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Algebra Structures on $Hom(C, L)$

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Abstract

We consider the space of linear maps from a coassociative coalgebra C into a Lie algebra L. Unless C has a cocommutative coproduct, the usual symmetry properties of the induced bracket on $Hom(C, L)$ fail to hold. We define the concept of twisted domain (TD) algebras in order to recover the symmetries and also construct a modified Chevalley-Eilenberg complex in order to define the cohomology of such algebras.

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1 Introduction

The principal thrust of this note is to introduce new algebraic structures which are defined on certain vector spaces of mappings. Assume that C is a coassociative coalgebra and that L is an algebra which generally may be either a Lie algebra, an associative algebra, or a Poisson algebra. We consider the vector space Hom(C,L) of linear mappings from C into L. When C is cocommutative Hom(C,L) inherits whatever structure L possesses. On the other hand when C is not cocommutative one generally obtains a new structure on Hom(C,L) which we refer to as a TD algebra. Similar constructions may be employed to obtain TD modules.

We are particularly interested in the case that L is a Lie algebra and in this case we are also interested in developing a cohomology theory for TD Lie algebras which parallels the development of Chevalley-Eilenberg cohomology for Lie algebras. Our focus on these issues is driven by the fact that such algebraic structures and their cohomologies arise naturally in the study of Lagrangian field theories in physics.

More specifically, certain physical theories, such as gauge field theories, may be formulated in terms of a function called the Lagrangian of the theory. From a mathematical point of view a Lagrangian is a smooth function defined on an appropriate jet bundle JE of some fiber bundle $E \longrightarrow M$. If L is a Lagrangian, its corresponding action S is defined by $S = \int_M L \; d(vol_M)$. A symmetry of the action is a "lift" of a generalized vector field on E to JE such that the flow of the lifted vector field leaves S invariant. Such symmetries are closed under the Lie brackets of generalized vector fields and are called variational symmetries of L. A gauge symmetry of the Lagrangian is a single member of a family of vector fields, each of which is a symmetry of S. This family is uniquely defined by the 'Noether identities' of the Lagrangian. We interpret such a family as a mapping R from an appropriate vector space $\mathcal{P}$ of "parameters" of the theory into the space Var of variational symmetries of S. Viewed this way, if R and T are such mappings and ϵ, η are in $\mathcal{P}$, then $[R(\epsilon), T(\eta)]$ is a gauge symmetry. If $[R, T]$ is the mapping from $\mathcal{P} \otimes \mathcal{P}$ into Var defined by $\epsilon \otimes \eta \longrightarrow [R(\epsilon), T(\eta)]$, we see that all possible commutators of gauge symmetries may be realized as images of mappings from $\mathcal{P} \otimes \mathcal{P}$ into Var. In general, if we permit mappings with domain the entire tensor coalgebra $C = \bigoplus_n \mathcal{P}^{\otimes n}$, then all possible iterated commutators of gauge symmetries are encoded as images of such mappings. These considerations lead us to the study of the TD Lie structure of $Hom(C, Var)$.

2 Preliminaries

Let (C, Δ) be a coassociative coalgebra and let (L, ϕ) be an algebra over a common field k. Let $Hom(C, L)$ denote the vector space of linear maps $L \longrightarrow C$. We explore the properties of the convolution product on $Hom(C, L)$ that result from various symmetry properties of Δ and ϕ. Recall that the convolution product [3] on $Hom(C, L)$ is given by

$$\Phi(f, g)(c) = \phi \circ (f \otimes g) \circ \Delta(c)$$

A useful alternative to this description of Φ arises from the utilization of the $Hom - \otimes$ interchange map [1] which is given by

$$\lambda : Hom(C, L) \otimes Hom(C', L') \longrightarrow Hom(C \otimes C', L \otimes L')$$

where

$$[\lambda(f \otimes f')](c \otimes c') = f(c) \otimes f'(c').$$

Also recall that given linear maps $\alpha : C' \longrightarrow C$ and $\beta : L \longrightarrow L'$, we have the induced maps $\alpha^* : Hom(C, L) \longrightarrow Hom(C', L)$ and $\beta_* : Hom(C, L) \longrightarrow Hom(C, L')$. We will make repeated use of the equality

$$\alpha^* \circ \beta_* = \beta_* \circ \alpha^* : Hom(C, L) \longrightarrow Hom(C', L')$$

in what follows.

We thus have the convolution product described by the composition

$$Hom(C, L) \otimes Hom(C, L) \xrightarrow{\lambda} Hom(C \otimes C, L \otimes L) \xrightarrow{\phi_* \circ \Delta^*} Hom(C, L),$$

i.e.

$$\Phi = \phi_* \circ \Delta^* \circ \lambda \tag{1}$$

We record some useful elementary properties of λ in the following two lemmas.

Lemma 1 *(Naturality)*

a) Given a map $\zeta : A \longrightarrow C_2$ *we have the commutative diagram*

$$\begin{array}{ccc} Hom(C_1, L_1) \otimes Hom(C_2, L_2) & \xrightarrow{\lambda} & Hom(C_1 \otimes C_2, L_1 \otimes L_2) \\ 1 \otimes \zeta^* \downarrow & & \downarrow (1 \otimes \zeta)^* \\ Hom(C_1, L_1) \otimes Hom(A, L_2) & \xrightarrow{\lambda} & Hom(C_1 \otimes A, L_1 \otimes L_2). \end{array}$$

b) Given a map $\psi : L_2 \longrightarrow B$*, we have the commutative diagram*

$$\begin{array}{ccc} Hom(C_1, L_1) \otimes Hom(C_2, L_2) & \xrightarrow{\lambda} & Hom(C_1 \otimes C_2, L_1 \otimes L_2) \\ 1 \otimes \psi_* \downarrow & & \downarrow (1 \otimes \psi)_* \\ Hom(C_1, L_1) \otimes Hom(C_2, B) & \xrightarrow{\lambda} & Hom(C_1 \otimes C_2, L_1 \otimes B). \end{array}$$

Proof. For a) we have

$$[\lambda \circ (1 \otimes \zeta^*)(f \otimes g)](c_1 \otimes a) = [\lambda(f \otimes g \circ \zeta)](c_1 \otimes a) = f(c_1) \otimes g(\zeta(a)).$$

On the other hand,

$$[(1 \otimes \zeta)^* \circ \lambda(f \otimes g)](c_1 \otimes a) = \lambda(f \otimes g) \circ (1 \otimes \zeta)(c_1 \otimes a) = f(c_1) \otimes g(\zeta(a)).$$

The proof of b) follows from a similar calculation. □

It is evident that the same result holds if we replace $1 \otimes \zeta^*$ by $\zeta^* \otimes 1$ and $1 \otimes \psi_*$ by $\psi_* \otimes 1$.

We are interested in the relationship between the symmetries of ϕ and Δ and of the convolution product Φ. The next lemma will provide us with a key link between them. Let

$$\Lambda : Hom(C_1, L_1) \otimes \ldots \otimes Hom(C_n, L_n) \longrightarrow Hom(C_1 \otimes \ldots \otimes C_n, L_1 \otimes \ldots \otimes L_n)$$

be given by $\Lambda = \lambda \circ (1 \otimes \lambda) \circ \ldots \circ (1^{n-2} \otimes \lambda)$. Here we are using the natural associativity of the tensor product. When it is helpful to the exposition, we will write $\Lambda = \Lambda^{n-1}$.

Lemma 2 *(Symmetry) Let $\sigma \in S_n$. The following diagram commutes.*

$$\begin{array}{ccc} Hom(C_1, L_1) \otimes \ldots \otimes Hom(C_n, L_n) & \stackrel{\Lambda}{\longrightarrow} & Hom(C_1 \otimes \ldots \otimes C_n, L_1 \otimes \ldots \otimes L_n) \\ \sigma \downarrow & & (\sigma^{-1})^* \circ \sigma_* \downarrow \\ Hom(C_{\sigma(1)}, L_{\sigma(1)}) \otimes \ldots \otimes Hom(C_{\sigma(n)}, L_{\sigma(n)}) & \stackrel{\Lambda}{\longrightarrow} & Hom(C_{\sigma(1)} \otimes \ldots \otimes C_{\sigma(n)}, L_{\sigma(1)} \otimes \ldots \otimes L_{\sigma(n)}) \end{array}$$

Proof. Let $f_1 \otimes \ldots \otimes f_n \in Hom(C_1, L_1) \otimes \ldots \otimes Hom(C_n, L_n)$. Then by definition

$$\sigma(f_1 \otimes \ldots \otimes f_n) = f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}$$

and further

$$[\Lambda(\sigma(f_1 \otimes \ldots \otimes f_n))](c_{\sigma(1)} \otimes \ldots \otimes c_{\sigma(n)}) = f_{\sigma(1)}(c_{\sigma(1)}) \otimes \ldots \otimes f_{\sigma(n)}(c_{\sigma(n)}).$$

On the other hand,

$$[(\sigma^{-1})^* \circ \sigma_* \circ \Lambda(f_1 \otimes \ldots \otimes f_n)](c_{\sigma(1)} \otimes \ldots \otimes c_{\sigma(n)}) = [\sigma \circ \Lambda(f_1 \otimes \ldots \otimes f_n) \circ \sigma^{-1}](c_{\sigma(1)} \otimes \ldots \otimes c_{\sigma(n)})$$

$$= [\sigma \circ \Lambda(f_1 \otimes \ldots \otimes f_n)](c_1 \otimes \ldots \otimes c_n) = \sigma(f_1(c_1) \otimes \ldots \otimes f_n(c_n))$$

$$= f_{\sigma(1)}(c_{\sigma(1)}) \otimes \ldots \otimes f_{\sigma(n)}(c_{\sigma(n)}). \ \Box$$

As permutations will play a major role in the remainder of this note, we remark that we will use the standard notation for cycles in the symmetric groups. Recall that the symbol $(i_1\, i_2\, \ldots\, i_k)$ means the permutation that sends i_1 to i_2, i_2 to i_3, ..., i_k to i_1 (or, if we wish, objects that are indexed by the integers i_1,...,i_k).

3 Twisted Domain Skew Maps

In this section we examine maps $Hom(C, L)^{\otimes n} \longrightarrow Hom(C, V)$ where L and V are arbitrary vector spaces.

To begin let (C, Δ) be a coassociative coalgebra and denote the iterated coproduct by

$$\Delta^{(n-1)} = (\Delta \otimes 1 \otimes \ldots \otimes 1) \circ \ldots \circ (\Delta \otimes 1) \circ \Delta : C \longrightarrow C^{\otimes n}.$$

Let $\phi : L^{\otimes n} \longrightarrow V$ be a linear map and $\Phi : Hom(C, L)^{\otimes n} \longrightarrow Hom(C, V)$ be given by

$$\Phi = \phi_* \circ \Delta^{(n-1)*} \circ \Lambda.$$

We say that Φ is the map that is induced by the map ϕ. Define also the map $\Phi^\sigma : Hom(C, L)^{\otimes n} \longrightarrow Hom(C, V)$ for each $\sigma \in S_n$ by

$$\Phi^\sigma = \phi_* \circ \Delta^{(n-1)*} \circ \sigma^* \circ \Lambda.$$

The statement contained in the next lemma is a useful fact to which we will appeal on several occasions in what follows.

Lemma 3 *Let $\Phi = \phi_* \circ \Delta^{(n-1)*} \circ \Lambda : Hom(C, L)^{\otimes n} \longrightarrow Hom(C, V)$ be the map induced by $\phi : L^{\otimes n} \longrightarrow V$. Then, for $\sigma \in S_n$, the map induced by $\phi \circ \sigma : L^{\otimes n} \longrightarrow V$ may be written as $\Phi^\sigma \circ \sigma : Hom(C, L)^{\otimes n} \longrightarrow Hom(C, V)$.*

Proof. The map induced by $\phi \circ \sigma$ is given by $(\phi \circ \sigma)_* \circ \Delta^{(n-1)*} \circ \Lambda$ which is equal to

$$\begin{aligned} &\phi_* \circ \sigma_* \circ \Delta^{(n-1)*} \circ \Lambda \\ &= \phi_* \circ \Delta^{(n-1)*} \circ \sigma_* \circ \Lambda \\ &= \phi_* \circ \Delta^{(n-1)*} \circ \sigma^* \circ (\sigma^{-1})^* \circ \sigma_* \circ \Lambda \\ &= \phi_* \circ \Delta^{(n-1)*} \circ \sigma^* \circ \Lambda \circ \sigma = \Phi^\sigma \circ \sigma. \ \square \end{aligned}$$

Definition 4 *We say that the linear map $\Phi : Hom(C, L)^{\otimes n} \longrightarrow Hom(C, V)$ is* **twisted domain skew (TD skew)** *if*

$$\Phi \circ \sigma = (-1)^\sigma \Phi^{\sigma^{-1}} \text{ for all } \sigma \in S_n.$$

Although we will use the formulation of TD skew that is given in the definition, we may clarify the situation by considering several alternate presentations. For $c \in C$, denote the iterated coproduct $\Delta^{(n-1)}(c) = c_1 \otimes \ldots \otimes c_n$. Then for $f_1 \otimes \ldots \otimes f_n \in Hom(C, L)^{\otimes n}$, we have

$$\Phi(f_1 \otimes \ldots \otimes f_n)(c) = \phi(f_1(c_1) \otimes \ldots \otimes f_n(c_n))$$

and

$$\Phi^\sigma(f_1 \otimes \ldots \otimes f_n)(c) = \phi(f_1(c_{\sigma(1)}) \otimes \ldots \otimes f_n(c_{\sigma(n)})).$$

Then the TD skew condition may be phrased as

$$\Phi(f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)})(c) = (-1)^\sigma \Phi^{\sigma^{-1}}(f_1 \otimes \ldots \otimes f_n)(c)$$

or

$$\phi(f_{\sigma(1)}(c_1) \otimes \ldots \otimes f_{\sigma(n)}(c_n)) = (-1)^\sigma \phi(f_1(c_{\sigma^{-1}(1)}) \otimes \ldots \otimes f_n(c_{\sigma^{-1}(n)})).$$

We note here that if the coalgebra (C, Δ) is cocommutative, then the concept of TD skew symmetry is identical to that of skew symmetry.

Example: The example which provides the motivation for this definition is the bracket of a Lie algebra. Let $\phi : L \otimes L \longrightarrow L$ denote this bracket. The skew symmetry of ϕ may be expressed as $\phi \circ \tau = -\phi$ where τ is the transposition. The map $\Phi : Hom(C, L) \otimes Hom(C, L) \longrightarrow Hom(C, L)$ that is given by $\Phi = \phi_* \circ \Delta^* \circ \lambda$ is not in general skew symmetric. Indeed if we write $\Delta(c) = x \otimes y$ and $\Phi(f \otimes g)(x \otimes y) = [f(x), g(y)]$, we have $\Phi \circ \tau(f \otimes g)(x \otimes y) = [g(x), f(y)] = -[f(y), g(x)] = -\Phi(f \otimes g)(y \otimes x) \neq -\Phi(f \otimes g)(x \otimes y)$. However, in this example, we have $\Phi^\tau = \phi_* \circ \Delta^* \circ \tau^* \circ \lambda$ and $\Phi \circ \tau = -\Phi^\tau$.

The next proposition will play a central role in the rest of this note.

Proposition 5 *If $\phi : L^{\otimes n} \longrightarrow V$ is skew symmetric, i.e. if $\phi \circ \sigma = (-1)^\sigma \phi$ for all $\sigma \in S_n$, then $\Phi = \phi_* \circ \Delta^{(n-1)*} \circ \Lambda$ is TD skew symmetric.*

Proof. We calculate

$$\Phi \circ \sigma = \phi_* \circ \Delta^{(n-1)*} \circ \Lambda \circ \sigma$$

$$= \phi_* \circ \Delta^{(n-1)*} \circ (\sigma^{-1})^* \circ \sigma_* \circ \Lambda \text{ (by the symmetry lemma)}$$

$$= (\phi \circ \sigma)_* \circ \Delta^{(n-1)*} \circ (\sigma^{-1})^* \circ \Lambda$$

$$= (-1)^\sigma \phi_* \circ \Delta^{(n-1)*} \circ (\sigma^{-1})^* \circ \Lambda \text{ (because } \phi \text{ is skew)}$$

$$= (-1)^\sigma \Phi^{\sigma^{-1}}. \ \square$$

Later in this note we will be confronted with various compositions of maps that are defined on tensor products of Hom-sets; we would like to place such maps into the context of maps that are induced by maps defined on the underlying vector spaces. To be precise, we have

Proposition 6 *Suppose that we have a collection of vector spaces L_p, M_q, V, and W together with linear maps*

$$\phi : M_1 \otimes \ldots \otimes M_k \longrightarrow V$$

and

$$\psi : L_1 \otimes \ldots \otimes L_{i-1} \otimes V \otimes L_i \otimes \ldots \otimes L_n \longrightarrow W.$$

Suppose also that ϕ and ψ induce linear maps

$$\Phi : \bigotimes_{q=1}^{k} Hom(C, M_q) \longrightarrow Hom(C, V)$$

and

$$\Psi : \bigotimes_{p=1}^{i-1} Hom(C, L_p) \otimes Hom(C, V) \otimes \bigotimes_{p=i}^{n} Hom(C, L_p) \longrightarrow Hom(C, W),$$

i.e., $\Phi = \phi_* \circ \Delta^{(k-1)*} \circ \Lambda^{k-1}$ and $\Psi = \psi_* \circ \Delta^{(n)*} \circ \Lambda^n$. Then the composition

$$\Psi \circ (1^{\otimes(i-1)} \otimes \Phi \otimes 1^{\otimes(n-i+1)}) :$$

$$\bigotimes_{p=1}^{i-1} Hom(C, L_p) \otimes \bigotimes_{q=1}^{k} Hom(C, M_q) \otimes \bigotimes_{p=i}^{n} Hom(C, L_p) \longrightarrow Hom(C, W)$$

is induced by $\psi \circ (1^{\otimes(i-1)} \otimes \phi \otimes 1^{\otimes(n-i+1)})$, i.e.

$$\Psi \circ (1^{\otimes(i-1)} \otimes \Phi \otimes 1^{\otimes(n-i+1)}) = [\psi \circ (1^{\otimes(i-1)} \otimes \phi \otimes 1^{\otimes(n-i+1)})]_* \circ \Delta^{(n+k-1)*} \circ \Lambda^{n+k-1}.$$

Proof. The coassociativity of Δ gives us the equality

$$\Delta^{(n+k-1)} = (1^{\otimes(i-1)} \otimes \Delta^{(k-1)} \otimes 1^{(n-i+1)}) \circ \Delta^{(n)}$$

and the associativity of the tensor product gives us

$$\Lambda^{n+k-1} = \Lambda^n \circ (1^{\otimes(i-1)} \otimes \Lambda^{k-1} \otimes 1^{\otimes(n-i+1)}).$$

As a result, we have

$$[\psi \circ (1^{\otimes(i-1)} \otimes \phi \otimes 1^{\otimes(n-i+1)})]_* \circ \Delta^{(n+k-1)*} \circ \Lambda^{n+k-1}$$

$$= \psi_* \circ (1^{\otimes(i-1)} \otimes \phi \otimes 1^{\otimes(n-i+1)})_* \circ ((1^{\otimes(i-1)} \otimes \Delta^{(k-1)} \otimes 1^{\otimes(n-i+1)}) \circ \Delta^{(n)})^*$$
$$\circ \Lambda^n \circ (1^{\otimes(i-1)} \otimes \Lambda^{k-1} \otimes 1^{\otimes(n-i+1)})$$

$$= \psi_* \circ \Delta^{(n)*} \circ (1^{\otimes(i-1)} \otimes \phi \otimes 1^{\otimes(n-i+1)})_* \circ ((1^{\otimes(i-1)} \otimes \Delta^{(k-1)} \otimes 1^{\otimes(n-i+1)})^*$$
$$\circ \Lambda^n \circ (1^{\otimes(i-1)} \otimes \Lambda^{k-1} \otimes 1^{\otimes(n-i+1)})$$

$$= \psi_* \circ \Delta^{(n)*} \circ \Lambda^n \circ (1^{\otimes(i-1)} \otimes \phi_* \otimes 1^{\otimes(n-i+1)}) \circ ((1^{\otimes(i-1)} \otimes \Delta^{(k-1)*} \otimes 1^{\otimes(n-i+1)})$$
$$\circ (1^{\otimes(i-1)} \otimes \Lambda^{k-1} \otimes 1^{\otimes(n-i+1)})$$

(by several applications of Lemma 1)

$$= (\psi_* \circ \Delta^{(n)*} \circ \Lambda^n) \circ (1^{\otimes(i-1)} \otimes \phi_* \circ \Delta^{(k-1)*} \circ \Lambda^{k-1} \otimes 1^{\otimes(n-i+1)})$$

$$= \Psi \circ (1^{\otimes(i-1)} \otimes \Phi \circ 1^{\otimes(n-i+1)}). \ \square$$

If we have $\sigma \in S_{n+k}$, we will denote the map

$$[\psi \circ (1^{\otimes(i-1)} \otimes \phi \otimes 1^{(n-i+1)})]_* \circ \Delta^{(n+k-1)*} \circ \sigma^* \circ \Lambda^{n+k-1}$$

by $(\Psi \circ (1^{\otimes(i-1)} \otimes \Phi \circ 1^{\otimes(n-i+1)}))^\sigma$, or by

$$(\Psi \circ (1^{\otimes(i-1)} \otimes \Phi \circ 1^{\otimes(n-i+1)}))^\sigma(g_1, \ldots, g_{n+k})$$
$$= (\Psi(g_1, \ldots, g_{i-1}, \Phi(g_i, \ldots, g_{i+k-1}), g_{i+k}, \ldots, g_{n+k}))^\sigma.$$

4 TD Algebras

When the product on an algebra L satisfies some (skew) symmetry relations, the induced product on $Hom(C, L)$ will satisfy the corresponding twisted domain symmetries. Indeed if (L, ϕ) is an algebra with relations given by permutations, i.e. $F(\phi^i_\alpha, \phi^i_\alpha \circ \sigma_\beta) = 0$, then the corresponding TD algebra will have relations $F(\Phi^i_\alpha, (\Phi^i_\alpha)^{\sigma_\beta} \circ \sigma_\beta) = 0$ where the ϕ^i_α's are various iterates of ϕ and $1 \otimes \ldots \phi \ldots \otimes 1$.

We also remark that relations in an algebra that do not involve permutations carry over intact to the convolution product. The fundamental example, of course, is associativity; i.e. if the algebra (L, ϕ) is associative and the coalgebra (C, Δ) is coassociative, then the algebra $(Hom(C, L), \Phi)$ is associative.

In this section we examine these TD symmetries for several types of algebras. Again, we assume that (C, Δ) is a fixed coassociative coalgebra. We are primarily interested in the example where (L, ϕ) is a Lie algebra. Recall that the basic relations in a Lie algebra may be phrased as follows:

$$\text{skew symmetry: } \phi \circ \tau = -\phi \tag{2}$$

and

$$\text{Jacobi identity: } \phi \circ (1 \otimes \phi) + \phi \circ (1 \otimes \phi) \circ \xi + \phi \circ (1 \otimes \phi) \circ \xi^2 = 0 \tag{3}$$

where $\tau : L \otimes L \longrightarrow L \otimes L$ is the transposition and $\xi : L \otimes L \otimes L \longrightarrow L \otimes L \otimes L$ is the cyclic permutation (123).

Proposition 7 *Let (L, ϕ) be a Lie algebra and let $\Phi = \phi_* \circ \Delta^* \circ \lambda$. Then $(Hom(C, L), \Phi)$ has the following symmetry properties:*

$$\Phi \circ \tau = -\Phi^\tau \tag{4}$$

and

$$\Phi \circ (1 \otimes \Phi) + \Phi \circ (1 \otimes \Phi)^\xi \circ \xi + \Phi \circ (1 \otimes \Phi)^{\xi^2} \circ \xi^2 = 0 \tag{5}$$

Proof. Equation 4 follows directly from Proposition 5. Next we rewrite the left hand side of Equation 5 as

$$\begin{aligned} &\phi_* \circ (1 \otimes \phi)_* \circ ((1 \otimes \Delta) \circ \Delta)^* \circ \Lambda \\ +&\phi_* \circ (1 \otimes \phi)_* \circ (\xi \circ (1 \otimes \Delta) \circ \Delta)^* \circ \Lambda \circ \xi \\ +&\phi_* \circ (1 \otimes \phi)_* \circ (\xi^2 \circ (1 \otimes \Delta) \circ \Delta)^* \circ \Lambda \circ \xi^2 \end{aligned}$$

$$= \phi_* \circ (1 \otimes \phi)_* \circ ((1 \otimes \Delta) \circ \Delta)^* \circ \Lambda$$
$$+\phi_* \circ (1 \otimes \phi)_* \circ (\xi \circ (1 \otimes \Delta) \circ \Delta)^* \circ (\xi^2)^* \circ \xi_* \circ \Lambda$$
$$+\phi_* \circ (1 \otimes \phi)_* \circ (\xi^2 \circ (1 \otimes \Delta) \circ \Delta)^* \circ \xi^* \circ (\xi^2)_* \circ \Lambda$$

by Lemma 2, and

$$= \phi_* \circ (1 \otimes \phi)_* \circ ((1 \otimes \Delta) \circ \Delta)^* \circ \Lambda$$
$$+\phi_* \circ (1 \otimes \phi)_* \circ \xi_* \circ (\xi^2 \circ \xi \circ (1 \otimes \Delta) \circ \Delta)^* \circ \Lambda$$
$$+\phi_* \circ (1 \otimes \phi)_* \circ \xi_*^2 \circ (\xi \circ \xi^2 \circ (1 \otimes \Delta) \circ \Delta)^* \circ \Lambda$$

$$= \phi_* \circ (1 \otimes \phi)_* \circ (1 + \xi_* + \xi_*^2) \circ ((1 \otimes \Delta) \circ \Delta)^* \circ \Lambda$$
$$= [\phi \circ (1 \otimes \phi) \circ (1 + \xi + \xi^2)]_* \circ ((1 \otimes \Delta) \circ \Delta)^* \circ \Lambda = 0$$

because ϕ satisfies the Jacobi identity. We also used the fact that $\xi^{-1} = \xi^2$ and that the iterated convolution product

$$\Phi \circ (1 \otimes \Phi) = \phi_* \circ (1 \otimes \phi)_* \circ \Delta^* \circ (1 \otimes \Delta)^* \circ \Lambda$$

by Proposition 6. □

When the coalgebra (C, Δ) is cocommutative, the usual Lie algebra structure on $Hom(C, L)$ is evident.

Corollary 8 *If the coassociative coalgebra (C, Δ) is cocommutative, i.e. $\Delta = \tau \circ \Delta$, then Φ is a Lie algebra structure for $Hom(C, L)$.*

Proof. The cocommutativity of Δ implies that $\xi \circ (1 \otimes \Delta) \circ \Delta = (1 \otimes \Delta) \circ \Delta$. We may write $\xi = (\tau \otimes 1) \circ (1 \otimes \tau)$ and then $\xi \circ (1 \otimes \Delta) \circ \Delta = (\tau \otimes 1) \circ (1 \otimes \tau) \circ (1 \otimes \Delta) \circ \Delta = (\tau \otimes 1) \circ (1 \otimes \Delta) \circ \Delta = (\tau \otimes 1) \circ (\Delta \otimes 1) \circ \Delta = (\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$. Replacement of $\tau \circ \Delta$ by Δ in equation 4 yields the skew symmetry relation; replacement of $\xi \circ (1 \otimes \Delta) \circ \Delta$ and $\xi^2 \circ (1 \otimes \Delta) \circ \Delta$ by $(1 \otimes \Delta) \circ \Delta$ yields the usual Jacobi identity. □

Another interesting example is the following:

Corollary 9 *If the coassociative coalgebra (C, Δ) is skew cocommutative, i.e. $\tau \circ \Delta = -\Delta$, then the bracket Φ is symmetric and satisfies the Jacobi identity.*

Proof. From the proof of Proposition 5, we have

$$\Phi \circ \tau = (\phi \circ \tau)_* \circ (\tau \circ \Delta)^* \circ \lambda$$

$$= -\phi_* \circ (\tau \circ \Delta)^* \circ \lambda = \phi_* \circ \Delta^* \circ \lambda$$

For the Jacobi identity we have almost as in the previous proof

$$\xi \circ (1 \otimes \Delta) \circ \Delta = (\tau \otimes 1) \circ (1 \otimes \tau) \circ (1 \otimes \Delta) \circ \Delta$$

$$= -(\tau \otimes 1) \circ (1 \otimes \Delta) \circ \Delta = -(\tau \otimes 1) \circ (\Delta \otimes 1) \circ \Delta$$
$$= (\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta.$$

A similar calculation holds for $\xi^2 \circ (1 \otimes \Delta) \circ \Delta$. $\square$

Remark: Let us write $\Phi(f \otimes g) = fg$ and then the conditions in Corollary 9 imply the equalities $fg = gf$ and $(f^2g)f + f^2(gf) = 0$. These are the defining equations of a **Jordan algebra**. To see that the second equation holds, we write the Jacobi identity as $(fg)h + (hf)g + (gh)f = 0$; if we first let $g = h = f$ we obtain the equality$(f^2)f + (f^2)f + (f^2)f = 0$, i.e.$(f^2)f = 0$ when characteristic $k \neq 3$. Next replace h by f and f by f^2 in the Jacobi identity and obtain the equation $(f^2g)f + (ff^2)g + (gf)f^2 = 0$. The middle term drops out and the remainder (after several applications of commutativity) yields the result.

Of course, the coalgebras that we have in mind for the three previous results are the tensor coalgebra $T^C V$ generated by the vector space V and the subcoalgebras of symmetric tensors and skew symmetric tensors.

We conclude this section with one more example. Recall that the vector space L is a **Poisson algebra** if it possesses a Lie bracket ϕ as well as an associative, commutative multiplication μ; also required is the property that the bracket ϕ acts as a derivation with respect to the operation μ.

This derivation property is usually written as

$$[a, bc] = [a, b]c + b[a, c]$$

where $\phi(a,b) = [a,b]$ and $\mu(a,b) = ab$. We will however use the commutativity of μ to rewrite this relation as

$$[a, bc] = c[a,b] + b[a,c] \tag{6}$$

The reason for this is that we can better describe this relation in functional notation together with permutations as

$$\phi \circ (1 \otimes \mu) = \mu \circ (1 \otimes \phi) \circ \xi + \mu \circ (1 \otimes \phi) \circ (\tau \otimes 1) \tag{7}$$

where as before ξ is the permutation on $L \otimes L \otimes L$ given by (123) and $\tau \otimes 1$ is the permutation given by (12).

Proposition 10 *Let (L, ϕ, μ) be a Poisson algebra and let (C, Δ) be a coassociative coalgebra. Let $\Phi = \phi_* \circ \Delta^* \circ \lambda$ and $M = \mu_* \circ \Delta^* \circ \lambda$. Then $(Hom(C,L), \Phi, M)$ has the structure of a TD Poisson algebra; i.e. Φ satisfies the Lie relations up to permutation as in Proposition 3, M is commutative up to permutation, and the derivation property holds up to permutation.*

Proof. The TD Lie structure for Φ was shown in Proposition 3. For the TD commutativity we have

$$M \circ \tau = \mu_* \circ \Delta^* \circ \lambda \circ \tau = \mu_* \circ \Delta^* \circ \tau_* \circ \tau^* \circ \lambda$$

$$= (\mu \circ \tau)_* \circ (\tau \circ \Delta)^* \circ \lambda = \mu_* \circ (\tau \circ \Delta)^* \circ \lambda = M^\tau.$$

For the derivation property we have

$$\Phi \circ (1 \otimes M) = \phi_* \circ (1 \otimes \mu)_* \circ \Delta^* \circ (1 \otimes \Delta)^* \circ \Lambda$$

$$= (\mu \circ (1 \otimes \phi) \circ \xi)_* \circ \Delta^* \circ (1 \otimes \Delta)^* \circ \Lambda + (\mu \circ (1 \otimes \phi) \circ (\tau \otimes 1))_* \circ \Delta^* \circ (1 \otimes \Delta)^* \circ \Lambda$$

(because (L, ϕ, μ) is a Poisson algebra)

$$= (\mu \circ (1 \otimes \phi))_* \circ \Delta^* \circ (1 \otimes \Delta)^* \circ \xi_* \circ \Lambda$$

$$+ (\mu \circ (1 \otimes \phi))_* \circ \Delta^* \circ (1 \otimes \Delta)^* \circ (\tau \otimes 1)_* \circ \Lambda$$

$$= (\mu \circ (1 \otimes \phi))_* \circ \Delta^* \circ (1 \otimes \Delta)^* \circ \xi^* \circ \Lambda \circ \xi$$

$$+ (\mu \circ (1 \otimes \phi))_* \circ \Delta^* \circ (1 \otimes \Delta)^* \circ (\tau \otimes 1)^* \circ \Lambda \circ (\tau \otimes 1)$$

which is the derivation property except for the presence of the permutations ξ^* and $(\tau \otimes 1)^*$ in the final two lines. □

These examples will be studied further in future work.

5 TD Module Structures and Cohomology

5.1 TD modules

Let (L, ϕ) be a Lie algebra and let B be a module over L via the linear map $\psi : L \otimes B \longrightarrow B$ which satisfies the equality

$$\psi(\phi(x_1, x_2), b) = \psi(x_1, \psi(x_2, b)) - \psi(x_2, \psi(x_1, b))$$

where $x_1, x_2 \in L$ and $b \in B$. Let us write this relation as

$$\psi \circ (\phi \otimes 1) = \psi \circ (1 \otimes \psi) - \psi \circ (1 \otimes \psi) \circ \tau$$

where $\tau = (12) \in S_3$. We may use this formulation to make the following definition.

Definition 11 *Let $(Hom(C, L), \Phi)$ be a TD Lie algebra. Then the vector space $Hom(C, B)$ is a module over $Hom(C, L)$ if there is a linear map*

$$\Psi : Hom(C, L) \otimes Hom(C, B) \longrightarrow Hom(C, B)$$

induced by $\psi : L \otimes B \longrightarrow B$ that satisfies the equation

$$\Psi \circ (\Phi \otimes 1) = \Psi \circ (1 \otimes \Psi) - (\Psi \circ (1 \otimes \Psi))^{\tau} \circ \tau.$$

The next proposition follows immediately from the above definition.

Proposition 12 *Let $\psi : L \otimes B \longrightarrow B$ give B the structure of a Lie module over L. Then $\Psi = \psi_* \circ \Delta^* \circ \lambda$ gives $Hom(C, B)$ the structure of a $Hom(C, L)$ module.*

Example: The fundamental example (indeed the example that motivates the definition) of a TD Lie module is the following: let $(Hom(C, L), \Phi)$ be a TD Lie algebra. Then $Hom(C, L)$ is a module over itself with structure map given by Φ. The proof of this is exactly the same as the Lie algebra case except for the twisting in the Jacobi identity.

5.2 Cohomology

We next consider the cohomology of a TD Lie algebra with coefficients in a module over it. We begin by reviewing the Chevalley - Eilenberg complex for Lie algebras. Let (L, ϕ) be a Lie algebra and B a Lie module over L with structure map ψ. Let $Alt^n(L, B)$ denote the vector space of skew- symmetric linear maps $L^{\otimes n} \longrightarrow B$. Define a linear map

$$d : Alt^n(L, B) \longrightarrow Alt^{n+1}(L, B)$$

for $f_n \in Alt^n(L, B)$ by

$$df_n(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1}\psi(x_i, f_n(x_1, \ldots, \hat{x_i}, \ldots, x_{n+1}))$$
$$+ \sum_{j<k}(-1)^{j+k} f_n(\phi(x_j, x_k), x_1, \ldots, \hat{x_j}, \ldots, \hat{x_k}, \ldots, x_{n+1}).$$

One can check that $d^2 = 0$ and thus we have a cochain complex. A useful fact about this differential is that each of the two summands is a skew symmetric map. This may be verified by using the next two lemmas.

Lemma 13 *Suppose that $f_n : L^{\otimes n} \longrightarrow V$ is a skew symmetric map. Then the extension of f_n to the map $f : L^{\otimes n+1} \longrightarrow V$ that is given by $f(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n}(-1)^{i+1} x_i \otimes f_n(x_1, \ldots, \hat{x_i}, \ldots, x_{n+1})$ is skew symmetric.*

Proof. We verify the claim for the transposition $(p\, p+1)$.

$$f(x_1, \ldots, x_{p+1}, x_p, \ldots, x_{n+1}) = \sum_{i \neq p, p+1} (-1)^{i+1} x_i \otimes f_n(x_1, \ldots, \hat{x_i}, \ldots, x_{p+1}, x_p, \ldots, x_{n+1})$$

$+(-1)^{p+1} x_{p+1} \otimes f_n(x_1, \ldots, \hat{x_{p+1}}, x_p, \ldots, x_{n+1})$ (x_{p+1} is the p-th coordinate)
$+(-1)^{p+2} x_p \otimes f_n(x_1, \ldots, x_{p+1}, \hat{x_p}, \ldots, x_{n+1})$ (x_p is the $p+1$-st coordinate)

$$= - \sum_{i \neq p, p+1} (-1)^{i+1} x_i \otimes f_n(x_1, \ldots, \hat{x_i}, \ldots, x_p, x_{p+1}, \ldots, x_{n+1})$$
$$-(-1)^{p+2} x_{p+1} \otimes f_n(x_1, \ldots, \hat{x}_{p+1}, \ldots, x_{n+1})$$
$$-(-1)^{p+1} x_p \otimes f_n(x_1, \ldots, \hat{x_p}, \ldots, x_{n+1})$$

$$= -f(x_1, \ldots, x_p, x_{p+1}, \ldots, x_{n+1}). \square$$

Lemma 14 *Suppose that* $f_k : L^{\otimes k} \longrightarrow L$ *and* $f_{n-k+1} : L^{\otimes n-k+1} \longrightarrow L$ *are skew symmetric maps. Then the map* $f : L^{\otimes n} \longrightarrow L$ *given by*

$$f(x_1, \dots, x_n) = \sum_{\sigma} (-1)^{\sigma} f_{n-k+1}(f_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}), \dots, x_{\sigma(n)})$$

where σ *runs through all* $(k,\, n-k)$ *unshuffles, is skew symmetric.*

Proof. Again, we verify the claim for the transposition $(p\, p+1)$ and show that

$$f(x_1, \dots, x_{p+1}, x_p, \dots, x_n) = -f(x_1, \dots, x_p, x_{p+1}, \dots, x_n).$$

In the expansion of f, there are three situations to consider. In the first, the indices $p,\ p+1 \in \{\sigma(1), \dots \sigma(k)\}$ and in the second, $p,\ p+1 \in \{\sigma(k+1), \dots, \sigma(n)\}$. The skew symmetry of f_k takes care of the first case while the skew symmetry of f_{n-k+1} takes care of the second. The remaining case occurs when p and $p+1$ are in different sets that are given by the unshuffle decomposition. However, such a term pairs off with the negative of the corresponding term in the expansion of $f(x_1, \dots, x_p, x_{p+1}, \dots, x_n)$ because the permutations that lead to these terms differ only by the product with the transposition $(p\, p+1)$. □

The first summand in the differential is the composition of a linear map, ψ, with the extension of the map f_n using $(1, n)$ - unshuffles; the second is the composition of the skew symmetric map f_n with the extension of the bracket, ϕ, using $(2, n-1)$ - unshuffles. Consequently, we have the alternate description of the differential in the Chevalley-Eilenberg complex given by

$$df = \sum_{\sigma} (-1)^{\sigma} \psi \circ (1 \otimes f) \circ \sigma - \sum_{\sigma'} (-1)^{\sigma'} f \circ (\phi \otimes 1) \circ \sigma'$$

where σ runs through all $(1, n)$ - unshuffles and σ' runs through all $(2, n-1)$ - unshuffles.

We now construct a cochain complex that may be used to calculate the cohomology of the TD Lie algebra $(Hom(C, L), \Phi)$ with coefficients in the module $Hom(C, B)$ with structure map Ψ. Let $TDalt^n(Hom(C, L), Hom(C, B))$ be the vector space of TD skew symmetric maps $(Hom(C, L))^{\otimes n} \longrightarrow Hom(C, B)$. For $n = 0$, $TDalt^0(Hom(C, L), Hom(C, B))$ consists of constant maps from $Hom(C, L)$ to the constant maps from C to B. Moreover, these maps must be induced by constant maps from L to B. Consequently, we define $TDalt^0$

to be the vector space B. We also define $TDalt^1(Hom(C,L), Hom(C,B))$ to be the vector space of linear maps that are induced by linear maps $L \longrightarrow B$

Define a linear map

$$\delta : TDalt^n(Hom(C,L), Hom(C,B)) \longrightarrow TDalt^{n+1}(Hom(C,L), Hom(C,B))$$

as follows: for $F_n \in TDalt^n$ that is induced from $f_n \in Alt^n$,

$$\delta F_n(g_1, \dots\ g_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \Psi(g_i, F_n(g_1, \dots, \hat{g}_i, \dots, g_{n+1}))^\sigma$$

$$- \sum_{j<k} (-1)^{j+k} F_n(\Phi(g_j, g_k), g_1, \dots, \hat{g}_j, \dots, \hat{g}_k, \dots, g_{n+1})^{\sigma'}$$

where σ is the $(1,n)$ unshuffle that in effect interchanges the first and the i-th coordinates and σ' is the $(2, n-1)$ unshuffle that replaces the first and second coordinates by the j-th and k-th coordinates.

When $n = 0$, we have $(\delta f_0)(g)(x) = \Psi(g, f_0)(x) = \psi(g(x), f_0)$ for $g \in Hom(C,L), x \in C$. Here we used the identification $F_0 = f_0 \in B$.

Let us write δ in the form

$$\delta F = \sum_\sigma (-1)^\sigma (\Psi \circ (1 \otimes F))^\sigma \circ \sigma - \sum_{\sigma'} (-1)^{\sigma'} (F \circ (\Phi \otimes 1))^{\sigma'} \circ \sigma'.$$

Proposition 15 *δ is well defined and is induced by d.*

Proof. We first show that δ is induced by d. Begin by considering the case in which both σ and σ' equal the identity element in S_n. Then the map induced by $\psi \circ (1 \otimes f)$, for $\psi \in Hom(L \otimes B, B)$ and $f \in Hom(L^{\otimes n}, B)$, is the map $\Psi \circ (1 \otimes F)$ by Proposition 6. Here, we assume that F is induced by f and Ψ is induced by ψ. Similarly, the map induced by $f \circ (\phi \otimes 1)$ is the map $F \circ (\Phi \otimes 1)$.

Now for general σ, σ', the map $\psi \circ (1 \otimes f) \circ \sigma$ induces the map $\Psi \circ (1 \otimes F)^\sigma \circ \sigma$ and the map $f \circ (\phi \otimes 1) \circ \sigma'$ induces the map $F \circ (\Phi \otimes 1)^{\sigma'} \circ \sigma'$ by Lemma 3. Because the maps $\sum_\sigma (-1)^\sigma \psi \circ (1 \otimes f) \circ \sigma$ and $\sum_{\sigma'} (-1)^{\sigma'} f \otimes (\phi \otimes 1) \sigma'$ are skew symmetric, the induced maps $\sum_\sigma (-1)^\sigma \Psi \circ (1 \otimes F)^\sigma \circ \sigma$ and $\sum_{\sigma'} (-1)^{\sigma'} F \circ (\Phi \otimes 1)^{\sigma'} \circ \sigma'$ are TD skew symmetric by Proposition 5 and it follows that δ is well defined. □

Corollary 16 $\delta^2 = 0$

Proof. This now follows directly from the fact that $d^2 = 0$. □

Of course, we now define the cohomology of $Hom(C, L)$ to be the cohomology of this complex. Just as in the Lie algebra setting, $H^0(Hom(C, L), Hom(C, B))$ will equal the subspace of invariant elements $\beta \in Hom(C, B) = B$ in the sense that $\Psi(\alpha, \beta) = 0$ for all $\alpha \in Hom(C, L)$.

6 Twisted Domain Lie - Rinehart

In the classical version of Lie Rinehart cohomology [2], the situation involves a Lie algebra (L, ϕ) and an associative algebra (B, ν) which are interrelated by the following data. First of all, L is a left B-module via a structure map $\mu : B \otimes L \longrightarrow L$; we usually write $\mu(a, x) = ax$. Next we have that B is an L module via a structure map $\psi : L \otimes B \longrightarrow B$; moreover, we assume that for fixed $x \in L$, ψ is a derivation on B; i.e. $\psi(x, ab) = a\psi(x, b) + \psi(x, a)b$ where we write $\nu(a, b) = ab$. Recall that this means that the adjoint ψ^* of ψ is a Lie homomorphism from L to the Lie algebra of derivations on B.

These structures are further related by the following two conditions:

LRa: $\psi \circ (\mu \otimes 1) = \nu \circ (1 \otimes \psi)$

LRb: $\phi \circ (1 \otimes \mu) = \mu \circ [(1 \otimes \phi) \circ \tau + (\psi \otimes 1)]$

where $\tau = (1\,2) \in S_3$. A pair (L, B) that satisfies all of the above conditions is called a **Lie - Rinehart pair**.

The property labeled LRa may be rephrased as $\psi(ax, b) = a\psi(x, b)$ for $x \in L$ and $a, b \in B$ and says that the map ψ is B-linear.

The property labeled LRb may be rephrased as $[x, ay] = \psi(x, a)y + a[x, y]$ for $a \in B$ and $x, y \in L$ and describes the relationship between the bracket on L with the two module structures.

Such structures are of interest when one wishes to study the maps in $Alt^n(L, B)$ that are B-linear. The conditions required for a Lie - Rinehart pair will guarantee that the subspace of $Alt^n(L, B)$ of B-linear maps together with the restriction of the Chevalley-Eilenberg differential is in fact a subcomplex of the Chevalley-Eilenberg complex.

We next examine the structures on the pair $(Hom(C,L), Hom(C,B))$ that are induced by a Lie-Rinehart structure on (L,B).

As in the previous section we have the twisted Lie algebra $(Hom(C,L),\Phi)$ and the module $Hom(C,B)$ with structure map Ψ. The associative algebra structure on (B,ν) induces an associative (no twist) algebra structure on $(Hom(C,B),\bar{\nu})$. The left B-module structure on L induces a left $Hom(C,B)$-module structure on $Hom(C,L)$ (again, no twist needed)

$$\bar{\mu} : Hom(C,B) \otimes Hom(C,L) \longrightarrow Hom(C,L)$$

such that $\bar{\mu} \circ (1 \otimes \bar{\mu}) = \bar{\mu} \circ (\bar{\nu} \otimes 1)$. Moreover, Ψ is a twisted derivation on $Hom(C,B)$; i.e.

$$\Psi \circ (1 \otimes \bar{\nu}) = (\bar{\nu} \circ (\Psi \otimes 1))^{\tau} \circ \tau + \bar{\nu} \circ (\Psi \otimes 1)$$

where $\tau = (1\,2) \in S_3$, or, for fixed $g \in Hom(C,L)$,

$$\Psi(g, \bar{\nu}(\alpha,\beta)) = \bar{\nu}(\alpha, \Psi(g,\beta))^{\tau} + \bar{\nu}(\Psi(g,\alpha),\beta).$$

The appropriate requirements for twisted Lie-Rinehart now take the form

TDLRa: $\Psi \circ (\bar{\mu} \otimes 1) = \bar{\nu} \circ (1 \otimes \Psi)$
TDLRb: $\Phi \circ (1 \otimes \bar{\mu}) = [\bar{\mu} \circ (1 \otimes \Phi)]^{\tau} \circ \tau + \bar{\mu} \circ (\Psi \otimes 1)$

where $\tau = (1\,2) \in S_3$. We say that $(Hom(C,L), Hom(C,B))$ is a **TD Lie Rinehart** pair.

Observe that by using skew symmetry, condition LRb may be written in the form

$$\phi \circ (\mu \otimes 1) = \mu \circ (1 \otimes \phi) - \mu \circ (\psi \otimes 1) \circ \sigma$$

where $\sigma = (1\,2\,3) \in S_3$. The corresponding TDLRb thus assumes the form

$$\Phi \circ (\bar{\mu} \otimes 1) = \bar{\mu} \circ (1 \otimes \Phi) - \bar{\mu} \circ (\Psi \otimes 1)^{\sigma} \circ \sigma$$

which in turn may be written as

$$\Phi(\bar{\mu}(\beta, g_1), g_2) = \bar{\mu}(\beta, \Phi(g_1, g_2)) - \bar{\mu}(\Psi(g_2,\beta), g_1)^{\sigma}$$

where $g_1, g_2 \in Hom(C,L), \beta \in Hom(C,B)$ and $\sigma = (1\,2\,3) \in S_3$.

We next consider elements α of $TDalt^n(Hom(C,L), Hom(C,B))$ that are **Hom(C, B)-linear**. By this we mean that

$$\alpha\circ(1^{\otimes(i-1)}\otimes\bar{\mu}\otimes 1^{\otimes(n-i+1)})(g_1,\ldots,g_{i-1},\beta,g_i,\ldots,g_n) = (\bar{\nu}\circ(1\otimes\alpha))^{\sigma}(\beta,g_1,\ldots,g_n)$$

or

$$\alpha(g_1,\ldots,g_{i-1},\bar{\mu}(\beta,g_i),g_{i+1},\ldots,g_n) = \bar{\nu}(\beta,\alpha(g_1,\ldots,g_n))^{\sigma}$$

for each i, where σ is the cyclic permutation $(0\,1\ldots(i-1))$ in S_{n+1} regarded as acting on the ordered set $\{0,1,\ldots,n\}$, $\beta \in Hom(C,B)$, and $(g_1,\ldots,g_n) \in Hom(C,L)^{\otimes n}$.

The point here is that just as in the Lie algebra case where the Lie Rinehart complex is a subcomplex of the Chevalley-Eilenberg complex, the TD Lie-Rinehart complex is a subcomplex of the TD Chevalley-Eilenberg complex.

To see this, we need only verify that the Chevalley-Eilenberg differential δ preserves $Hom(C,B)$ linearity. We summarize this in

Theorem 17 *If $(Hom(C,L), Hom(C,B))$ is a TD Lie-Rinehart pair, then the TD Lie-Rinehart complex is a subcomplex of the TD Chevalley-Eilenberg complex.*

Proof. We claim that

$$\delta F_n(g_1,\ldots,\bar{\mu}(\beta,g_i),\ldots,g_{n+1}) = \bar{\nu}(\beta,\delta F_n(g_1,\ldots,g_i,\ldots,g_{n+1}))^{\xi}$$

for each $\xi = (0\,1\ldots i-1) \in S_{n+2}$. Here, we regard the permutations as acting on the set of integers $\{0,1,\ldots,n+1\}$. We first verify this for the case $i=1$ and then use skew symmetry to complete the proof.

To begin, we must show that

$$\delta F_n(\bar{\mu}(\beta,g_1),g_2,\ldots,g_{n+1}) = \bar{\nu}(\beta,\delta F_n(g_1,\ldots,g_i,\ldots,g_{n+1})).$$

We use the definition of δ to write

$$\delta F_n(\bar{\mu}(\beta,g_1),g_2,\ldots,g_{n+1}) =$$

$$\sum_{i\neq 1}(-1)^{i+1}\Psi(g_i,F_n(\bar{\mu}(\beta,g_1),g_2,\ldots,\hat{g}_i,\ldots,g_{n+1}))^{\sigma}$$

$$+\Psi(\bar{\mu}(\beta,g_1),F_n(g_2,\ldots,g_{n+1}))$$

$$+\sum_{\substack{p<q\\ p\neq 1}}(-1)^{p+q}F_n(\Phi(g_p,g_q),\bar{\mu}(\beta,g_1),\ldots,\hat{g}_p,\ldots,\hat{g}_q,\ldots,g_{n+1}))^{\sigma'}$$

$$+\sum_{q>1}(-1)^{1+q}F_n(\Phi(\bar{\mu}(\beta,g_1),g_q),g_2,\ldots,\hat{g}_q,\ldots,g_{n+1})^{\sigma''}$$

where $\sigma = (i\,0\,1\,\ldots(i-1))$, $\sigma' = (p\,0\,1\ldots(p-1))(q\,1\,2\ldots(q-1))$ and $\sigma'' = (q\,2\,3\ldots(q-1))$. We apply the property of $Hom(C,B)$ linearity to F_n and to Ψ in the first three summands to rewrite the above sum as

$$\sum_{i\neq 1}(-1)^{i+1}\Psi(g_i,\bar{\nu}(\beta,F_n(g_1,\ldots,\hat{g}_i,\ldots,g_{n+1}))^{\sigma}$$

$$+\bar{\nu}(\beta,\Psi(g_1,F_n(g_2,\ldots,g_{n+1})))$$

$$+\sum_{\substack{p<q\\ p\neq 1}}(-1)^{p+q}\bar{\nu}(\beta,F_n(\Phi(g_p,g_q),g_1,\ldots,\hat{g}_p,\ldots,\hat{g}_q,\ldots,g_{n+1}))^{\sigma'\gamma}$$

$$+\sum_{q>1}(-1)^{1+q}F_n(\Phi(\bar{\mu}(\beta,g_1),g_q),g_2,\ldots,\hat{g}_q,\ldots,g_{n+1})^{\sigma''}$$

with $\gamma = (2\,0\,1) \in S_{n+2}$. We next apply the derivation property of Ψ to the first summand and the TD Lie-Rinehart property to the last summand and obtain

$$\sum_{i\neq 1}(-1)^{i+1}\bar{\nu}(\beta,\Psi(g_i,F_n(g_1,\ldots,\hat{g}_i,\ldots,g_{n+1}))^{\sigma\tau}$$

$$+\sum_{i\neq 1}(-1)^{i+1}\bar{\nu}(\Psi(g_i,\beta),F_n(g_1,\ldots,\hat{g}_i,\ldots,g_{n+1}))^{\sigma}$$

$$+\bar{\nu}(\beta,\Psi(g_1,F_n(g_2,\ldots,g_{n+1}))$$

$$+\sum_{\substack{p<q\\ p\neq 1}}(-1)^{p+q}\bar{\nu}(\beta,F_n(\Phi(g_p,g_q),g_1,\ldots,\hat{g}_p,\ldots,\hat{g}_q,\ldots,g_{n+1}))^{\sigma'\gamma}$$

$$+\sum_{q>1}(-1)^{1+q}F_n(\bar{\mu}(\beta,\Phi(g_1,g_q)),g_2,\ldots,\hat{g}_q,\ldots,g_{n+1})^{\sigma''}$$

$$-\sum_{q>1}(-1)^{1+q}F_n(\bar{\mu}(\Psi(g_q,\beta),g_1),g_2,\ldots,\hat{g}_q,\ldots,g_{n+1})^{\sigma''\gamma'}$$

with $\tau = (0\,1)$ and $\gamma' = (0\,1\,2)$.

Finally, we apply the $Hom(C, B)$ linearity property to the last summand to write it in the form

$$-\sum_{q>1}(-1)^{1+q}\bar{\nu}(\Psi(g_q, \beta), F_n(g_1, \ldots, \hat{g}_q, \ldots, g_{n+1}))^{\sigma''\gamma'}$$

which will then cancel with the second summand after we note that $\sigma''\gamma' = \sigma$. In the first summand, we have $\sigma\tau = (1\,2\ldots q)$ which we may regard as a cyclic permutation in S_{n+1}. It then follows that all of the remaining summands yield the desired $\bar{\nu}(\beta, \delta F_n(g_1, \ldots, g_i, \ldots, g_{n+1}))$.

For the general case, we first recall that

$$\delta F_n \circ (1^{\otimes(i-1)} \otimes \bar{\mu} \otimes 1^{\otimes(n-i+1)})$$

is induced by the map

$$df_n \circ (1^{\otimes(i-1)} \otimes \mu \otimes 1^{\otimes(n-i+1)})$$

by Proposition 6. We use the skew symmetry of df_n to rewrite this map as

$$(-1)^{\sigma} df_n \circ (\mu \otimes 1^{\otimes n}) \circ \sigma \circ \xi$$

where $\xi = (0\,1\,\ldots\,(i-1))$ and $\sigma = (1\,2\,\ldots\,i)$. This map induces the map

$$(-1)^{\sigma}(\delta F_n \circ (\bar{\mu} \otimes 1^{\otimes n})^{\xi\sigma}) \circ \sigma \circ \xi$$

which is equal to

$$(-1)^{\sigma}(\bar{\nu} \circ (1 \otimes \delta F_n))^{\xi\sigma} \circ \sigma \circ \xi$$

by the first part of this proof. However, this last map is induced by the map

$$(-1)^{\sigma}(\nu \circ (1 \otimes df_n)) \circ \sigma \circ \xi$$

which is equal to

$$(-1)^{\sigma}(-1)^{\sigma^{-1}}(\nu \circ (1 \otimes df_n)) \circ \sigma^{-1} \circ \sigma \circ \xi$$

by the skew symmetry of df_n. This last map is equal to

$$\nu \circ (1 \otimes df_n) \circ \xi$$

which induces the map

$$\bar{\nu} \circ (1 \otimes \delta F_n)^{\xi} \circ \xi$$

and the proof is complete. □

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