CONNECTIONS BETWEEN CONJECTURES OF ALON-TARSI, HADAMARD-HOWE, AND INTEGRALS OVER THE SPECIAL UNITARY GROUP

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Abstract. We show the Alon-Tarsi conjecture on Latin squares is equivalent to a very special case of a conjecture made independently by Hadamard and Howe, and to the non-vanishing of some interesting integrals over $\mathrm{SU}(n)$. Our investigations were motivated by geometric complexity theory.

1. Introduction

We first describe the conjectures of Alon-Tarsi, Hadamard-Howe, integrals over the special unitary group, and a related conjecture of Foulkes. We then state the equivalences (Theorem 1.9) and prove them.

1.1. **Combinatorics I: The Alon-Tarsi conjecture.** Call an $n \times n$ array of natural numbers a *Latin square* if each row and column consists of $[n] := \{1, \cdots, n\}$. Each row and column of a Latin square defines a permutation σ of n, where the ordered entries of the row (or column) are $\sigma(1), \cdots, \sigma(n)$. Define the sign of the row/column to be the sign of this permutation. Define the *column sign* of the Latin square to be the product of all the column signs (which is 1 or -1, respectively called *column even* or *column odd*), the *row sign* of the Latin square to be the product of the row signs and the *sign* of the Latin square to be the product of the row sign and the column sign.

Conjecture 1.1. [1] *[Alon-Tarsi] Let n be even, then the number of even Latin squares of size n does not equal the number of odd Latin squares of size n.*

Conjecture 1.1 is known to be true when $n = p \pm 1$, where p is an odd prime; in particular, it is known to be true up to $n = 24$ [10, 8].

The Alon-Tarsi conjecture is known to be equivalent to several other conjectures in combinatorics. For our purposes, the most important is the following due to Huang and Rota:

Conjecture 1.2. [15] *[Column-sign Latin square conjecture] Let n be even, then the number of column even Latin squares of size n does not equal the number of column odd Latin squares of size n.*

Theorem 1.3. [15, Identities 8,9] *The difference between the number of column even Latin squares of size n and the number of column odd Latin squares of size n equals the difference between the number of even Latin squares of size n and the number of odd Latin squares of size n, up to sign. In particular, the Alon-Tarsi conjecture holds for n if and only if the column-sign Latin square conjecture holds for n.*

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Remark 1.4. It is easy to see that for n odd, the number of even Latin squares of size n equals the number of odd Latin squares of size n.

1.2. **The Hadamard-Howe conjecture.** Let V be a finite dimensional complex vector space, let $V^{\otimes n}$ denote the space of multi-linear maps $V^* \times \cdots \times V^* \to \mathbb{C}$, the space of *tensors*. The permutation group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the inputs of the map. Let $S^n V \subset V^{\otimes n}$ denote the subspace of symmetric tensors, the tensors invariant under $\mathfrak{S}_n$, which we may also view as the space of homogeneous polynomials of degree n on V^*. Let $\mathrm{Sym}(V) := \oplus_d S^d V$, which is an algebra under multiplication of polynomials. Let $\mathrm{GL}(V)$ denote the general linear group of invertible linear maps $V \to V$. Consider the $\mathrm{GL}(V)$-module map

$$h_{d,n} : S^d(S^n V) \to S^n(S^d V)$$

given as follows: Include $S^d(S^n V) \subset V^{\otimes nd}$. Write $V^{\otimes nd} = (V^{\otimes n})^{\otimes d}$, as d groups of n vectors reflecting the inclusion. Now rewrite $V^{\otimes nd} = (V^{\otimes d})^{\otimes n}$ by grouping the first vector space in each group of n together, then the second vector space in each group etc.. Next symmetrize within each group of d to obtain an element of $(S^d V)^{\otimes n}$, and finally symmetrize the groups to get an element of $S^n(S^d V)$.

For example $h_{d,n}((x_1)^n \cdots (x_d)^n) = (x_1 \cdots x_d)^n$ and $h_{3,2}\big((x_1x_2)^3\big) = \frac{1}{4}x_1^3x_2^3 + \frac{3}{4}(x_1^2x_2)(x_1x_2^2)$.

The map $h_{d,n}$ was first considered by Hermite [13] who proved that, when $\dim V = 2$, the map is an isomorphism. It had been conjectured by Hadamard [12] and tentatively conjectured by Howe [14] (who wrote "it is not unreasonable to expect") that $h_{d,n}$ is always of maximal rank, i.e., injective for $d \le n$ and surjective for $d \ge n$. A consequence of the theorem of [23] (explained below) is that, contrary to the expectation above, $h_{5,5}$ is not an isomorphism.

For any $n \ge 1$, define the *Chow variety*

$$\mathrm{Ch}_n(V^*) := \{P \in S^n V^* \mid P = \ell_1 \cdots \ell_n \text{ for some } \ell_j \in V^*\}.$$

(This is a special case of a Chow variety, namely of the zero cycles in projective space, but it is the only one that we discuss in this article.) In [3, 4], Brion (and independently Weyman and Zelevinsky) observed that $\oplus_d S^n(S^d V)$ is the coordinate ring of the normalization of the Chow variety. (Given an irreducible affine variety Z, its *normalization* $\tilde{Z}$ is an irreducible affine variety whose ring of regular functions is integrally closed and such that there is a regular, finite, birational map $\tilde{Z} \to Z$, see e.g., [26, §II.5].)

Lemma 1.5. *(Hadamard, see e.g.* [19, §8.6]*) The kernel of the* $\mathrm{GL}(V)$*-module map*

$$\oplus h_{d,n} : \mathrm{Sym}(S^n V) := \oplus_d S^d(S^n V) \to \oplus_d S^n(S^d V)$$

is the ideal of the Chow variety.

Brion also showed that for d exponentially large with respect to n, $h_{d,n}$ is surjective [4]. McKay [22] showed that if $h_{d,n}$ is surjective, then $h_{d',n}$ is surjective for all $d' > d$, using $h_{d,n:0}$ defined below. It is also known that if $h_{d,n}$ is surjective, then $h_{n,d}$ is injective, see [16].

The $\mathrm{GL}(V)$-modules appearing in the tensor algebra of V are indexed by partitions $\pi = (p_1 \ge p_2 \ge \cdots \ge p_q \ge 0)$, $q \le \dim V$, and denoted $S_\pi V$. If π is a partition of d, i.e., $p_1 + \cdots + p_q = d$, the module $S_\pi V$ appears in $V^{\otimes d}$ and in no other degree. We will use the notation $s\pi := (sp_1, \cdots, sp_q)$. Repeated numbers in partitions are sometimes expressed as exponents when there is no danger of confusion, e.g., $(3,3,1,1,1,1) = (3^2,1^4)$. Let $\mathrm{SL}(V)$ be the subgroup of $\mathrm{GL}(V)$ consisting of determinant 1 elements, and let $\mathfrak{sl}(V)$ denote its Lie algebra.

This paper addresses a very special case of the general problem of determining the $\mathrm{GL}(V)$-module $\ker h_{d,n}$: simply to determine whether or not the module $S_{(d^n)}V$ is in the kernel.

Conjecture 1.6. [17] *For all d and n, $S_{(d^n)}V$ is not in the kernel of $h_{d,n} : S^d(S^n V) \to S^n(S^d V)$.*

1.3. **Combinatorics II: Foulke's conjecture.** The dimension of V, as long as it is at least d, is irrelevant for the $\mathrm{GL}(V)$-module structure of the kernel of $h_{d,n}$. *In this section we assume* $\dim V = dn$. Choose a linear isomorphism $V \simeq \mathbb{C}^{nd}$. The Weyl group $\mathcal{W}_V$ of $\mathrm{GL}(V) = \mathrm{GL}(nd)$, which can be thought of as the subgroup of $\mathrm{GL}(nd)$ consisting of the permutation matrices (in particular, it is isomorphic to $\mathfrak{S}_{dn}$), acts on $V^{\otimes dn}$ by acting on each factor. (We write $\mathcal{W}_V$ to distinguish this from the $\mathfrak{S}_{dn}$-action permuting the factors.) An element $x \in V^{\otimes dn}$ has $\mathfrak{sl}(V)$-weight zero if, in the standard basis $\{e_i\}_{1\le i\le dn}$ of V induced from the identification $V \simeq \mathbb{C}^{nd}$, x is a sum of monomials $x = \sum_{I=(i_1,\dots,i_{nd})} x^I e_{i_1}\otimes\cdots\otimes e_{i_{nd}}$, where I runs over the orderings of $[nd]$. If one restricts $h_{d,n}$ to the $\mathfrak{sl}(V)$-weight zero subspace, one obtains a $\mathcal{W}_V$-module map

$$h_{d,n:0} : S^d(S^n V)_0 \to S^n(S^d V)_0.$$

These $\mathcal{W}_V$-modules are as follows: Let $\mathfrak{S}_n \wr \mathfrak{S}_d \subset \mathfrak{S}_{dn}$ denote the wreath product, which, by definition, is the normalizer of $\mathfrak{S}_n^{\times d}$ in $\mathfrak{S}_{dn}$. It is the semi-direct product of $\mathfrak{S}_n^{\times d}$ with $\mathfrak{S}_d$, where $\mathfrak{S}_d$ acts by permuting the factors of $\mathfrak{S}_n^{\times d}$, see e.g., [20, p 158]. Since $\dim V = dn$, we get $S^d(S^n V)_0 = \mathrm{Ind}_{\mathfrak{S}_n\wr\mathfrak{S}_d}^{\mathfrak{S}_{dn}} \mathrm{triv}$, where triv denotes the trivial $\mathfrak{S}_n \wr \mathfrak{S}_d$-module.

We obtain a $\mathcal{W}_V = \mathfrak{S}_{dn}$-module map

$$h_{d,n:0} : \mathrm{Ind}_{\mathfrak{S}_n\wr\mathfrak{S}_d}^{\mathfrak{S}_{dn}} \mathrm{triv} \to \mathrm{Ind}_{\mathfrak{S}_d\wr\mathfrak{S}_n}^{\mathfrak{S}_{dn}} \mathrm{triv}\,.$$

Moreover, since every irreducible module appearing in $S^d(S^n V)$ has a non-zero $\mathrm{SL}(V)$-weight zero subspace, $h_{d,n}$ is the unique $\mathrm{SL}(V)$-module extension of $h_{d,n:0}$.

The map $h_{d,n:0}$ was defined purely in terms of combinatorics in [2] as a path to try to prove the following conjecture of Foulkes:

Conjecture 1.7. [9] *Let $d > n$, let π be a partition of dn and let $[\pi]$ denote the corresponding $\mathfrak{S}_{dn}$-module. Then,*

$$\mathrm{mult}([\pi], \mathrm{Ind}_{\mathfrak{S}_n\wr\mathfrak{S}_d}^{\mathfrak{S}_{dn}} \mathrm{triv}) \ge \mathrm{mult}([\pi], \mathrm{Ind}_{\mathfrak{S}_d\wr\mathfrak{S}_n}^{\mathfrak{S}_{dn}} \mathrm{triv}).$$

Conjecture 1.7 was shown to hold asymptotically by L. Manivel in [21], in the sense that for any partition μ, the multiplicity of the partition $(dn - |\mu|, \mu)$ is the same in $S^d(S^n V)$ and $S^n(S^d V)$ as soon as d and n are at least $|\mu|$. Conjecture 1.7 is still open in general. However, the map $h_{5,5:0}$ was shown not to be injective in [23], and thus $h_{5,5}$ is not injective. The $\mathrm{GL}(V)$-module structure of the kernel of $h_{5,5}$ was determined by C. Ikenmeyer and S. Mrktchyan as part of a 2012 AMS MRC program:

Proposition 1.8 (Ikenmeyer and Mkrtchyan, unpublished). *The kernel of $h_{5,5} : S^5(S^5\mathbb{C}^5) \to S^5(S^5\mathbb{C}^5)$ consists of irreducible modules corresponding to the following partitions:*

$$\{(14,7,2,2), (13,7,2,2,1), (12,7,3,2,1), (12,6,3,2,2),$$
$$(12,5,4,3,1), (11,5,4,4,1), (10,8,4,2,1), (9,7,6,3)\}.$$

All these occur with multiplicity one in the kernel, but not all occur with multiplicity one in $S^5(S^5\mathbb{C}^5)$. In particular, the kernel is not an isotypic component.

1.4. **Integration over** $\mathrm{SU}(n)$**.** Let $d\mu$ denote the Haar measure on $\mathrm{SU}(n)$ with volume one. Let W be any $\mathrm{SU}(n)$-module and let $W^{\mathrm{SU}(n)}$ be its subspace of invariants. Consider the $\mathrm{SU}(n)$-module projection map $\pi : W \to W^{\mathrm{SU}(n)}$. Then, the projection π is explicitly realized as the integration:

$$\int_{\mathrm{SU}(n)} : W \to W^{\mathrm{SU}(n)}, \;\; v \mapsto \int_{g\in\mathrm{SU}(n)} g\cdot v\, d\mu.$$

Assume further that $W^{\mathrm{SU}(n)}$ is one dimensional. Take the unique (up to a scalar multiple) nonzero element in the dual space $P \in {W^*}^{\mathrm{SU}(n)}$. Then,

$$\pi(v) \neq 0 \iff \langle P, \int_{g\in \mathrm{SU}(n)} g\cdot v\, d\mu\rangle \neq 0 \iff \langle P, v\rangle \neq 0. \tag{1}$$

In particular, take the vector space $V = \mathbb{C}^n$. Then, $\mathrm{End}(V)$ is a $\mathrm{GL}(V)$ module under the left multiplication. For $\delta = dn$ (for any $d \geq 0$), there is a unique (up to scale) $\mathrm{SL}(V)$-invariant in $S^\delta(\mathrm{End}(V))$, namely $\det_n^{\otimes d}$ and there is none otherwise, see e.g., [11, Thm. 5.6.7]. We denote $\det_n \in S^n(\mathrm{End}(V))$ by $\det_n^V$. Similarly, since $\mathrm{End}(V)$ is canonically isomorphic with the dual $\mathrm{End}(V)^*$, we can think of $\det_n \in S^n(\mathrm{End}(V)^*)$. To distinguish, when thinking of $\det_n \in S^n(\mathrm{End}(V)^*)$, we denote it by $\det_n^{V^*}$.

These integrals have been extensively studied in the free probability and mathematical physics literature, see, e.g., [6, 7]. Despite this, the integrals that arose in our study do not appear to be known.

1.5. **The equivalences.** The following is the main result of this note.

Theorem 1.9. *Fix n even. Let $V = \mathbb{C}^n$ and write $\mathrm{End}(V) = \mathrm{Mat}_n$, where Mat_n is the space of $n\times n$ matrices. Let $d\mu$ denote the Haar measure on $\mathrm{SU}(n)$ and let $\mathrm{SU}(n)$ act on $\mathrm{End}(V)$ by left multiplication. Write g^i_j for the coordinate functions on $\mathrm{End}(V)$. The following are equivalent:*
(a) The Alon-Tarsi conjecture for n.
(b) Conjecture 1.6 for n with $d = n$.
(c) $\int_{g\in \mathrm{SU}(n)} g\cdot (\mathrm{perm}_n^{V^*})^n d\mu \neq 0$.
(d) $\langle (\mathrm{perm}_n^{V^*})^n, (\det_n^V)^n\rangle \neq 0$.
(e) $\int_{g\in \mathrm{SU}(n)} \big(\Pi_{1\leq i,j\leq n} g^i_j\big) d\mu \neq 0$.
(f) $\langle \Pi_{i,j} g^i_j, (\det_n^V)^n\rangle \neq 0$.

The pairings in (d) and (f) may also be thought of as a pairing between homogeneous polynomials of degree n^2 and homogeneous differential operators of order n^2.

Rectangular versions of these equivalences can be formulated as well.

1.6. **Motivation from geometric complexity theory.** In geometric complexity theory, see [24, 25, 5, 18], one looks for modules that are in the ideal of the orbit closure $\overline{\mathrm{GL}_{n^2}\cdot \det_n} \subset S^n(\mathrm{End}(V)^*)$ of the determinant polynomial. One approach to this search is to find modules in $\mathrm{Sym}(S^n(\mathrm{End}(V)))$ that do not occur in the coordinate ring of the orbit $\mathrm{GL}_{n^2}\cdot \det_n$, which can in principle be determined from representation theory, see [5]. The following observations are from [17] (where they are explained in detail): Since $\mathrm{Ch}_n(\mathrm{End}(V)^*) \subset \overline{\mathrm{GL}_{n^2}\cdot \det_n}$, any polynomial not in the ideal of $\mathrm{Ch}_n(\mathrm{End}(V)^*)$ cannot be in the ideal of $\overline{\mathrm{GL}_{n^2}\cdot \det_n}$. Thus, if $S_{(n^d)}(\mathbb{C}^{n^2}) \notin I(\mathrm{Ch}_n(\mathbb{C}^{n^2}))$, for all $1 \leq d \leq n$, then for any partition π with at most n parts, the module $S_{n\pi}\mathbb{C}^{n^2}$ occurs at least once in $\mathbb{C}[\overline{\mathrm{GL}_{n^2}\cdot \det_n}]$; in particular, the symmetric Kronecker coefficient $s_{n\pi,d^n,d^n}$ is non-vanishing (cf. [17, § 6]).

2. Construction of the invariant

Let $V = \mathbb{C}^d$ and let $\Omega \in \Lambda^d V^*$ be non-zero. Then, for any even n, the one-dimensional module $S_{(n^d)}V^*$ occurs with multiplicity one in $S^d(S^nV^*)$ (cf. [14, Proposition 4.3]). Write $\overline{\Omega}$ when considering Ω as a multi-linear form on V, and write Ω when using it as an element of the dual space $\Lambda^d V^*$ to $\Lambda^d V$.

Proposition 2.1. *Let n be even. Choosing the scale appropriately, the unique (up to scale) polynomial $P \in S_{(n^d)}V^* \subset S^d(S^nV^*)$ evaluates on*

$$x = (v_1^1 \cdots v_n^1)(v_1^2 \cdots v_n^2)\cdots(v_1^d \cdots v_n^d) \in S^d(S^nV), \text{ for any } v_j^i \in V,$$

to give

$$\langle P, x\rangle = \sum_{\sigma_1,\cdots,\sigma_d \in \mathfrak{S}_n} \overline{\Omega}(v^1_{\sigma_1(1)},\cdots,v^d_{\sigma_d(1)})\cdots\overline{\Omega}(v^1_{\sigma_1(n)},\cdots,v^d_{\sigma_d(n)}). \tag{2}$$

Proof. Let $\bar{P} \in (V^*)^{\otimes nd}$ be defined by the identity (2) (with P replaced by $\bar{P}$). It suffices to check that
(i) $\bar{P} \in S^d(S^nV^*)$,
(ii) $\bar{P}$ is $\mathrm{SL}(V)$ invariant, and
(iii) $\bar{P}$ is not identically zero.

Observe that (iii) follows from the identity (2) by taking $v_j^i = e_i$ where $e_1,\cdots,e_d$ is the standard basis of V, and (ii) follows because $\mathrm{SL}(V)$ acts trivially on Ω.

To see (i), we show (ia) $\bar{P} \in S^d((V^*)^{\otimes n})$ and (ib) $\bar{P} \in (S^nV^*)^{\otimes d}$ to conclude. To see (ia), it is sufficient to show that exchanging two adjacent factors in parentheses in the expression of x will not change (2). Exchange v_j^1 with v_j^2 in the expression for $j = 1,\cdots,n$. Then, each individual determinant will change sign, but there are an even number of determinants, so the right hand side of (2) is unchanged. To see (ib), it is sufficient to show the expression is unchanged if we swap v_1^1 with v_2^1 in (2). If we multiply by $n!$, we may assume $\sigma_1 = \mathrm{Id}$, i.e.,

$$\langle \bar{P}, x\rangle =$$
$$n! \sum_{\sigma_2,\cdots,\sigma_d \in \mathfrak{S}_n} \overline{\Omega}(v_1^1, v^2_{\sigma_2(1)},\cdots,v^d_{\sigma_d(1)})\overline{\Omega}(v_2^1, v^2_{\sigma_2(2)},\cdots,v^d_{\sigma_d(2)})\cdots\overline{\Omega}(v_n^1, v^2_{\sigma_2(n)},\cdots,v^d_{\sigma_d(n)}).$$

With the two elements v_1^1 and v_2^1 swapped, we get

$$n! \sum_{\sigma_2,\cdots,\sigma_d \in \mathfrak{S}_n} \overline{\Omega}(v_2^1, v^2_{\sigma_2(1)},\cdots,v^d_{\sigma_d(1)})\overline{\Omega}(v_1^1, v^2_{\sigma_2(2)},\cdots,v^d_{\sigma_d(2)})\cdots\overline{\Omega}(v_n^1, v^2_{\sigma_2(n)},\cdots,v^d_{\sigma_d(n)}). \tag{3}$$

Now right compose each σ_s in (3) by the transposition $(1,2)$. The expressions become the same. □

3. Proof of the equivalences in Theorem 1.9

Let $V = \mathbb{C}^n$ with the standard basis $\{e_1,\dots,e_n\}$. The equivalences (c)⇔ (d) and (e) ⇔ (f) follow from the identity (1) applied to the $\mathrm{SU}(n)$-module $W = S^{n^2}(\mathrm{End}(V)^*)$. (To prove the equivalence (e) ⇔ (f), we have used the fact that any $\mathrm{SU}(n)$-invariant polynomial $Q \in S^{n^2}(\mathrm{End}(V)^*)$ is non-zero if and only if it does not vanish at $\mathrm{Id} \in \mathrm{End}(V)$.)

We now prove the other equivalences. We have two natural bases of $S^n(S^nV)_0$ to work with, the *monomial basis* consisting of products in the e_j such that the $\mathfrak{sl}(V)$-weight of the expression is zero, and a *weight basis*. To obtain a weight basis, first decompose $S^n(S^nV)$ into irreducible $\mathrm{GL}(V)$-modules and then take a basis of the $\mathfrak{sl}(V)$-weight zero subspace of each module. A weight basis is the collection of the vectors in these spaces. (Observe that this basis is not unique.) The polynomial $P \in S_{(n^n)}(V^*) \subset S^n(S^n(V^*))$ will have a non-zero evaluation on $(e_1\cdots e_n)^n$ (equivalently, not be in the ideal of the Chow variety) if and only if, when expanding P in the monomial basis obtained from the basis $y_1,\cdots,y_n$ dual to $e_1,\cdots,e_n$, the coefficient of $(y_1\cdots y_n)^n$ is non-zero.

To see (a)$\Leftrightarrow$(b) (which was already shown in [17]), by the identity (2) for $d = n$,

$$\langle P, (e_1 \cdots e_n)^n \rangle = \sum_{\sigma_1, \cdots, \sigma_n \in \mathfrak{S}_n} \overline{\Omega}(e_{\sigma_1(1)}, \cdots, e_{\sigma_n(1)}) \cdots \overline{\Omega}(e_{\sigma_1(n)}, \cdots, e_{\sigma_n(n)}). \tag{4}$$

A term in the summation is non-zero if and only if the permutations $\sigma_1, \cdots, \sigma_n$ give rise to a Latin square by putting the values of σ_i in the i-th row, and the contribution of the term is the column sign of the square. This proves the equivalence of (a) and (b) by Lemma 1.5.

Now,

$$\begin{aligned}
\int_{g \in \mathrm{SU}(n)} g \cdot (e_1 \cdots e_n)^n d\mu &= \int_{g \in \mathrm{SU}(n)} ((g \cdot e_1) \cdots (g \cdot e_n))^n d\mu \\
&= \sum_{1 \le i_q^p \le n} \int_{g \in \mathrm{SU}(n)} (g_1^{i_1^1} \cdots g_n^{i_n^1}) \cdots (g_1^{i_1^n} \cdots g_n^{i_n^n})(e_{i_1^1} \cdots e_{i_n^1}) \cdots (e_{i_1^n} \cdots e_{i_n^n}) d\mu \\
&= \sum_{1 \le i_q^p \le n} [\int_{g \in \mathrm{SU}(n)} (g_1^{i_1^1} \cdots g_n^{i_n^1}) \cdots (g_1^{i_1^n} \cdots g_n^{i_n^n}) d\mu](e_{i_1^1} \cdots e_{i_n^1}) \cdots (e_{i_1^n} \cdots e_{i_n^n}) \\
&= [\int_{g \in \mathrm{SU}(n)} \sum_{\{i_1^j, \cdots, i_n^j\} = [n] \forall j} (g_1^{i_1^1} \cdots g_n^{i_n^1}) \cdots (g_1^{i_1^n} \cdots g_n^{i_n^n}) d\mu](e_1 \cdots e_n)^n + \\
&\quad [\int_{g \in \mathrm{SU}(n)} \sum_{\sigma \in \mathfrak{S}_n : \{i_1^j, \cdots, i_n^j\} = \sigma(j)} (g_1^{i_1^1} \cdots g_n^{i_n^1}) \cdots (g_1^{i_1^n} \cdots g_n^{i_n^n}) d\mu](e_1)^n \cdots (e_n)^n + x \\
&= [\int_{g \in \mathrm{SU}(n)} (\mathrm{perm}(g))^n d\mu](e_1 \cdots e_n)^n + \\
&\quad [\int_{g \in \mathrm{SU}(n)} (\Pi_{1 \le i, j \le n} g_j^i) d\mu](e_1)^n \cdots (e_n)^n + x,
\end{aligned} \tag{5}$$

where $x \in S^n(S^n(V))_0$ is in the span of the monomial basis not involving $(e_1 \cdots e_n)^n$ and $(e_1)^n \cdots (e_n)^n$.

Consider the projection $\int_{\mathrm{SU}(n)} : W \to W^{\mathrm{SU}(n)}$ as in §1.4 for the $\mathrm{SU}(n)$-module $S^n(S^n(V))$ and for the unique $\mathrm{SU}(n)$-invariant $P^* \in S^n(S^n(V))$. It implies

$$\int_{g \in \mathrm{SU}(n)} g \cdot (e_1 \cdots e_n)^n d\mu = \alpha P^*, \text{ for some } \alpha \in \mathbb{C}. \tag{6}$$

From this, together with the identity (1), we get the equivalence of (b) and the non-vanishing of $\int_{g \in \mathrm{SU}(n)} g \cdot (e_1 \cdots e_n)^n d\mu$. Thus, the identity (5) shows that (e) implies (b). Further, assuming (b), the identity (6) implies $\int_{g \in \mathrm{SU}(n)} g \cdot (e_1 \cdots e_n)^n d\mu$ is a non-zero multiple of P^*. But, by the proof of Proposition 2.1, P^* contains the monomial $(e_1^n) \cdots (e_n^n)$ with non-zero coefficient. Thus, identity (5) implies (e). This shows the equivalence of (b) and (e).

It is easy to see that (c) is equivalent to $\int_{g \in \mathrm{SU}(n)} (\mathrm{perm}(g))^n d\mu \neq 0$. Thus, by the identity (5) , (c) implies (b). Further, from the identity (2) for $d = n$, it is easy to see that if (a) (equivalently (b)) is true, P^* contains the monomial $(e_1 \cdots e_n)^n$ with non-zero coefficient. Thus, from the identities (5) and (6), we get that (b) implies the non-vanishing of $\int_{g \in \mathrm{SU}(n)} (\mathrm{perm}(g))^n d\mu$, and hence (c). Thus (b) and (c) are equivalent. This proves the theorem. □

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