow \mathbb{P}(\text{Poi}(\mu_k(x, y)) = 0), \end{aligned}$$

where, by (4.22), $\mu_k(x, y) = \lim_{n \rightarrow \infty} \mu_k^{(n)}(x, y) = \lim_{n \rightarrow \infty} \mathbb{E}[N_k^{(n)}] + \mathbb{E}[M_k^{(n)}] = \lambda_k(x) + \lambda_{k+1}(y)$, and where we define

$$\lambda_l(z) = a_l e^z, \quad l \geq 1, z \in \mathbb{R}. \quad (4.33)$$

Thus, comparing with (4.23),

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}(W_k(n) > z_n(x), W_{k+1}(n) > z_n(y)) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}(W_k(n) > z_n(x)) \lim_{n \rightarrow \infty} \mathbb{P}(W_{k+1}(n) > z_n(y)), \end{aligned} \quad (4.34)$$

and consequently, we see that the events $\{W_k(n) > z_n(x)\}$ and $\{W_{k+1}(n) > z_n(y)\}$ are asymptotically independent. It is then straightforward to conclude that the minimum of the normalized pair $(W_k(n), W_{k+1}(n))$, where the normalization is as in the left-hand side of part (a) of Theorem 2.2 converges in distribution to the minimum of the *independent* pair

$$(\Xi_k/(k-1), \Xi_{k+1}/k).$$

The lower bound for $(\log n)^s W_k(n)$ of Theorem 4.2 and the tightness of H_n (see (4.7)) again completes the proof of part (b), the hopcount part, and subsequently also part (a), of Theorem 2.2. \square

In order to prove (4.32), we again rely on the Poisson approximation in [5]. Set $\mathcal{T}_k(n) = \mathcal{S}_k(n) \cup \mathcal{S}_{k+1}(n)$, the index set of all paths from 1 to n having either k or $k+1$ edges, where, as before, $k = \lfloor s_j + 1 \rfloor$. To denote that the length of a path is equal to k , we give it a subscript k and write α_k for an element of $\mathcal{S}_k(n)$. For $\alpha_k \in \mathcal{S}_k(n)$, we denote by

$$I_{\alpha_k} = I_{\alpha_k}(z_n(x)) = \mathbb{1}_{\{w_s(\alpha_k) \leq z_n(x)\}},$$

whereas for a path $\alpha_{k+1} \in \mathcal{S}_{k+1}(n)$, we define

$$I_{\alpha_{k+1}} = I_{\alpha_{k+1}}(z_n(y)) = \mathbb{1}_{\{w_s(\alpha_{k+1}) \leq z_n(y)\}},$$

so that

$$\begin{aligned} p_k^{(n)}(x) &= \mathbb{P}(w_s(\alpha_k) \leq z_n(x)) = F_k(z_n(x)) \\ p_{k+1}^{(n)}(y) &= \mathbb{P}(w_s(\alpha_{k+1}) \leq z_n(y)) = F_{k+1}(z_n(y)). \end{aligned}$$

Writing α for α_k or α_{k+1} , we denote by $\mathcal{I}^*(\alpha) \subseteq \mathcal{T}_k(n)$ the set of paths (not including α) which have at least one edge in common to α , and by $\mathcal{S}^*(\alpha) \subseteq \mathcal{T}_k(n)$ the set of paths that do not overlap on any edge with α . Finally, let

$$Z_\alpha = \sum_{\beta_k \in \mathcal{I}^*(\alpha)} \mathbb{1}_{\{w_s(\beta_k) \leq z_n(x)\}} + \sum_{\beta_{k+1} \in \mathcal{I}^*(\alpha)} \mathbb{1}_{\{w_s(\beta_{k+1}) \leq z_n(y)\}}.$$

The total variation distance in (4.32) is bounded by

$$\begin{aligned} & \frac{\sum_{\alpha \in \mathcal{S}_k(n)} [(p_k^{(n)}(x))^2 + p_k^{(n)}(x) \mathbb{E}[Z_\alpha] + \mathbb{E}[I_\alpha Z_\alpha]]}{\mu_k^{(n)}(x, y)} \\ & + \frac{\sum_{\alpha \in \mathcal{S}_{k+1}(n)} [(p_{k+1}^{(n)}(y))^2 + p_{k+1}^{(n)}(y) \mathbb{E}[Z_\alpha] + \mathbb{E}[I_\alpha Z_\alpha]]}{\mu_k^{(n)}(x, y)}. \end{aligned}$$

Since

$$\mu_k^{(n)}(x, y) = p_k^{(n)}(x)|\mathcal{S}_k(n)| + p_{k+1}^{(n)}(y)|\mathcal{S}_{k+1}(n)| \geq \max\{p_k^{(n)}(x)|\mathcal{S}_k(n)|, p_{k+1}^{(n)}(y)|\mathcal{S}_{k+1}(n)|\},$$

we conclude from the proof of Proposition 4.4 that

$$\frac{\sum_{\alpha \in \mathcal{S}_k(n)} [(p_k^{(n)}(x))^2 + p_k^{(n)}(x)\mathbb{E}[Z_\alpha]]}{\mu_k^{(n)}(x, y)} + \frac{\sum_{\alpha \in \mathcal{S}_{k+1}(n)} [(p_{k+1}^{(n)}(y))^2 + p_{k+1}^{(n)}(y)\mathbb{E}[Z_\alpha]]}{\mu_k^{(n)}(x, y)} \rightarrow 0.$$

Hence, it remains to prove that

$$\frac{\sum_{\alpha \in \mathcal{S}_k(n)} \mathbb{E}[I_\alpha Z_\alpha] + \sum_{\alpha \in \mathcal{S}_{k+1}(n)} \mathbb{E}[I_\alpha Z_\alpha]}{\mu_k^{(n)}(x, y)} \rightarrow 0. \quad (4.35)$$

We next decompose $\mathbb{E}[I_\alpha Z_\alpha]$ into the part where β has k or $k+1$ edges, that is,

$$\mathbb{E}[I_\alpha Z_\alpha] = \sum_{\beta_k \in \mathcal{I}^*(\alpha)} \mathbb{P}(I_\alpha = 1, I_{\beta_k} = 1) + \sum_{\beta_{k+1} \in \mathcal{I}^*(\alpha)} \mathbb{P}(I_\alpha = 1, I_{\beta_{k+1}} = 1).$$

By making this decomposition, as well as differentiating between the number of edges of α , the numerator in (4.35) splits into 4 different double sums. The two double sums running over the index sets $\alpha \in \mathcal{S}_k(n), \beta_k \in \mathcal{I}^*(\alpha)$ and $\alpha \in \mathcal{S}_{k+1}(n), \beta_{k+1} \in \mathcal{I}^*(\alpha)$ are treated in the proof of Proposition 4.4, apart from the small change that $z_n(x)$ and $z_n(y)$ are now possibly *different*. Since $z_n(x)/z_n(y) \rightarrow 1$, it is straightforward to adapt the argument. Below, we will show that

$$\frac{\sum_{\alpha \in \mathcal{S}_k(n)} \sum_{\beta_{k+1} \in \mathcal{I}^*(\alpha)} \mathbb{P}(I_\alpha = 1, I_{\beta_{k+1}} = 1)}{\mu_k^{(n)}(x, y)} \rightarrow 0. \quad (4.36)$$

The terms with $\alpha \in \mathcal{S}_{k+1}(n)$ and $\beta_k \in \mathcal{I}^*(\alpha)$ are identical, apart from the fact that x and y are interchanged. Thus, (4.36) completes the proof of (4.32).

To prove (4.36), we write, as in (4.28),

$$\sum_{\beta_{k+1} \in \mathcal{I}^*(\alpha)} \mathbb{P}(I_\alpha = 1, I_{\beta_{k+1}} = 1) = \sum_{j=1}^{k-1} |\mathcal{I}_{k+1,j}^*(\alpha)| p_{k+1,j}^{(n)}(x, y),$$

where $\mathcal{I}_{k+1,j}^*(\alpha) \subset \mathcal{T}_k(n)$ consists of the set of paths of length $k+1$ which overlap with α , which has length k , in exactly j edges, while

$$p_{k+1,j}^{(n)}(x, y) = \mathbb{P}(X_{k,k} \leq z_n(x), X_{k+1,j} \leq z_n(y)),$$

where, similarly as in the proof of Proposition 4.4, we now write $X_{k,k} = \sum_{r=1}^k E_r^{-s}$, while $X_{k+1,j} = \sum_{r=1}^j E_r^{-s} + \sum_{r=j+1}^{k+1} \tilde{E}_r^{-s}$, $1 \leq j \leq k-1$.

By adapting the Laplace method-type argument used in the proof of Lemma 4.6, it is readily verified that for $z_1, z_2 \downarrow 0$ such that $\lim_{z_1 \rightarrow 0} z_2/z_1 = 1$, and $k \geq 3$ and $1 \leq j \leq k-1$, we have

$$\mathbb{P}(X_{k,k} \leq z_1, X_{k+1,j} \leq z_2) = \exp(-z_1^{-p}[(k-j)\nu + j + 1]^{p+1}(1 + o(1))), \quad (4.37)$$

where, as before, $\nu = 2^{1/(p+1)}$. By (4.30) $(z_n(x))^{-p} \sim (k-1) \log n / (k^{1+p})$. Further, since $s \in \mathcal{S}$, we have that $(z_n(y))^{-p} \sim \log n / ((k+1)^{1+p}) = (k-1) \log n / (k^{1+p})$. Thus, $\lim_{n \rightarrow \infty} z_n(y)/z_n(x) = 1$, as required.

We conclude that in order for (4.36) to hold, we need to show that $n^{2k-j-1} p_{k+1,j}^{(n)}(t) \rightarrow 0$, or equivalently that

$$\exp\left(-\log n \left[(k-1) \left(\left(1 - \frac{j}{k}\right) \nu + \frac{j+1}{k} \right)^{p+1} - (2k-j-1) \right]\right) \rightarrow 0, \quad 1 \leq j \leq k-1.$$

This follows from the convexity of the function $x \mapsto x^{p+1}$ and the facts that $1 < \nu = 2^{1/(p+1)} < 2$ and $p > 0$, since, for $1 \leq j \leq k-1$,

$$\begin{aligned} \frac{(2k-j-1)}{k-1} &= \left(2 - \frac{j-1}{k-1}\right) \leq \left(\left(1 - \frac{j-1}{k}\right) \nu + \frac{j-1}{k}\right)^{p+1} \\ &= \left(\left(1 - \frac{j}{k}\right) \nu + \frac{j+1}{k} + \frac{\nu-2}{k}\right)^{p+1} < \left(\left(1 - \frac{j}{k}\right) \nu + \frac{j+1}{k}\right)^{p+1}. \end{aligned}$$

This proves (4.36), and thus completes the proof of (4.32).

4.6. Multipoint distance limits

In this section, we indicate how to prove Corollary 2.3 for 2 multipoint distances. The case for general m follows similarly.

More precisely, let $\tilde{W}_n^{(12)}, \tilde{W}_n^{(13)}$ denote the recentered and rescaled optimal weights between 1 and 2 and 1 and 3. Recall, for any fixed $t \in \mathbb{R}$, the function $z_n(t)$ from (4.19), where we take $k = k^*(s)$. For $j = 2, 3$ and any $t \in \mathbb{R}$, let $N_k^{j,(n)}(z_n(t))$ denote the number of paths between 1 and j having k edges whose weight is less than $z_n(t)$.

The proof of Corollary 2.3 will be an adaptation of the proof of Theorem 2.2 in Section 4.5, and we start by recalling some results we have proved and shall rely on. Recall that we have already proved that, as $n \rightarrow \infty$,

$$\lambda_k^{(n)}(t) = \mathbb{E}[N_k^{j,(n)}(z_n(t))] \rightarrow \lambda_k(t),$$

where $\lambda_k(t) = a_k e^t$ is defined in (4.33) and

$$d_{\text{TV}}(N_k^{j,(n)}(z_n(t)), \text{Poi}(\lambda_k(t))) \rightarrow 0.$$

For any fixed $x, y \in \mathbb{R}$, define

$$N_n^* = N_k^{2,(n)}(z_n(x)) + N_k^{3,(n)}(z_n(y)). \quad (4.38)$$

Below, we shall show that

$$N_n^* \xrightarrow{d} \text{Poi}(\lambda_k(x) + \lambda_k(y)). \quad (4.39)$$

Then the argument leading to (4.34) implies that $\tilde{W}_n^{(12)}$ and $\tilde{W}_n^{(13)}$ are asymptotically independent, so that

$$\lim_{n \rightarrow \infty} \mathbb{P}(\tilde{W}_n^{(12)} > x, \tilde{W}_n^{(13)} > y) \rightarrow \exp(-\lambda_k(x) - \lambda_k(y)),$$

establishing the result we want. We next sketch how to prove (4.39).

Sketch of proof of (4.39). Fix any path α with k edges between 1 and 2 and path β with k edges between 1 and 3. Since the argument is quite close to the proof of Theorem 2.2, we shall keep the discussion brief and focus on the differences. We again rely on the total variation bound in Theorem 4.5 that implies

$$d_{\text{TV}}(N_n^*, \text{Poi}(\lambda_k^{(n)}(x) + \lambda_k^{(n)}(y))) \leq \frac{\text{(I)} + \text{(II)} + \text{(III)}}{\lambda_k^{(n)}(x) + \lambda_k^{(n)}(y)}.$$

Here,

$$\begin{aligned} \text{(I)} &= p_k^{(n)}(x)\lambda_k^{(n)}(x) + p_k^{(n)}(y)\lambda_k^{(n)}(y), \\ \text{(II)} &= \lambda_k^{(n)}(x)(\mathbb{E}[Z_\alpha^{(1,2)}] + \mathbb{E}[Z_\alpha^{(1,3)}]) + \lambda_k^{(n)}(y)(\mathbb{E}[Z_\beta^{(1,2)}] + \mathbb{E}[Z_\beta^{(1,3)}]), \\ \text{(III)} &= \mathbb{E}[I_\alpha Z_\alpha^{(1,2)} + I_\alpha Z_\alpha^{(1,3)} + I_\alpha Z_\beta^{(1,2)} + I_\alpha Z_\beta^{(1,3)}], \end{aligned}$$

where, as in (4.25),

$$I_\alpha = \mathbb{1}_{\{|w_s(\alpha)| \leq z_n(x)\}}, \quad I_\beta = \mathbb{1}_{\{|w_s(\beta)| \leq z_n(y)\}},$$

while, writing $\mathcal{I}_{1,2}^*(\alpha)$ for the set of paths from 1 to 2, which overlap with α , and $\mathcal{I}_{1,3}^*(\alpha)$ for the set of paths from 1 to 3, which overlap with α (and similarly for β),

$$Z_\alpha^{(1,2)} = \sum_{\gamma \in \mathcal{I}_{1,2}^*(\alpha)} \mathbb{1}_{\{|w_s(\gamma)| \leq z_n(x)\}}, \quad \text{and} \quad Z_\alpha^{(1,3)} = \sum_{\gamma \in \mathcal{I}_{1,3}^*(\alpha)} \mathbb{1}_{\{|w_s(\gamma)| \leq z_n(y)\}},$$

and similarly for $Z_\beta^{(1,2)}$ and $Z_\beta^{(1,3)}$. Now, we have already shown that the terms (I) and (II) divided by $\lambda_k^{(n)}(x) + \lambda_k^{(n)}(y)$ vanish as $n \rightarrow \infty$. Thus, to complete the proof we just need to show that, as $n \rightarrow \infty$,

$$\frac{\mathbb{E}[I_\alpha Z_\alpha^{(1,2)}]}{\lambda_k^{(n)}(x) + \lambda_k^{(n)}(y)} \rightarrow 0,$$

as well as the corresponding other three terms of (III). This is a minor adaptation of the proof of (4.36), and we omit the details. \square

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References

- [1] ALDOUS, D. and STEELE, J.M. (2004). The objective method: Probabilistic combinatorial optimization and local weak convergence. In *Probability on Discrete Structures. Encyclopaedia Math. Sci.* **110** 1–72. Berlin: Springer. [MR2023650](#)
- [2] ALDOUS, D.J. (2001). The $\zeta(2)$ limit in the random assignment problem. *Random Structures Algorithms* **18** 381–418. [MR1839499](#)
- [3] ALDOUS, D.J. (2010). More uses of exchangeability: Representations of complex random structures. In *Probability and Mathematical Genetics. London Mathematical Society Lecture Note Series* **378** 35–63. Cambridge: Cambridge Univ. Press. [MR2744234](#)
- [4] ALDOUS, D.J., MCDIARMID, C. and SCOTT, A. (2009). Uniform multicommodity flow through the complete graph with random edge-capacities. *Oper. Res. Lett.* **37** 299–302. [MR2573437](#)
- [5] BARBOUR, A.D., HOLST, L. and JANSON, S. (1992). *Poisson Approximation. Oxford Studies in Probability* **2**. New York: The Clarendon Press Oxford Univ. Press. Oxford Science Publications. [MR1163825](#)
- [6] BHAMIDI, S. and VAN DER HOFSTAD, R. (2012). Weak disorder asymptotics in the stochastic mean-field model of distance. *Ann. Appl. Probab.* **22** 29–69.
- [7] BHAMIDI, S., VAN DER HOFSTAD, R. and HOOGHIEMSTRA, G. (2010). First passage percolation on random graphs with finite mean degrees. *Ann. Appl. Probab.* **20** 1907–1965. [MR2724425](#)
- [8] BRAUNSTEIN, L.A., BULDYREV, S.V., COHEN, R., HAVLIN, S. and STANLEY, H.E. (2003). Optimal paths in disordered complex networks. *Phys. Rev. Lett.* **91** 168701.
- [9] BRAUNSTEIN, L.A., WU, Z., CHEN, Y., BULDYREV, S.V., KALISKY, T., SREENIVASAN, S., COHEN, R., LÓPEZ, E., HAVLIN, S. and STANLEY, H.E. (2007). Optimal path and minimal spanning trees in random weighted networks. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **17** 2215–2255. [MR2349740](#)
- [10] FRIEZE, A.M. (1985). On the value of a random minimum spanning tree problem. *Discrete Appl. Math.* **10** 47–56. [MR0770868](#)
- [11] GRIMMETT, G. and KESTEN, H. (1984). Random electrical networks on complete graphs. *J. London Math. Soc. (2)* **30** 171–192. [MR0760886](#)
- [12] HAVLIN, S., BRAUNSTEIN, L.A., BULDYREV, S.V., COHEN, R., KALISKY, T., SREENIVASAN, S. and STANLEY, H.E. (2005). Optimal path in random networks with disorder: A mini review. *Phys. A* **346** 82–92.

- [13] HOWARD, C.D. (2004). Models of first-passage percolation. In *Probability on Discrete Structures. Encyclopaedia Math. Sci.* **110** 125–173. Berlin: Springer. [MR2023652](#)
- [14] JANSON, S. (1999). One, two and three times $\log n/n$ for paths in a complete graph with random weights. *Combin. Probab. Comput.* **8** 347–361. [MR1723648](#)
- [15] KESTEN, H. (1986). Aspects of first passage percolation. In *École D'été de Probabilités de Saint-Flour, XIV—1984. Lecture Notes in Math.* **1180** 125–264. Berlin: Springer. [MR0876084](#)
- [16] SMYTHE, R.T. and WIERMAN, J.C. (1978). *First-passage Percolation on the Square Lattice. Lecture Notes in Math.* **671**. Berlin: Springer. [MR0513421](#)
- [17] SREENIVASAN, S., KALISKY, T., BRAUNSTEIN, L.A., BULDYREV, S.V., HAVLIN, S. and STANLEY, H.E. (2004). Effect of disorder strength on optimal paths in complex networks. *Phys. Rev. E* **70** 46133.

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