CRITICAL POINTS AND RESONANCE OF HYPERPLANE ARRANGEMENTS

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ABSTRACT. If Φ_λ is a master function corresponding to a hyperplane arrangement $\mathcal{A}$ and a collection of weights λ, we investigate the relationship between the critical set of Φ_λ, the variety defined by the vanishing of the one-form $\omega_\lambda = \mathrm{d}\log\Phi_\lambda$, and the resonance of λ. For arrangements satisfying certain conditions, we show that if λ is resonant in dimension p, then the critical set of Φ_λ has codimension at most p. These include all free arrangements and all rank 3 arrangements.

1. Introduction

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in $V = \mathbb{C}^\ell$, with complement $M = M(\mathcal{A}) = V \setminus \bigcup_{j=1}^n H_j$. Fix coordinates $\mathbf{x} = (x_1, \dots, x_\ell)$ on V, and for each hyperplane H_j of $\mathcal{A}$, let f_j be a linear polynomial for which $H_j = \{\mathbf{x} \mid f_j(\mathbf{x}) = 0\}$. A collection $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ of complex weights determines a *master function*

$$\Phi_\lambda = \prod_{j=1}^n f_j^{\lambda_j}, \tag{1.1}$$

a multi-valued holomorphic function with zeros and poles on the variety $\bigcup_{j=1}^n H_j$ defined by $\mathcal{A}$. The master function Φ_λ determines a one-form

$$\omega_\lambda = \mathrm{d}\log\Phi_\lambda = \sum_{j=1}^n \lambda_j \frac{\mathrm{d}f_j}{f_j} \tag{1.2}$$

in the Orlik-Solomon algebra $A(\mathcal{A}) \cong H^\bullet(M;\mathbb{C})$, a quotient of an exterior algebra.

Two focal points in the recent study of arrangements are the cohomology $H^\bullet(A(\mathcal{A}),\omega_\lambda)$ of the Orlik-Solomon algebra with differential given by multiplication by ω_λ, and the critical set of the master function Φ_λ, the variety $V(\omega_\lambda) \subset M$ defined by the vanishing of the one-form ω_λ. We shall denote the latter by Σ_λ. The cohomology $H^\bullet(A(\mathcal{A}),\omega_\lambda)$ arises in the study of local systems on M. Under certain conditions on the weights λ, the inclusion of $(A(\mathcal{A}),\omega_\lambda)$ in the twisted de Rham complex $(\Omega^\bullet(*\mathcal{A}), \mathrm{d}+\omega_\lambda)$ induces an isomorphism $H^\bullet(A(\mathcal{A}),\omega_\lambda) \cong H^\bullet(M;\mathcal{L}_\lambda)$, where $\mathcal{L}_\lambda$

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is the complex, rank one local system on M with monodromy $\exp(-2\pi\sqrt{-1}\,\lambda_j)$ about the hyperplane H_j. See [OT01] for discussion of these results and applications to hypergeometric integrals. The critical set of the master function is also of interest in mathematical physics. For instance, for certain arrangements, the critical equations of the Φ_λ coincide with the Bethe ansatz equations for the Gaudin model associated with a complex simple Lie algebra $\mathfrak{g}$, see [RV95, Var06].

Assume that $\mathcal{A}$ contains ℓ linearly independent hyperplanes, and note that M has the homotopy type of an ℓ-dimensional cell complex. For generic weights λ, the cohomology groups $H^q(A(\mathcal{A}), \omega_\lambda)$ vanish in all dimensions except possibly $q = \ell$, and $\dim H^\ell(A(\mathcal{A}), \omega_\lambda) = |\chi(M)|$, where $\chi(M)$ is the Euler characteristic of M, see Yuzvinsky [Yuz95]. Those weights λ for which the cohomology does not vanish (in dimension $q \neq \ell$) are said to be resonant, and comprise the resonance varieties

$$R^q_p(A(\mathcal{A})) = \{\lambda \in \mathbb{C}^n \mid \dim H^q(A(\mathcal{A}), \omega_\lambda) \geq p\}, \quad 0 < q < \ell,\ 0 < p.$$

In [Var95], Varchenko conjectured that, for generic weights λ, the master function Φ_λ has $|\chi(M)|$ nondegenerate critical points in M, and proved this result in the case where the hyperplanes of $\mathcal{A}$ are defined by real linear polynomials f_j. Varchenko's conjecture was established for an arbitrary arrangement $\mathcal{A}$ by Orlik and Terao [OT95a]. See Damon [Dam99] and Silvotti [Sil96] for generalizations. For generic, or nonresonant, weights λ, the critical set of Φ_λ was used to construct a basis for the local system homology group $H_\ell(M; \mathcal{L}_\lambda)$ by Orlik and Silvotti [OS02].

Let $z = (z_1, \ldots, z_n)$ be an n-tuple of distinct complex numbers, $z_i \neq z_j$ for $i \neq j$, $m = (m_1, \ldots, m_n)$ an n-tuple of nonnegative integers, and $\kappa \in \mathbb{C}^*$ generic. The master function

$$\Phi_{\ell,n} = \prod_{i=1}^{\ell}\prod_{j=1}^{n}(x_i - z_j)^{-m_j/\kappa} \prod_{1\leq p<q\leq \ell}(x_p - x_q)^{2/\kappa} \tag{1.3}$$

defines a local system on the complement of the Schechtman-Varchenko discriminantal arrangement $\mathcal{A}_{\ell,n}$ corresponding to the $\mathfrak{sl}_2$ KZ differential equations, see [SV91]. The critical set of $\Phi_{\ell,n}$ was determined by Scherbak and Varchenko [SV03]. Let $|m| = \sum_{j=1}^n m_j$. If m satisfies $0 \leq |m| - \ell + 1 < \ell$, then for generic z, the critical set of $\Phi_{\ell,n}$ consists of a certain number, say k, of curves in V, see [SV03, Thm. 1]. Let $\lambda = (\ldots, -m_j/\kappa, \ldots, 2/\kappa, \ldots)$ denote the associated collection of weights. The Orlik-Solomon cohomology $H^\bullet(A(\mathcal{A}_{\ell,n}), \omega_\lambda)$ was subsequently studied by Cohen and Varchenko [CV03]. Under the same conditions on m, this cohomology is nontrivial in codimension one, $H^{\ell-1}(A(\mathcal{A}_{\ell,n}), \omega_\lambda) \neq 0$. Furthermore, the dimension of the subspace of skew-symmetric cohomology classes under the natural action of the symmetric group S_ℓ is equal to k, the number of components of the critical set, see [CV03, Thm. 1.1]. Mukhin and Varchenko [MV04, MV05] also describe interesting multidimensional critical sets for master functions generalizing those of (1.3) to other root systems.

These results suggest a relationship between the critical set Σ_λ and the resonance, or nonvanishing, of $H^\bullet(A(\mathcal{A}), \omega_\lambda)$. The main purpose of this note is to establish such a relationship for tame arrangements, defined below. Our main result, Theorem 4.1, insures that if $H^p(A(\mathcal{A}), \omega_\lambda) \neq 0$, then the codimension of the critical set of the master function Φ_λ is at most p, as long as one of the following conditions holds: $\mathcal{A}$ is free; $\mathcal{A}$ has rank 3; $\mathcal{A}$ is tame and $p \leq 2$.

Some of the results presented here were announced in [Den07] and [Fal07]. These reports inspired Dimca [Dim08] to find other conditions which insure that the codimension of $Z(\omega_\lambda)$ is at most p, where $Z(\omega_\lambda)$ is the zero set of ω_λ in a good compactification of M and it is additionally assumed that $H^j(A(\mathcal{A}), \omega_\lambda) = 0$ for $j < p$.

Our main result is proven in two steps. First, in §2, we develop some properties of a variety $\Sigma(\mathcal{A}) \subseteq M \times \mathbb{C}^n$ that parameterizes all critical sets for a fixed arrangement $\mathcal{A}$. Its closure in affine space $\mathbb{C}^\ell \times \mathbb{C}^n$, denoted by $\overline{\Sigma}(\mathcal{A})$, can be described in terms of logarithmic derivations; we show that the variety is a complete intersection if and only if $\mathcal{A}$ is free. We also find that $\Sigma(\mathcal{A})$ is arithmetically Cohen-Macaulay if $\mathcal{A}$ is tame, although we do not know if the converse holds or not.

Let R be the coordinate ring of $V = \mathbb{C}^\ell$, and identify R with the polynomial ring $\mathbb{C}[x_1, \ldots, x_\ell]$. Assume that $\mathcal{A}$ is a central arrangement, so that each hyperplane of $\mathcal{A}$ passes through the origin in V. We will see (§2.1) that this assumption causes no loss of generality. The polynomials f_j defining the hyperplanes of $\mathcal{A}$ are then linear forms, and a defining polynomial $Q = \prod_{j=1}^n f_j$ of $\mathcal{A}$ is homogeneous of degree $n = |\mathcal{A}|$. For any k-algebra T, let $\operatorname{Der}_k(T)$ denote the T-module of k-linear derivations on T. Let $\operatorname{Der}(\mathcal{A})$ denote the module of logarithmic derivations on $M(\mathcal{A})$:

$$\operatorname{Der}(\mathcal{A}) = \{\theta \in \operatorname{Der}_{\mathbb{C}}(R) \colon \theta(Q) \in (Q)\}\,. \tag{1.4}$$

The arrangement $\mathcal{A}$ is said to be free if the module $\operatorname{Der}(\mathcal{A})$ is a free R-module.

The notion of a tame arrangement first arose in [OT95b] and subsequently appeared in [TY95, WY97]. Tame arrangements include generic arrangements, free arrangements (hence discriminental arrangements), and all arrangements of dimension less than 4. The precise definition appears in the next section: see Definition 2.2.

In Section §3, we use a complex of logarithmic differential forms to resolve the defining ideal of $\overline{\Sigma}(\mathcal{A})$. For free arrangements, this is simply a Koszul complex. The general case is more awkward, since the resolution is not in general free, and we require the tame hypothesis to show that it is exact. Nevertheless, this provides a link relating the codimension of a critical set and nonvanishing of the cohomology $H^\bullet(\Omega^\bullet(\mathcal{A}), \omega_\lambda)$ of the complex of logarithmic forms with poles along $\mathcal{A}$ (Theorem 3.5 and corollaries).

The second step is to show that $H^p(\Omega^\bullet(\mathcal{A}), \omega_\lambda) \neq 0$ implies that $H^p(A(\mathcal{A}), \omega_\lambda) \neq 0$, which we do in §4. The argument combines a result of Wiens and Yuzvinsky [WY97] (which requires the "tame" hypothesis again) with a spectral sequence due to Farber [Far04]. In §5, we give some examples which show, in particular, that the reverse implication does not hold in general.

2. Geometry of the critical set

In this section, we introduce and compare several slightly different algebraic descriptions of critical sets of master functions. In particular, we recall that for each arrangement $\mathcal{A}$ of n hyperplanes, there exists a manifold of dimension n that parameterizes the critical sets of all master functions on $\mathcal{A}$.

2.1. **Central and irreducible arrangements.** We will want to make two reductions to the class of arrangements considered in the arguments that follow. First, it is sufficient to consider arrangements which are central. For this, if $\mathcal{A} = \{H_j\}_{j=1}^n$ is a noncentral arrangement in $\mathbb{C}^{\ell-1}$ with master function Φ_λ, we homogenize the equations $\{f_j\}$ by adding a new variable x_0, and introduce a new hyperplane H_0 defined by $f_0 = x_0$ with weight $\lambda_0 = -\sum_{i=1}^n \lambda_i$. This yields a central arrangement $\mathcal{A}'$ in $\mathbb{C}^\ell$ (the cone of $\mathcal{A}$), with weights $\lambda' = (\lambda_0, \lambda_1, \ldots, \lambda_n)$, and corresponding master function $\Phi_{\lambda'}$. If $\Sigma_{\lambda'}$ is the critical set of $\Phi_{\lambda'}$, then Σ_λ can be identified with $\mathbb{P}\Sigma_{\lambda'}$ by restricting to the affine chart of $\mathbb{P}^{\ell-1}$ with $x_0 \neq 0$. Accordingly, the codimensions of Σ_λ in $\mathbb{C}^{\ell-1}$, of $\Sigma_{\lambda'}$ in $\mathbb{C}^\ell$, and of $\mathbb{P}\Sigma_{\lambda'}$ in $\mathbb{P}^{\ell-1}$ are all equal.

On the other hand, the Orlik-Solomon complexes for $\mathcal{A}'$ and $\mathcal{A}$ are related by

$$(A(\mathcal{A}'), \omega_{\lambda'}) \cong (A(\mathcal{A}), \omega_\lambda) \otimes_{\mathbb{C}} (\mathbb{C} \xrightarrow{0} \mathbb{C}).$$

Then the least p for which $H^p(A(\mathcal{A}), \omega_\lambda) \neq 0$ is the same as that for which $H^p(A(\mathcal{A}'), \omega_{\lambda'}) \neq 0$.

Second, recall that an arrangement $\mathcal{A}$ in V is said to be reducible if there exist subspaces V_1 and V_2 with $V \cong V_1 \oplus V_2$ and a nontrivial partition $P_1 \sqcup P_2 = [n]$ for which $f_i \in V_j^*$ if and only if $i \in P_j$. If $\mathcal{A}$ is reducible, write $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, where $\mathcal{A}_j$ is the arrangement in V_j of hyperplanes indexed by P_j. Otherwise, $\mathcal{A}$ is said to be irreducible.

2.2. **Complexes of forms.** Fix a central arrangement $\mathcal{A}$ of n hyperplanes in $V = \mathbb{C}^\ell$, with defining polynomial Q. We assume that $\mathcal{A}$ is essential, that is, contains a subarrangement of ℓ linearly independent hyperplanes. Recall that R is the coordinate ring of V. The localization R_Q is the coordinate ring of the hyperplane complement M.

Let $C = C(\mathcal{A}) = \mathbb{C}[a_1, \ldots, a_n]$, where $a_1, \ldots, a_n$ will be interpreted as weights on the hyperplanes, and let $S = C \otimes R$. For each p and k-algebra T, let $\Omega^p_{T/k}$ be the T-module of k-valued Kähler p-forms over T, so that $\Omega^p_{R/\mathbb{C}}$ and $\Omega^p_{S/C}$ are $\mathbb{C}$- and C-valued polynomial p-forms on V, respectively. For $T = R, S$, let $\Omega^p_{T/k}(*\mathcal{A}) = \Omega^p_{T_Q/k}$, the T_Q-module of k-valued, rational p-forms with poles on the hyperplanes $\mathcal{A}$. Write $\Omega^p(*\mathcal{A}) = \Omega^p_{R/\mathbb{C}}(*\mathcal{A})$ for short. Similarly, the T-module $\Omega^p_{T/k}(\mathcal{A})$ of logarithmic p-forms with poles along $\mathcal{A}$ is defined by

$$\Omega^p_{T/k}(\mathcal{A}) = \left\{ \eta \in \Omega^p_{T/k}(*\mathcal{A}) \colon Q\eta \in \Omega^p_{T/k} \text{ and } Q\mathrm{d}\eta \in \Omega^{p+1}_{T/k} \right\}, \tag{2.1}$$

and again write $\Omega^p(\mathcal{A}) = \Omega^p_{R/\mathbb{C}}(\mathcal{A})$. In particular, $\Omega^p_{T/k}(\mathcal{A}) = 0$ if $p < 0$ or $p > \ell$.

For any $\eta \in \Omega^k(\mathcal{A})$, by definition, $Q\eta \in \Omega^k_{R/\mathbb{C}}$. If η is homogeneous, we say its total degree is m and write $\operatorname{tdeg}(\eta) = m$ if

$$Q\eta = \sum f_I \mathrm{d}x_I \text{ and } m = k + \deg f_I - \deg Q = k + \deg f_I - n.$$

Let $\Omega^\bullet(\mathcal{A})_m = \{\eta \in \Omega^\bullet(\mathcal{A}) \mid \operatorname{tdeg}(\eta) = m\}$.

For a $\mathbb{Z}$-graded module N and integer r, define the shift $N(r)$ by $N(r)_q = N_{r+q}$, for $q \in \mathbb{Z}$. Then $R(n-\ell) \cong \Omega^\ell(\mathcal{A})$ via the map $1 \mapsto Q^{-1}\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_\ell$.

We recall that $\Omega^1(\mathcal{A})$ is the R-dual of $\operatorname{Der}(\mathcal{A})$: see [OT92, 4.75]. Moreover, $\mathcal{A}$ is free if and only if $\Omega^1(\mathcal{A})$ is a free R-module. The logarithmic forms themselves are self-dual:

Lemma 2.1. *For each p, $0 \le p \le \ell$, we have* $\operatorname{Hom}_R(\Omega^p(\mathcal{A}), R) \cong \Omega^{\ell-p}(\mathcal{A})(\ell - n)$.

Proof. Exterior multiplication gives a map $\Omega^p(\mathcal{A}) \otimes_R \Omega^{\ell-p}(\mathcal{A}) \to R(n-\ell)$ from [OT92, 4.79]. By comparing with the regular forms, it is straightforward to check this is a nondegenerate pairing. □

The following turns out to be an interesting weakening of freeness.

Definition 2.2. Say that an arrangement $\mathcal{A}$ in V is tame if the projective dimension of each module of logarithmic forms is bounded by cohomological degree: that is, $\operatorname{pd}_R \Omega^p(\mathcal{A}) \le p$ for all p with $0 \le p \le \ell$.

We will make use of several choices of differential on the graded vector spaces $\Omega^\bullet(*\mathcal{A})$ and $\Omega^\bullet(\mathcal{A})$. First, the exterior derivative $\mathrm{d}\colon \Omega^p(*\mathcal{A}) \to \Omega^{p+1}(*\mathcal{A})$ restricts to the logarithmic forms $\Omega^\bullet(\mathcal{A})$, making both $(\Omega^\bullet(*\mathcal{A}), \mathrm{d})$ and $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$ ($\mathbb{C}$-)cochain complexes. Also, for $(T, k) = (R, \mathbb{C})$ or (S, C), for any $\omega \in \Omega^1_{T/k}(\mathcal{A})$, we shall denote by $(\Omega^\bullet_{T/k}(*\mathcal{A}), \omega)$ and $(\Omega^\bullet_{T/k}(\mathcal{A}), \omega)$ the cochain complexes of T_Q- and T-modules, respectively, obtained by using (left)-multiplication by ω as a differential. Last, for $t \in \mathbb{C}$, let $\nabla_t = \mathrm{d} + t\omega$, and $\nabla = \nabla_1$. As long as $\mathrm{d}\omega = 0$, this gives a third choice of differential.

Observe that the log complex decomposes into complexes of finite dimensional vector spaces

$$(\Omega^\bullet(\mathcal{A}), \omega_\lambda) = \bigoplus_{m \in \mathbb{Z}} (\Omega^\bullet(\mathcal{A})_m, \omega_\lambda), \text{ resp., } (\Omega^\bullet(\mathcal{A}), \nabla) = \bigoplus_{m \in \mathbb{Z}} (\Omega^\bullet(\mathcal{A})_m, \nabla).$$

2.3. **Localizations.** If $\mathfrak{p}$ is a prime ideal of R, following [OT92, 4.6], let $X(\mathfrak{p})$ denote the subspace in $L(\mathcal{A})$ of least dimension containing $V(\mathfrak{p})$. Then $\Omega^p(\mathcal{A})_\mathfrak{p} = \Omega^p(\mathcal{A}_X)_\mathfrak{p}$ where $X = X(\mathfrak{p})$: in particular, the localization $\Omega^1(\mathcal{A})_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module if and only if $\mathcal{A}_X$ is a free arrangement.

Recall a central arrangement $\mathcal{A}$ is said to be *locally free* if $\mathcal{A}_X$ is free for all $X \neq \{0\}$: see [MS01]. In this case, $\Omega^1(\mathcal{A})_\mathfrak{p}$ is free for all prime ideals not equal to the homogeneous maximal ideal R_+. Since all rank 2 arrangements are free, the locus on which $\Omega^1(\mathcal{A})_\mathfrak{p}$ is not a free module has codimension at least 3.

2.4. **The meromorphic ideal.** Recall that our goal is to understand the solutions to the ℓ equations given by $\omega_\lambda = 0$ as $\lambda \in \mathbb{C}^n$ varies, where the 1-form ω_λ is defined in (1.2). It is natural, then, to consider the "universal" 1-form.

Definition 2.3. Let

$$\omega_\mathbf{a} = \sum_{i=1}^n a_i \frac{\mathrm{d}f_i}{f_i} \in \Omega^1_{S/C}(\mathcal{A}), \tag{2.2}$$

and let I_{mer} be the ideal of S_Q defined by the ℓ equations $\omega_{\mathbf{a}} = 0$. We will call I_{mer} the *meromorphic ideal* of critical sets for $\mathcal{A}$.

In coordinates, if the hyperplanes of $\mathcal{A}$ are defined by equations $f_i = \sum_{j=1}^{\ell} c_{ij} x_j$ for $1 \leq i \leq n$, then

$$\omega_{\mathbf{a}} = \sum_{i,j} \frac{a_i c_{ij}}{f_i} \mathrm{d}x_j, \tag{2.3}$$

and the meromorphic ideal I_{mer} is generated by the elements $\{d_j : 1 \leq j \leq \ell\}$, where $d_j = \sum_i a_i c_{ij}/f_i$. Thus, I_{mer} is the image of the duality pairing $\langle \mathrm{Der}_C(S), \omega_{\mathbf{a}} \rangle$ in S_Q.

For $\omega_\lambda \in A^1(\mathcal{A}) \cong \mathbb{C}^n$, the degree-1 part of the Orlik-Solomon algebra, let

$$\Sigma_\lambda = V(\omega_\lambda) \subseteq M \tag{2.4}$$

denote the critical set of the master function Φ_λ. Further let

$$\Sigma = \Sigma(\mathcal{A}) = \left\{ (x, \omega) \in M \times A^1 : \omega_{\mathbf{a}}(x) = 0 \right\}, \tag{2.5}$$

and note that $\Sigma \cong V(I_{\mathrm{mer}})$. Denote by π_1^*, π_2^* the two projections

$$V \xleftarrow{\pi_1^*} V \times \mathbb{C}^n \xrightarrow{\pi_2^*} \mathbb{C}^n \tag{2.6}$$

induced by the inclusions of coordinate rings

$$R \xrightarrow{\pi_1} S \xleftarrow{\pi_2} C.$$

Proposition 2.4 (Proposition 4.1, [OT95a])**.** *If $\mathcal{A}$ is an arrangement of rank ℓ, then Σ is a codimension-ℓ complex manifold embedded in $V \times \mathbb{C}^n$.*

More precisely, one has the following. (See [HKS05, Theorem 4] for a related result.)

Proposition 2.5. *The restriction of the projection $\pi_1^* \colon \Sigma \to M$ gives Σ the structure of a trivial vector bundle over M of rank $n - \ell$.*

Proof. Let $W = \{\lambda \in \mathbb{C}^n \colon \sum_{i=1}^n \lambda_i f_i = 0\}$, a codimension-$\ell$ subspace. Now define a map

$$s \colon M \times W \to M \times \mathbb{C}^n \tag{2.7}$$

by setting $\pi_1^* \circ s(x, \lambda) = x$ and $\pi_2^* \circ s(x, \lambda) = \sum_{i=1}^n \lambda_i f_i(x) e_i$ for $1 \leq i \leq n$, where e_i denotes the ith coordinate vector in $\mathbb{C}^n$. Since $\sum_{i=1}^n \lambda_i \mathrm{d}f_i = 0$ for $\lambda \in W$, it follows from (1.2) that the image of s actually lies in Σ. Since $f_i(x) \neq 0$ for $x \in M$, the map s is invertible:

$$\Sigma \cong M \times W. \tag{2.8}$$

$\square$

So for each $x \in M$, the fibre $\pi_1^{*-1}(x)$ is a $n - \ell$-dimensional vector space W of weights λ for which $x \in \Sigma_\lambda$. The fibres of the other projection, $\pi_2^* \colon \Sigma \to A^1$, are the critical sets: $\Sigma_\lambda = \pi_2^{*-1}(\lambda)$ for each $\lambda \in A^1$. We can also see the limit behaviour of critical sets near the origin in V. Let $\overline{\Sigma}$ denote the closure of Σ in $V \times \mathbb{C}^n$.

Proposition 2.6. *If $\mathcal{A}$ is an irreducible arrangement, then*

$$\overline{\Sigma} \cap \left(\pi_1^{*-1}(0)\right) = \Big\{(0,\lambda) \in V \times \mathbb{C}^n \colon \sum_{i=1}^{n} \lambda_i = 0\Big\}.$$

Proof. The second coordinate of the map s from (2.7) lies in the hyperplane $H := \{\lambda \in \mathbb{C}^n \colon \sum_{i=1}^n \lambda_i = 0\} = \operatorname{span}(e_i - e_j \colon 1 \leq i,j \leq n)$, so the projection of $\overline{\Sigma}$ onto $\mathbb{C}^n$ also lies in H.

To show equality, let J_s be the Jacobian of s, and use calculus to check that the limit of the image of J_s at $x = 0$ contains a set of vectors which span H. Reordering the hyperplanes if necessary, suppose that $\{f_1, \ldots, f_{r+1}\}$ form a circuit in $\mathcal{A}$. By definition, any r of the set are linearly independent, and there exist nonzero scalars $c_1, \ldots, c_{r+1}$, for which $\sum_{i=1}^{r+1} c_i f_i = 0$. Regarding f_{r+1} as a function of $\{f_i \colon 1 \leq i \leq r\}$, we have

$$\frac{\partial f_{r+1}}{\partial f_i} = -c_i/c_{r+1}, \tag{2.9}$$

for each $1 \leq i \leq r$. Let $\lambda = \sum_{i=1}^{r+1} c_i e_i$: by construction, $\lambda \in W$. Now evaluate J_s at (x,λ). Consider partial derivatives of the jth coordinate of $\pi_2^* \circ s$, for $1 \leq j \leq r+1$:

$$\frac{\partial}{\partial f_i} \lambda_j f_j(x) = \begin{cases} c_i & \text{if } j = i; \\ c_{r+1}(-c_i/c_{r+1}) & \text{if } j = r+1; \\ 0 & \text{otherwise.} \end{cases}$$

Since the coefficients c_i are nonzero, this implies $(0, e_i - e_{r+1})$ is in the limit of the image of J_s for each $1 \leq i \leq r$. By linearity, $(0, e_i - e_j) \in \overline{\Sigma}$ whenever the hyperplanes indexed by i and j are contained in a common circuit. Since $\mathcal{A}$ is irreducible, its underlying matroid is connected, so any two hyperplanes are contained in a common circuit. It follows that the closure of Σ over $x = 0$ equals H. $\square$

2.5. **The logarithmic ideal.** The critical variety Σ becomes more tractible when it is extended to the affine space V. We indicate two natural ways to do this which turn out to coincide.

As in [OT95a], we may apply the logarithmic derivations $\operatorname{Der}(\mathcal{A})$ to obtain critical equations in the polynomial ring S. Let $I = I(\mathcal{A}) = (\langle \operatorname{Der}_C(\mathcal{A}), \omega_{\mathbf{a}} \rangle)$ be the image of the duality pairing. It follows from (1.4) that I is actually an ideal in the polynomial ring S, rather than just the localization S_Q. We will call $I(\mathcal{A})$ the *logarithmic ideal* of critical sets for $\mathcal{A}$.

If the arrangement $\mathcal{A}$ is free, one can write generators of I explicitly as follows. First, $\operatorname{Der}(\mathcal{A})$ is a free R-module with some homogeneous basis $\{D_1, \ldots, D_\ell\}$. Then

$$D_i = \sum_{j=1}^{\ell} g_{ij} \partial/\partial x_j \tag{2.10}$$

for some polynomials $\{g_{ij}\}$. Let m_i denote the (total) degree of D_i, for each i, ordering $D_1, \ldots, D_\ell$ so that $m_1 \leq \cdots \leq m_\ell$. We may assume D_1 is the Euler derivation, and $m_1 = 0$. The numbers $\{m_i\}$ are classically called the exponents of $\mathcal{A}$.

Proposition 2.7. *If $\mathcal{A}$ is a free arrangement, then the ideal I has homogeneous generators in the exponents of $\mathcal{A}$.*

Proof. Apply the derivations (2.10) to $\omega_{\mathbf{a}}$, introduced in (2.3). Explicitly,

$$I = \Big(\sum_{j=1}^{\ell} g_{ij} d_j : 1 \leq i \leq \ell \Big). \tag{2.11}$$

Since each d_j is a rational function with simple poles and each g_{ij} has degree $m_i + 1$, the polynomial $\sum_{j=1}^{\ell} g_{ij} d_j$ is homogeneous of degree m_i. □

If $\mathcal{A}$ is not free, only the generators of $\mathrm{Der}(\mathcal{A})$ in minimal degree are easily understood. In particular, if $\mathcal{A}$ is irreducible, the Euler derivation generates $\mathrm{Der}(\mathcal{A})_0$, which gives the following.

Proposition 2.8. *If $\mathcal{A}$ is an irreducible arrangement, then the degree 0 part of I is generated by $\sum_{i=1}^{n} a_i$.*

Theorem 2.9. *For any arrangement $\mathcal{A}$, $V(I(\mathcal{A}))$ is the closure of $\Sigma(\mathcal{A})$ in $V \times \mathbb{C}^n$.*

Accordingly, we will write $\overline{\Sigma} = V(I)$. We defer the proof to §3.4.

Corollary 2.10. *For any arrangement $\mathcal{A}$ of rank ℓ, the variety $\overline{\Sigma}$ is irreducible of codimension ℓ.*

Proof. By Theorem 2.9, the vanishing ideal of $\overline{\Sigma}$ is $\mathrm{rad}(I)$. This is the contraction of $I_Q = I_{\mathrm{mer}}$, and I_{mer} is prime by Proposition 2.4. It follows that the radical of I is also prime. □

In general, the ideal I need not be radical (Example 5.3). However, we will see that if $\mathcal{A}$ is tame, then I is actually prime (Corollary 3.8.)

2.6. **A naive ideal.** For purposes of comparison, let $I' = (Qd_j \colon 1 \leq j \leq \ell)$, the ideal of S obtained by clearing denominators in Definition 2.3. From a geometric point of view, this ideal should be replaced by an ideal quotient by Q. It turns out that doing so recovers the logarithmic ideal. We note that a closely related result appears in the algorithm of [HKS05]: in that setting, the weights a_i are specialized to natural numbers, while the polynomial Q is generalized to an arbitrary homogeneous ideal.

Proposition 2.11. *For any arrangement $\mathcal{A}$, we have $(I' : Q) = I$.*

Proof. By definition, $(I' : Q) = \{s \in S \colon sQ \in I'\}$. To show $I \subseteq (I' : Q)$, write any $\theta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{A})$ as $\theta = \sum_{j=1}^{\ell} r_j \partial / \partial x_j$ for some coefficients $r_j \in S$. Then

$$Q \left\langle \theta, \omega_{\mathbf{a}} \right\rangle = Q \sum_{j=1}^{\ell} r_j d_j = \sum_{j=1}^{\ell} r_j (Q d_j) \in I',$$

so $\langle \theta, \omega_{\mathbf{a}} \rangle \in (I' : Q)$. To show the other inclusion, suppose $f \in (I' : Q)$. We may write

$$fQ = \sum_{j=1}^{\ell} r_j Q d_j$$

for some polynomials $r_j \in S$; that is, $f = \sum_j r_j d_j \in S$.

Form the derivation $\theta = \sum_{j=1}^{\ell} r_j \frac{\partial}{\partial x_j}$. Since $\langle \theta, \omega_{\mathbf{a}} \rangle = f$, to show $f \in I$ it is enough to show $\theta \in \mathrm{Der}(\mathcal{A}) \otimes S$. In turn, since $\langle \theta, \mathrm{d}f_i \rangle = \theta(f_i)$, we need to prove $\langle \theta, \mathrm{d}f_i \rangle \in (f_i)$ for each i, $1 \leq i \leq n$ (by [OT92, Prop. 4.8]).

For this, use (2.2) to write

$$fQ = \langle \theta, \omega_{\mathbf{a}} \rangle \, Q = \sum_i \langle \theta, \mathrm{d}f_i \rangle \, a_i Q / f_i.$$

Since $fQ \in (Q)$, the image of fQ under the map $S \to S/(f_1) \times \cdots \times S/(f_n)$ is zero. Since Q/f_i is divisible by all f_j, $j \neq i$, it follows the image of $\langle \theta, \mathrm{d}f_i \rangle \, a_i Q/f_i$ is also zero for each i. Since $a_i Q/f_i \neq 0$ in $S/(f_i)$, and (f_i) is a prime ideal, $\langle \theta, \mathrm{d}f_i \rangle = 0$ in $S/(f_i)$, and $f \in I$ as claimed. □

2.7. **Complete intersections.** It follows from Proposition 2.4 together with Corollary 2.10 that the codimension of I and I_{mer} both equal ℓ. Since S and therefore S_Q are Cohen-Macaulay, the depth of I and I_{mer} are also both ℓ, see [Eis95, Theorem 18.7]. Since I_{mer} is generated by ℓ elements of S_Q, we obtain the following.

Lemma 2.12. *The ideal I_{mer} is a complete intersection.*

The logarithmic critical set ideal behaves more subtly.

Theorem 2.13. *The ideal I is a complete intersection if and only if $\mathcal{A}$ is free.*

Proof. If $\mathcal{A}$ is free, then I is generated by ℓ elements, (2.11). Since I has codimension ℓ, by Corollary 2.10, I is a complete intersection. On the other hand, suppose that I is a complete intersection. Then it has some ℓ homogeneous generators $f_1, f_2, \ldots, f_\ell$. For each i, $1 \leq i \leq \ell$, let $\theta_i \in \mathrm{Der}_C(\mathcal{A})$ be a derivation for which $\theta_i(\omega_{\mathbf{a}}) = f_i$. By definition of I, the module $\mathrm{Der}_C(\mathcal{A})$ is generated by $\theta_1, \ldots, \theta_\ell$.

Since the number of generators is equal to its rank and S is a domain, $\mathrm{Der}_C(\mathcal{A})$ is a free S-module. It follows $\mathrm{Der}_{\mathbb{C}}(\mathcal{A})$ is a flat R-module. Since $\mathrm{Der}_{\mathbb{C}}(\mathcal{A})$ is a finitely-generated graded module, it is free. □

Example 2.14. The arrangement X_3, defined by $Q = xyz(x+y)(x+z)(y+z)$, is not free. The ideal I is minimally generated by 4 generators, not 3.

Example 2.15 (Pencils)**.** An arrangement $\mathcal{A}$ of n lines in $\mathbb{C}^2$ is free ([OT92, Example 4.20]). $\mathrm{Der}_{\mathbb{C}}(\mathcal{A})$ has a basis $\{D_1, D_2\}$ where $D_2 = Q/f_1(\frac{\partial f_1}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial}{\partial x_1})$, and D_1 is the Euler derivation. Then I is a complete intersection generated in degrees $0, n-2$:

$$I = \left(\sum_{H \in \mathcal{A}} a_H, \sum_{i=2}^{n} a_i \frac{Q}{f_1 f_i} \left(\frac{\partial f_1}{\partial x_1}\frac{\partial f_i}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_i}{\partial x_1} \right) \right).$$

3. Koszul complexes

In this section, we indicate how to determine the codimension of a critical set using the complexes of forms from §2.2. The key idea is that $(\Omega^\bullet_{S/C}(*\mathcal{A}), \omega_{\mathbf{a}})$ is the Koszul complex for the defining ideal of Σ. In the tame case, $(\Omega^\bullet_{S/C}(\mathcal{A}), \omega_{\mathbf{a}})$ is a resolution, not necessarily free, of the defining ideal of $\overline{\Sigma}$.

3.1. **Meromorphic forms.** Recall from §2.2 that $\Omega^\bullet(*\mathcal{A})$ is the space of meromorphic forms with poles on the hyperplanes. We may regard this as the exterior algebra on R_Q. Then $(\Omega^\bullet_{S/C}(*\mathcal{A}), \omega_{\mathbf{a}})$ is the Koszul complex of the generators $\{d_j\}$ of I_{mer}, by definition of $\omega_{\mathbf{a}}$. Since the depth of I_{mer} is equal to ℓ, we obtain the following.

Proposition 3.1. *If $\mathcal{A}$ is a central, essential arrangement,*

$$0 \to \Omega^0_{S/C}(*\mathcal{A}) \overset{\omega_{\mathbf{a}}}{\to} \Omega^1_{S/C}(*\mathcal{A}) \overset{\omega_{\mathbf{a}}}{\to} \cdots \overset{\omega_{\mathbf{a}}}{\to} \Omega^\ell_{S/C}(*\mathcal{A}) \to S_Q/I_{\mathrm{mer}} \to 0 \tag{3.1}$$

is an exact complex of S_Q-modules.

Definition 3.2. For $\lambda \in \mathbb{C}^n$, let $R_\lambda = R$, regarded as an S-module via the homomorphism that maps a_i to λ_i, for each i, $1 \le i \le n$.

Corollary 3.3. *For any $\lambda \in \mathbb{C}^n$ and $0 \le p \le \ell$,*

$$H^p(\Omega^\bullet(*\mathcal{A}), \omega_\lambda) \cong \mathrm{Ext}^p_{S_Q}(S_Q/I_{\mathrm{mer}}, (R_\lambda)_Q). \tag{3.2}$$

If the critical set Σ_λ is nonempty, then the codimension of Σ_λ is the smallest p for which $H^p(\Omega^\bullet(*\mathcal{A}), \omega_\lambda) \neq 0$.

Proof. By Proposition 3.1, the complex (3.1) is a free resolution of S_Q/I_{mer}. Now applying $\mathrm{Hom}_{S_Q}(-, (R_\lambda)_Q)$ to the complex (3.1) gives $(\Omega^\bullet(*\mathcal{A}), \omega_\lambda)$, since Koszul complexes are self-dual. Taking cohomology, we obtain (3.2).

Now $I_{\mathrm{mer}} \otimes (R_\lambda)_Q$ is the vanishing ideal of Σ_λ, and $(\Omega^\bullet(*\mathcal{A}), \omega_\lambda)$ its Koszul complex. Since R_Q is Cohen-Macaulay, the codimension of the ideal is equal to its depth, which is the least p for the cohomology of the Koszul complex (3.2) is nonzero. (In particular, if the critical set is empty, then $H^p(\Omega^\bullet(*\mathcal{A}), \omega_\lambda) = 0$ for all p.) □

3.2. **Logarithmic forms.** Analogous statements hold for the complex $\Omega^\bullet(\mathcal{A})$, provided that the arrangement $\mathcal{A}$ is tame. The advantage is that, since $\mathcal{A}$ is a central arrangement, $\Omega^\bullet(\mathcal{A})$ is a graded R-module.

Since localization is exact and $\Omega^\bullet(\mathcal{A})_Q \cong \Omega^\bullet(*\mathcal{A})$, Corollary 3.3 yields the following.

Proposition 3.4. *For any $\lambda \in \mathbb{C}^n$, if Σ_λ is nonempty, then the codimension of Σ_λ is the least p for which*

$$H^p(\Omega^\bullet(\mathcal{A}), \omega_\lambda)_Q \neq 0.$$

For tame arrangements, it is possible to make a more precise analysis. The proof of the following is deferred to the next section.

Theorem 3.5. *If $\mathcal{A}$ is free, then the "universal" log-complex is a free resolution of $(S/I)(n-\ell)$ as a graded S-module. More generally, for any tame arrangement $\mathcal{A}$, the complex*

$$0 \to \Omega^0_{S/C}(\mathcal{A}) \overset{\omega_{\mathbf{a}}}{\to} \Omega^1_{S/C}(\mathcal{A}) \overset{\omega_{\mathbf{a}}}{\to} \cdots \overset{\omega_{\mathbf{a}}}{\to} \Omega^\ell_{S/C}(\mathcal{A}) \to (S/I)(n-\ell) \to 0 \tag{3.3}$$

is exact.

By analogy with Corollary 3.3, we have the following.

Corollary 3.6. *Suppose $\mathcal{A}$ is tame. For any $\lambda \in \mathbb{C}^n$ and $0 \le p \le \ell$, then*

$$H^p(\Omega^\bullet(\mathcal{A}), \omega_\lambda) \cong \operatorname{Tor}^S_{\ell-p}(S/I, R_\lambda)(n-\ell), \tag{3.4}$$

where R_λ is the specialization from Definition 3.2.

Proof. If $\mathcal{A}$ is free, the statement follows directly. More generally, since the complex (3.3) is exact, the first hyper-Tor spectral sequence degenerates:

$$\operatorname{Tor}^S_\bullet(S/I, M)(n-\ell) \cong \mathbf{Tor}^S_\bullet(\Omega^{\ell-\bullet}_{S/C}(\mathcal{A}), M), \tag{3.5}$$

for any S-module M. Since, we also have $\operatorname{Tor}^S_q(\Omega^p_{S/C}(\mathcal{A}), R_\lambda) = \operatorname{Tor}^R_q(\Omega^p_{R/\mathbb{C}}(\mathcal{A}), R) = 0$ for $q > 0$, by flat base change, the second hyper-Tor spectral sequence also degenerates, giving the isomorphism claimed. □

Recall that a complete intersection is an example of a Cohen-Macaulay ring, for which we refer to [Eis95]. Theorem 2.13 can be extended as follows.

Theorem 3.7. *If $\mathcal{A}$ is a tame arrangement, then the affine coordinate ring S/I of $\overline{\Sigma}$ is Cohen-Macaulay.*

Note that the coordinate ring S/I is not Cohen-Macaulay for all arrangements (Example 5.3).

Proof. Since $\overline{\Sigma}$ has codimension ℓ (Corollary 2.10), the depth of I is ℓ. It follows that $\operatorname{pd}_S(S/I) \ge \ell$. The ring S/I is Cohen-Macaulay if and only if this is an equality. Using the isomorphism (3.5) for $M = \mathbb{C}$, we have a spectral sequence

$$E^1_{pq} = \operatorname{Tor}^S_q(\Omega^{\ell-p}_{S/C}(\mathcal{A}), \mathbb{C}) \Rightarrow \operatorname{Tor}^S_{p+q}(S/I, \mathbb{C})(n-\ell).$$

The tame hypothesis is equivalent to having $E^1_{pq} = 0$ for $p + q > \ell$. It follows that $\operatorname{pd}_S(S/I) \le \ell$, as required. □

This allows a sharpening of Theorem 2.9.

Corollary 3.8. *If $\mathcal{A}$ is a tame arrangement, then I is the vanishing ideal of $\overline{\Sigma}$. In particular, I is prime.*

Proof. By Theorem 2.9, the vanishing ideal of $\overline{\Sigma}$ is $\operatorname{rad}(I)$. This is prime, by Corollary 2.10. Suppose $\mathcal{A}$ is tame. By Theorem 3.7, the ideal I has no embedded primes, so I is primary. Since $Q \notin \operatorname{rad}(I)$, it follows $(I : Q) = I$. Then I is the contraction of $I_Q = I_{\mathrm{mer}}$ under the inclusion $S \hookrightarrow S_Q$, so $I = \operatorname{rad}(I)$. □

Now suppose Φ_λ is a master function with $\omega_\lambda = \mathrm{d}\log\Phi_\lambda$. Then $S/I \otimes_S R_\lambda = R/I_\lambda$, where I_λ is the ideal generated by $\langle \mathrm{Der}(\mathcal{A}), \omega_\lambda\rangle$. Let

$$\overline{\Sigma}_\lambda = V(I_\lambda) \subseteq \mathbb{C}^\ell. \tag{3.6}$$

Clearly $\overline{\Sigma}_\lambda \cap M = \Sigma_\lambda$. However, it is not the case in general that $\overline{\Sigma}_\lambda$ is the closure of Σ_λ (see Example 5.1.) The next result prepares for our main Theorem 4.1.

Proposition 3.9. *If $\mathcal{A}$ is a tame arrangement, then the codimension of $\overline{\Sigma}_\lambda$ is the smallest p for which $H^p(\Omega^\bullet(\mathcal{A}), \omega_\lambda) \neq 0$, provided that either $\mathcal{A}$ is free, $\mathcal{A}$ has rank 3, or $p \leq 2$.*

Proof. The codimension of $\overline{\Sigma}_\lambda$ is equal to the depth of I_λ, or equivalently the depth of I on R_λ, which is the least p for which $\mathrm{Ext}^p_S(S/I, R_\lambda) \neq 0$. By the same argument as in (3.5), using Theorem 3.5 with hyper-Ext gives a spectral sequence

$$E_1^{pq} = \mathrm{Ext}^q_S(\Omega^{\ell-p}_{S/C}(\mathcal{A}), R_\lambda) \Rightarrow \mathrm{Ext}^{p+q}_S(S/I, R_\lambda),$$

suppressing the degree shift. Then $E_1^{pq} \cong \mathrm{Ext}^q_R(\Omega^{\ell-p}_{R/\mathbb{C}}(\mathcal{A}), R)$, and in particular $E_1^{p0} \cong \Omega^p_{R/\mathbb{C}}(\mathcal{A})$ by self-duality. Consequently, $E_2^{p0} \cong H^p(\Omega^\bullet(\mathcal{A}), \omega_\lambda)$. If $\mathcal{A}$ is free, $E_1^{pq} = 0$ for $q > 0$, and the conclusion follows.

In general, since Ω^ℓ is a free module, $E_1^{0q} = 0$ for $q > 0$, which means $E_2^{p0} = E_\infty^{p0}$ for $p \leq 2$. That is,

$$\mathrm{Ext}^p_S(S/I, R_\lambda) \cong H^p(\Omega^\bullet(\mathcal{A}), \omega_\lambda) \tag{3.7}$$

for $0 \leq p \leq 2$. The claims for rank-3 arrangements and codimensions $p \leq 2$ follow. □

3.3. **Proof of Theorem 3.5.** The purpose of this section is to show that the complex (3.3) is exact if the arrangement $\mathcal{A}$ is tame. We begin with a reduction to irreducible arrangements. Suppose $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, and let S_j and C_j be corresponding coordinate rings for $j = 1, 2$, and let $\omega_{\mathbf{a}_j} \in \Omega^1_{S_j/C_j}(\mathcal{A}_j)$ as defined in (2.2). The following lemma is routine and we omit the proof: a similar result appears as [OT92, Proposition 4.14].

Lemma 3.10. *If $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, the decomposition induces an isomorphism of cochain complexes*

$$(\Omega^\bullet_{S/C}(\mathcal{A}), \omega_{\mathbf{a}}) \cong (\Omega^\bullet_{S_1/C_1}(\mathcal{A}_1), \omega_{\mathbf{a}_1}) \otimes_{\mathbb{C}} (\Omega^\bullet_{S_2/C_2}(\mathcal{A}_2), \omega_{\mathbf{a}_2}).$$

Moreover, $\Sigma(\mathcal{A}) \cong \Sigma(\mathcal{A}_1) \times \Sigma(\mathcal{A}_2)$ and $I(\mathcal{A}) = S_2 I(\mathcal{A}_1) + S_1 I(\mathcal{A}_2)$.

We now argue induction on the number of hyperplanes. Clearly if $|\mathcal{A}| = 1$, the arrangement is free, and (3.3) is exact. We may assume $\mathcal{A}$ is irreducible: if not, by Lemma 3.10, the complex (3.3) decomposes as a tensor product of complexes for arrangements with strictly fewer hyperplanes. By induction and the Künneth formula, then, (3.3) is exact.

Since an S-module N is zero if and only if $N_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$, we wish to show that the localization of (3.3) is exact for every $\mathfrak{m}$. It is enough to consider just those ideals $\mathfrak{p} = S\mathfrak{m}$, for maximal ideals $\mathfrak{m}$ of R.

Let $X = X(\mathfrak{m})$, in the notation of §2.3. First, consider the case where $X \neq 0$. By assumption, $\mathcal{A}$ is essential, so some hyperplane does not contain X. Without loss of generality, assume $X \not\subseteq \ker f_n$, and let $\mathcal{A}'$ denote the arrangement obtained from $\mathcal{A}$ by deleting the last hyperplane.

Since $\mathcal{A}$ is irreducible, $\mathcal{A}'$ is an essential arrangement in V. Let $C' = \mathbb{C}[a_i \colon 1 \leq i \leq n-1]$ and $S' = C' \otimes_{\mathbb{C}} R$. Similarly, let $\omega_{\mathbf{a}'} = \sum_{i=1}^{n-1} a_i \mathrm{d}f_i/f_i$. Consider, for $0 \leq p \leq \ell$, the inclusion

$$i \colon \Omega^p_{S'/C'}(\mathcal{A}') \otimes_{\mathbb{C}} \mathbb{C}[a_n] \hookrightarrow \Omega^p_{S/C}(\mathcal{A}).$$

Since f_n is a unit in $S_{\mathfrak{p}}$, the map i localizes to an isomorphism of $S_{\mathfrak{p}}$-modules. Since $\mathcal{A}$ is irreducible, we may write $f_n = \sum_{i=1}^{n-1} c_i f_i$ for some scalars $c_i \in \mathbb{C}$.

Then we have an isomorphism of cochain complexes

$$\begin{aligned} (\Omega^\bullet_{S/C}(\mathcal{A}), \omega_{\mathbf{a}})_{\mathfrak{p}} &\cong (\Omega^\bullet_{S'/C'}(\mathcal{A}') \otimes_{\mathbb{C}} \mathbb{C}[a_n], \omega_{\mathbf{a}'} + a_n \frac{\mathrm{d}f_n}{f_n})_{\mathfrak{p}} \\ &\cong (\Omega^\bullet_{S'/C'}(\mathcal{A}') \otimes_{\mathbb{C}} \mathbb{C}[a_n], \eta)_{\mathfrak{p}}, \end{aligned} \tag{3.8}$$

where the differential is given by multiplication by

$$\eta = \sum_{i=1}^{n} (a_i + \frac{c_i f_i a_n}{f_n}) \frac{\mathrm{d}f_i}{f_i}.$$

Define a homomorphism $\phi \colon S_{f_n} \to S_{f_n}$ by setting $\phi(x_i) = x_i$ for $1 \leq i \leq \ell$ and $\phi(a_i) = a_i - c_i f_i a_n / f_n$ for $1 \leq i \leq n-1$, and $\phi(a_n) = a_n$. Note ϕ is an isomorphism, so it localizes to an isomorphism of local rings $S_{\mathfrak{p}} \to S_{\mathfrak{p}'}$, where $\mathfrak{p}' = (\phi^{-1})^*(\mathfrak{p})$.

By construction, ϕ induces an isomorphism on forms with $\phi(\eta) = \omega_{\mathbf{a}'}$, taking (3.8) to the cochain complex

$$(\Omega^\bullet_{S'/C'}(\mathcal{A}') \otimes_{\mathbb{C}} \mathbb{C}[a_n], \omega_{\mathbf{a}'})_{\mathfrak{p}'}.$$

The cohomology of this complex is concentrated in top degree, by induction, so the complex (3.3) is exact at $\mathfrak{m}$: in fact, we have

$$\big(S/I(\mathcal{A})\big)_{\mathfrak{p}} \cong \big(S/SI(\mathcal{A}')\big)_{\mathfrak{p}'}. \tag{3.9}$$

It remains to consider the case where $X = 0$, i.e., $\mathfrak{m} = R_+$. From the previous argument, (3.3) is exact at all other maximal primes, so some power of R_+ annihilates $H^q := H^q(\Omega^\bullet_{S/C}(\mathcal{A}), \omega_{\mathbf{a}})$, for $0 \leq q < \ell$. The following lemma shows that power must be zero, completing the proof of exactness.

Lemma 3.11. *Suppose $\mathcal{A}$ is a tame arrangement. Let q be the least integer for which $H^q \neq 0$. If $q < \ell$, then H^q is R_+-saturated.*

Proof. Equivalently, we wish to show that the local cohomology group $H^0_{R_+}(H^q) = 0$. For this, consider the two hypercohomology spectral sequences of local cohomology, writing $\Omega^\bullet$ in place of $\Omega^\bullet_{S/C}(\mathcal{A})$:

$$\begin{aligned} {}'E_2^{pq} = H^p(H^q_{R_+}(\Omega^\bullet)) &\Rightarrow \mathbb{H}^{p+q}_{R_+}(\Omega^\bullet), \quad \text{and} \\ {}''E_2^{pq} = H^p_{R_+}(H^q(\Omega^\bullet)) &\Rightarrow \mathbb{H}^{p+q}_{R_+}(\Omega^\bullet). \end{aligned}$$

The tame hypothesis implies that $H^q_{R_+}(\Omega^p) = 0$ for $0 \le q < \ell - p$. Then, from the first spectral sequence, we obtain $\mathbb{H}^k_{R_+}(\Omega^\bullet) = 0$ for $0 \le k < \ell$.

On the other hand, consider the least q for which $H^q = H^q(\Omega^\bullet) \neq 0$. Then if $q < \ell$, we must have $''E^{0q}_\infty = {}''E^{0q}_2 = 0$. So $H^0_{R_+}(H^q) = 0$, as required. □

Remark 3.12. The hypothesis that $\mathcal{A}$ is tame was required only to show that the complex (3.3) was exact when localized at R_+; other localizations followed by induction. Theorem 3.5 can then be extended slightly as follows.

Theorem 3.13. *If $\mathcal{A}$ is an essential arrangement for which all proper subarrangements $\mathcal{A}_X$ are tame, then the complex of coherent sheaves on $\mathbb{P}^{\ell-1} \times \mathbb{P}^{n-1}$*

$$0 \to \widetilde{\Omega}^0_{S/C}(\mathcal{A}) \to \widetilde{\Omega}^1_{S/C}(\mathcal{A}) \to \cdots \to \widetilde{\Omega}^\ell_{S/C}(\mathcal{A}) \to \mathcal{O}_{\mathbb{P}\overline{\Sigma}}(n-\ell) \to 0$$

is exact.

3.4. **Proof of Theorem 2.9.** The argument that the variety of the logarithmic ideal $I(\mathcal{A})$ equals the closure of $\Sigma(\mathcal{A})$ is parallel to the proof of Theorem 3.5, so we include it here to avoid unnecessary repetition. We note, however, that the arrangement $\mathcal{A}$ is not assumed to be tame here.

Proof. Again, argue by induction on n, the number of hyperplanes. If $n = 1$, then $\overline{\Sigma} = V(I) = \{(0,0)\}$. If $n > 1$, it suffices to consider irreducible arrangements, using the induction hypothesis and Lemma 3.10. Clearly $\Sigma \subseteq V(I)$. If $(x,\lambda) \in V(I) - \Sigma$, we argue that it has a neighborhood that intersects Σ.

First, consider the case where $x = 0$. Since $\mathcal{A}$ is irreducible, by Proposition 2.8, $V(I)$ is given in a neighborhood of $x = 0$ by the equation $\sum_{i=1}^n a_i = 0$. Comparing with Proposition 2.6 establishes the claim.

Otherwise, since $\mathcal{A}$ is assumed to be essential, we assume again that the last hyperplane of $\mathcal{A}$ does not contain the point x. Let $\mathcal{A}'$ denote the deletion, following the notation of §3.3. From (3.9), ϕ^* gives a homeomorphism between neighborhoods of $(x,\lambda) \in V(I(\mathcal{A}))$ and $(x,\lambda') \in V(I(\mathcal{A}')) \times \mathbb{C}$, where $\lambda'_i = \lambda_i + c_i f_i(x)\lambda_n/f_n(x)$ for $1 \le i \le n-1$ and $\lambda'_n = \lambda_n$. By the induction hypothesis, the neigborhood of (x,λ') meets $\Sigma(\mathcal{A}') \times \mathbb{C}$, which means the neighborhood of (x,λ) in $V(I)$ meets $\Sigma(\mathcal{A})$, as required. □

4. Resonant 1-forms have high-dimensional zero loci

The purpose of this section is to establish the following result.

Theorem 4.1. *Let $\mathcal{A}$ be a tame arrangement of n hyperplanes in V. If $\lambda \in \mathbb{C}^n$ is a vector of weights for which $H^p(A(\mathcal{A}), \omega_\lambda) \neq 0$, then the codimension of the critical set $\overline{\Sigma}_\lambda$ is at most p, provided either $\mathcal{A}$ is free or $p \le 2$.*

Nonzero λ for which $H^1(A, \omega_\lambda) \neq 0$ have been studied extensively: see [FY07]. For such λ, if A is tame, then we see $\overline{\Sigma}_\lambda$ is a hypersurface.

Since rank 3 arrangements are tame [WY97], we also find:

Corollary 4.2. *If $\mathcal{A}$ has rank 3 and $\lambda \in \mathbb{C}^n$ is a collection of weights for which $H^p(A(\mathcal{A}), \omega_\lambda) \neq 0$, then the codimension of $\overline{\Sigma}_\lambda$ is at most p.*

To prove the theorem, we first show that resonance in dimension p implies that the cohomology of the log complex $\Omega^\bullet(\mathcal{A})$, with differential $\nabla = \mathrm{d} + \omega_\lambda$, is also nontrivial in dimension p.

Proposition 4.3. *For each $t \in \mathbb{C}^*$, the inclusion $(A(\mathcal{A}), t\omega_\lambda) \to (\Omega^\bullet(\mathcal{A}), \nabla_t)$ induces a monomorphism $H^\bullet(A(\mathcal{A}), t\omega_\lambda) \to H^\bullet(\Omega^\bullet(\mathcal{A}), \nabla_t)$. If $H^p(A(\mathcal{A}), \omega_\lambda) \neq 0$, then $H^p(\Omega^\bullet(\mathcal{A}), \nabla) \neq 0$.*

Proof. For $t \neq 0$, let $\nabla_t = \mathrm{d} + t\omega_\lambda$. For t sufficiently small, the inclusion $(A(\mathcal{A}), t\omega_\lambda) \hookrightarrow (\Omega^\bullet(*\mathcal{A}), \nabla_t)$ is a quasi-isomorphism, by [SV91, Theorem 4.6]. Consequently, the sequence of inclusions

$$(A(\mathcal{A}), t\omega_\lambda) \hookrightarrow (\Omega^\bullet(\mathcal{A}), \nabla_t) \hookrightarrow (\Omega^\bullet(*\mathcal{A}), \nabla_t)$$

implies that the map

$$H^\bullet(A(\mathcal{A}), t\omega_\lambda) \to H^\bullet(\Omega^\bullet(\mathcal{A}), \nabla_t)$$

in cohomology is a monomorphism. Since $H^\bullet(A(\mathcal{A}), t\omega_\lambda) = H^\bullet(A(\mathcal{A}), \omega_\lambda)$, if $H^p(A(\mathcal{A}), \omega_\lambda) \neq 0$, then $H^p(\Omega^\bullet(\mathcal{A}), \nabla_t) \neq 0$. The results then follows from the upper semicontinuity with respect to t of $H^p(\Omega^\bullet(\mathcal{A}), \nabla_t)$. ☐

In light of this result, to prove Theorem 4.1, it suffices to show that the nonvanishing of $H^p(\Omega^\bullet(\mathcal{A}), \nabla)$ implies that of $H^p(\Omega^\bullet(\mathcal{A}), \omega_\lambda)$. For this, we will use a spectral sequence, following Farber [Far01, Far04].

If C is a cochain complex equipped with two differentials d and δ satisfying $d \circ \delta + \delta \circ d = 0$, then for each $t \in \mathbb{C}$, $(C, d + t\delta)$ is a cochain complex. In [Far04], Farber constructs a spectral sequence converging to the cohomology $H^\bullet(C, d + t\delta)$, with E_1 term given by $E_1^{p,q} = H^{p+q}(C, d)$ for all $q \geq 0$, and $d_1 : H^{p+q}(C, d) \to H^{p+q+1}(C, d)$ induced by δ. For r sufficiently large, the differential d_r vanishes, and $E_\infty^{p,q} \cong H^{p+q}(C, d + t\delta)$ for all but finitely many $t \in \mathbb{C}$, see [Far04, §10.8].

For each $m \in \mathbb{Z}$, we use this construction to analyze the cohomology of the complex

$$(\Omega^\bullet(\mathcal{A})_m, \nabla_t) = (\Omega^\bullet(\mathcal{A})_m, \mathrm{d} + t\omega_\lambda). \tag{4.1}$$

In many cases, the monomorphism of Proposition 4.3 is actually an isomorphism. In [WY97], Wiens and Yuzvinsky show that if $\mathcal{A}$ is a tame arrangement (Definition 2.2), then $H^\bullet(\Omega^\bullet(\mathcal{A}), \mathrm{d}) \cong A(\mathcal{A})$. That is, the logarithmic forms compute the cohomology of the complement.

Proposition 4.4. *Suppose that $\mathcal{A}$ is a tame arrangement. If $m \neq 0$, then we have $H^\bullet(\Omega^\bullet(\mathcal{A})_m, \mathrm{d} + t\omega_\lambda) = 0$. Furthermore, for $m = 0$, the inclusion $(A(\mathcal{A}), t\omega_\lambda) \hookrightarrow (\Omega^\bullet(\mathcal{A})_0, \mathrm{d} + t\omega_\lambda)$ induces an isomorphism in cohomology for all but finitely many t.*

Proof. By the main theorem of [WY97], $H^\bullet(\Omega^\bullet(\mathcal{A}), \mathrm{d}) = A(\mathcal{A})$. Consequently, in the Farber spectral sequence for the complex (4.1), we have

$$E_1^{p,q} = H^{p+q}(\Omega^\bullet(\mathcal{A})_m, \mathrm{d}) = \begin{cases} 0 & \text{for } m \neq 0, \\ A^{p+q}(\mathcal{A}) & \text{for } m = 0. \end{cases}$$

The first assertion follows immediately. For $m = 0$, since multiplication by ω_λ induces the differential $d_1\colon E_1^{p,q} \to E_1^{p+1,q}$, the E_2-term of the spectral sequence is $E_2^{p,q} = H^{p+q}(A(\mathcal{A}), \omega_\lambda)$. By Proposition 4.3, the vector space $E_2^{p,q}$ is a subspace of $E_\infty^{p,q} = H^{p+q}(\Omega^\bullet(\mathcal{A})_0, \nabla_t)$ for large p. The result follows. $\square$

Proof of Theorem 4.1. It suffices to show that if $H^p(A(\mathcal{A}), \omega_\lambda) \neq 0$ does not vanish, then $H^p(\Omega^\bullet(\mathcal{A}), \omega_\lambda) \neq 0$ as well.

For $t \in \mathbb{C}^*$, the map $\phi\colon \big(\Omega^\bullet(\mathcal{A}), \mathrm{d} + t\omega_\lambda\big) \to \big(\Omega^\bullet(\mathcal{A}), \omega_\lambda + \frac{1}{t}\mathrm{d}\big)$ defined by $\phi(\eta) = \big(\frac{1}{t}\big)^q \eta$ for $\eta \in \Omega^q(\mathcal{A})$ is a cochain map, and is an isomorphism. This fact, together with Proposition 4.3, implies that $H^\bullet(\Omega^\bullet(\mathcal{A})_m, \omega_\lambda + t'\mathrm{d}) = 0$ for $m \neq 0$ and that $H^\bullet(\Omega^\bullet(\mathcal{A})_0, \omega_\lambda + t'\mathrm{d}) \cong H^\bullet(A(\mathcal{A}), \omega_\lambda)$ for all but finitely many t'.

The Farber spectral sequence of the complex $\big(\Omega^\bullet(\mathcal{A})_0, \omega_\lambda + t'd\big)$ has E_1-term $H^\bullet(\Omega^\bullet(\mathcal{A})_0, \omega_\lambda)$, and abuts to $H^\bullet(\Omega^\bullet(\mathcal{A})_0, \omega_\lambda + t'd) \cong H^\bullet(A(\mathcal{A}), \omega_\lambda)$ for generic t'. Consequently, the assumption that $H^p(A(\mathcal{A}), \omega_\lambda) \neq 0$ implies that $H^p(\Omega^\bullet(\mathcal{A})_0, \omega_\lambda) \neq 0$ as well. Hence, $H^p(\Omega^\bullet(\mathcal{A}), \omega_\lambda) \neq 0$. Now use Proposition 3.9: if $\mathcal{A}$ is free or $p \leq 2$, the codimension of $\overline{\Sigma}_\lambda$ is at most p. $\square$

5. Examples and Counterexamples

If Φ_λ is a master function, recall that $\mathcal{L}_\lambda$ denotes the corresponding complex, rank one local system on the complement M of the underlying arrangement $\mathcal{A}$. As noted in the Introduction, for sufficiently generic weights λ, the inclusion of the Orlik-Solomon complex $(A(\mathcal{A}), \omega_\lambda)$ in the twisted de Rham complex $(\Omega^\bullet(*\mathcal{A}), \mathrm{d} + \omega_\lambda)$ induces an isomorphism $H^\bullet(A(\mathcal{A}), \omega_\lambda) \cong H^\bullet(M; \mathcal{L}_\lambda)$. See [ESV92, STV95] for conditions on λ which insure that this isomorphism holds.

In light of this relationship between the Orlik-Solomon cohomology $H^\bullet(A(\mathcal{A}), \omega_\lambda)$ and the local system cohomology $H^\bullet(M; \mathcal{L}_\lambda)$, one might expect a correspondence between the non-vanishing of local system cohomology and the codimension of the critical set of Φ_λ, analogous to that established in Theorem 4.1. Such a correspondence does not hold, as the following family of examples illustrate.

Example 5.1. Let r be a natural number, and α, β, γ complex numbers. The master function

$$\Phi = x_1^{r\alpha} x_2^{r\beta} (x_1^r - x_2^r)^\gamma (x_1^r - x_3^r)^\beta (x_2^r - x_3^r)^\alpha$$

determines a local system $\mathcal{L}$ on the complement M of the arrangement $\mathcal{A}$ with defining polynomial $Q(\mathcal{A}) = x_1 x_2 (x_1^r - x_2^r)(x_1^r - x_3^r)(x_2^r - x_3^r)$. Note that $\mathcal{A}$ has $3r+2$ hyperplanes, and let $\lambda \in \mathbb{C}^{3r+2}$ denote the collection of weights corresponding to Φ. The one-form $\omega_\lambda = \mathrm{d}\log\Phi$ is given by $\omega_\lambda = d_1 \mathrm{d}x_1 + d_2 \mathrm{d}x_2 + d_3 \mathrm{d}x_3$, where

$$d_1 = \frac{r\alpha}{x_1} + \frac{r x_1^{r-1}\gamma}{x_1^r - x_2^r} + \frac{r x_1^{r-1}\beta}{x_1^r - x_3^r},\ d_2 = \frac{r\beta}{x_2} + \frac{r x_2^{r-1}\gamma}{x_2^r - x_1^r} + \frac{r x_2^{r-1}\alpha}{x_2^r - x_3^r},\ d_3 = \frac{r x_3^{r-1}\alpha}{x_3^r - x_2^r} + \frac{r x_3^{r-1}\beta}{x_3^r - x_1^r},$$

and the critical set $\Sigma_\lambda = V(\omega_\lambda)$ by $\Sigma_\lambda = V(d_1, d_2, d_3) \subseteq M$.

The arrangement $\mathcal{A}$ is supersolvable, hence free. The module $\mathrm{Der}(\mathcal{A})$ has basis

$$D_1 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}, \qquad D_2 = x_1^{r+1}\frac{\partial}{\partial x_1} + x_2^{r+1}\frac{\partial}{\partial x_3} + x_3^{r+1}\frac{\partial}{\partial x_3},$$
$$D_3 = x_1x_2(x_1x_2x_3)^{r-1}\left(x_1^{1-r}\frac{\partial}{\partial x_1} + x_2^{1-r}\frac{\partial}{\partial x_3} + x_3^{1-r}\frac{\partial}{\partial x_3}\right),$$

see [OT92, Prop. 6.85]. Consequently, the ideal I_λ is generated by $d_i' = \langle D_i, \omega_\lambda\rangle$, $1 \le i \le 3$, where

$$d_1' = r(2\alpha+2\beta+\gamma),\ d_2' = r(\alpha+\beta+\gamma)(x_1^r+x_2^r)+r(\alpha+\beta)x_3^r,\ d_3' = r(\beta x_1^r+\alpha x_2^r)x_3^{r-1},$$

and $\overline{\Sigma}_\lambda = V(I_\lambda) = V(d_1', d_2', d_3') \subseteq \mathbb{C}^3$. Observe that if $2\alpha+2\beta+\gamma \neq 0$, then $\overline{\Sigma}_\lambda = \emptyset$ is empty, and hence $\Sigma_\lambda = \overline{\Sigma}_\lambda \cap M = \emptyset$ is empty as well.

Let q be a natural number with $1 \le q \le r-1$, and assume that α, β, γ satisfy $\alpha+\beta+\gamma \in \mathbb{Z}$ and $\gamma = -q/r$. In this instance, it is known that the first local system cohomology group is non-zero, $H^1(M; \mathcal{L}_\lambda) \neq 0$, while the first Orlik-Solomon cohomology group vanishes, $H^1(A(\mathcal{A}), \omega_\lambda) = 0$, see [Coh02, Suc02]. However, for such α, β, γ, one has $2\alpha+2\beta+\gamma \neq 0$, so $\Sigma_\lambda = \emptyset$ and $\overline{\Sigma}_\lambda = \emptyset$ as noted above.

Other choices of α, β, γ may be used to illustrate that the variety $\overline{\Sigma}_\lambda$ is not, in general, the closure of Σ_λ, in contrast to the result of Theorem 2.9 for the variety Σ. This is the case, for example, if $\alpha+\beta = 0$ and $\gamma = 0$. Here, $\overline{\Sigma}_\lambda = V((x_1^r - x_2^r)x_3)$, while $\Sigma_\lambda = V(x_3)$.

The last example above may also be used to show that a converse of Theorem 4.1 cannot hold. That is, a master function with positive-dimensional critical set need not, in general, correspond to weights which are resonant in the corresponding dimension.

Example 5.2. Let r be a natural number, and α, β complex numbers. The master function

$$\Phi = x_1^{r\alpha}x_2^{r\beta}(x_1^r - x_3^r)^\beta(x_2^r - x_3^r)^\alpha$$

determines a local system $\mathcal{L}$ on the complement M of the arrangement $\mathcal{A}$ with defining polynomial $Q(\mathcal{A}) = x_1x_2(x_1^r - x_3^r)(x_2^r - x_3^r)$. Note that $\mathcal{A}$ has $2r+2$ hyperplanes, and let $\lambda \in \mathbb{C}^{2r+2}$ denote the collection of weights corresponding to Φ. The arrangement $\mathcal{A} \subset \mathbb{C}^3$ is not free, but is tame.

The one-form $\omega_\lambda = \mathrm{d}\log\Phi$ is given by $\omega_\lambda = d_1\mathrm{d}x_1 + d_2\mathrm{d}x_2 + d_3\mathrm{d}x_3$, where

$$d_1 = \frac{r\alpha}{x_1} + \frac{rx_1^{r-1}\beta}{x_1^r - x_3^r},\ d_2 = \frac{r\beta}{x_2} + \frac{rx_2^{r-1}\alpha}{x_2^r - x_3^r},\ d_3 = \frac{rx_3^{r-1}\alpha}{x_3^r - x_2^r} + \frac{rx_3^{r-1}\beta}{x_3^r - x_1^r},$$

and the critical set $\Sigma_\lambda = V(\omega_\lambda)$ by $\Sigma_\lambda = V(d_1, d_2, d_3) \subseteq M$. If $\alpha+\beta = 0$, it is readily checked that $\Sigma_\lambda = V(x_3) \subset M$ is one-dimensional. However, if $\alpha \neq 0$, then $H^1(A(\mathcal{A}), \omega_\lambda) = 0$, and if the local system $\mathcal{L}_\lambda$ corresponding to λ is nontrivial, then $H^1(M; \mathcal{L}_\lambda) = 0$.

Example 5.3. Consider the arrangement in $\mathbb{P}^3$ given by the nine linear forms x_1, x_2, x_3, $x_i + x_4$ for $1 \le i \le 3$, and $x_i + x_j + x_4$, for $1 \le i < j \le 3$. A computation with Macaulay 2 [GS] shows that S/I is not Cohen-Macaulay: the projective dimension of S/I is 5, while the codimension is 4. It follows that the arrangement is not tame, which can also be verified directly. Accordingly, the ideal I has an embedded prime (x_1, x_2, x_3, x_4), so we see that Corollary 3.8 requires the hypothesis that $\mathcal{A}$ is tame.

On the other hand, further calculation shows that the complex (3.3) is exact for this arrangement, in contrast to Example 5.6 of [OT95b]. It would be interesting to know, then, if Theorem 3.5 holds without hypothesis. For this example, the logarithmic comparison isomorphism $H^\bullet(\Omega^\bullet(\mathcal{A}), \mathrm{d}) \cong A(\mathcal{A})$ holds, since the rank is 4, by [WY97, Corollary 6.3]. However, we also do not know if this isomorphism holds in general.

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