Positivity of Entropy in the Semi-Classical Theory of Black Holes and Radiation

David Hochberg and Thomas W. Kephart

Department of Physics and Astronomy, Vanderbilt University

Nashville, TN 37235

and

James W. York, Jr.

Institute of Field Physics and

Theoretical Astrophysics and Relativity Group

Department of Physics and Astronomy

University of North Carolina, Chapel Hill, NC 27599-3255

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ABSTRACT

Quantum stress-energy tensors of fields renormalized on a Schwarzschild background violate the classical energy conditions near the black hole. Nevertheless, the associated equilibrium thermodynamical entropy ΔS by which such fields augment the usual black hole entropy is found to be positive. More precisely, the derivative of ΔS with respect to radius, at fixed black hole mass, is found to vanish at the horizon for *all* regular renormalized stress-energy quantum tensors. For the cases of conformal scalar fields and U(1) gauge fields, the corresponding second derivative is positive, indicating that ΔS has a local minimum there. Explicit calculation shows that indeed ΔS increases monotonically for increasing radius and is positive. (The same conclusions hold for a massless spin 1/2 field, but the accuracy of the stress-energy tensor we employ has not been confirmed, in contrast to the scalar and vector cases). None of these results would hold if the back-reaction of the radiation on the spacetime geometry were ignored; consequently, one must regard ΔS as arising from both the radiation fields and their effects on the gravitational field. The back-reaction, no matter how "small", is therefore always significant in describing thermal properties of the spacetime geometries and fields near black holes.

I Introduction

A black hole can exist in thermodynamical equilibrium provided that it is surrounded by radiation with a suitable distribution of stress-energy. In the semi-classical approach, such radiation is characterized by the expectation value of a stress-energy tensor obtained by renormalization of a quantum field on the classical spacetime geometry of a black hole. One can use such a stress-energy tensor as a source in the semi-classical Einstein equation,

$$G^{\mu}_{\nu} = 8\pi < T^{\mu}_{\nu} >_{renormalized},\tag{1}$$

to calculate the change effected by the stress-energy tensor in the black hole's spacetime metric. This is the "back-reaction" problem associated with the spacetime geometry of a black hole in equilibrium.

In this paper we use solutions of back-reaction problems of the above type to compute the thermodynamical entropy ΔS by which quantum fields augment the usual Bekenstein-Hawking black hole entropy $S_{BH} = (1/4)A_H\hbar^{-1}$, where A_H is the area of the event horizon (Units are chosen such that $G = c = k_B = 1$, but $\hbar \neq 1$.). We consider explicitly the case of a Schwarzschild black hole surrounded by either a massless conformal scalar field or a U(1) gauge field (Maxwell field). (A massless spin 1/2 field is treated in the Appendix, but the accuracy of its stress-energy tensor has not to our knowledge been checked, in contrast to the conformal scalar and vector fields.) We show in all these cases that ΔS is positive.

Our investigation shows rigorously that for *all* possible regular stress-energy tensors, the radial derivative of ΔS vanishes at the horizon, for fixed black-hole mass; that is, ΔS has there a local extremum with respect to radius. The form of the second derivative gives the criterion for a local minimum, which indeed occurs in all cases we have considered. Then by explicit calculation we show that ΔS is positive and monotonically increasing for increasing radius. Therefore the local minimum of ΔS at the horizon is the only one and is its global minimum. As a consequence, the entropy is amenable to statistical interpretation. None of these features holds if the back-reaction of the fields on the spacetime metric is ignored. In this sense, ΔS must be regarded as arising from both the quantized radiation fields *and* from their effects on the gravitational field.

We shall see, from the properties of the renormalized stress-energy tensors we employ and of the semi-classical Einstein equation, that we can obtain accurate fractional corrections

to the metric only in $O(\epsilon)$, where $\epsilon = \hbar M^{-2}$, $M_{Pl} = \hbar^{1/2}$ is the Planck mass and M is the mass of the black hole. Because the usual black hole entropy $S_{BH} = (4\pi M^2)\hbar^{-1} = O(\epsilon^{-1})$, corrections to S_{BH} can be obtained in $O(\epsilon^0) = O(1)$ from fractional corrections of $O(\epsilon)$ in the metric. It turns out that these corrections are of the same order as the naive flat space radiation entropy $(4/3)a\,T_H^3 V$, where $a = (\pi^2/15\hbar^3)$, $T_H = \hbar(8\pi M)^{-1}$ is the uncorrected Hawking temperature of a Schwarzschild black hole, and V is the flat space volume. From this fact alone it follows that the back-reaction cannot be ignored.

II Stress-Energy Tensors

Stress-energy tensors renormalized on a Schwarzschild background have been obtained in exact form for conformal scalar fields and for U(1) gauge fields, respectively, by Howard [1] and by Jensen and Ottewill [2]. Both results can be written in the form

$$< T_\nu^\mu >_{renormalized} = < T_\nu^\mu >_{analytic} + \left(\frac{\hbar}{\pi^2 (4M)^4}\right) \Delta_\nu^\mu, \qquad (2)$$

where the analytic piece, in the case of a conformal scalar field, was given by Page [3]. The term Δ_ν^μ is obtained from a numerical evaluation of a mode sum. The numerical piece is small compared to the analytic piece, and we do not include it in the calculations in this paper. This does not change any of our results qualitatively because both pieces separately obey the required regularity and consistency conditions. The analytic piece has the exact trace anomaly in both cases.

The stress-energy tensors satisfy $\hat{\nabla}_\mu < T_\nu^\mu >= 0$ on the Schwarzschild background with metric

$$\hat{g}_{\mu\nu} = \mathrm{diag}\left[-(1 - \frac{2M}{r}), (1 - \frac{2M}{r})^{-1}, r^2, r^2 \sin^2\theta\right]. \qquad (3)$$

These tensors represent the stress-energy distribution required to equilibrate the black hole with its own Hawking radiation. Each satisfies $< T_t^t >=< T_r^r >$ at the horizon $r = 2M$, which is required for regularity of the spacetime geometry [3]. Each has the asymptotic form of a flat spacetime radiation stress-energy tensor at the uncorrected Hawking temperature at infinity of an ordinary Schwarzschild black hole, denoted here by $T_H = \hbar(8\pi M)^{-1}$.

Dropping the angular brackets and displaying the analytic piece, one has for the confor-

mal scalar field [3]

$$T_t^t = -\frac{1}{3}aT_H^4(\frac{1}{2})\left(3 + 6w + 9w^2 + 12w^3 + 15w^4 + 18w^5 - 99w^6\right),\tag{4}$$

$$T_r^r = \frac{1}{3}aT_H^4(\frac{1}{2})\left(1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 + 15w^6\right),\tag{5}$$

$$T_\theta^\theta = T_\phi^\phi = \frac{1}{3}aT_H^4(\frac{1}{2})\left(1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 - 9w^6\right),\tag{6}$$

where $w \equiv 2M/r$. We have displayed the factor $(1/2)$ explicitly because the scalar field has one helicity state while the vector field below has two. It is convenient in what follows to write

$$\frac{1}{3}aT_H^4 = \frac{\epsilon}{48\pi KM^2},\tag{7}$$

where $K = 3840\pi$. For the U(1) vector field, we have [2]

$$T_t^t = -\frac{1}{3}aT_H^4\left(3 + 6w + 9w^2 + 12w^3 - 315w^4 + 78w^5 - 249w^6\right),\tag{8}$$

$$T_r^r = \frac{1}{3}aT_H^4\left(1 + 2w + 3w^2 - 76w^3 + 295w^4 - 54w^5 + 285w^6\right),\tag{9}$$

$$T_\theta^\theta = T_\phi^\phi = \frac{1}{3}aT_H^4\left(1 + 2w + 3w^2 + 44w^3 - 305w^4 + 66w^5 - 579w^6\right).\tag{10}$$

In both cases $T_r^r > 0$ and the energy density $-T_t^t$ is negative in the vicinity of the event horizon, thus violating the weak energy condition. For the scalar field, the energy density is negative from $r = 2M$ to $r \approx 2.34M$ and for the vector field from $r = 2M$ to $r \approx 5.14M$. Both tensors also violate the dominant energy condition in a region surrounding and bordering on the horizon.

III Back-reaction on the Metric

We obtain fractional corrections h_ν^α to the metric by setting

$$g_{\mu\nu} = \hat{g}_{\alpha\mu}[\delta_\nu^\alpha + \epsilon\, h_\nu^\alpha]\tag{11}$$

in the semi-classical Einstein equation (1). We work in linear order in ϵ as required by $\hat{\nabla}_\mu T_\nu^\mu = 0$ and $\hat{\nabla}_\mu(\delta G_\nu^\mu) = 0$, where δG_ν^μ is the Einstein operator linearized on a background satisfying $\hat{G}_\nu^\mu = 0$. The corrected geometry will be taken to be static and spherically symmetric. Working out the equations as in [4], we find the corrected metric can be written as

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right)(1 + 2\epsilon\bar{\rho}(r))\, dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\omega^2,\tag{12}$$

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where $d\omega^2$ is the standard metric of a normal round unit sphere. To obtain $m(r)$ and $\bar{\rho}(r)$ requires only simple radial integrals involving T^t_t and T^r_r. The angular components enter linearized Einstein equations that hold automatically by virtue of $\hat{\nabla}_\mu T^\mu_\nu = 0$ in a static spherical geometry.

The mass function $m(r)$ has the form

$$m(r) = M(1 + \epsilon\,\mu(r) + \epsilon\,CK^{-1}), \tag{13}$$

with

$$\mu(r) = \frac{1}{\epsilon M}\int_{2M}^r (-T^t_t)\,4\pi\tilde{r}^2\,d\tilde{r}, \tag{14}$$

so $\mu(r)$ vanishes at the horizon. In (13), C is an undetermined integration constant that inspection of (12) shows is to be absorbed into M to obtain a renormalized mass for the black hole. Thus, setting $g^{rr} = 0$ shows that $r = 2m = 2M(1 + \epsilon\,CK^{-1}) = 2M_{renormalized}$ locates the event horizon. Note that, to the order we are working, we can write $m(r) = M(1 + \epsilon\,CK^{-1})(1 + \epsilon\,\mu(r)) \equiv M_{ren}(1 + \epsilon\,\mu(r))$. The renormalized mass will not be distinguished notationally from the original Schwarzschild mass M in what follows, as the bare Schwarzschild mass has no physical meaning in the back-reaction problem. Therefore, we write

$$m(r) = M(1 + \epsilon\,\mu(r)) \equiv M + M_{rad}(r) \tag{15}$$

where, using (14), we see that $M_{rad} = \epsilon\,M\,\mu$ is the usual expression for the effective mass of a spherical source.

For the scalar field, denoted where necessary by a subscript "s", one finds [4]

$$K\,\mu_s = \frac{1}{2}(\frac{2}{3}w^{-3} + 2w^{-2} + 6w^{-1} - 8\ln(w) - 10w - 6w^2 + 22w^3 - \frac{44}{3}). \tag{16}$$

For the vector field, denoted by a subscript "v", one finds [5]

$$K\,\mu_v = \frac{2}{3}w^{-3} + 2w^{-2} + 6w^{-1} - 8\ln(w) + 210w - 26w^2 + \frac{166}{3}w^3 - 248. \tag{17}$$

In both (16) and (17), we note that the first term on the right, multiplied by $\epsilon\,MK^{-1}$, gives the naive flat-space value $a\,T_H^4 V$ for radiation energy.

The metric is completed by a determination of $\bar{\rho}$ which, like μ, can be found from an elementary integration. Defining

$$K\,\bar{\rho} \equiv K\,\rho + k, \tag{18}$$

where k is a constant of integration, we have

$$\rho = \frac{1}{\epsilon} \int_{2M}^{r} (T_r^r - T_t^t)(\tilde{r} - 2M)^{-1} 4\pi \tilde{r}^2 \, d\tilde{r}. \tag{19}$$

For the scalar field, one finds [4] $(K\bar{\rho}_s = K\rho_s + k_s)$

$$K\rho_s = \frac{1}{2}\left(\frac{2}{3}w^{-2} + 4w^{-1} - 8\ln(w) - \frac{40}{3}w - 10w^2 - \frac{28}{3}w^3 + \frac{84}{3}\right). \tag{20}$$

Note that at the horizon $r = 2M$, or $w = 1$, we have $\rho_s(1) = 0$. The constant k for the scalar (vector) is denoted k_s (k_v) and will be determined below by a boundary condition. Similarly, for the vector field we have $K\,\bar{\rho}_v = K\,\rho_v + k_v$, where [5]

$$K\,\rho_v = \frac{2}{3}w^{-2} + 4w^{-1} - 8\ln(w) + \frac{40}{3}w + 10w^2 + 4w^3 - 32, \tag{21}$$

and $\rho_v(1) = 0$ at $w = 1$.

Because both radiation stress-energy tensors are asymptotically constant, it is clear that the system composed of black hole plus equilibrium radiation must be put in a finite "box". Otherwise, the fractional corrections $\epsilon\, h_\nu^\alpha$ to the metric would not remain small for sufficiently large radius. Physically, this means that the radiation in a box that is too large would collapse onto the black hole, producing a larger one. Hence, we must choose the radius r_o of the box such that it is less than the second positive root r_* for r in $g^{rr} = 0$ (the first zero corresponds to the horizon $r = 2M$). We shall also assume that the box radius r_o is sufficiently large that the stress-energy tensors we employ, which were constructed for infinite asymptotically flat spacetime, are a good approximation. Clearly, a finite radius would cut out some of the radial modes that were used in these calculations. However, if r_o is somewhat greater than the longest wavelength characteristic of Hawking radiation, which in turn is associated with the least-damped quasi-normal mode of lowest angular momentum for the field in question, then this effect should be negligible. This wavelength λ_* is about $42M$ for the conformal scalar field and is smaller for the higher-spin massless fields. Also, if $r_o > \lambda_*$, the explicit nature of the walls of the box (e.g., adiabatic versus diathermic) should not be important. For these reasons we shall assume throughout the remainder of this work that $\lambda_* < r_o < r_*$. (Of course, one must also assume that $M \gtrsim M_{Pl}$, in any treatment based on (1).) If the radius r_o were to approach the horizon, then explicit size and boundary

effects would have to be taken into account in the construction of $< T^\mu_\nu >$, as shown in the work of Elster [6,7].

One convenient way to fix the constants k_s and k_v is to impose a microcanonical boundary condition [4]. We fix r_o and imagine placing there an ideal massless perfectly reflecting wall. Outside r_o, we then have an ordinary Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{2m(r_o)}{r}\right)dt^2 + \left(1 - \frac{2m(r_o)}{r}\right)^{-1}dr^2 + r^2 d\omega^2, \tag{22}$$

for $r \geq r_o$. Continuity of the three-metric induced by metrics (12) and (22) on the world tube $r = r_o$ fixes the constant k, i.e., k_s or k_v, in $\bar\rho$ by the relation

$$k = -K\,\rho(r_o). \tag{23}$$

There are finite discontinuities in the extrinsic curvature of the world tube $r = r_o$ [4], but these, and other properties of the box wall, are of no interest in the present analysis, as we argued above. The spacetime geometry, including back-reaction, is now completely determined by (22) for $r \geq r_o$, and for $r \leq r_o$ by

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right)[1 + 2\epsilon\,(\rho(r) - \rho(r_o))]dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1}dr^2 + r^2\,d\omega^2. \tag{24}$$

IV Temperature

If we release a small packet of energy from a closed box containing a black hole through a long thin radial tube, it will undergo a red-shift and approach the asymptotic temperature

$$T_\infty = \frac{\kappa_H \hbar}{2\pi}, \tag{25}$$

where κ_H is the surface gravity of the event horizon. For an ordinary Schwarzschild black hole (ignoring the radiation), one finds $\kappa_H = (4M)^{-1}$ and $T_\infty = T_H = \hbar(8\pi M)^{-1}$. However, the stress-energy of the radiation changes the surface gravity of the horizon to

$$\kappa_H = \frac{1}{4M} \left[1 + \epsilon(\bar{\rho} - \mu) + 8\pi r^2 \, T_t^t \right] |_{r=2M}, \tag{26}$$

as a straightforward calculation shows [4]. With the microcanonical boundary conditions, we can use (23) to obtain from (25) and (26)

$$T_\infty = \frac{\hbar}{8\pi M} \left[1 - \epsilon \, \rho(r_o) + \epsilon \, nK^{-1} \right], \tag{27}$$

where n takes the value $n_s = 12$ for the scalar field and $n_v = 304$ for the vector field. The local temperature at the boundary of the box is obtained by blue-shifting (27) from infinity back to r_o. We find from

$$T_{loc} = T_\infty [-g_{tt}(r_o)]^{-1/2}, \tag{28}$$

that

$$T_{loc}(r_o) = \frac{\hbar}{8\pi M} \left[1 - \epsilon \, \rho(r_o) + \epsilon \, nK^{-1} \right] \left[1 - \frac{2m(r_o)}{r_o} \right]^{-1/2}. \tag{29}$$

The temperature T_{loc}, unlike T_∞, is actually *independent* of the boundary condition that determines the constant k, as explained in detail in [4]. Indeed, it can be readily verified by the reader that k cancels out in $O(\epsilon)$ in the expression (28) for T_{loc}. *Either* measure of temperature, T_∞ or T_{loc}, can be used to calculate the same entropy in conjunction with an appropriate measure of energy. This is quite important: it means that the specific boundary condition chosen does *not* affect the calculated entropy, as we shall see below.

V Thermodynamical Entropy

One way to calculate the entropy is as follows. Fix the radius r_o of a closed box. The measure of energy in the box conjugate to the asymptotic inverse temperature $\beta_\infty \equiv T_\infty^{-1}$

is then the Arnowitt-Deser-Misner (ADM) mass $m(r_o)$ determined at spatial infinity. The first law of thermodynamics for slightly differing equilibrium configurations tells us that

$$dS = \beta_\infty \, dm \qquad (dr_o = 0), \tag{30}$$

where $S(r_o)$ is the total entropy in the box. By this method we seem to obtain only the total entropy $S(r_o)$ rather than the distribution of entropy in the given box, $S(r)$, for $r \le r_o$, where $S(r)$ denotes the total entropy inside the radius r. However, the latter can be obtained by using the quasi-local energy E [8-11], which for static spherical metrics like those treated here is given for any radius $r \le r_o$ by

$$E(r) = r - r[g^{rr}(r)]^{1/2}, \tag{31}$$

with $g^{rr}(r)$ determined by (24), the metric for $r \le r_o$. This energy, unlike m, does not depend on asymptotic flatness in its definition, nor even on the existence of an asymptotically flat region [10,11]. Furthermore, even the "normalization" of the zero of energy [10,11] that is incorporated in E as given in (31) does not affect the calculated entropy, as it certainly should not. (This "normalization" is intended to make E approach the ADM mass in an asymptotically flat region, if such a region exists.) Similarly, the inverse local temperature $\beta(r) \equiv T_{loc}^{-1}(r)$, $r \le r_o$, is independent of the boundary conditions as mentioned above. Hence, *the value of the entropy depends neither on the zero of energy nor on the existence of an asymptotic region.*

Therefore, to obtain $S(r)$, in place of (30) we can write

$$dS = \beta \, dE \qquad (dr = 0, \, r \le r_o). \tag{32}$$

Choosing M and r as independent variables, and fixing r, we can readily integrate (32) to obtain S up to a function of r and a constant. From (29) we have

$$\beta(r) = \frac{8\pi M}{\hbar} \left[1 + \epsilon \, \rho(r) - \epsilon \, n K^{-1} \right] \left[1 - \frac{2m(r)}{r} \right]^{1/2}, \tag{33}$$

and from (15), (24), and (31), holding r fixed,

$$dE = \left[1 - \epsilon \, \mu + \epsilon \, M \frac{\partial \mu}{\partial M} \right]^{-1/2} \left[1 - \frac{2m(r)}{r} \right]^{-1/2} dM. \tag{34}$$

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One can see directly for any $r \leq r_o$ that $\beta_\infty dm = \beta \, dE$ where, of course, one replaces r_o by r in the formulas for β_∞ and m to establish this result. This equality means that we can calculate $S(r)$ for any $r \leq r_o$. The key point of this discussion is that one can think of adding layer upon layer of entropy, associated with the black hole and a given $< T^\mu_\nu >$ that is valid from $r = 2M$ to $r = r_o$, beginning at $r = 2M$ and ending at $r = r_o$. (Additivity of entropy in configurations analogous to this case is established in [12], but our method here establishes it independently.)

Observe that from fractional changes of $O(\epsilon)$ in the metric, which affect the surface gravity and temperature in this order, we are able to calculate from (32) departures of $O(\epsilon^0) = O(1)$ from the usual black hole entropy $S_{BH} = (4\pi M^2)\hbar^{-1} = 4\pi\epsilon^{-1}$. But in fact all of the corrections to the entropy are of the *same* order as the naive flat-space entropy itself:

$$\frac{4}{3}a \, T_H^3 V = \frac{4}{3}\left(\frac{\pi^2}{15\hbar^3}\right)\left(\frac{\hbar}{8\pi M}\right)^3\left(\frac{4}{3}\pi r^3\right) = \frac{8\pi}{K}\left(\frac{8}{9}w^{-3}\right) = O(1) \times w^{-3}. \tag{35}$$

The $\hbar$'s in (35) cancel out, leaving only a function of $w = 2Mr^{-1}$.

Combining (33) and (34) yields

$$dS = \frac{8\pi M}{\hbar}dM + 8\pi\left[w^{-1}(\rho - \mu) + \frac{\partial\mu}{\partial w} - n\,K^{-1}w^{-1}\right]dw, \tag{36}$$

with $dr = 0$. Integration of (36) gives an expression of the form

$$S = \frac{4\pi M^2}{\hbar} + \Delta S(w) + f\left(\frac{r}{\hbar^{1/2}}\right), \qquad (1 \leq w \leq w_o = 2M/r_o) \tag{37}$$

where the first term is the usual Bekenstein-Hawking expression S_{BH} for the black hole entropy, the second term is a function of w determined up to an additive integration constant by the second term on the right of (36), and f is a dimensionless function of r that does not depend on M. The appearance of a function f in (37) can be understood as follows. Since our problem involves three mass or length scales $M_{Planck} = \hbar^{1/2}$, the mass of the black hole, M, and a radius $r \leq r_o$, there are, for a given r, exactly three dimensionless parameters one can define, namely, $\epsilon = \hbar M^{-2}$, $w = 2M/r$ and $r/\hbar^{1/2}$. However, the first two terms on the right of (37) depend only on ϵ and w, respectively. Thus, if the entropy S depends on $r/\hbar^{1/2}$, it can only do so through a separate function of this parameter.

Let us first dispose of the dimensionless function f, which clearly can depend only on $(r/\hbar^{1/2})$, where $\hbar^{1/2}$ is the Planck length in our units. It seems that such a term could only

arise in a theory taking quantum gravity into explicit account because the semi-classical theory has incorporated the dimensionless terms involving $\hbar/M^2$ and $2M/r$. (Of course, quantum gravity could modify terms of these latter two types quantitatively.) On dimensional grounds, therefore, we take $f = 0$ in the semi-classical theory. (A formal argument that $f = 0$ based on [9] can be constructed [13].) The possibility of an additive constant will be discussed when we treat ΔS below.

In considering ΔS, which will be given explicitly below, we first note the significant property that

$$\frac{\partial(\Delta S)}{\partial w} = 8\pi \left[w^{-1}(\rho - \mu) + \frac{\partial \mu}{\partial w} - n\,K^{-1}w^{-1} \right] \tag{38}$$

vanishes at the horizon $w = 1$. Therefore, for a fixed black hole mass M, the derivative with respect to r of ΔS vanishes at the horizon. Thus ΔS has a local extremum with respect to r at the horizon. This result follows from several general features that will be enjoyed by *all* regular renormalized stress-energy tensors on the Schwarzschild background and the back-reactions they induce, not just the cases analyzed here. First, μ vanishes at the horizon by virtue of the black hole's mass having been suitably renormalized. Second, ρ vanishes at the horizon, as follows from (19) and the regularity condition $T_t^t = T_r^r$ at the horizon [3]. More precisely, we have that

$$\lim_{w \to 1^+} \left(\frac{T_t^t - T_r^r}{1 - w} \right) \qquad \text{exists.} \tag{39}$$

Third, the last two terms on the right of (38) add to zero at the horizon because there the Hamiltonian constraint ($G_t^t - 8\pi T_t^t = 0$) holds. Furthermore, note that if the fractional effects of $O(\epsilon)$ in the temperature induced by the back-reaction were neglected, the derivative (38) would not vanish at the horizon, a property that the reader can verify.

Is the local extremum of ΔS at the horizon a local minimum? To answer this we calculate

$$\frac{\partial^2(\Delta S)}{\partial w^2} = 8\pi \left[-w^{-2}(\rho - \mu) + w^{-1}(\frac{\partial \rho}{\partial w} - \frac{\partial \mu}{\partial w}) + \frac{\partial^2 \mu}{\partial w^2} + nK^{-1}w^{-2} \right], \tag{40}$$

which becomes, at the horizon $w = 1$,

$$\frac{\partial^2(\Delta S)}{\partial w^2}|_{w=1} = 8\pi \left(\frac{\partial \rho}{\partial w} + \frac{\partial^2 \mu}{\partial w^2} \right)|_{w=1} \tag{41}$$

or , equivalently, with M fixed,

$$\frac{\partial^2(\Delta S)}{\partial r^2}|_{r=2M} = \frac{32\pi^2 M^2}{\hbar} \left[4M \frac{\partial(-T_t^t)}{\partial r} - 8T_r^r - (\frac{T_r^r - T_t^t}{1 - 2M/r}) \right]|_{r=2M}. \tag{42}$$

Hence we need only examine the stress-tensors. In all the cases we consider (conformal scalar, vector, massless fermion), (41) and (42) are positive so that ΔS takes a local minimum with respect to radius at the horizon. This suggests, but does not prove, that ΔS is non-negative.

The local minimum of ΔS at the horizon and the fact that S_{BH} in the expression (37) for the total entropy S contains the *renormalized* mass M of the hole motivate the choice of the remaining additive constant in ΔS, which can only be a pure number, to be such that $\Delta S = 0$ at $w = 1$. For $w = 1$, with no "room" for the fields to contribute anything further, one then obtains only the Bekenstein-Hawking entropy $(1/4)A_H\hbar^{-1}$, as would be expected. With the choice $\Delta S(w = 1) = 0$, we obtain for the conformal scalar field [14,15]

$$\Delta S_s = \frac{8\pi}{K}(\frac{1}{2})\left(\frac{8}{9}w^{-3} + \frac{8}{3}w^{-2} + 8w^{-1} + \frac{32}{3}\ln(w) - \frac{40}{3}w - 8w^2 + \frac{104}{9}w^3 - \frac{16}{9}\right) \qquad (43)$$

for $1 \geq w \geq w_o$. Similarly, for the electromagnetic or U(1) gauge field we find

$$\Delta S_v = \frac{8\pi}{K}\left(\frac{8}{9}w^{-3} + \frac{8}{3}w^{-2} + 8w^{-1} - 96\ln(w) + \frac{40}{3}w - 8w^2 + \frac{344}{9}w^3 - \frac{496}{9}\right). \qquad (44)$$

In both expressions, the naive flat-space radiation entropy term (35) appears as the first term on the right. Both ΔS_s and ΔS_v are positive for $1 \geq w \geq w_o > w_* = 2Mr_*^{-1}$ and vanish at $w = 1$. Hence, in that they are positive, both are amenable to arguments relating thermodynamical and statistical entropy. It has not heretofore been evident that this desirable feature would be present in the semi-classical theory. The reader can verify, by omitting the back-reaction terms in the inverse temperature (33), that not only is the vanishing slope of ΔS at $w = 1$ lost, but also that the value of the resulting "ΔS", normalized as above, is no longer positive for the range $1 \geq w \geq w_o$. In this fundamental sense, we conclude that the back-reaction, however small quantitatively in its effects on the metric near a black hole, can never be regarded as negligible.

Acknowledgments

References

[1]. K.W. Howard, Phys. Rev. D**30**, 2532 (1984).

[2]. B.P. Jensen and A. Ottewill, Phys. Rev. D**39**, 1130 (1989).

[3]. D.N. Page, Phys. Rev. D**25**, 1499 (1982).

[4]. J.W. York, Phys. Rev. D**31**, 775 (1985).

[5]. D. Hochberg and T.W. Kephart, Phys. Rev. D**47**, 1465 (1993).

[6]. T. Elster, J. Phys. A: Math. Gen. **16**, 989 (1983).

[7]. T. Elster, Class. Quantum Grav. **1**, 43 (1984).

[8]. J.W. York, Phys. Rev. D**33**, 2092 (1986).

[9]. J.D. Brown, G.L. Comer, E.A. Martinez, J. Melmed, B.F. Whiting
and J.W. York, Class. Quantum Grav. **7**, 1433 (1990).

[10]. J.D. Brown and J.W. York, Contemporary Mathematics, **132**, 129 (1992).

[11]. J.D. Brown and J.W. York, Phys. Rev. D**47**, 1407 (1993).

[12]. E.A. Martinez and J.W. York, Phys. Rev. D**40**, 2124 (1989).

[13]. G.L. Comer and J.W. York, *Integrability of the entropy*, (unpublished, 1991).

[14]. J.W. York, *Entropy of a conformal scalar field and a black hole*, (unpublished, 1985).

[15]. G.L. Comer, *The thermodynamic stability of systems containing black holes*, University
of North Carolina doctoral dissertation (unpublished, 1990).

[16]. M.R. Brown, A.C. Ottewill and D.N. Page, Phys. Rev. D**33**, 2840 (1986).

[17]. C.F.P. Lee, preprint SUSSEX-TH92/10-21 (1992).

Appendix

Here we outline the calculation of ΔS for a massless spin $1/2$ field. We use the stress-energy tensor given in [16]. As far as we have been able to determine, its accuracy has not been verified by an exact numerical analysis, unlike the two cases we treated in the body of the text. This tensor has also been used in a calculation similar to the one presented here in [17], where qualitatively different results were obtained for the entropy ΔS.

The stress-energy tensor is given by

$$T_t^t = -\frac{1}{3}aT_H^4 \left(\frac{7}{8}\right)\left(3 + 6w + 9w^2 + 12w^3 + \frac{135}{7}w^4 + \frac{186}{7}w^5 - 69w^6\right), \qquad (A1)$$

$$T_r^r = \frac{1}{3}aT_H^4 \left(\frac{7}{8}\right)\left(1 + 2w + 3w^2 - \frac{52}{7}w^3 - 5w^4 - \frac{18}{7}w^5 + \frac{15}{7}w^6\right), \qquad (A2)$$

$$T_\theta^\theta = T_\phi^\phi = \frac{1}{3}aT_H^4 \left(\frac{7}{8}\right)\left(1 + 2w + 3w^2 - \frac{45}{7}w^3 - \frac{45}{7}w^4 + \frac{62}{7}w^5 + 23w^6\right). \qquad (A3)$$

We find for μ and ρ

$$K\mu_f = \frac{7}{8}\left(\frac{2}{3}w^{-3} + 2w^{-2} + 6w^{-1} - 8\ln(w) - \frac{90}{7}w - \frac{62}{7}w^2 + \frac{46}{3}w^3 - \frac{16}{7}\right), \qquad (A4)$$

$$K\rho_f = \frac{7}{8}\left(\frac{2}{3}w^{-2} + 4w^{-1} - 8\ln(w) - \frac{200}{21}w - \frac{50}{7}w^2 + \frac{52}{7}w^3 + \frac{32}{7}\right), \qquad (A5)$$

where the subscript "f" denotes "fermion". The formulas for temperature and inverse temperature have the same form as before with $n_f = -4$. The quantity ΔS enjoys all the same basic properties as for the conformal scalar and vector fields. It is given by

$$\Delta S_f = \frac{8\pi}{K}\left(\frac{7}{8}\right)\left(\frac{8}{9}w^{-3} + \frac{8}{3}w^{-2} + 8w^{-1} + \frac{24}{7}\ln(w) - \frac{200}{21}w - \frac{56}{7}w^2 + \frac{800}{63}w^3 - \frac{424}{63}\right),$$
$$(A6)$$

and is positive.