Can a combination of the conformal thin-sandwich and puncture methods yield binary black hole solutions in quasi-equilibrium?

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We consider combining two important methods for constructing quasi-equilibrium initial data for binary black holes: the conformal thin-sandwich formalism and the puncture method. The former seeks to enforce stationarity in the conformal three-metric and the latter attempts to avoid internal boundaries, like minimal surfaces or apparent horizons. We show that these two methods make partially conflicting requirements on the boundary conditions that determine the time slices. In particular, it does not seem possible to construct slices that are quasi-stationary and avoid physical singularities and simultaneously are connected by an everywhere positive lapse function, a condition which must obtain if internal boundaries are to be avoided. Some relaxation of these conflicting requirements may yield a soluble system, but some of the advantages that were sought in combining these approaches will be lost.

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I. INTRODUCTION

Binary black holes are expected to produce some of the strongest gravitational wave signals that lie within the pass-bands of both the new generation of terrestrial gravitational wave observatories, such as LIGO, GEO, TAMA, and VIRGO, and the space-based LISA detector. All aspects of the orbital decay and signal history of such decaying binaries are of interest, both for enhancing the prospects of source detection and for determining source parameters [1, 2]. With this need in mind, an intense worldwide theoretical effort is being made to model different phases of the merger of binary black holes.

Considerable interest attaches to the phase between early inspiral and loss of orbital stability and plunge. For astrophysical reasons many binary black holes are expected to enter this phase in quasi-circular orbital motion, with the residual orbital eccentricity following merely from the gradual radiative loss of energy and angular momentum. Hence, early on these quasi-circular orbits are well approximated by positing circular motion and ignoring radiation reaction.

The first numerical models of binary black holes in quasi-circular orbit were obtained by Cook [3] (hereafter C94), who used an effective-potential method to identify those particular data sets that most closely approximated binaries in circular orbits. The initial data were constructed using the conformal-imaging formalism. In that approach, the momentum constraints decouple from the Hamiltonian constraint and can be solved analytically, yielding the Bowen-York conformal extrinsic curvature [4]. The conformal-imaging approach takes as given a conformally-flat metric, a maximal time slice, and a two-sheeted topology where two identical universes are connected by one or more black holes. The identification of the universes manifests as an isometry condition that restricts the form of the solutions. This isometry condition can be used to obtain boundary conditions on the black-hole throats. In particular, a minimal surface boundary condition is obtained for the conformal factor and the Hamiltonian constraint need only be solved on one of the two isometric sheets of the hypersurface. The solution of the Hamiltonian constraint is (near completely) restricted to the exterior of the black holes in one of the two universes and the calculation avoids both coordinate and physical singularities in the black hole interiors.

Brandt and Brügmann [5] suggested an alternative *puncture method* for the construction of spacetimes containing multiple black holes. The momentum constraint is again solved analytically on the conformal manifold, yielding the Bowen-York extrinsic curvature. The conformal factor, however, is split into a sum of an analytically-known singular term and a correction. The analytic term corresponds to the conformal factor of superposed static black holes in isotropic coordinates and captures the topological character of the black holes. The Hamiltonian constraint is then written as an equation for the correction term, which exists for non-vanishing extrin-

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sic curvature. In this method, though, isometry across the throats is not assumed or used. Instead the computational domain is all of $\mathbb{R}^3$ minus the puncture points where the conformal factor is singular. However, by construction, the terms in the Hamiltonian constraint are regular everywhere, allowing the punctures to be ignored. By eliminating the need to excise the black hole interiors and to impose boundary conditions on the black hole throats, the puncture method provides a much simpler computational domain. Baumgarte [6] (hereafter B00) adopted this approach to construct binary black holes in quasi-circular orbits and found very similar results to those of C94. This is not surprising, as the resulting spacetimes only differ in terms of the underlying topology (the puncture method gives rise to $N+1$ asymptotically flat regions with N throats), and the effects can be estimated to be fairly small.

Gourgoulhon, Grandclément and Bonazolla [7, 8] (hereafter GGB02) adopted a conformal thin-sandwich decomposition of the initial value problem (e.g. [9, 10]) to construct binary black hole data. Unlike the methods described above, the conformal thin-sandwich decomposition has a link to dynamics. This is seen in the appearance of the lapse function and shift vector in the conformal thin-sandwich equations, and in the fact that the time rate of change of the conformal three-geometry is part of the freely-specifiable data. In the conformal thin-sandwich approach, the momentum constraint does not yield the Bowen-York extrinsic curvature as a solution. Instead, it becomes an elliptic system on the shift vector, which once solved indirectly determines the extrinsic curvature. As before, the Hamiltonian constraint is an elliptic equation for the conformal factor but the conformal thin-sandwich approach must be supplemented with a condition on the lapse. GGB02 adopted maximal slicing, which is a natural choice for the construction of equilibrium data. They also take the time rate of change of the conformal three geometry to vanish which again is a natural choice for equilibrium data. GGB02 also follow C94 in assuming an isometry across the throats, which allows the black hole interiors to be excised. While this approach is more involved than those of C94 and B00, it has been suggested that the conformal thin-sandwich decomposition is more natural for the construction of binaries in quasi-circular orbit (e.g. [10]), and in fact the results of GGB02 agree more closely with post-Newtonian calculations than do those of C94 and B00 [11, 12]. There is furthermore good evidence that the differences between the results of GGB02 and those of C94 and B00 are due in fact to the differences in initial value decompositions, in particular the behavior of the extrinsic curvature [12, 13, 14].

As we will discuss in more detail in Section III, the imposition of the isometry on the throats leads to a technical problem in the conformal thin-sandwich approach. GGB02 had to regularize their solutions on the throats in a way that introduced a small inconsistency (i.e. a small violation of the constraints) into their solutions. It is thought that this problem may be circumvented with a different set of boundary conditions on the throats [15]. This issue, however, suggests that one might try to combine the thin-sandwich approach with the puncture approach, with the goal of eliminating the need for interior boundary conditions.

We have attempted to combine the conformal thin-sandwich and puncture methods, but have concluded that this approach leads to some conceptional problems and some likely serious numerical obstacles. As we will argue below the approach requires several simultaneous conditions: (1) The construction of quasi-equilibrium data requires the existence of a quasi-stationary coordinate system. (2) The puncture method requires physical fields in the vicinity of each puncture such that each puncture represents a separate spatial infinity on a multiply-connected manifold (i.e., the slice(s) must thread each wormhole and avoid the physical singularities). (3) Finally, the conformal thin-sandwich decomposition (usually) requires that the lapse be everywhere positive, to avoid the existence of internal boundaries necessitating regularity conditions. Unfortunately, even in Schwarzschild spacetime such a slicing cannot exist, suggesting that the construction of black hole equilibrium data with a combined conformal thin-sandwich and puncture method is either impossible or will involve numerical complications that it was initially intended to avoid.

The remainder of the paper is organized as follows. In Section II we review briefly the conformal thin-sandwich decomposition. Section III shows the choice of freely-specifiable data that represents quasi-equilibrium, giving the form of the conformal thin-sandwich equations that must be solved. Section IV then briefly reviews the original puncture method. Based on that discussion, Section V presents how we argue that singular terms can be factored out of the conformal factor and lapse function in the conformal thin-sandwich equations of Section III, allowing those equations to be solved as if the manifold were free of punctures. Freedom exists in the choice of lapse and the remainder of this section focuses on the potential inconsistencies or numerical difficulties in solving the momentum constraints. Finally, Section VI summarizes our conclusions and suggests some avenues for further exploration.

II. THE CONFORMAL THIN-SANDWICH DECOMPOSITION

The spacetime line element can be written in the 3+1 form

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (1)$$

where α is the lapse function, β^i the shift vector, and γ_{ij} the induced metric on a spatial slice Σ. The spatial metric γ_{ij} and the slice's extrinsic curvature K_{ij} satisfy the Hamiltonian constraint

$$R + K^2 - K_{ij}K^{ij} = 0 \quad (2)$$

and the momentum constraint

$$\nabla_j K^{ij} - \gamma^{ij}\nabla_j K = 0 \tag{3}$$

on each slice Σ. Here ∇_i, R_{ij}, and $R = \gamma^{ij}R_{ij}$ are the spatial covariant derivative, Ricci tensor, and scalar curvature associated with γ_{ij} (we have no occasion in this paper to use the analogous four-dimensional quantities). We have assumed vanishing matter sources ($T_{\mu\nu} = 0$). The metric and extrinsic curvature evolve in accordance with

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + \nabla_i\beta_j + \nabla_j\beta_i \tag{4}$$

and

$$\begin{aligned}\partial_t K_{ij} \;&=\; -\nabla_i\nabla_j\alpha + \alpha(R_{ij} - 2K_{ik}K^k_{\ j} + KK_{ij}) \\ &\quad +\beta^k\nabla_k K_{ij} + K_{ik}\nabla_j\beta^k + K_{kj}\nabla_i\beta^k. \end{aligned}\tag{5}$$

The construction of initial-data solutions requires specifying the spatial metric and extrinsic curvature that satisfy the constraint equations (2) and (3). The approach based on conformal decomposition [16] provides a straightforward process to solving these coupled equations. The spatial metric is conformally scaled by

$$\gamma_{ij} \equiv \psi^4\tilde{\gamma}_{ij} \tag{6}$$

(see [17, 18]). Here ψ is the conformal factor and $\tilde{\gamma}_{ij}$ is the conformal or background metric. The extrinsic curvature K_{ij} is split into its trace K and the traceless part A_{ij}, and the latter is then also conformally transformed

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K = \psi^{-2}\tilde{A}_{ij} + \frac{1}{3}\psi^4\tilde{\gamma}_{ij}K. \tag{7}$$

There is no conformal scaling of K. The Hamiltonian constraint (2) can now be written as

$$\tilde{\nabla}^2\psi = \frac{1}{8}\psi\tilde{R} + \frac{1}{12}\psi^5K^2 - \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij}, \tag{8}$$

where $\tilde{\nabla}$, $\tilde{R}_{ij}$, and $\tilde{R} = \tilde{\gamma}^{ij}\tilde{R}_{ij}$ denote the covariant derivative, conformal Ricci tensor, and curvature scalar compatible with the conformal metric $\tilde{\gamma}_{ij}$.

In the conformal thin-sandwich decomposition, both the background metric $\tilde{\gamma}_{ij}$, its time derivative

$$\tilde{u}_{ij} \equiv \partial_t\tilde{\gamma}_{ij}, \tag{9}$$

and the trace of extrinsic curvature K are considered freely-specifiable quantities. The traceless part of the evolution equation (4) for γ_{ij} then yields an equation for $\tilde{A}_{ij}$

$$\tilde{A}_{ij} = \frac{1}{2\tilde{\alpha}}\left[(\tilde{\mathbb{L}}\beta)_{ij} - \tilde{u}_{ij}\right], \tag{10}$$

where $\tilde{\mathbb{L}}$ is a longitudinal operator whose action $(\tilde{\mathbb{L}}\beta)^{ij}$ on the shift yields the symmetrized trace-free gradient,

$$(\tilde{\mathbb{L}}\beta)^{ij} = \tilde{\nabla}^i\beta^j + \tilde{\nabla}^j\beta^i - \frac{2}{3}\tilde{\gamma}^{ij}\tilde{\nabla}_k\beta^k, \tag{11}$$

and where we have introduced the densitized lapse (or slicing function)

$$\tilde{\alpha} = \psi^{-6}\alpha. \tag{12}$$

Inserting (10) into the momentum constraint (3) yields

$$\tilde{\Delta}_{\mathbb{L}}\beta^i - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j\ln\tilde{\alpha} = \frac{4}{3}\tilde{\alpha}\psi^6\tilde{\nabla}^iK + \tilde{\alpha}\tilde{\nabla}_j\left(\frac{1}{\tilde{\alpha}}\tilde{u}_{ij}\right), \tag{13}$$

which can be regarded as an elliptic system for the shift vector β^i. Here the operator $\tilde{\Delta}_{\mathbb{L}}$ (a vector Laplacian) is the divergence of the longitudinal operator $\tilde{\mathbb{L}}$, with the effect

$$\tilde{\Delta}_{\mathbb{L}}\beta^i \equiv \tilde{\nabla}_j(\tilde{\mathbb{L}}\beta)^{ij} = \tilde{\nabla}^2\beta^i + \frac{1}{3}\tilde{\nabla}^i(\tilde{\nabla}_k\beta^k) + \tilde{R}^i_j\beta^j. \tag{14}$$

The shift vector components are conformally invariant ($\beta^i = \tilde{\beta}^i$). Equations (8) and (13) are incomplete in that we have yet to specify how the lapse α is chosen. The condition on the lapse is intertwined with the choice of free data that yield a quasi-equilibrium state, and we take up these issues next.

III. QUASI-EQUILIBRIUM DATA IN THE THIN-SANDWICH APPROACH

The construction of binary black holes in quasi-circular orbit assumes that the emission of gravitational radiation is negligible, so that the spacetime is in quasi-equilibrium when viewed from a co-rotating reference frame. The time coordinate of this reference frame is an approximate helical Killing vector [19].

Our goal is therefore to construct initial data which, when evolved in such a co-rotating reference frame, would lead to metric components that are independent of time. The conformal thin-sandwich decomposition provides a natural framework for the construction of quasi-equilibrium data, since we can explicitly set the time derivative of the conformal metric to zero

$$\tilde{u}_{ij} = 0. \tag{15}$$

We follow C94, B00 and GGB02 and assume conformal flatness, i.e.

$$\tilde{\gamma}_{ij} = f_{ij}, \tag{16}$$

where f_{ij} is a flat metric. It is furthermore reasonable to assume that the time-derivative of the trace of the extrinsic curvature vanishes, $\partial_t K = 0$. With this assumption, the trace of (5) combined with (2) yields

$$\tilde{\nabla}^2(\psi^7\tilde{\alpha}) = \tilde{\alpha}\psi^7\left(\frac{7}{8}\psi^{-8}\tilde{A}_{ij}\tilde{A}^{ij} + \frac{5}{12}\psi^4K^2\right) + \psi^5\beta^l\tilde{\nabla}_lK. \tag{17}$$

In this equation, $\tilde{A}^{ij}$ is now given by

$$\tilde{A}^{ij} = \frac{1}{2\tilde{\alpha}}(\tilde{\mathbb{L}}\beta)^{ij}. \tag{18}$$

We could further follow GGB02 and choose maximal slicing, $K = 0$, but the argument is more general and we merely assume that a sufficiently smooth choice for K is made that is consistent with asymptotic flatness and spatial slices.

Our choices of $\tilde{\gamma}_{ij} = f_{ij}$, $\tilde{u}_{ij} = 0$ and K completely determine all freely-specifiable variables. With these choices, the Hamiltonian constraint (8) reduces to

$$\tilde{\nabla}^2\psi = -\frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{12}\psi^5 K^2 \tag{19}$$

and the momentum constraint (13) reduces to

$$\tilde{\alpha}\tilde{\nabla}_j\left[\frac{1}{\tilde{\alpha}}(\tilde{\mathbb{L}}\beta)^{ij}\right] = \tilde{\Delta}_{\mathbb{L}}\beta^i - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j \ln\tilde{\alpha} = \frac{4}{3}\tilde{\alpha}\psi^6\tilde{\nabla}^i K. \tag{20}$$

Equations (17) through (20) together with appropriate boundary conditions now determine the conformal factor ψ, the lapse α and the shift β^i. This set of equations was derived independently by [20, 21] and has been used in several applications, in particular for the construction of binary neutron stars (see [10, 22] for recent reviews).

The choice of $\tilde{u}_{ij} = 0$ and $\partial_t K = 0$ guarantees a limited sense of stationarity. However, there is no assurance that the other components of the extrinsic curvature or the conformal factor have a vanishing first time derivative. (Were this the case, it would imply exact equilibrium.) Furthermore, there is no guarantee that these time derivatives are even small, in some appropriate norm. There are a couple of issues to consider.

Firstly, the boundary conditions on the problem must be made consistent with the limited stationarity conditions given above. Boundary conditions can be imposed in different ways (e.g., in the puncture method via free parameters at the punctures–see below) but must be reflective of two black holes in instantaneous transverse motion or vanishing motion in a rotating frame. Even then, however, parameter freedom exists and only some parameter choices will lead to black holes in instantaneous force balance as seen in the rotating frame. Most parameter choices will imply an acceleration of the holes away from or toward each other, which will be reflected in the time derivative of $\tilde{A}_{ij}$. All methods to date involve a parameter search and some equilibrium criterion, such as an effective potential minimization or a virial theorem argument.

Secondly, given a solution for ψ, α and β^i following such a parameter search, one could *a posteriori* compute the right hand side of (5) and the trace of (4) to check whether these time derivatives are sufficiently small in an appropriate sense. If they are small, then one has in fact constructed a quasi-equilibrium orbit. If, however, these time derivatives are not small, the initial data may still represent a quasi-equilibrium orbit but in a *dynamical slicing*. Even on a stationary spacetime with a timelike Killing vector, an arbitrary choice of the lapse or shift would lead generically to apparent time-dependence of the metric. In such a dynamical slicing the Killing vector representing the symmetry (a helical Killing vector in the case of binaries) would have an unknown functional form and it is unclear how the symmetry would be established or exploited. For the construction of quasi-equilibrium orbits it therefore seems crucial that the solution be represented in a stationary slicing. The slicing condition (17) may be consistent with stationarity but stationarity also depends on boundary conditions. It is also an open question what advantage might arise in specifying some $K \neq 0$ but any such choice would also need to be consistent with stationarity.

In constructing their binary black hole solutions, GGB02 employed a method that is essentially the conformal thin-sandwich decomposition as outlined in Section II and this section. They demanded additionally that their solutions be inversion symmetric, i.e., that the isometry conditions across the throats be satisfied as in C94. One choice for inversion symmetry of the lapse is to require, as did GGB02, that $\tilde{\alpha} = 0$ on the throats. This condition clearly leads to problems in constructing the extrinsic curvature via Eq. (18) unless $(\tilde{\mathbb{L}}\beta)^{ij} = 0$ on the throats as well. This latter condition is not the natural isometry condition on the shift, nor is it the boundary condition used in GGB02. Rather, GGB02 enforced this extra condition, over-determining the shift, and leading to a small inconsistency in their numerical solution. It may be that these problems can be resolved by adopting an alternative set of boundary conditions representing black holes in equilibrium [15].

Alternatively, one might hope to avoid interior boundary conditions altogether by combining the conformal thin-sandwich approach with an extension of the original puncture method. To set the stage for that discussion, we first review briefly the puncture method.

IV. THE ORIGINAL PUNCTURE METHOD

The puncture method can be motivated by considering the Brill-Lindquist [23] solution for multiple black holes at a moment of time symmetry ($K = 0, \tilde{A}_{ij} = 0$). In isotropic coordinates $\{x^i\}$, the conformal factor can be written as

$$\psi = 1 + \sum_n^N \frac{\mathcal{M}_n}{2|\mathbf{x} - \mathbf{x}_n|} = 1 + \sum_n^N \frac{\mathcal{M}_n}{2r_n}. \tag{21}$$

This is a solution of the Hamiltonian constraint (19) for N black holes at locations $\mathbf{x}_n$, each parameterized by a mass $\mathcal{M}_n$. For convenience we define

$$\frac{1}{\rho} \equiv \sum_n^N \frac{\mathcal{M}_n}{2r_n} \tag{22}$$

so that the above solution is simply $\psi = 1 + 1/\rho$. Brandt and Brügmann [5] noted that for black holes with nonzero linear and angular momentum ($\tilde{A}_{ij} \neq 0$), the conformal factor can be written as

$$\psi = 1 + \frac{1}{\rho} + u, \tag{23}$$

where u is a correction term (our definition of u corresponds to $u-1$ in [5] and B00). Having factored out $1/\rho$, the Hamiltonian constraint becomes an equation for u,

$$\nabla^2 u = -\frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{12}\psi^5 K^2. \tag{24}$$

Provided the trace-free extrinsic curvature $\tilde{A}^{ij}$ diverges at each puncture no faster than $1/r_n^3$, the term $\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij}$ will be everywhere finite. A similar restriction on the second term on the right hand side of (24) (Brandt and Brügmann had assumed $K = 0$) requires that K vanish at least as fast as r_n^3 at each puncture. For such choices of $\tilde{A}^{ij}$ and K, u will be at least a C^2 function everywhere. This allows the Hamiltonian constraint to be solved as if the conformal manifold were $\mathbb{R}^3$, ignoring the punctures. In this way, there is no need to provide inner boundary conditions on minimal surfaces (throats) or apparent horizons [13, 15]. This greatly simplifies the solution procedure. B00 applied the puncture method with a (modified) Bowen-York extrinsic curvature [4] (the $1/r_n^4$ terms that guarantee isometry are dropped for the reason given above). Baumgarte's initial value problem consisted of this analytic solution of the momentum constraints and a numerical solution of (24). Thus he was able to redo the calculation of C94 with the isometry assumption replaced by the puncture approach, constructing a sequence of black-hole binaries in quasi-circular orbit. Quite similar physical results were obtained.

The puncture approach relies upon the spatial slice Σ passing smoothly through each wormhole and reaching a separate spatial infinity associated with each black hole. The particular form of the unbounded growth in ψ (assuming u is finite) as each puncture is approached $\mathbf{x} \to \mathbf{x}_n$ is indicative of separate compactified asymptotic infinities. The intent is to avoid physical singularities, which could be verified by computing a Riemann curvature scalar or, as is typical, using a set of coordinate transformations (inversion through spheres about each puncture) and verifying that ψ and $\tilde{A}_{ij}$ are consistent with asymptotic flatness at the punctures.

With the success of the puncture method in yielding results similar to the conformal imaging approach, it seemed worth considering a combination of the puncture method and the conformal thin-sandwich formalism. Such a method has the potential of conferring computational simplicity and yet avoiding the ambiguities in the boundary conditions on the throats in the calculations of GGB02. The next section outlines the idea and its potential shortcomings.

V. A COMBINATION OF THE CONFORMAL THIN-SANDWICH FORMALISM AND THE PUNCTURE METHOD

In order to apply the puncture method to the conformal thin-sandwich decomposition, as a first step we need to examine how the lapse equation (17) and lapse function α behave in the vicinity of the punctures. From the similarities between the lapse equation (17) and the Hamiltonian constraint (8) we can anticipate splitting $\tilde{\alpha}\psi^7$ into a sum

$$\tilde{\alpha}\psi^7 = 1 + \sum_n^N \frac{\mathcal{C}_n}{2r_n} + v, \tag{25}$$

where the $\mathcal{C}_n$ are constants and v is a correction. The term v is expected to be finite and sufficiently smooth that the domain once again can be taken as Euclidean. As with the Hamiltonian constraint, for this to be true conditions must be imposed at the punctures on the behavior of $\tilde{A}^{ij}$, K, and now β^i.

The possible values of the $\mathcal{C}_n$ correspond to different boundary conditions on the lapse at the other N asymptotic infinities. Each unique set $\{\mathcal{C}_n\}$ gives rise to a different development between time slices. The full generality of these boundary conditions need not be considered here. It suffices for our present purpose to distinguish two principal classes.

To understand these two choices, consider a single Schwarzschild black hole described by isotropic spatial coordinates. One might choose to describe the spacetime with static Schwarzschild time slices, for which the lapse function is

$$\alpha = \frac{1 - M/2r}{1 + M/2r}. \tag{26}$$

Given that $\psi = 1 + M/2r$ in these coordinates, we have

$$\alpha\psi = \tilde{\alpha}\psi^7 = 1 - \frac{M}{2r}, \tag{27}$$

which satisfies equation (17) and the lapse serves to link successive maximal slices. As is well known, these slices are frozen on the throat ($r = M/2$), have negative lapse on the second sheet, and $\alpha \to -1$ at the other spatial infinity. In analogy with the arguments of Section IV this suggests that the generalization

$$\tilde{\alpha}\psi^7 = 1 - \frac{1}{\rho} + v \tag{28}$$

be used for multiple black holes. In this case the lapse would also be negative on each of the other N sheets and limit on -1.

Alternatively, one might choose to use the opposite sign on the singular terms, to wit

$$\tilde{\alpha}\psi^7 = 1 + \frac{1}{\rho} + v. \tag{29}$$

For a single Schwarzschild hole this reduces to $\tilde{\alpha}\psi^7 = 1 + M/2r$ and hence to $\alpha = 1$. The unit lapse does connect two slices in the foliation of maximal slices of Schwarzschild found by Estabrook *et. al.* [24]. Each pair of slices in that foliation has $\alpha > 0$ everywhere, though in general $\partial_i\alpha \neq 0$. In applying this boundary condition to two or more black holes, we certainly expect $\partial_i\alpha \neq 0$ but, with a sufficient bound on v, will have $\alpha > 0$ and $\alpha \to 1$ at infinity on the other sheets.

These two choices illustrate the primary dichotomy in the choice of boundary conditions on the lapse: Either the lapse reverses sign in the interior of the black holes, which in the puncture method means that each puncture is surrounded by a two-surface on which $\alpha = 0$, or the lapse remains positive everywhere, including on each *then dynamic* throat. In a nutshell, therein lie the potential shortcomings of combining the puncture method with the conformal thin-sandwich approach.

In the former case (28), the vanishing of the lapse on some surface around each puncture is very problematic for the solution of equation (20) or for the construction of the extrinsic curvature from equation (18). Examination of (20) reveals that $\tilde{\alpha} = 0$ is a critical surface for the differential equations. A regular solution for β^i may be obtained by imposing the regularity condition that $m_j(\tilde{\mathbb{L}}\beta)^{ij} = 0$ on this surface, where m_j is the normal to the level surface $\tilde{\alpha} = 0$. However, even if a regular solution for β^i can be obtained, handling such an internal boundary is precisely the type of task that the puncture method seeks to avoid. Furthermore, a regular solution for β^i does not ensure a finite extrinsic curvature, as reference to equation (18) indicates. Stipulating the vanishing of all five algebraically-independent components of $(\tilde{\mathbb{L}}\beta)^{ij}$ on the critical surface appears to entail more freedom than is available. This is the difficulty that GGB02 faced in requiring both isometry on the shift and regularity in $\tilde{A}^{ij}$.

So, if the use of the boundary conditions implied by equation (28) is doubtful, what of the latter choice expressed by equation (29)? Here the problem is more subtle. As we discussed in Section III, it seems undesirable to construct a quasi-equilibrium solution through some means that leads to a dynamical slicing. At minimum, this would make it difficult to verify whether or not one has in fact constructed a quasi-equilibrium solution. This requisite and the previous one lead to the question of whether it is possible to find a pair of slices (maximal or non-maximal) on which the fields appear stationary, that give rise to a lapse that is positive everywhere, and which end at spatial infinity on all sheets. The second and third conditions are requirements on the applicability and usefulness of the puncture method. The first condition attempts to ensure that the quasi-equilibrium nature of data constructed with the thin-sandwich formalism actually appear in equilibrium. As we will argue below, such a slicing does not even exist for a Schwarzschild spacetime.

The argument is easy to make. Consider a t equal constant surface of Schwarzschild and $\alpha = 1$ consistent with (29). A quick evaluation of the evolution equation (5) indicates that the time derivative of $\tilde{A}^{ij}$ will be non-vanishing. In crossing the hole with a positive lapse we necessarily begin to see the pinch-off of the throat.

This argument can be made more general by considering a standard Penrose diagram of Schwarzschild, as in Figure 1. The curved lines with arrows represent the Killing flow associated with the Killing vector that is timelike in the black hole exterior. The Killing vectors are tangent to the $r =$ const curves. The two dashed lines represent two spacelike hypersurfaces that have no special properties other than ending at spatial infinity in both universes. In particular, there is no reason that they should be maximal. In this example, the upper surface represents a slice that stems from evolution off the lower slice with a lapse that is everywhere positive.

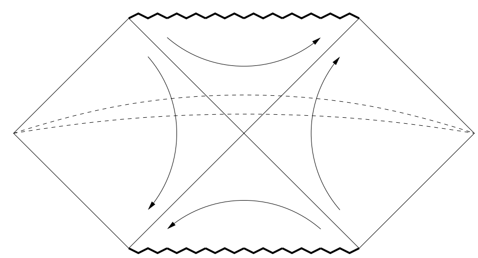

FIG. 1: Penrose diagram of the Schwarzschild spacetime. The curved solid lines are a Killing flow and the dashed lines are spacelike hypersurfaces.

Figure 2 focuses on the black hole interior and just one of the hypersurfaces. Here we begin with no *a priori* assumption about the second slice or whether the lapse changes sign or not in crossing the black hole interior. Point "B" represents the minimal surface or throat in the hypersurface, as it corresponds to the smallest value of the radial coordinate (e.g., areal coordinate). Any slicing that ends at spatial infinity in both universes must have such a point.

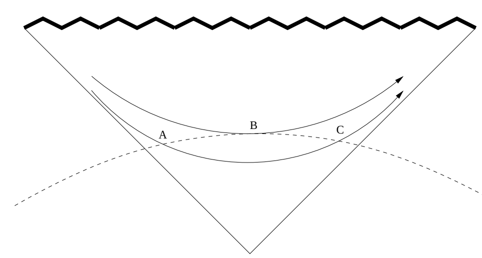

FIG. 2: Penrose diagram of the black hole interior. Point B represents the minimal two-surface.

This leads to the following crucial observation. As we are interested in stationary slicings, the time four-vector t^μ $(\partial/\partial t)$ for evolving to the next slice should be coincident with the Killing vector. The spatial slice is tangent to the $r =$ const surface at the throat. But the Killing vector is tangent to the $r = \text{const}$ surface everywhere. So, in particular, the Killing vector is tangent to the spatial slice at the throat. The time four-vector can be written in terms of the lapse and shift as $t^\mu = \alpha n^\mu + \beta^\mu$. At the location in question, since the Killing vector is tangent to the hypersurface, the lapse must vanish at the throat. This is unavoidable and establishes that we cannot choose a slicing that connects two spatial infinities, is time independent, and has an everywhere positive lapse. The lapse must at least vanish at the minimal surface if the slicing is stationary.

The argument can be extended to show that if stationarity is assumed the lapse must become negative interior to the throat. At point "C" in the figure the Killing vector points in the upper right direction so the lapse must be positive there. But, at point "A", the Killing vector points in the lower right direction, so the lapse will be negative at "A". Alternatively, if the lapse is positive definite, the slice must be non-stationary at least in some neighborhood of the black hole throat.

An added consequence of these arguments is that a slicing of Schwarzschild that is stationary and has an everywhere positive lapse can never be tangent to an $r = \text{const}$ surface. This requires that one end of the slice terminate at the physical singularity, which obviates the application of the puncture method.

VI. SUMMARY

Several arguments point to the conformal thin-sandwich decomposition as a promising approach for constructing binary black holes in quasi-circular orbits (cf. GGB02). However, assuming an isometry and using the black hole throats as inner boundaries leads to some mathematical inconsistencies (cf. Ref. [15]). This experience suggested that one might combine the conformal thin-sandwich approach with the puncture method [5], which seeks to avoid interior boundaries altogether.

Combining the conformal thin-sandwich decomposition with the puncture method for the construction of equilibrium data imposes three requirements on the slicing. Equilibrium data can most easily be identified when they are presented in stationary slices. The puncture method also requires that the slices connect the spatial infinities of the multiple black holes, as opposed to ending on a physical singularity. Finally, the conformal thin-sandwich decomposition appears to require an everywhere positive lapse, to ensure a finite extrinsic curvature and, at minimum, to avoid internal boundaries. However, even a Schwarzschild spacetime admits no stationary slicing that has an everywhere positive lapse and ends at spatial infinity in both universes.

The above arguments do not mean that it is impossible to construct quasi-equilibrium data with a combination of the conformal thin-sandwich approach and the puncture method. One might resign to constructing data in a slicing that cannot be everywhere stationary. For example, a construction might yield slicings that are quasi-stationary exterior to a neighborhood of the throat but become dynamical near the throat. The issue then is establishing whether the resulting data are in fact in quasi-equilibrium. Not enough is known yet to rule out the success of this technique. Alternatively one might consider allowing the lapse to cross zero. It is not yet completely clear what regularity conditions can be imposed or how the extrinsic curvature behaves in the neighborhood of these surfaces. At minimum this undertaking potentially involves a difficult numerical implementation at the critical surfaces; the need to do so defeating the purpose of the puncture method. We conclude that the use of a combination of the conformal thin-sandwich and puncture methods for constructing quasi-equilibrium black hole initial data is less straightforward than it first appeared and may in fact be impractical.

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