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Compact Group Actions, Spherical Bessel Functions, and Invariant Random Variables*

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The theory of compact group actions on locally compact abelian groups provides a unifying theory under which different invariance conditions studied in several contexts by a number of statisticians are subsumed as special cases. For example, Schoenberg's characterization of radially symmetric characteristic functions on $\mathbb{R}^n$ is extended to this general context and the integral representations are expressed in terms of the generalized spherical Bessel functions of Gross and Kunze. These same Bessel functions are also used to obtain a variant of the Lévy–Khinchine formula of Parthasarathy, Ranga Rao, and Varadhan appropriate to invariant distributions.

0. INTRODUCTION

The theory of compact group actions on locally compact abelian groups, as developed in [8], provides a general framework for the study of invariant distributions and the unification of a variety of different kinds of

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invariance considered in the literature. For example, Schoenberg's characterization [15] of radially symmetric distributions, when considered from our viewpoint, may be regarded as classifying the positive definite functions on $\mathbb{R}^n$ that are invariant under the natural action of the group $O(n)$ of orthognal $n \times n$ matrices. In this classical situation, Schoenberg showed that the characteristic function of a radially symmetric distribution can be represented by an integral of certain Bessel functions.

With this in mind we generalize Schoenberg's results from the orthogonal action on $\mathbb{R}^n$ to a compact group U acting on a locally compact abelian topological group X. Our work, therefore, gives a general context for the investigations of Dawid [5], Anderson and Fang [1], James [10], and other statisticians who have considered specific orthogonal actions on Euclidean spaces as they pertain to invariant distributions.

In the extension of Schoenberg's results to the compact transformation group (U, X), the kernel for realizing the characteristic function of a U-invariant distribution is the *spherical Bessel function* for the transformation group (U, X) that was defined and studied by Gross and Kunze [8].

This spherical Bessel function also relates to other statistical problems. For example, it allows us to adapt the work of Parthasarathy, Ranga Rao, and Varadhan [11] to prove a Lévy–Khinchine formula for infinitely divisible U-invariant characteristic functions on X.

This paper is organized as follows. The context for the first eight chapters is the general theory of a compact topological transformation group (U, X) in which the compact group U acts by automorphisms on a locally compact abelian topological group X. In particular, in Chapters 3 and 4 we study the orbit space $U\backslash X$ and describe the decomposition theorem for a U-invariant measure, a result crucial to much of what follows. Chapter 5 and 6 deal with the spherical Bessel function and the generalized Schoenberg theorem for (U, X). Our Lévy–Khinchine formula is the subject of Chapters 7 and 8.

The last four chapters are specialized to the case in which X is a finite-dimensional real inner product space and U is an orthogonal action on X. Chapter 9 deals with a general form of "polar coordinates" for (U, X), and Chapter 10 shows how the abstract theory of these orthogonal transformation groups applies to a variety of examples including the classical context of Schoenberg; its generalization to matrix spaces considered by James, Richards, and others; the "rotatable" distributions of Dawid; and the invariance models of Anderson and Fang. The paper concludes with a general result in Chapters 11 and 12 on stochastic representations for U-invariant random variables.

1. Fourier Analysis on Locally Compact Abelian Groups

Throughout the paper X denotes a locally compact abelian (Hausdorff) topological group. We recall that the *dual group* $\hat{X}$ consists of all continuous homomorphisms of X into the multiplicative group of complex numbers of unit modulus. Then $\hat{X}$ itself becomes a locally compact abelian (Hausdorff) group relative to the compact-open topology. For $\beta \in \hat{X}$ and $x \in X$, we write $(\beta \mid x)$ for $\beta(x)$. Pontryagin duality identifies $(\hat{X})\hat{}$ with X through the formula $(x \mid \beta) = (\beta \mid x)$ for $x \in X$ and $\beta \in \hat{X}$.

For a finite positive Borel measure m on X, its *Fourier transform* is the function $\hat{m}$ on $\hat{X}$ given by

$$\hat{m}(\beta) = \int_X (\beta \mid x)\, dm(x), \qquad \beta \in \hat{X}. \tag{1.1}$$

The mapping $m \to \hat{m}$ is one-to-one. If m is a probability measure then $\hat{m}$ is called its *characteristic function*. We refer to [14, 16] for details.

2. Compact Actions

Throughout, U denotes a compact group acting on X by automorphisms. That is to say, (U, X) is a *compact transformation group*. Thus, with $u(x)$ denoted by ux, the following properties hold: The mapping $(u, x) \to ux$ is continuous from $U \times X$ to X; $(u_1 u_2)\, x = u_1(u_2 x)$ for all x in X and u_1, u_2 in U; and $x \to ux$ is a continuous automorphism of the group X for each $u \in U$. By duality, $(U, \hat{X})$ is also a compact transformation group with the action defined by

$$(u\beta \mid x) = (\beta \mid u^{-1}x) \tag{2.1}$$

for $(u, \beta, x) \in U \times \hat{X} \times X$.

3. The Orbit Space and Invariance

The *orbit space* for the action of U on X, denoted $U\backslash X$, consists of all *orbits* $Ux = \{ux: u \in U\}$. Equipped with the quotient topology the orbit space is a locally compact Hausdorff space. The canonical quotient map $\pi: x \to Ux$ is open and continuous from X to $U\backslash X$.

A function f on X is called *U-invariant* if $f(ux) = f(x)$ for all $(u, x) \in U \times X$. Equivalently, such an f is constant on orbits. This means that a U-invariant function f can be regarded as a function $\tilde{f}$ on $U\backslash X$, where

$$\tilde{f} \circ \pi = f. \tag{3.1}$$

In order to construct U-invariant functions, let f be any continuous function f on X. Define

$$f_1(x) = \int_U f(ux)\, du, \tag{3.2}$$

where du is normalized Haar measure on U. By the right invariance of Haar measure, f_1 is U-invariant. By standard measure theory, formula (3.2) is defined almost surely in X for $f \in L^1(X, m)$.

Similarly, a measure m on X is *U-invariant* if $m(uB) = m(B)$ for all $u \in U$ and Borel sets B in X. The following theorem shows that there is a one-to-one correspondence between invariant measures m on X and measures $\tilde{m}$ on $U\backslash X$. See [3, Sect. 2; 2, Sect. 3; or 8, Sect. 2].

4. Theorem

Let m be a U-invariant measure on X. Then there exists a unique measure $\tilde{m}$ on $U\backslash X$ such that

$$\int_X f(x)\, dm(x) = \int_{U/X} \left(\int_U f(ux)\, du \right) d\tilde{m}(\pi(x)) \tag{4.1}$$

for all $f \in L^1(X, m)$.

In other words,

$$\int_X f(x)\, dm(x) = \int_{U/X} \tilde{f}_1(\pi(x))\, d\tilde{m}(\pi(x)). \tag{4.2}$$

The measure $\tilde{m}$ on $U\backslash X$ is called the *quotient measure* corresponding to the U-invariant measure m. This "decomposition" of an invariant measure is crucial to later results.

5. The Spherical Bessel Function and Schoenberg's Theorem

The *spherical Bessel function* for the transformation group (U, X) is the function J_0 on $\tilde{X} \times X$ defined by

$$J_0(\beta, x) = \int_U (\beta \mid ux)\, du \tag{5.1}$$

for $(\beta, x) \in \tilde{X} \times X$. Note that J_0 is U-invariant in each variable; that is,

$$J_0(\beta, x) = J_0(u\beta, x) = J_0(\beta, ux) \tag{5.2}$$

for all $(u, \beta, x) \in U \times \hat{X} \times X$.

The spherical Bessel functions were defined in [8] in connection with the Fourier analysis of (U, X). To briefly indicate their significance, note that for a U-invariant finite measure m on X,

$$\hat{m}(\beta)=\int_X J_0(\beta, x)\, dm(x) \tag{5.3}$$

for all $\beta \in \hat{X}$. Indeed, since m is U-invariant, so is $\hat{m}$; for if $u \in U$ and $\beta \in \hat{X}$ then

$$\hat{m}(u\beta)=\int_X (u\beta \mid x)\, dm(x)=\int_X (\beta \mid u^{-1}x)\, dm(x)$$

$$=\int_X (\beta \mid x)\, dm(ux)=\int_X (\beta \mid x)\, dm(x)=\hat{m}(\beta).$$

Therefore,

$$\hat{m}(\beta)=\int_U \hat{m}(u^{-1}\beta)\, du=\int_U \left(\int_X (u^{-1}\beta \mid x)\, dm(x)\right) du$$

$$=\int_X \left(\int_U (\beta \mid ux)\, du\right) dm(x)=\int_X J_0(\beta, x)\, dm(x).$$

The following theorem generalizes the decomposition theorem for $\mathbb{R}^n$, first proved by Schoenberg [14].

6. Theorem

Let m be a U-invariant probability measure on X and let $\tilde{m}$ be the corresponding quotient measure on $U\backslash X$. Then $\tilde{m}$ is a probability measure, and

$$\hat{m}(\beta)=\int_{U\backslash X} J_0(\beta, x)\, d\tilde{m}(\pi(x))$$

for $\beta \in \hat{X}$.

Proof. We apply Theorem 4 to (5.3). Thus, for U-invariant m,

$$\hat{m}(\beta)=\int_X J_0(\beta, x)\, dm(x)$$

$$=\int_{U\backslash X} \left(\int_U J_0(\beta, ux)\, du\right) d\tilde{m}(\pi(x))$$

$$=\int_{U\backslash X} J_0(\beta, x)\, d\tilde{m}(\pi(x))$$

by (5.2) and the fact that du has mass 1. All that remains is to verify, by the substitution $f=1$ in Theorem 4, that $\tilde{m}$ is a probability measure if m is a probability measure.

7. The Spherical Bessel Function and the Lévy–Khinchine Formula

We relate the spherical Bessel function for (U, X) to infinitely divisible probability measures. Our version of the Lévy–Khinchine representation relies upon the work of Parthasarathy, Ranga Rao, and Varadhan [11], but in part is motivated by the variant of their theorem due to Gangolli [6].

Let X be separable and metric and μ a probability measure on X. Following [11], we say μ is *idempotent* if $\mu * \mu = \mu * \delta_x$ for some x in X; where $*$ denotes convolution and δ_x is the Dirac measure at x. The measure μ_1 is said to be a *factor* of μ if $\mu = \mu_1 * \mu_2$ for some measure μ_2. Also, μ is called *infinitely divisible* if for each positive integer n, there exist x_n in X and a measure λ_n such that $\mu = \lambda_n^n * \delta_{x_n}$, where $\lambda_n^n = \lambda_n * \lambda_n * \cdots * \lambda_n$ (n convolutions). The proof of the following Lévy–Khinchine theorem is derived from Theorem 7.1 of [11] by applying the "radializing" formula, (3.2), above.

8. Theorem

Let X be a locally compact abelian separable metric group, U be as before, and m be an infinitely divisible, U-invariant probability measure on X. If m has no idempotent factors and $\hat{m}$ is real-valued, then $\hat{m}$ has the representation

$$\hat{m}(\beta) = \exp\left[\int_X \{J_0(\beta, x) - 1\}\, dF(x) - \Phi(\beta)\right], \qquad \beta \in \hat{X}.$$

The measure F is σ-finite on X; it has finite mass outside every neighborhood of the identity in X; and

$$\int_X [1 - J_0(\beta, x)]\, dF(x) < \infty$$

for all $\beta \in \hat{X}$. Also, $\Phi(\beta)$ is a non-negative, continuous U-invariant function satisfying

$$\Phi(\beta_1 + \beta_2) + \Phi(\beta_1 - \beta_2) = 2[\Phi(\beta_1) + \Phi(\beta_2)]$$

for all β_1, $\beta_2 \in \hat{X}$.

9. Orthogonal Actions and Polar Coordinates

Suppose now that X is a finite dimensional real vector space with an inner product and that the elements of U act orthogonally on X; i.e., $\|ux\| = \|x\|$ for all $(u, x) \in U \times X$, where $\|\cdot\|$ is the norm derived from the inner product. In this case one calls (U, X) an *orthogonal transformation group.*

Throughout the remainder of the paper we assume (U, X) is an orthogonal transformation group.

By polar coordinates for (U, X) we mean a decomposition $\Sigma \times C$ for a dense open subset X^r of X whose complement has Lebesgue measure zero. More specifically, we suppose that $\Sigma = U/U_0$ for a fixed closed subgroup U_0 of U, C is an open subset of the subspace X_0 of points in X fixed by U_0, for each $r \in C$ the orbit Ur is equivalent to Σ, and the mapping $(uU_0, r) \to ur$ is a diffeomorphism of $\Sigma \times C$ onto X^r. The points in X^r are called *regular.*

In the presence of polar coordinates, the orbit space (except for a null set) can be identified with C, and the spherical Bessel function is determined more explicitly on C. The following examples illustrate this phenomenon.

10. Examples

(A) Let $X = \mathbb{R}^n$ with the usual inner product $\langle \cdot | \cdot \rangle$ and $U = O(n)$, the group of $n \times n$ orthogonal matrices. U acts on X by left matrix multiplication, $x \to ux$, $u \in U$, $x \in X$. The dual group $\hat{X}$ may be identified with $\mathbb{R}^n$ via the inner product, since every element of $\hat{X}$ is of the form $\exp(i < \beta, \cdot \rangle)$, $\beta \in \mathbb{R}^n$. Here, we have the usual polar coordinates: $X^r = \{x \in \mathbb{R}^n : x \neq 0\}$; Σ is the unit sphere, realized as the set $U\sigma_0$, where $\sigma_0 = (1, 0, \dots, 0)'$ and $U_0 = \{u \in U : u\sigma_0 = \sigma_0\} \cong O(n-1)$; and $C = \{r\sigma_0 \in \mathbb{R}^n : r > 0\} \cong (0, \infty)$. Thus, for $x \neq 0$ in X, $\pi(x)$ is identified with the number $r = \|x\|$.

If $\beta, x \in \mathbb{R}^n$, then

$$J_0(\beta, x) = \int_{O(n)} \exp(i\langle u^{-1}\beta, x\rangle)\, du$$

$$= \Omega_n(\|\beta\| \cdot \|x\|),$$

where $\Omega_n(t) = 2^\nu \Gamma(n/2)\ t^{-\nu} J_\nu(t)$, $t > 0$. Here, $\nu = (n-2)/2$, and J_ν is the classical Bessel function of the first kind of order ν. Putting these facts together, we recover Schoenberg's classical result from Theorem 6: If m is an $O(n)$-invariant probability measure on $\mathbb{R}^n$, then $\hat{m}(x) = \phi(\|x\|)$, where

$$\phi(t) = \int_0^\infty \Omega_n(tr)\, d\tilde{m}(r), \qquad t \geqslant 0,$$

and $\hat{m}$ is a probability measure on $[0, \infty)$.

Here, we call the reader's attention to the fact that the function J_0, originally defined on $\mathbb{R}^n \times \mathbb{R}^n$ is ultimately determined by a function on $\mathbb{R}$.

(B) For $n \geqslant k$, let $X = \mathbb{R}^{n \times k}$, the space of $n \times k$ real matrices, with inner product $\langle x, y \rangle = \mathrm{tr}(x'y)$, where x' denotes the transpose of x, and $U = O(n)$. The group U acts "one-sidedly" on X by left matrix multiplication, $x \to ux$, $u \in U$, $x \in X$. $\hat{X}$ may be identified with X since the homomorphisms in $\hat{X}$ are of the form $\exp(i\, \mathrm{trace}(\beta' x))$. Here, $X^r = \{x \in X: x \text{ has maximal rank, } k\}$, $\Sigma = \Sigma_{n,k}$ is the Stiefel manifold $\{\sigma \in \mathbb{R}^{n \times k}: \sigma'\sigma = I_k\}$ which is realized as $U\sigma_0$, where σ_0 is the $n \times k$ matrix with $(\sigma_0)_{jj} = 1$ for all $j = 1, ..., k$ and all other entries are zero; $U_0 = \{u \in U: u\sigma_0 = \sigma_0\} \cong O(n-k)$; and $C = \{\sigma_0 r \in X: r \text{ is a } k \times k \text{ (symmetric) positive definite matrix}\}$. Thus, for $x \in \mathbb{R}^{n \times k}$ such that $r = x'x$ is invertible, $\pi(x)$ is identified with the positive definite $k \times k$ matrix $r^{1/2}$.

If β, $x \in \mathbb{R}^{n \times k}$, then

$$\begin{aligned} J_0(\beta, x) &= \int_{O(n)} \exp(i\, \mathrm{trace}(\beta' u x))\, du \\ &= \text{constant} \times A_{-1/2}(\tfrac{1}{4} x' x \beta' \beta), \end{aligned}$$

where $A_{-1/2}(\cdot)$ is the Bessel function of matrix argument defined by Herz [9]; see also [8]. The corresponding generalization of Schoenberg's theorem now reads: If m is an $O(n)$-invariant probability measure on $\mathbb{R}^{n \times k}$, then $\hat{m}(x) = \phi(x'x)$, where for any $t \in \{t \geqslant 0\}$, the cone of symmetric positive definite $k \times k$ matrices,

$$\phi(t) = \text{constant} \times \int_{r \geqslant 0} A_{-1/2}(tr)\, d\tilde{m}(r),$$

and $\tilde{m}$ is a probability measure on $\{r \geqslant 0\}$.

We remark that Example A is the special case $k = 1$. Observe, again, the phenomenon that the function J_0, originally defined for $n \times k$ matrices is determined by a function on the lower dimensional space of $k \times k$ matrices.

We note in passing that random matrices having distributions of the type treated in this example have been studied in the work of Richards [13] and others.

(C) Let $X = \mathbb{R}^{n \times n}_{\text{sym}}$, the space of $n \times n$ real symmetric matrices. The group $U = O(n)$ acts "two-sidedly" on X via $x \to uxu^{-1}$, $u \in U$, $x \in X$. As in the previous example, it may be shown that $\hat{X}$ can be identified with X. In addition, $U \backslash X$ may be identified with the set Δ_n of $n \times n$ real diagonal matrices. For β, $x \in X$,

$$J_0(\beta, x) = \int_{O(n)} \exp(i \operatorname{trace}(\beta u x u^{-1}))\, du = {}_0F_0(i\beta, x),$$

where ${}_0F_0$ is the hypergeometric function of two matrix arguments defined by James [10]. It is straightforward to write down the generalized Schoenberg theorem for this example.

Random matrices having two-sided invariant distributions have been studied by Dawid [5] who calls such distributions *rotatable.*

(D) Let $X = \hat{X} = \mathbb{R}^{n \times k}$ and $U = O(n) \times \cdots \times O(n)$, the direct product of k copies of $O(n)$. Let $x_1, \ldots, x_k$ be the columns of $x \in X$ and $u = (u_1, \ldots, u_n)$ be a typical element of U, where $u_i \in O(n)$ for all $i = 1, \ldots, k$. The group U acts on X via $x = [x_1, \ldots, x_k] \to [u_1 x_1, u_2 x_2, \ldots, u_k x_k]$. This is just k copies of the Euclidean example A.

If $\beta = [\beta_1, \ldots, \beta_k] \in \hat{X}$, and $x = [x_1, \ldots, x_k] \in X$, then

$$\begin{aligned} J_0(\beta, x) &= \int_{O(n) \times \cdots \times O(n)} \exp\left(i \sum_{j=1}^{k} \beta_j' u_j x_j\right) du_1 \cdots du_k \\ &= \prod_{j=1}^{k} \Omega_n(\|\beta_j\| \cdot \|x_j\|), \end{aligned}$$

where $\Omega_n(\cdot)$ is defined in Example A. From this, we immediately derive a Schoenberg-type theorem obtained previously by Anderson and Fang [1].

We also note that examples analogous to those above can be constructed for matrices with entries in any real finite-dimensional division algebra $\mathbb{F}$. When $\mathbb{F} = \mathbb{C}$, the spherical Bessel functions are related to the complex generalized hypergeometric functions of James [10]; see also [7, 8].

11. Stochastic Representations

In the classical situation, Schoenberg's theorem is sometimes restated in terms of stochastic representations. We shall now extend this concept to the group setting using polar coordinates for the orthogonal transformation group (U, X).

12. Theorem

Let $\mathscr{X}$ be an X-valued random variable with associated probability measure m. Further, assume that m is supported on the set X^r of regular elements of X, where $\mathscr{X} = \Theta R$ is the unique decomposition of $\mathscr{X}$ through polar coordinates. Then m is U-invariant if and only if Θ and R are independent, and Θ is uniformly distributed with respect to the normalized U-invariant measure on $\Sigma = U/U_0$.

Here, we write $\pi_\Sigma(x) = \sigma$ and $\pi_C(x) = r$ for $x = \sigma r \in X^r$. Then $\Theta = \pi_\Sigma(\mathscr{X})$ and $R = \pi_C(\mathscr{X})$. If $\pi: X \to U\backslash X$ is the canonical quotient map, then the assumption on the support of m assures $\pi(\mathscr{X}) = \pi_C(\mathscr{X})$ a.s. The proof of Theorem 11 now follows directly from (4.1). In particular, if $m = m_{\mathscr{X}}$ is U-invariant then $m_R = \tilde{m}$, $m_\Theta = d\sigma$ is uniformly distributed, and $m = m_\Theta \times m_R$ so Θ and R are independent. Conversely, if $m = m_\Theta \times m_R$ and m_Θ is uniformly distributed, then $m_\Theta = d\sigma$ and clearly m is U-invariant.

In the classical case, the stochastic representations have been used (cf. Cambanis *et al.* [4]) to develop the theory of elliptically contoured distributions under minimal regularity conditions. Theorem 12 opens the way towards generalizations of the classical results to orthogonal transformation groups. These problems will be treated in a later article.

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