ON ALGEBRAIC EQUATIONS SATISFIED BY HYPERGEOMETRIC CORRELATORS IN WZW MODELS. II.

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1. Introduction

1.1. Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra; let $(\ ,\ )$ be an invariant scalar product on $\mathfrak{g}$ normalized in such a way that $(\theta,\theta)=2$, θ being the highest root. Fix a positive integer k. Let $L_1,\dots,L_{n+1}$ be irreducible representations of $\mathfrak{g}$ with highest weights $\Lambda_1,\dots,\Lambda_{n+1}$. Suppose that $(\Lambda_i,\theta)\le k$ for all i.

Consider a complex affine n-dimensional affine space $\mathbb{A}^n$ with fixed coordinates $\mathbf{z}=(z_1,\dots,z_n)$. Consider the space $X_n=\mathbb{A}^n-\cup_{i,j}\Delta_{ij}$ where $\Delta_{ij}=\{(z_1,\dots,z_n)|z_i=z_j\}$ are diagonals. According to Conformal field theory, one can define a remarkable finite dimensional holomorhic vector bundle $\mathcal{C}(\Lambda_1,\dots,\Lambda_{n+1})$ over X_n equipped with a flat connection (with logarithmic singularities along Δ_{ij}). (We imply that the last representation "lives" at the point $z_{n+1}=\infty$.)

More precisely, consider a trivial bundle over X_n with a fiber $(L_1\otimes\dots\otimes L_{n+1})_{\mathfrak{g}}$. Here we denote by $M_{\mathfrak{g}}$ the space of coinvariants $M/\mathfrak{g}M$ of a $\mathfrak{g}$-module M. Let us denote this bundle by $\mathcal{B}(\Lambda_1,\dots,\Lambda_{n+1})$; it is equipped with a flat connection given by a system of Knizhnik-Zamolodchikov (KZ) differential equations, [KZ]. The bundle $\mathcal{C}(\Lambda_1,\dots,\Lambda_{n+1})$ is a certain quotient of $\mathcal{B}(\Lambda_1,\dots,\Lambda_{n+1})$ stable under KZ connection.

Classically this quotient is described in terms of certain coinvariants of the tensor product $\mathbf{L}_1\otimes\dots\otimes\mathbf{L}_{n+1}$ where $\mathbf{L}_i$ is the irreducible representation of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ corresponding to L_i and having the central charge k (see for example [KL] or Sect. 2 below). The first goal of the present paper is a precise description of $\mathcal{B}(\Lambda_1,\dots,\Lambda_{n+1})$ in terms of finite dimensional representations L_i. More precisely, the fiber of this bundle at a point $\mathbf{z}=(z_1,\dots,z_n)$ may be described as follows.

Let $f_\theta\in\mathfrak{g}$ be a root vector of weight $-\theta$. Consider the operator

$$(1)\qquad \mathbf{z}\cdot f_\theta=\sum_{i=1}^{n} z_i f_\theta^{(i)}: L_1\otimes\dots\otimes L_n\longrightarrow L_1\otimes\dots\otimes L_n$$

where $f_\theta^{(i)}$ denotes operator acting as f_θ on i-th factor and as identity on the other factors. For a weight λ let M_λ denote the weight component of a $\mathfrak{g}$-module M. The map (1) induces an

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operator

$$(\mathbf{z}\cdot f_\theta)^{k-(\Lambda_{n+1},\theta)+1} : (L_1\otimes\ldots\otimes L_n)_{s_0(\bar\Lambda_{n+1})} \longrightarrow (L_1\otimes\ldots\otimes L_n)_{\bar\Lambda_{n+1}} \tag{2}$$

where $\bar\Lambda_{n+1}$ is the highest weight of the dual module L^*_{n+1}, and $s_0(\bar\Lambda_{n+1}) = \bar\Lambda_{n+1}+(k-(\Lambda_{n+1},\theta)+1)\theta$. Let us denote this operator by $\mathbf{T}(\mathbf{z})$. We prove (see 2.10):

Theorem. *One has a canonical isomorphism*

$$\mathcal{C}(\Lambda_1,\ldots,\Lambda_{n+1})_{\boldsymbol{z}} \cong \text{Coker } \boldsymbol{T}(\boldsymbol{z}).$$

1.2. The second goal of this paper is a construction of a natural map from $\mathcal{C}(\Lambda_1,\ldots,\Lambda_{n+1})$ to a certain bundle of "geometric" origin.

More precisely, let $\Lambda_{n+1} = \sum_{s=1}^n \Lambda_s - \sum_{i=1}^r k_i\alpha_i$, α_i being simple roots of $\mathfrak{g}$. All k_i are non-negative integers (otherwise $(L_1\otimes\ldots\otimes L_{n+1})_{\mathfrak{g}} = 0$). Set $N=\sum_{i=1}^r k_i$. Let us consider the space $X_{n+N} = \mathbb{C}_{n+N} - \cup_{i,j=1}^{n+N}\Delta_{ij}$; let us denote coordinates in X_{n+N} by $z_1,\ldots,z_n,t_1,\ldots,t_N$. We have a projection to the first coordinates $p_N : X_{n+N}\longrightarrow X_n$.

Following [SV], define the flat connection on the trivial one-dimensional vector bundle over X_{n+N} by the 1-form

$$\omega = \sum_{s>s'} \frac{(\Lambda_s,\Lambda_{s'})}{k+g}\,\text{dlog}(z_s-z_{s'}) + \sum_{s,i} -\frac{\alpha_{\pi(i)},\Lambda_s}{k+g}\,\text{dlog}(t_i-z_s) + \sum_{i>j}\frac{(\alpha_{\pi(i)},\alpha_{\pi(j)})}{k+g}\,\text{dlog}(t_i-t_j)$$

where g is the dual Coxeter number of $\mathfrak{g}$, $\pi : \{1,\ldots,N\}\longrightarrow\{1,\ldots,r\}$ is any map with $\text{card}(\pi^{-1}(i)) = k_i$ for all i. Let us denote the trivial bundle equipped with this connection by $\mathcal{L} = \mathcal{L}(\Lambda_1,\ldots,\Lambda_{n+1})$. The product of symmetric groups $\Sigma = \Sigma_{k_1}\times\ldots\times\Sigma_{k_r}$ acts naturally fiberwise on the pair $(X_{N+n},\mathcal{L})$.

For each $\mathbf{z}\in X_n$ consider the De Rham cohomology $H^N(p_N^{-1}(\mathbf{z}),\mathcal{L}_{p_N^{-1}(\mathbf{z})})$; these spaces form a vector bundle $R^Np_{N*}\mathcal{L}$ over X_n equipped with a flat Gauss-Manin connection. In [SV] certain maps compatible with the connections

$$\omega : \mathcal{B}(\Lambda_1,\ldots,\Lambda_{n+1}) \longrightarrow R^Np_{N*}\mathcal{L}(\Lambda_1,\ldots,\Lambda_{n+1})^{\Sigma,-} \tag{3}$$

where constructed (here the superscript "$\Sigma,-$" denotes the subbundle of skew invariants). The second main result of the present paper is (see 4.3.1):

Theorem. *The map ω passes through the projection $\mathcal{B}(\Lambda_1,\ldots,\Lambda_{n+1})\longrightarrow\mathcal{C}(\Lambda_1,\ldots,\Lambda_{n+1})$ and thus induces the map*

$$\bar\omega : \mathcal{C}(\Lambda_1,\ldots,\Lambda_{n+1}) \longrightarrow R^Np_{N*}\mathcal{L}(\Lambda_1,\ldots,\Lambda_{n+1})^{\Sigma,-}$$

There are reasons to expect that the map $\bar\omega$ is injective. It would be very interesting to define its image in topological terms; if the above expectation is true, we would have a topological description of the bundle of conformal blocks.

1.3. The paper goes as follows. Section 1 is devoted to the proof of Theorem 2.10.

The main aim of Sections 2 and 3 is to define the map (3). This map actually was introduced in [SVB], [SV]. However, [SVB] did not contain proofs, and the result [SV] was formulated in a greater generality, and we need here some important details not formulated explicitely in *loc.cit.* We include these details in Section 2. In this Section we discuss the beautifull interrelation between certain spaces of rational functions on affine spaces, graphs, and free Lie algebras. We believe that the contents of this Section might be of independent interest.

At the end of Section 3 we formulate Theorem 4.3.1.

Sections 4 and 5 are devoted to the proof of Theorem 4.3.1. This result is equivalent to the claim that certain differential forms are exact. In Theorem 5.8 we write down certain identity between differential forms, which is more general and precise than the above claim. We call it **Resonance identity**. In Section 5 we prove it.

1.4. The results of this paper have been announced in [FSVA]. The proof for the case $\mathfrak{g} = sl(2)$ is given in [FSV1].

Although the present paper heavily depends on the main construction of [SV], we regard it as practically self-contained. In fact, we tried to include into Sections 2 and 3 all the results from *loc.cit.* which we need; the proofs are either given or straightforward. We hope that this alternative exposition is usefull also for a better understanding of a more general framework of *loc.cit.*

We are greately indebted to Michael Finkelberg for his permission to include his proof of the key point of Theorem 2.10. Our initial proof was more complicated.

2. Spaces of conformal blocks

2.1. Throughout the paper we fix a complex finite dimensional simple Lie algebra $\mathfrak{g}$ with a chosen system of Chevalley generators $f_i, e_i, h_i,\ i = 1, \ldots, r$. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the corresponding Cartan decomposition; $\alpha_1, \ldots, \alpha_r \in \mathfrak{h}^*$ the simple roots. (For a vector space V, V^* will always denote the dual vector space.)

Let $\omega : \mathfrak{g} \overset{\sim}{\longrightarrow} \mathfrak{g}$ denote *Chevalley involution* of $\mathfrak{g}$ - the Lie algebra isomorphism such that $\omega(f_i) = -e_i,\ \omega(e_i) = -f_i,\ \omega(h_i) = -h_i$.

$$\theta = \sum_{i=1}^{r} a_i \alpha_i \tag{4}$$

will denote the highest root;

$$\theta^\vee = \sum_{i=1}^{r} a_i^\vee h_i \tag{5}$$

the highest root of the dual root system. Here all $a_i, a_i^\vee$ are positive integers. We set

$$g = \sum_{i=1}^{r} a_i^\vee + 1$$

- it is the dual Coxeter number of $\mathfrak{g}$, [K], 6.0.

Let

$$\nu : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^* \tag{6}$$

denote the isomorphism such that $\nu(h_i) = a_i^{\vee -1} a_i \alpha_i$ (cf. [K], 6.2.2).Let

$$(,) : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{C}$$

denote the corresponding bilinear form; it is non-degenerate and symmetric. We denote by the same symbol the bilinear form on $\mathfrak{h}^*$ induced by means of ν. We have $(\theta, \theta) = 2$. Evidently, $\nu(\theta^\vee) = \theta$.

We extend $(,)$ to the symmetric non-degenerate invariant bilinear form on $\mathfrak{g}$ as in [K], ch. 2.

$$\Omega \in \mathfrak{g} \otimes \mathfrak{g} \tag{7}$$

will denote the corresponding invariant symmetric tensor.

2.2. If M is a representation of $\mathfrak{h}$, $\lambda \in \mathfrak{h}^*$, we set $M_\lambda = \{x \in M | hx =< \lambda, h > x \text{ for all } h \in \mathfrak{h}\}$. We will consider only $\mathfrak{h}$-*diagonalizable* representations of $\mathfrak{g}$, i.e. such that $M = \oplus_\lambda M_\lambda$.

Set $M^0 = \oplus_\lambda M_\lambda^*$; introduce an action of $\mathfrak{g}$ on M^0 by the formula $< cx^*, x >=< x^*, -\omega(c)x >$ for $x \in M,\ x^* \in M^0,\ c \in \mathfrak{g}$. This $\mathfrak{g}$-module is called *the contragradient* to M.

Given $\Lambda \in \mathfrak{h}^*$, $M(\Lambda)$ will denote the Verma module over $\mathfrak{g}$, generated by the vacuum vector v_Λ subject to defining relations $\mathfrak{n}_+ v_\Lambda = 0,\ hv_\Lambda =< \Lambda, h > v_\Lambda$. $L(\Lambda)$ will denote the unique maximal irreducible quotient of $M(\Lambda)$; by abuse of the notations, we will denote by v_Λ the image of v_Λ in $L(\Lambda)$ too.

There is a unique $\mathfrak{g}$-module morphism

$$S : M(\Lambda) \longrightarrow M(\Lambda)^0 \tag{8}$$

such that $< S(v_\Lambda), v_\Lambda >= 1$. We have $L(\Lambda) = M(\Lambda)/\ker(S)$.

We can also consider S as a bilinear form $M(\Lambda) \times M(\Lambda) \longrightarrow \mathbb{C}$; it is called the *Shapovalov form*.

The weight Λ is called *dominant integral* if all $< \Lambda, h_i >$ are non-negative integers. This is equivalent to finite dimensionality of $L(\Lambda)$.

Let W denote the Weyl group of $\mathfrak{g}$, $w_0 \in W$ the longest element. For a dominant integral Λ, $w_0(\Lambda)$ is the lowest weight of $L(\Lambda)$ ([B], ch. VIII, §7, no. 2, Remark 2). We shall denote

$$\bar{\Lambda} = -w_0(\Lambda)$$

It is again a dominant integral weight, and we have

$$L(\bar{\Lambda}) = L(\Lambda)^*$$

2.3. Let T be an independent variable, $\mathbb{C}[[T]]$ the ring of formal power series, $\mathbb{C}((T))$ the field of Laurent power series. For $f(T),\ g(T) \in \mathbb{C}((T))$, introduce the notation

$$\operatorname{res}_0(f(T)dg(T)) = \text{ coefficient at } T^{-1} \text{ of } f(T)g'(T).$$

Set $\mathfrak{g}[[T]] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[T]] \subset \mathfrak{g}((T)) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((T))$. These are Lie algebras with the bracket $[c \otimes f(T), c' \otimes g(T)] = [c, c'] \otimes f(T)g(T),\ c, c' \in \mathfrak{g}$. Define $\hat{\mathfrak{g}}$ as a central extension of $\mathfrak{g}((T))$

$$\hat{\mathfrak{g}} = \mathfrak{g}((T)) \oplus \mathbb{C} \cdot \mathbf{1}$$

where $\mathbf{1}$ lies in the centrum of $\hat{\mathfrak{g}}$, and

$$[c \otimes f(T), c' \otimes g(T)] = [c, c'] \otimes f(T)g(T) + (c, c') \operatorname{res}_0(f(T)dg(T)) \cdot \mathbf{1}$$

Set

$$\tilde{\mathfrak{g}} = \mathfrak{g}[T, T^{-1}] \oplus \mathbb{C} \cdot \mathbf{1}$$

It is a Lie subalgebra of $\hat{\mathfrak{g}}$.

We have natural embeddings $\mathfrak{g} \subset \tilde{\mathfrak{g}} \subset \hat{\mathfrak{g}}$; we will identify $\mathfrak{g}$ with its image in these algebras. We denote by $\tilde{\mathfrak{g}}^+$ (resp., $\hat{\mathfrak{g}}^+$) the Lie subalgebra $\mathfrak{g}[T] \oplus \mathbb{C} \cdot \mathbf{1} \subset \tilde{\mathfrak{g}}$ (resp., $\mathfrak{g}[[T]] \oplus \mathbb{C} \cdot \mathbf{1} \subset \hat{\mathfrak{g}}$).

Let us choose an element e_θ in the root subspace $\mathfrak{g}_\theta$ such that $(e_\theta, -\omega(e_\theta)) = 1$; set $f_\theta = \omega(e_\theta)$. We have $[e_\theta, f_\theta] = \theta^\vee$.

Set $e_0 = f_\theta T,\ f_0 = e_\theta T^{-1}$. The elements $e_0, \ldots, e_r,\ f_0, \ldots f_r$ form a system of generators of $\tilde{\mathfrak{g}}$, [K], ch. 7. We have

$$[e_0, f_0] = \mathbf{1} - \theta^\vee \tag{9}$$

2.4. All representations V of $\tilde{\mathfrak{g}}$ we will consider will have the following finiteness property: for every $x \in V$ there exists $n \in \mathbb{Z}$ such that $cT^{n'}x = 0$ for all $n' \geq n,\ c \in \mathfrak{g}$. For such representations, the action of $\tilde{\mathfrak{g}}$ may be extended uniquely to the action of $\hat{\mathfrak{g}}$, cf. [KL], no. 1.

We fix a positive integer k; unless specified otherwise, $\mathbf{1}$ will act as the multiplication by k on all our representations. We set $\kappa = k + g$.

Let M be a representation of $\mathfrak{g}$. Consider M as a $\tilde{\mathfrak{g}}^+$-module, by setting $T\mathfrak{g}[T]$ to act as zero, and $\mathbf{1}$ as k. Set

$$\tilde{M} = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}^+)} M$$

This $\tilde{\mathfrak{g}}$-module is called *the generalized Weyl module* associated to M. We have a natural embedding $M \subset \tilde{M}$.

If $M = L(\Lambda)$, we denote $\tilde{M}$ by $\tilde{L}(\Lambda)$. This module has a unique irreducible quotient which will be denoted by $\mathbf{L}(\Lambda)$. We have an embedding $L(\Lambda) \subset \mathbf{L}(\Lambda)$.

A weight Λ is called *small* if it is dominant integral and $(\Lambda, \theta) \leq k$. The set of all small weights will be denoted by $C \subset \mathfrak{h}^*$. Define $s_0 : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ as

$$s_0(\lambda) = \lambda + (k - (\lambda, \theta) + 1)\theta$$

For $\Lambda \in C$, the irreducible module $\mathbf{L}(\Lambda)$ is the quotient of the Weyl module $\tilde{L}(\Lambda)$ by the $\tilde{\mathfrak{g}}$-submodule L' generated by the singular vector $f_0^{k-(\Lambda,\theta)+1}v_\Lambda$. These irreducible modules will be most important in the sequel. We have an exact sequence

$$\tilde{L}(s_0(\Lambda)) \longrightarrow \tilde{L}(\Lambda) \longrightarrow \mathbf{L}(\Lambda) \longrightarrow 0 \tag{10}$$

where the first map sends $v_{s_0(\Lambda)}$ to $f_0^{k-(\Lambda,\theta)+1}v_\Lambda$.

2.5. Spaces of coinvariants. For a positive integer n denote by $\hat{\mathfrak{g}}^n$ the central extension of the n-th cartesian power $\mathfrak{g}[[T]]^n$

$$\mathfrak{g}[[T]]^n \oplus \mathbb{C}\cdot\mathbf{1}$$

with the bracket

$$[(c_i\otimes f_i(T))_{1\le i\le n}, (c_i'\otimes g_i(T))_{1\le i\le n}] = ([c_i,c_i']\otimes f_i(T)g_i(T))_{1\le i\le n} \oplus \sum_{i=1}^{n}(c_i,c_i')\operatorname{res}_0(f(T)dg(T))\cdot\mathbf{1}$$

(it is *not* the n-th cartesian power of $\hat{\mathfrak{g}}^n$). The Lie subalgebra $\tilde{\mathfrak{g}}^n \subset \hat{\mathfrak{g}}^n$ is defined analogously.

Consider the Riemann sphere $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$, with a fixed coordinate z. Let us pick $n+1$ distinct points $z = z_i,\ i = 1,\dots,n+1$. We suppose that $z_i \in \mathbb{C}$ for $1 \le i \le n$, and $z_{i+1} = \infty$.

Let $\mathfrak{g}(z_1,\dots,z_{n+1})$ denote by the Lie algebra of rational functions on $\mathbb{P}^1$ with values in $\mathfrak{g}$, regular outside $z_1,\dots,z_{n+1}$. We have local coordinates at our punctures: $z - z_i$ for $1 \le i \le n$, and $1/z$ at ∞. The Laurent expansions at our points define an embedding

$$\mathfrak{g}(z_1,\dots,z_{n+1}) \hookrightarrow \hat{\mathfrak{g}}^{n+1} \tag{11}$$

(see for example [KL]II, 9.9 or [FSV1], 2.3).

For a Lie algebra $\mathfrak{a}$ and an $\mathfrak{a}$-module M, we will denote by $M_{\mathfrak{a}}$ the space of coinvariants $M/\mathfrak{a}M$.

Given $n+1$ $\hat{\mathfrak{g}}$-modules $M_1,\dots,M_{n+1}$, the algebra $\hat{\mathfrak{g}}^n$ acts naturally on the tensor product $M_1\otimes\ldots\otimes M_{n+1}$ (recall that $\mathbf{1}$ acts as k on each M_i by our assumtion). Using (11), we regard this tensor product as a $\mathfrak{g}(z_1,\dots,z_{n+1})$-module, and can consider the space of coinvariants

$$(M_1\otimes\ldots\otimes M_{n+1})_{\mathfrak{g}(z_1,\dots,z_{n+1})}$$

2.6. Lemma. *Let $M_1,\dots,M_{n+1}$ be $\mathfrak{g}$-modules. The embeddings $M_i \hookrightarrow \tilde{M}_i$ induce an isomorphism of coinvariants*

$$(M_1\otimes\ldots\otimes M_{n+1})_{\mathfrak{g}} \cong (\tilde{M}_1\otimes\ldots\otimes \tilde{M}_{n+1})_{\mathfrak{g}(z_1,\dots,z_{n+1})}$$

Proof. See [FSV1], 2.3.1 or [KL]II, 9.15. $\square$

2.7. Let $L(\Lambda)$ be a finite dimensional irreducible representation of $\mathfrak{g}$. Pick a lowest vector $v^0 \in L(\Lambda)$(i.e. such that $\mathfrak{n}_- v_0 = 0$; it is unique up to a constant). We have $v^0 \in L(\Lambda)_{w_0(\Lambda)}$.

Let X be a finite dimensional representation of $\mathfrak{g}$. Let us consider the maps

$$X_{\bar{\Lambda}} \hookrightarrow X \otimes L(\Lambda) \longrightarrow (X \otimes L(\Lambda))_{\mathfrak{g}}$$

where the first one sends x to $x \otimes v^0$, and the second one is the projection. They induce map

$$(X_{\mathfrak{n}_-})_{\bar{\Lambda}} \longrightarrow (X \otimes L(\Lambda))_{\mathfrak{g}} \tag{12}$$

2.7.1. **Lemma.** *The map (12) is an isomorphism.*

Proof. Easy. (Cf. [FSV1], 2.3.3). □

2.8. Let $\Lambda_1, \dots, \Lambda_{n+1}$ be dominant integral weights.

Introduce a notation

$$W = (L(\Lambda_1) \otimes \ldots \otimes L(\Lambda_n))_{\mathfrak{n}_-}.$$

As usually, W_λ will denote a weight component.

2.8.1. **Corollary.** *The map*

$$W_{\bar{\Lambda}_{n+1}} \longrightarrow (\tilde{L}(\Lambda_1) \otimes \ldots \otimes \tilde{L}(\Lambda_{n+1}))_{\mathfrak{g}(z_1, \dots, z_{n+1})} \tag{13}$$

sending $x_1 \otimes \ldots \otimes x_n$ to $x_1 \otimes \ldots \otimes x_n \otimes v^0_{n+1}$, where $v^0_{n+1} \in L(\Lambda_{n+1})$ is a lowest vector, is an isomorphism.

Proof. Follows from 2.6 and 2.7.1 applied to $X = L(\Lambda_1) \otimes \ldots \otimes L(\Lambda_n)$. □

2.9. Suppose now that all $\Lambda_i \in C$. Set for brevity $L = L(\Lambda_1) \otimes \ldots \otimes L(\Lambda_n)$. Consider the operator

$$\mathbf{z} \cdot f_\theta = \sum_{i=1}^{n} z_i f_\theta^{(i)} : L \longrightarrow L,$$

where $f_\theta^{(i)}$ acts as f_θ on the factor $L(\Lambda_i)$ and as the identity on other factors. It gives rise to the operator

$$(\mathbf{z} \cdot f_\theta)^{k-(\Lambda_{n+1}, \theta)+1} : L_{s_0(\bar{\Lambda}_{n+1})} \longrightarrow L_{\bar{\Lambda}_{n+1}}$$

which induces the operator

$$W_{s_0(\bar{\Lambda}_{n+1})} \longrightarrow W_{\bar{\Lambda}_{n+1}}$$

Let us denote this operator by $\mathbf{T}(\mathbf{z})$. We set

$$W(z_1, \dots, z_n) = W_{\bar{\Lambda}_{n+1}} / \operatorname{Im}(\mathbf{T}(\mathbf{z}))$$

2.10. Theorem. *Suppose that $\Lambda_1, \dots, \Lambda_{n+1} \in C$. The isomorphism (13) induces an isomorphism*

$$W(z_1, \dots, z_n) \cong (\boldsymbol{L}(\Lambda_1) \otimes \ldots \otimes \boldsymbol{L}(\Lambda_{n+1}))_{\mathfrak{g}(z_1, \dots, z_{n+1})}$$

The proof will follow some lemmas.

2.11. Suppose we have $\Lambda \in C$. One sees easily that the weight $s_0(\Lambda)$ is dominant integral. Let us choose lowest vectors $v^0_\Lambda \in L(\Lambda)$ and $v^0_{s_0(\Lambda)} \in L(s_0(\Lambda))$.

2.11.1. **Lemma.** *One has an exact sequence*

$$\tilde{L}(s_0(\Lambda)) \longrightarrow \tilde{L}(\Lambda) \longrightarrow \boldsymbol{L}(\Lambda) \longrightarrow 0 \tag{14}$$

where the first map sends $v^0_{s_0(\Lambda)}$ *to* $f_\theta^{k-(\Lambda,\theta)+1} v^0_\Lambda$.

Proof. There exists a unique involution

$$\tilde{\omega} : \tilde{\mathfrak{g}} \longrightarrow \tilde{\mathfrak{g}} \tag{15}$$

such that $\tilde{\omega}(c) = \omega(c)$ for $c \in \mathfrak{g}$, and $\tilde{\omega}(f_0) = f_\theta T^{-1},\ \tilde{\omega}(e_0) = e_\theta T$.

For a $\tilde{\mathfrak{g}}$-module M, denote by ${}^{\tilde{\omega}}M$ the $\tilde{\mathfrak{g}}$-module obtained from M by the restriction of scalars using (15). We have an isomorphism

$${}^{\tilde{\omega}}\tilde{L}(\Lambda) \xrightarrow{\sim} \tilde{L}(\bar{\Lambda})$$

sending v^0_Λ to $v_{\bar{\Lambda}}$. (Note that ${}^{\omega}L(\Lambda) \cong L(\bar{\Lambda})$).

Since θ is the highest weight of the adjoint representation, $\bar{\theta} = \theta$. It follows from the W-invariance of (,) that $\overline{s_0(\Lambda)} = s_0(\bar{\Lambda})$.

Now, if we apply ${}^{\tilde{\omega}}$ to (10), we get (14). □

2.12. Corollary. *The isomorphism (13) induces an isomorphism*

$$W(z_1, \dots, z_n) \cong (\tilde{L}(\Lambda_1) \otimes \dots \otimes \tilde{L}(\Lambda_n) \otimes \boldsymbol{L}(\Lambda_{n+1}))_{\mathfrak{g}(z_1,\dots,z_{n+1})}$$

Proof. Apply the functor $(L \otimes \cdot)_{\mathfrak{g}(z_1,\dots,z_{n+1})}$ to (14). □

The rest of the argument is due to M.Finkelberg.

2.13. Lemma. *Let $\Lambda \in C$, let Y be the kernel of the projection $\tilde{L}(\Lambda) \longrightarrow \boldsymbol{L}(\Lambda)$. The operator $e_0 = f_\theta T$ is surjective on Y.*

Proof. Let us denote by $\mathcal{O}$ the category of $\tilde{\mathfrak{g}}$-modules M which are:

(a) $\mathfrak{h}$-diagonalizable, and such that for all $\lambda \in \mathfrak{h}^*$, $\dim(M_\lambda) < \infty$. **1** acts as a multiplication by k.
(b) The subalgebra $\tilde{\mathfrak{n}}^+ = \mathfrak{n}^+ \oplus T\mathfrak{g}[T]$ acts locally nilpotently on M, i.e. for every $x \in M,\ e \in \tilde{\mathfrak{n}}^+$, $e^N x = 0$ for $N >> 0$.

Denote by $\omega' : \tilde{\mathfrak{g}} \xrightarrow{\sim} \tilde{\mathfrak{g}}$ the Lie algebra involution that sends e_i to $-f_i$, and f_i to $-e_i$ for $i = 0, \dots, r$. Let us define the duality functor

$$D : \mathcal{O} \longrightarrow \mathcal{O}$$

as follows. For $M \in \mathcal{O}$, consider the space $M' = \oplus_{\lambda \in \mathfrak{h}} M^*_\lambda$ with the $\tilde{\mathfrak{g}}$-action $< xm^*, m >=< m^*, -\omega'(x)m >$. By definition, $D(M) \subset M'$ is the maximal submodule on which $\tilde{\mathfrak{n}}^+$ acts locally nilpotently.

D is an exact contravariant functor, and $DD \cong \mathrm{Id}$.

Let us return to the lemma. Suppose that e_0 is not surjective on Y. Then f_0 is not injective on $D(Y)$. Let $y \in \mathrm{Ker}(f_0)$. Let $Z \subset D(Y)$ be the $\tilde{\mathfrak{g}}$-submodule generated by y. All operators $f_i,\ i = 1, \ldots , r$, and $e_i,\ i = 0, \ldots , r$, act locally nilpotently on $D(Y)$, hence Z is an integrable $\tilde{\mathfrak{g}}$-module (which means by definition that all e_i, f_i are locally nilpotent on it). Hence, $D(Z)$ is a non-zero integrable quotient of Y. This contradicts to the fact that $\mathbf{L}(\Lambda)$ is the maximal integrable quotient of $\tilde{L}(\Lambda)$, (cf. [K], ch. 10). □

2.14. In the setup of our theorem, consider the tensor product

$$X = \tilde{L}(\Lambda_1) \otimes X_2 \otimes \ldots \otimes X_n \otimes \mathbf{L}(\Lambda_{n+1})$$

where $X_i = \tilde{L}(\Lambda_i)$ or $\mathbf{L}(\Lambda_i)$. As usually, we consider X as a $\mathfrak{g}(z_1, \ldots , z_{n+1})$-module. We have

$$f_\theta T y \otimes x \equiv -y \otimes Ax \mod \mathfrak{g}(z_1, \ldots , z_{n+1})X$$

for $y \in \tilde{L}(\Lambda_1),\ x \in X' := X_2 \otimes \ldots \otimes X_n \otimes \mathbf{L}(\Lambda_{n+1})$, where A is a linear operator on X' which acts as

$$A(x_2 \otimes \ldots \otimes x_{n+1}) = u_1 f_\theta x_2 \otimes \ldots \otimes x_{n+1} + \ldots + x_2 \otimes \ldots \otimes u_n f_\theta x_n \otimes x_{n+1} + x_2 \otimes \ldots \otimes x_n \otimes f_\theta T^{-1} x_{n+1}$$

for some $u_i \in \mathbb{C}$.

2.14.1. **Lemma.** *A is locally nilpotent on X'.*

In fact, it is easy to see that f_θ is locally nilpotent on all X_i, and $f_\theta T^{-1}$ is locally nilpotent on $\mathbf{L}(\Lambda_{n+1})$ since this module is $\tilde{\mathfrak{g}}$-integrable (cf. [TK], 1.4.6). Hence, A is locally nilpotent on X'. □

Let $Y_1 = \ker(\tilde{L}(\Lambda_1) \longrightarrow \mathbf{L}(\Lambda_1))$. We have an exact sequence

$$(Y_1 \otimes X')_{\mathfrak{g}(z_1,\ldots,z_{n+1})} \longrightarrow (\tilde{L}(\Lambda_1) \otimes X')_{\mathfrak{g}(z_1,\ldots,z_{n+1})} \xrightarrow{\phi} (\mathbf{L}(\Lambda_1) \otimes X')_{\mathfrak{g}(z_1,\ldots,z_{n+1})} \longrightarrow 0.$$

Hence, any element in $\ker(\phi)$ is of the form $\sum_j y_j \otimes x_j$ with $y_j \in Y_1,\ x_j \in X'$. It follows from 2.13 and 2.14.1 that such an element must be zero.

It follows that

$$\phi : (\tilde{L}(\Lambda_1) \otimes X_2 \otimes \ldots \otimes X_n \otimes \mathbf{L}(\Lambda_{n+1}))_{\mathfrak{g}(z_1,\ldots,z_{n+1})} \longrightarrow (\mathbf{L}(\Lambda_1) \otimes X_2 \otimes \ldots \otimes X_n \otimes \mathbf{L}(\Lambda_{n+1}))_{\mathfrak{g}(z_1,\ldots,z_{n+1})}$$

is an isomorphism. Applying the same argument to other factors X_i instead of X_1, we get the statement of Theorem 2.10 from 2.12. □

3. Trees, rational functions and Lie algebras

3.1. Let us fix a finite set $\mathcal{I}$ and a set of distinct complex numbers $\mathbf{z} = \{z_1, \ldots , z_n\}$, $n \geq 1$.

In the sequel, given a positive integer m, we will use the notation $[m] := \{1, \ldots , m\}$.

For every subset $I \subset \mathcal{I}$ denote $\tilde{I} := I \cup [n]$.

Let us consider finite oriented graphs Γ whose set of vertices $\mathrm{Ver}(\Gamma)$ is identified with a subset of $\tilde{\mathcal{I}}$. We will denote by $\mathrm{Ar}(\Gamma)$ the set of arrows of Γ, and

$$b, e : \mathrm{Ar}(\Gamma) \rightrightarrows \mathrm{Ver}(\Gamma)$$

the source and target of arrows respectively. We will call *a support of* Γ the subset

$$\mathrm{Supp}(\Gamma) = b(\mathrm{Ar}(\Gamma)) \cup e(\mathrm{Ar}(\Gamma)) \subset \mathrm{Ver}(\Gamma) \subset \tilde{\mathcal{I}}.$$

The vertex $v \in \mathrm{Ver}(\Gamma)$ lying outside $\mathrm{Supp}(\Gamma)$ is called *isolated*.

We will suppose that:

(a) every pair of vertices is joined by not more than one arrow;

(b) Γ contains no loops.

Thus, Γ is a disjoint union of trees. We will draw Γ as a graph with vertices labeled by elements $i \in \mathcal{I}$ or $s \in [n]$, and we will not picture isolated vertices.

We will denote the set of all such graphs by Gr, and call its elements simply "graphs".

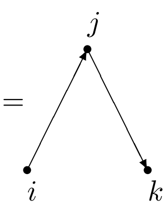
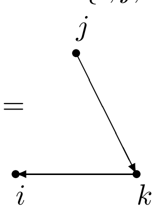
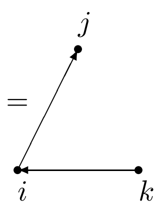
For a subset $I \subset \mathcal{I}$, $s \in [n]$, Gr_I (resp., $Gr_{I;s}$) will denote the subset of all *connected* graphs with support equal to I (resp., $I \cup \{s\}$).

3.2. Let us define a $\mathbb{C}$-vector space $\mathfrak{G}$ with generators $[\Gamma]$, $\Gamma \in Gr$, subject to the following relations.

3.2.1. If Γ' is obtained from Γ by removing an isolated vertex then $[\Gamma'] = [\Gamma]$.

3.2.2. If Γ' is obtained from Γ by reversing one arrow, then $[\Gamma'] = -[\Gamma]$.

3.2.3. *Triangle relation.* Let $\{i, j, k\} \subset \mathcal{I}$ be any three-element subset. Consider three graphs:

$\Gamma_i =$  $\Gamma_j =$  $\Gamma_k =$ 

Let Γ be a graph such that all three graphs $\Gamma_i \cup \Gamma$ belong to Gr. Then

$$[\Gamma_i \cup \Gamma] + [\Gamma_j \cup \Gamma] + [\Gamma_k \cup \Gamma] = 0$$

3.2.4. If Γ' is obtained from Γ by removing an arrow which begins at s and ends at s' where $s, s' \in [n]$, then

$$[\Gamma'] = (z_s - z_{s'})[\Gamma]$$

3.2.5. *Compositions.* For $I \subset \mathcal{I}$, we will denote by $\mathfrak{G}_I$, $\mathfrak{G}_{I;s}$ the subspace of $\mathfrak{G}$ generated by all $[\Gamma]$ where $\Gamma \in Gr_I$, $\Gamma \in Gr_{I;s}$ respectively.

Let us denote by $\mathfrak{G}_{\tilde{I}}$ the space generated by all $[\Gamma]$ where $\mathrm{Supp}(\Gamma) \subset \tilde{I}$, and each connected component of $\mathrm{Supp}(\Gamma)$ contains an element $s \in [n]$.

We have operations of partial multiplications

$$\cdot : \mathfrak{G}_I \otimes \mathfrak{G}_J \longrightarrow \mathfrak{G},$$

$$\cdot : \mathfrak{G}_{I;s} \otimes \mathfrak{G}_{J;s'} \longrightarrow \mathfrak{G},$$

$[\Gamma] \otimes [\Gamma'] \mapsto [\Gamma] \cdot [\Gamma'] := [\Gamma \cup \Gamma']$, defined iff $\mathrm{card}(I \cap J) \leq 1$. They are associative and commutative in the obvious sense.

If $\mathrm{card}(I \cap J) = 1$ then $\mathfrak{G}_I \cdot \mathfrak{G}_J \subset \mathfrak{G}_{I \cup J}$. If $I \cap J = \emptyset$ then $\mathfrak{G}_{I;s} \cdot \mathfrak{G}_{J;s} \subset \mathfrak{G}_{I \cup J;s}$.

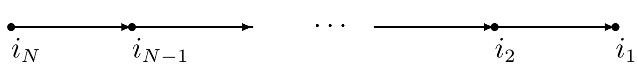
For $I \subset \mathcal{I}$ denote by $Seq(I)$ the set of all sequences

$$\vec{I} = (i_1, \dots, i_N) \tag{16}$$

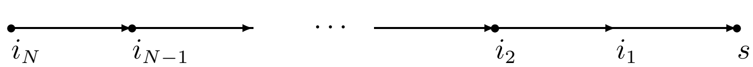
such that $I = \{i_1, \dots, i_N\}$. In other words, $Seq(I)$ is the set of all total orders on I. Evidently, $Seq(I)$ is an $\mathrm{Aut}(I)$-torsor.

Notational remark: if $\vec{I}$ is a sequence as above, we denote by I the set of its elements.

We define the graphs:

$\Gamma_{\vec{I}} =$ 

and

$\Gamma_{\vec{I};s} =$ 

for $s \in [n]$.

The following two lemmas are checked directly.

3.2.6. **Lemma.** *For a given $I \subset \mathcal{I}$, $s \in [n]$, elements $[\Gamma_{\vec{I};s}]$, $\vec{I} \in Seq(I)$, form a basis of $\mathfrak{G}_{I;s}$.* $\square$

3.2.7. **Lemma.** *Let us pick an element $i_0 \in I$. All elements $[\Gamma_{\vec{I}}]$ where $\vec{I} \in Seq(I)$ is such that its first element is i_0, form a basis of $\mathfrak{G}_I$.* $\square$

3.3. Let us consider a field $\mathcal{F} = \mathbb{C}(\mathbf{t})$ of rational functions of the set $\mathbf{t} = \{t_i\}_{i\in\mathcal{I}}$ of commuting independent variables indexed by $\mathcal{I}$. For a subset $I \subset \mathcal{I}$ denote $\mathbf{t}_I := \{t_i\}_{i\in I} \subset \mathbf{t}$.

Let us assign to each $\Gamma \in Gr$ a rational function

$$E(\Gamma) = \prod_{a\in \mathrm{Ar}(\Gamma)} \frac{1}{t_{b(a)} - t_{e(a)}}$$

where we agree that $t_s := z_s$ for $s \in [n]$. One sees easily that one gets a well defined map

$$E : \mathfrak{G} \longrightarrow \mathcal{F}$$

Moreover, E is compatible with multiplication in the evident sense.

For $I \subset \mathcal{I}$, let us define subspaces $R_I \subset R_I(\mathbf{z}) \subset \mathcal{F}$ as $R_I = E(\mathfrak{G}_I),\ R_I(\mathbf{z}) = E(\mathfrak{G}_{\tilde{I}})$.

3.3.1. **Lemma.** *The map E induces isomorphisms*

$$E : \mathfrak{G}_I \xrightarrow{\sim} R_I;\ E : \mathfrak{G}_{\tilde{I}} \xrightarrow{\sim} R_I(\boldsymbol{z}).\ \square$$

For $\vec{I}$ as in (16) will use the notations

$$E_{\vec{I}}(\mathbf{t}_I) = E(\Gamma_{\vec{I}}) = \prod_{j=2}^{N} \frac{1}{t_{i_j} - t_{i_{j-1}}}$$

$$E_{\vec{I};s}(\mathbf{t}_I; z_s) = E(\Gamma_{\vec{I};s}) = \frac{1}{t_{i_1} - z_s} \cdot \prod_{j=2}^{N} \frac{1}{t_{i_j} - t_{i_{j-1}}}$$

3.4. Shapovalov form. Suppose we are given a symmetric map

$$a : \mathcal{I} \times \mathcal{I} \longrightarrow \mathbb{C}$$

and numbers $a(i, s) \in \mathbb{C}$ for each $i \in \mathcal{I},\ s \in [n]$.

(i) Given two sequences $\vec{I} = (i_0, \dots, i_N),\ \vec{J} = (j_0, \dots, j_N) \in \mathcal{I}^{N+1}$ $(N \geq 0)$ such that $i_0 = j_0$, define a complex number $S(\vec{I}, \vec{J})$ by induction on N. Namely, set $S((i), (i)) = 1$, and

$$S(\vec{I}, \vec{J}) = \sum_{q\in[N]: i_q = j_N} (\sum_{p=0}^{q} a(i_p, j_N)) \cdot S((i_0, \dots, \hat{i}_p, \dots, i_N), (j_0, \dots, j_{N-1}))$$

for $N \geq 1$.

It is clear that $S(\vec{I}, \vec{J}) \neq 0$ only if $I = J$. One easily proves that

$$S(\vec{I}, \vec{J}) = S(\vec{J}, \vec{I})$$

(ii) Given arbitrary $\vec{I} = (i_1, \dots, i_N),\ \vec{J} = (j_1, \dots, j_N) \in \mathcal{I}^N$ $(N \geq 0)$ and $s \in [n]$, define a complex number $S(\vec{I}, \vec{J})^{(s)}$ by induction on N. Namely, set $S((), ())^{(s)} = 1$, and

$$S(\vec{I}, \vec{J})^{(s)} = \sum_{q\in[N]: i_q = j_N} (a(s, j_N) + \sum_{p=1}^{q} a(i_p, j_N)) \cdot S((i_1, \dots, \hat{i}_p, \dots, i_N), (j_1, \dots, j_{N-1}))^{(s)}$$

for $N \geq 1$.

Again $S(\vec{I},\vec{J})^{(s)} \neq 0$ only if $I = J$, and

$$S(\vec{I},\vec{J})^{(s)} = S(\vec{J},\vec{I})^{(s)}$$

3.5. Given $I \subset \mathcal{I}$, $\vec{I} = (i_1, \dots, i_N) \in Seq(I)$ and $s \in [n]$, define important rational functions

$$A_{\vec{I}}(\mathbf{t}_I) = \prod_{p=2}^{N} (\sum_{q=1}^{p-1} \frac{a(p,q)}{t_{i_p} - t_{i_q}}) \tag{17}$$

and

$$B_{\vec{I}}(\mathbf{t}_I; z_s) = \prod_{p=1}^{N} (\frac{a(p,s)}{t_{i_p} - z_s} + \sum_{q=1}^{p-1} \frac{a(p,q)}{t_{i_p} - t_{i_q}}) \tag{18}$$

3.5.1. **Key Lemma.** *We have*

$$B_{\vec{I}}(\boldsymbol{t}; z_s) = \sum_{\vec{J}=(j_1,\dots,j_N)\in Seq(I)} S(\vec{I},\vec{J})^{(s)} \cdot E_{\vec{I}}(\boldsymbol{t}_I; z_s)$$

Proof. This is a particular case of [SV], Thm. 6.6 (for $\cdot = 0$ and $\lambda = (1, \dots, 1)$). Or else, it may be checked directly. □

3.6. Let $\mathfrak{n} = \mathrm{Lie}(f_i)_{i\in\mathcal{I}}$ be a free Lie algebra on generators f_i, $i \in \mathcal{I}$ (over $\mathbb{C}$). Its enveloping algebra $U(\mathfrak{n})$ may be identified with a free associative $\mathbb{C}$-algebra with these generators.

For a sequence

$$\vec{I} = (i_1, \dots, i_N) \in \mathcal{I}^N \tag{19}$$

denote by $f_{\vec{I}} \in U(\mathfrak{n})$ the monomial

$$f_{\vec{I}} = f_{i_N} f_{i_{N-1}} \cdot \ldots \cdot f_{i_1},$$

and by $[f_{\vec{I}}] \in \mathfrak{n}$ the commutator

$$[f_{\vec{I}}] = \mathrm{ad}(f_{i_N}) \circ \mathrm{ad}(f_{i_{N-1}}) \circ \ldots \circ \mathrm{ad}(f_{i_2})(f_{i_1}),$$

where $\mathrm{ad}(x)(y) := [x, y]$. (Note the reverse order!)

Given $I \subset \mathcal{I}$, let $U(\mathfrak{n})_I \in U(\mathfrak{n})$ (resp., $\mathfrak{n}_I \in \mathfrak{n}$) be the $\mathbb{C}$-subspace generated by all monomials $f_{\vec{I}}$ (resp., by all commutators $[f_{\vec{I}}]$), $\vec{I} \in Seq(I)$. We set $U(\mathfrak{n})_\emptyset = \mathbb{C} \cdot 1$; $\mathfrak{n}_\emptyset = 0$.

Let us denote by $Seq^{(n)}(I)$ the set of all n-tuples of sequences $\vec{I}_1, \dots, \vec{I}_n$ such that I is a *disjoint* union $\cup_{s=1}^n I_s$.

We denote by $U(\mathfrak{n})_I^{\otimes n} \subset U(\mathfrak{n})^{\otimes n}$ the $\mathbb{C}$-subspace generated by all monomials of the form

$$x = f_{\vec{I}_1} \otimes \ldots \otimes f_{\vec{I}_n}$$

with $(\vec{I}_1, \dots, \vec{I}_n) \in Seq^{(n)}(I)$.

3.7. Suppose we have $(\vec{I}_1, \dots \vec{I}_n) \in Seq^{(n)}(I)$ as above. We set

$$B_{\vec{I}_1,\dots,\vec{I}_n}(\mathbf{t};\mathbf{z}) = B_{\vec{I}_1}(\mathbf{t}_{I_1};z_1)\cdot\ldots\cdot B_{\vec{I}_n}(\mathbf{t}_{I_n};z_n) \tag{20}$$

3.8. Lemma. *Pick $I \subset \mathcal{I}$.*

(i) The assignment $f_{\vec{I}} \mapsto B_{\vec{I}}(\boldsymbol{t}_I; z_s)$ defines the map

$$B_I(z_s):\ U(\mathfrak{n})_I \longrightarrow R_I(\boldsymbol{z}).$$

(ii) The assignment

$$x = f_{\vec{I}_1} \otimes \ldots \otimes f_{\vec{I}_n} \mapsto B_{(\vec{I}_1,\dots,\vec{I}_n)}(\boldsymbol{t};\boldsymbol{z})$$

defines the map

$$B_I(\boldsymbol{z}) : U(\mathfrak{n})_I^{\otimes n} \longrightarrow R_I(\boldsymbol{z}).$$

Proof. This is trivial since the above monomials form bases of the left hand sides. □

3.9. Lemma. *We have*

$$B_I(z_s)([f_{\vec{I}}]) = A_{\vec{I}}(\boldsymbol{t}_I)\cdot \sum_{i\in I}\frac{a(i,s)}{t_i - z_s}$$

Proof. The lemma is proved by induction on card(I), simultaneously with the following statement which is a particular case of 3.10.1 below.

Claim. *Suppose that $j \notin I$. Then*

$$B_{I\cup\{j\}}(z_s)([f_{\vec{I}}]f_j) = \frac{a(j,s)}{t_j - z_s}\cdot A_{\vec{I}}(\boldsymbol{t}_I)\cdot\sum_{i\in I}(\frac{a(i,j)}{t_i - t_j} + \frac{a(i,s)}{t_i - z_s}).$$

□

3.10. Suppose we have $n+1$ disjoint subsets $I_1, \dots, I_n, J \subset \mathcal{I}$, and sequences $\vec{I}_j \in Seq(I_j),\ j \in [n];\ \vec{J} \in Seq(J)$. Set $I = \cup_{j=1}^n I_j$.

For $x = x_1 \otimes \ldots \otimes x_n \in U(\mathfrak{n})^{\otimes n},\ s \in [n],\ y \in U(\mathfrak{n})$, denote

$$y^{(s)}x = x_1 \otimes \ldots \otimes yx_s \otimes \ldots \otimes x_n \tag{21}$$

3.10.1. **Lemma.** *We have*

$$B_{I\cup J}(\boldsymbol{z})([f_{\vec{J}}]^{(s)}\cdot f_{\vec{I}_1}\otimes\ldots\otimes f_{\vec{I}_n}) = B_{\vec{I}_1,\dots,\vec{I}_n}(\boldsymbol{t};\boldsymbol{z})\cdot A_{\vec{J}}(\boldsymbol{t}_J)\cdot(\sum_{i\in I_s; j\in J}\frac{a(j,i)}{t_j - t_i} + \sum_{j\in J}\frac{a(j,s)}{t_j - z_s})$$

Proof. Similar to the proof of Lemma 3.9 above, by induction on card(I). □

3.11. Lemma. *The assignment* $[f_{\vec{I}}] \mapsto A_{\vec{I}}(\boldsymbol{t}_I)$ *defines the map*

$$A_I : \mathfrak{n}_I \longrightarrow R_I.$$

Proof. Follows from 3.9. $\square$

3.12. Lemma. *Suppose we have two disjoint subsets* $I, J \subset \mathcal{I}$, *and* $\vec{I} \in Seq(I),\ \vec{J} \in Seq(J)$. *Then*

$$A_{I\cup J}([[f_{\vec{I}}], [f_{\vec{J}}]]) = A_I([f_{\vec{I}}]) \cdot A_J([f_{\vec{J}}]) \cdot \sum_{i\in I; j\in J} \frac{a(j,i)}{t_j - t_j}$$

Proof. Induction on $\mathrm{card}(J)$ using 3.10.1 for $n = 1$. $\square$

3.13. We will need a generalization of the previous constructions to the following situation. Let

$$\pi : \mathcal{I} \longrightarrow \mathcal{J}$$

be a surjective map between two finite sets. We set

$$k_j = \mathrm{card}(\pi^{-1}(j));\ \mathbf{k} = (k_j)_{j\in\mathcal{J}}$$

One can regard the pair $(\mathcal{J}, \mathbf{k})$ as a "weighted set", k_j being "the multiplicity" of j. We will say that the map π is *an unfolding* of $(\mathcal{J}, \mathbf{k})$.

We set $\Sigma_{\mathcal{I}} := \mathrm{Aut}(\mathcal{I})$ and

$$\Sigma_\pi := \prod_{j\in\mathcal{J}} \mathrm{Aut}(\pi^{-1}(j)) \subset \Sigma_{\mathcal{I}} \tag{22}$$

the last group is non-canonically isomorphic to the product of symmetric groups Σ_{k_j}.

We consider a Lie algebra $\mathfrak{n} = \mathrm{Lie}(f_j)_{j\in\mathcal{J}}$, and the field $\mathcal{F} = \mathbb{C}(t_i)_{i\in\mathcal{I}}$, with subspaces $R_I,\ R_I(\mathbf{z})$, $(I \subset \mathcal{I})$ as above.

The group $\Sigma_{\mathcal{I}}$ acts on $\mathcal{F}$ permuting generators t_i. For every $J \subset \mathcal{J}$ we have an induced action of Σ_π on $R_{\pi^{-1}(J)},\ R_{\pi^{-1}(J)}(\mathbf{z})$. We will denote by Sym_π the symmetrisation operator

$$\mathrm{Sym}_\pi = \sum_{\sigma\in\Sigma_\pi} \sigma$$

acting on these rings.

Pick $J \subset \mathcal{J}$; set $I = \pi^{-1}(J)$. Set $N := \mathrm{card}(I)$; evidently $N = \sum_{j\in J} k_j$.

Let us denote by $Seq(J; \mathbf{k})$ the set of all sequences

$$\vec{J} = (j_1, \ldots, j_N),\ j_i \in J,$$

such that for each $j \in J$ there are exactly k_j entries equal to j in $\vec{J}$. We will denote by $U(\mathfrak{n})_{\mathcal{J},\mathbf{k}}$ (resp., $\mathfrak{n}_{\mathcal{J},\mathbf{k}}$) the $\mathbb{C}$-space generated by all monomials $f_{\vec{J}}$ (resp., commutators $[f_{\vec{J}}]$) with $\vec{J} \in Seq(J; \mathbf{k})$.

For a sequence $\vec{I} = \{i_1, \ldots, i_N\}$ we write $\pi(\vec{I}) = (\pi(i_1), \ldots, \pi(i_N)\}$.

Given $\vec{J} \in Seq(J;\mathbf{k})$, let us call *an unfolding of $\vec{J}$ with respect to* π a sequence

$$\vec{I} = (i_1, \dots, i_N) \in \mathcal{I}^N,$$

such that $\pi(\vec{I}) = \vec{J}$ and all i_p are distinct. We denote the set of all unfoldings by $Unf(\vec{J})$.It is naturally a torsor over the group $\prod_{j\in J} \mathrm{Aut}(\pi^{-1}(j))$[1].

Suppose we are given a symmetric map $\mathcal{I}\times\mathcal{I} \longrightarrow \mathbb{C}$. Let us pick $\vec{I} \in Unf(\vec{J})$. Let us define rational functions

$$A_{\vec{J};\pi}(\mathbf{t}_I) = Sym_\pi\{A_{\vec{I}}(\mathbf{t}_I)\} \in R_I \tag{23}$$

$$B_{\vec{J};\pi}(\mathbf{t}_I; z_s) = Sym_\pi\{B_{\vec{I}}(\mathbf{t}_I; z_s)\} \in R_I(\mathbf{z}) \tag{24}$$

where the functions in figure brackets are defined in (17), (18).

Analogously, for any positive integer n, denote by $Seq^{(n)}(J;\mathbf{k})$ the set of all n-tuples of sequences $\vec{J}_1, \dots, \vec{J}_n$ such that their concatenation $\vec{J} = \vec{J}_1|\dots|\vec{J}_n$ belongs to $Seq(J;\mathbf{k})$. We will denote by $U(\mathfrak{n})^{\otimes n}_{J,\mathbf{k}} \subset U(\mathfrak{n})$ the subspace spanned by all monomials

$$x = f_{\vec{J}_1} \otimes \dots \otimes f_{\vec{J}_n},\ (\vec{J}_1, \dots, \vec{J}_n) \in Seq^{(n)}(J;\mathbf{k}).$$

Given such $(\vec{J}_1, \dots, \vec{J}_n)$, we will call its *unfolding* an n-tuple of sequences $(\vec{I}_1, \dots, \vec{I}_n)$ such that for all s $\vec{I}_s$ is an unfolding of $\vec{J}_s$, and all these sequences are *disjoint*, i.e. the corresponding sets $I_1, \dots, I_n$ do not intersect. The set of unfoldings will be denoted $Unf(\vec{J}_1, \dots, \vec{J}_n)$.

Pick an unfolding $(\vec{I}_1, \dots, \vec{I}_n)$. Define

$$B_{\vec{J}_1,\dots,\vec{J}_n;\pi}(\mathbf{t};\mathbf{z}) = \mathrm{Sym}_\pi\{B_{\vec{I}_1}(\mathbf{t}_{I_1}; z_1)\cdot\ldots\cdot B_{\vec{I}_n}(\mathbf{t}_{I_n}; z_n)\} \tag{25}$$

One can see that the above functions do not depend on a particular choice of unfoldings, as the notation suggests.

3.14. Lemma. *(i) The assignment*

$$f_{\vec{J}} \mapsto B_{\vec{J};\pi}(\boldsymbol{t}_I; z_s)$$

defines the map

$$B_{J;\boldsymbol{k}}(z_s) : U(\mathfrak{n})_{J;\boldsymbol{k}} \longrightarrow R_I(\boldsymbol{z})^{\Sigma_\pi}.$$

(ii) The assignment

$$x = f_{\vec{J}_1} \otimes \dots \otimes f_{\vec{J}_n} \mapsto B_{\vec{J}_1,\dots,\vec{J}_n;\pi}(\boldsymbol{t};\boldsymbol{z})$$

defines the map

$$B_{J;\boldsymbol{k}}(\boldsymbol{z}) : U(\mathfrak{n})^{\otimes n}_{J;\boldsymbol{k}} \longrightarrow R_I(\boldsymbol{z})^{\Sigma_\pi}.$$

(iii) The assignment

$$[f_{\vec{J}}] \mapsto A_{\vec{J};\pi}(\boldsymbol{t}_I)$$

defines the map

$$A_{J;\boldsymbol{k}} : \mathfrak{n}_{J;\boldsymbol{k}} \longrightarrow R_I^{\Sigma_\pi}.$$

[1]Recall that a torsor over a group G is set X equipped with a free and transitive action of G

Proof follows from the non-symmetrized case (Lemmas 3.8 and 3.11 above). Cf. also [SV], 5.11. □

4. Conformal blocks and de Rham cohomology

4.1.

4.1.1. Let us introduce some notations. For $\lambda \in \mathfrak{h}^*,\ \lambda = \sum_{i=1}^r q_i\alpha_i$, set $|\lambda| = \sum_{i=1}^r q_i$.

For $\lambda,\ \lambda' \in \mathfrak{h}^*$, we write $\lambda \leq \lambda'$ iff $\lambda' - \lambda = \sum_{i=1}^r q_i\alpha_i$ where all q_i are non-negative integers.

4.1.2. Let us fix weights $\Lambda_1, \ldots, \Lambda_n \in \mathfrak{h}^*$; set $\Lambda = \sum_{i=1}^n \Lambda_i$. Fix non-negative integers $k_1, \ldots, k_r$, and set $\mathcal{J} = \{i \in [r] | k_i > 0\}$; $\mathbf{k} = (k_j)_{j\in\mathcal{J}}$. Set $N = \sum_{i=1}^r k_i,\ \alpha = \sum_{i=1}^r k_i\alpha_i,\ \Lambda' = \Lambda - \alpha$.

Let us pick an unfolding of $(\mathcal{J}, \mathbf{k})$:

$$\pi : [N] \longrightarrow \mathcal{J}, \tag{26}$$

where $\mathrm{card}(\pi^{-1}(j)) = k_j$ for all $j \in \mathcal{J}$. We denote $\mathcal{I} := [N]$.

As in 3.13, one defines the symmetric group $\Sigma := \Sigma_\pi \subset \Sigma_{\mathcal{I}} = \Sigma_N$.

Recall that we have fixed a positive integer k, and we set $\kappa = k + g$ (see 2.4).

4.1.3. Let us consider the cartesian product of N projective lines, $X = (\mathbb{P}^1)^N$, with coordinates $(t_1, \ldots, t_N),\ t_i \in \mathbb{C}\cup\{\infty\}$. Fix n distinct complex numbers $z_1, \ldots, z_n$, and set $z_{n+1} = \infty$. Inside X, consider the following hyperplanes:

$$H_{ij}:\ t_i = t_j,\ i,j = 1, \ldots, N \text{ (so, } H_{ij} = H_{ji});\ H_{i;s}:\ t_i = z_s,\ i = 1, \ldots, N;\ s = 1, \ldots, n+1.$$

We denote by $\bar{\mathcal{C}}$ the set of all these hyperplanes. We set $\mathcal{C}_\infty = \{H_{i,n+1}\}_{i=1,\ldots,N}$, $\mathcal{C} = \bar{\mathcal{C}} - \mathcal{C}_\infty$.

Let us define the map $a : \bar{\mathcal{C}} \longrightarrow \mathbb{C}$ as follows. Set

$$a(H_{ij}) = (\alpha_{\pi(i)}, \alpha_{\pi(j)})/\kappa;\ a(H_{i;s}) = -(\Lambda_s, \alpha_{\pi(i)})/\kappa$$

if $i < n+1$. Finally, set

$$a(H_{i;n+1}) = (\Lambda - \sum_{j\neq i} \alpha_{\pi(j)}, \alpha_{\pi(i)})/\kappa$$

We will also use the notations

$$a(i,j) = a(H_{ij});\ a(i,s) = a(H_{i;s}).$$

These numbers are determined by the condition that for every line $L \cong \mathbb{P}^1 \hookrightarrow X$ defined by equations $t_j = z_{p_j},\ j \neq i$, (i being fixed), the sum $\sum a(H)$ over all $H \in \bar{\mathcal{C}}$ meeting L transversally, equals 0.

Set $U = X - \cup_{H\in\bar{\mathcal{C}}} H \subset X$. We will identify $X - \cup_{H\in\mathcal{C}_\infty} H$ with the affine space $\mathbb{A}^N$ with coordinates $t_1, \ldots, t_N$. For each $H \in \mathcal{C}$, we define the function f_H on $\mathbb{A}^N$ as follows: $f_{H_{ij}} = t_i - t_j;\ f_{H_{i;s}} = t_i - z_s$.

Let us define the following complex of vector spaces

$$\Omega^\bullet : 0 \longrightarrow \Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^N \longrightarrow 0 \tag{27}$$

By definition, Ω^i is the space of holomorphic i-forms on U. The differential d is the sum

$$d = d_{DR} + \omega_a \tag{28}$$

where d_{DR} is the de Rham differential and ω_a denotes the left multiplication by the closed 1-form

$$\omega_a = \sum_{H \in \mathcal{C}} a(H)\, \mathrm{dlog}(f_H)$$

where $\mathrm{dlog}(f_H) = d(f_H)/f_H$. We will write elements of Ω^i symbolically as

$$f(t_1, \dots , t_N) \cdot l \cdot dt_{p_1} \wedge \dots \wedge dt_{p_i} \tag{29}$$

where f is a holomorphic function, and

$$l = l(t_1, \dots , t_N) = \prod_{i,s} (t_i - z_s)^{a(i,s)} \prod_{i<j} (t_i - t_j)^{a(i,j)}$$

- this expression should be considered as a formal symbol. The formal differentiation of (29) gives the differential (28) since $\mathrm{dlog}(l) = \omega_a$.

The symmetric group Σ acts on $\Omega^\bullet$ by the rule

$$\sigma(f(t_1, \dots , t_N) \cdot l \cdot dt_{p_1} \wedge \dots \wedge dt_{p_i}) = f(t_{\sigma(1)}, \dots , t_{\sigma(N)}) \cdot l \cdot dt_{\sigma(p_1)} \wedge \dots \wedge dt_{\sigma(p_i)} \tag{30}$$

The geometric meaning of $\Omega^\bullet$ is as follows. The form ω_a defines an integrable connection $\nabla = d_{DR} + \omega_a$ on the sheaf $\mathcal{O}_U$ of holomorphic functions on U. $\Omega^\bullet$ is the complex of global sections of the holomorphic de Rham complex associated with ∇. It computes the cohomology $H^\cdot(U, \mathcal{S})$ of the locally constant sheaf $\mathcal{S}$ of horizontal sections of ∇.

4.2. Consider irreducibles $L_i = L(\Lambda_i)$; we denote by $v_i \in L_i$ the highest vector; set $L = L_1 \otimes \dots \otimes L_n$. In this Subsection we introduce, following [SV], a certain map

$$\omega : L_{\Lambda'} \longrightarrow \Omega^N$$

4.2.1. The subspace $L_{\Lambda'}$ is generated by all monomials of the form

$$x = f_{\vec{J}_1} v_1 \otimes \dots \otimes f_{\vec{J}_n} v_n$$

where $(\vec{J}_1, \dots , \vec{J}_n)$ runs through $Seq^{(n)}(\mathcal{J}; \mathbf{k})$.

Given such a monomial, we can consider the rational function $B_{\vec{J}_1,\dots,\vec{J}_n;\pi}(\mathbf{t}; \mathbf{z})$, as in (25). Set

$$\omega(x) = B_{\vec{J}_1,\dots,\vec{J}_n;\pi}(\mathbf{t}; \mathbf{z}) \cdot l(\mathbf{t}) \cdot d\mathbf{t} \in \Omega^N \tag{31}$$

where $d\mathbf{t} := dt_1 \wedge \dots \wedge dt_N$.

4.2.2. **Theorem.** *(i) The formula (31) correctly defines the map*

$$\omega : L_{\Lambda'} \longrightarrow \Omega^N$$

(ii) We have $\omega((\mathfrak{n}_- L)_{\Lambda'}) \subset d\Omega^{N-1}$. *Thus,* ω *induces the map*

$$\bar{\omega} : L_{\mathfrak{n}_-,\Lambda'} \longrightarrow H^N(U, \mathcal{S})$$

Proof. It is one of the main results of [SV], Part II. Cf. *loc. cit*, Cor. 6.13. The key result here is Lemma 3.5.1. □

4.3. Set $\Lambda_{n+1} = \bar{\Lambda}'$. Suppose that $\Lambda_{n+1} \in C$. Set $m_0 = k - (\Lambda_{n+1}, \theta) + 1$, $\Lambda'' = \Lambda' + m_0\theta$. Consider the operator

$$(\mathbf{z} \cdot f_\theta)^{m_0} : L_{\Lambda''} \longrightarrow L_{\Lambda'}$$

as in 2.9.

4.3.1. **Theorem.** *We have* $\omega(\mathrm{Im}((\boldsymbol{z} \cdot f_\theta)^{m_0})) \subset d\Omega^{N-1}$.

Consequently, if all $\Lambda_i \in C$, ω *induces the mapping*

$$\bar{\omega} : W(z_1, \ldots , z_{n+1}) \longrightarrow H^N(U, \mathcal{S})^{\Sigma,-}.$$

The rest of this work will be devoted to the proof of the first statement. Note that it is non-trivial only if $\Lambda'' \leq \Lambda$. The last statement follows from Thm. 2.10. □

4.4. Resonances. Keep the notations of 4.3.

Let us call *an edge* any non-empty intersection L of hyperplanes $H \in \bar{\mathcal{C}}$. Set

$$a(L) = \sum_{H \in \bar{\mathcal{C}} | H \supset L} a(H)$$

For instance, consider the point $L_\infty = \{(\infty, \ldots , \infty)\} \subset X$.

4.4.1. **Lemma.** *We have*

$$a(L_\infty) = -\sum_{H \in \mathcal{C}} a(H)$$

Proof. Easy. □

4.4.2. **Lemma.** *Suppose that* $\Lambda'' = \Lambda$. *Then*

$$\sum_{H\in\mathcal{C}} a(H) = -m_0$$

Proof. From our assumption it follows that $\Lambda' = \Lambda - m_0\theta$. Recall the notations from 2.1. Note that $m_0 = k - (\Lambda', \theta) + 1$ since $\Lambda' = \bar{\Lambda}_{n+1}$.

We have $(\Lambda', \theta) = (\Lambda - m_0\theta, \theta) = (\Lambda, \theta) - 2m_0$, so

$$m_0 = k - (\Lambda', \theta) + 1 = k - (\Lambda, \theta) + 2m_0 + 1,$$

hence

$$(\Lambda, \theta) = k + m_0 + 1 \tag{32}$$

On the other hand, one easily sees that

$$\sum_{1\le i<j\le N} a(i,j) = \frac{1}{2\kappa}((m_0\theta, m_0\theta) - m_0 \sum_i a_i(\alpha_i, \alpha_i)) = \frac{1}{\kappa}(m_0^2 - m_0(g-1))$$

since

$$(\alpha_i, \alpha_i) =< \nu^{-1}(\alpha_i), \alpha_i >=< a_i^\vee a_i^{-1} h_i, \alpha_i >= 2a_i^\vee a_i^{-1}$$

It follows that

$$\sum_{H\in C} a(H) = -\frac{m_0}{\kappa}(\Lambda, \theta) + \sum_{1\le i<j\le N} a(i,j) = -\frac{m_0}{\kappa}(k+g) = -m_0$$

(cf. (32)), and we are done. $\Box$

Now suppose that $\Lambda'' \le \Lambda$. Denote $\beta := \Lambda - \Lambda'' = \sum_{i=1}^r q_i\alpha_i,\ M := |\beta|$.

Let us fix maps

$$p : [M] \longrightarrow [r]$$

such that $\mathrm{card}(p^{-1}(i)) = q_i$ for all i, and

$$p' : [M+1, N] \longrightarrow [r]$$

such that $\mathrm{card}((p')^{-1}(i)) = m_0 a_i$ for all i (recall that $\theta = \sum a_i\alpha_i$). Let us define the map π, (26), as

$$\pi(j) = \begin{cases} p(j) & \text{if } 1 \le j \le M \\ p'(j) & \text{if } j > M. \end{cases}$$

The following lemma generalizes 4.4.2.

4.4.3. **Lemma.** *The sum of $a(H)$ over all $H = H_{ij} \in \mathcal{C}$ such that i or j is $> M$ or $H = H_{i;s} \in \mathcal{C}$ such that $i > M$, is equal to $-m_0$.*

Proof. Computation similar to the one in the proof of 4.4.2, shows that the sum in question is equal to

$$\frac{m_0}{\kappa}(-(\Lambda,\theta) + m_0 - g + 1 + (\beta,\theta)). \tag{33}$$

On the other hand, our assumptions imply that

$$(\Lambda',\theta) = (\Lambda - \beta - m_0\theta, \theta) = (\Lambda,\theta) - (\beta,\theta) + 2m_0,$$

hence

$$m_0 = k - (\Lambda',\theta) + 1 = k - (\Lambda,\theta) + (\beta,\theta) + 2m_0 + 1,$$

so

$$m_0 = -k + (\Lambda,\theta) - (\beta,\theta) + 1$$

Substituting this into (33), we get our claim. □

5. Resonance identity

5.1.

5.1.1. We fix $\Lambda_1, \dots, \Lambda_n \in \mathfrak{h}^*$; we set $\Lambda = \sum_{i=1}^n \Lambda_i$. We fix non-negative integers $k_1, \dots, k_r$, set $\alpha = \sum_{i=1}^r k_i\alpha_i,\ \Lambda' = \Lambda - \alpha$.

We fix a positive integer m, and set $\Lambda'' = \Lambda' + m\theta$. We suppose that $\Lambda' \le \Lambda'' \le \Lambda$. We set

$$\beta = \Lambda - \Lambda'' = \sum_{i=1}^r q_i\alpha_i;\ M = |\beta|$$

We fix a map

$$\pi_1 : [M] \longrightarrow [r] \tag{34}$$

such that $\mathrm{card}(\pi_1^{-1}(i)) = q_i$ for all i.

Recall that $\theta = \sum_{i=1}^r a_i\alpha_i$, and all $a_i > 0$. We set $A = |\theta|$. We fix a map

$$\pi_2 : [A] \longrightarrow [r] \tag{35}$$

such that $\mathrm{card}(\pi_2^{-1}(i)) = a_i$ for all i.

5.1.2. Let us introduce the following sets of independent variables: $\mathbf{u} = \{u_i\}_{1\le i\le M}$; $\mathbf{v}(i) = \{v_j(i)\}_{1\le j\le A}$, $1 \le i \le m$, $\mathbf{v} = \cup_i \mathbf{v}(i)$. Let us assign to every variable x a simple root $\alpha(x)$ as follows. We set:

$$\alpha(u_i) = \alpha_{\pi_1(i)};\ \alpha(v_j(i)) = \alpha_{\pi_2(j)}$$

Let us fix distinct complex numbers $z_1, \dots, z_n$. Let us define the complex numbers $a(x, y)$, or $a(x, z_s)$ where x, y are any two of our variables; we will call these numbers *exponents*. Namely, set

$$a(x, y) = (\alpha(x), \alpha(y))/\kappa;\ a(x, z_s) = -(\alpha(x), \Lambda_s)/\kappa$$

We set

$$e_{ij} = a(v_i(p), v_j(p))$$

for any $p \in [m]$ (these numbers do not depend on p).

Let us consider the following symbolic expressions.

$$l_{\Lambda''}(\mathbf{u}) = \prod_{i,s}(u_i - z_s)^{a(u_i,z_s)} \prod_{i,j:i>j}(u_i - u_j)^{a(u_i,u_j)}; \tag{36}$$

$$l_\theta(\mathbf{v}(p)) = \prod_{i\in[A],s\in[n]}(v_i(p) - z_s)^{a(v_i(p),z_s)} \prod_{i,j\in[A]:i>j}(v_i(p) - v_j(p))^{e_{ij}}; \tag{37}$$

$$l_{\Lambda'}(\mathbf{u}, \mathbf{v}) = l_{\Lambda'}(\mathbf{u}; \mathbf{v}(1), \dots, \mathbf{v}(m)) = l_{\Lambda''}(\mathbf{u}) \cdot l'(\mathbf{u}, \mathbf{v}) \tag{38}$$

where

$$l'(\mathbf{u}, \mathbf{v}) = \prod_p l_\theta(\mathbf{v}(p)) \cdot \prod_{i,j,p}(v_i(p) - u_j)^{a(v_i(p),u_j)} \cdot \prod_{i,j;\ p,q:\ p>q}(v_i(p) - v_j(q))^{e_{ij}} \tag{39}$$

Let us define the following numbers:

$$C(m) = m + \sum(\text{all exponents involved in } l'); \tag{40}$$

$$a = \sum_{i,s} a(v_i(p), z_s);\ b = \sum_{i,j:\ i>j} e_{ij};\ c = \sum_{i,j} e_{ij};\ d = \sum_{i,j} a(v_i(p), u_j), \tag{41}$$

in the defintion of a and d a number $p \in [m]$ is fixed; the value does not depend on it.

5.1.3. **Lemma.** $C(m) = m + ma + mb + \frac{m(m-1)}{2}c + d.$

Proof. Easy computation. □

5.2. Let $t_1, \ldots, t_n$ be independent variables. Let us define a differential $(n-1)$-form

$$\nu(t) = \nu(t_1, \ldots, t_n) = \sum_{i=1}^{n} (-1)^{i-1} t_i dt_1 \wedge \ldots \wedge \hat{dt_i} \wedge \ldots \wedge dt_n \tag{42}$$

For any function $f(t) = f(t_1, \ldots, t_n)$ we have

$$d(f(t)\nu(t)) = (nf(t) + \sum_{i=1}^{n} t \frac{\partial f}{\partial t_i})dt \tag{43}$$

where $dt := dt_1 \wedge \ldots \wedge dt_n$.

Let us consider a formal expresstion

$$l(t) = l(t_1, \ldots, t_n) = \prod_{i,s} (t_i - z_s)^{p_{is}} \prod_{i>j} (t_i - t_j)^{q_{ij}} \tag{44}$$

Differentiating formally, we get

$$d(l(t)\nu(t)) = (n + \sum_{i>j} q_{ij} + \sum_{i,s} \frac{p_{is} t_i}{t_i - z_s}) l(t) dt \tag{45}$$

5.3. It is known that all root spaces of our Lie algebra $\mathfrak{g}$ are one-dimensional. It follows that a root vector $f_\theta \in \mathfrak{g}_{-\theta}$ may be chosen in the form

$$f_\theta = c_\theta \cdot [f_{\vec{I}(\theta)}] \tag{46}$$

for some $\vec{I}(\theta) = (i_1, \ldots, i_A) \in [r]^A$ and a non-zero $c_\theta \in \mathbb{C}$. We fix such a representation once for all.

We also fix an unfolding of $\vec{I}(\theta)$ with respect to π_2:

$$\vec{J}(\theta) = (j_1, \ldots, j_A) \in [A]^A$$

5.4. Let us return to the setup of 5.1.1. Let $\mathbf{t} = \{t_1, \ldots, t_N\}$ denote the union

$$\mathbf{t} = \mathbf{u} \cup \mathbf{v}(1) \cup \ldots \cup \mathbf{v}(m) \tag{47}$$

and

$$\tilde{\mathbf{t}} = \mathbf{t} \cup \{z_1, \ldots, z_n\} \tag{48}$$

By definition, $N := \text{card}(\mathbf{t}) = M + mA$. We order the variables $t_1, \ldots, t_N$ by the natural left-to-right order following from (47). So,

$$(t_1, \ldots, t_M) = (u_1, \ldots, u_M),\ (t_{M+1}, \ldots, t_{M+A}) = (v_1(1), \ldots, v_A(1)),$$

etc.

The maps (34) and (35) induce the surjective map

$$\pi : [N] \longrightarrow [r] \tag{49}$$

with $\text{card}(\pi^{-1}(i)) = q_i + ma_i$ for all i. We set $\Sigma = \Sigma_\pi$ as in 3.13.

To each variable $t_i \in \mathbf{t}$ we have assigned a simple root $\alpha(t_i) = \alpha_{\pi(i)}$, and to a point z_s we assign the weight $-\Lambda_s$. We introduce notations

$$a(i,j) := a(t_i,t_j);\ a(i,s) := a(t_i,z_s).$$

5.5. Note that we can rewrite the expression (38) in the following form:

$$l_{\Lambda'}(\mathbf{u},\mathbf{v}) = l_{\Lambda''}(\mathbf{u}) \cdot l_\theta(\mathbf{v}(1)) \prod(\mathbf{v}(1),\mathbf{u}) \cdot l_\theta(\mathbf{v}(2)) \prod(\mathbf{v}(2),\mathbf{v}(1)) \prod(\mathbf{v}(2),\mathbf{u}) \cdot \ldots \cdot \\ l_\theta(\mathbf{v}(m)) \prod(\mathbf{v}(m),\mathbf{v}(m-1)) \cdot \ldots \cdot \prod(\mathbf{v}(m),\mathbf{v}(1)) \prod(\mathbf{v}(m),\mathbf{u}) \tag{50}$$

where

$$\prod(\mathbf{v}(i),\mathbf{u}) := \prod_{p,j} (v_p(i) - u_j)^{a(v_p(i),u_j)}$$

and

$$\prod(\mathbf{v}(i),\mathbf{v}(j)) := \prod_{p,q} (v_p(i) - v_q(j))^{a(v_p(i),v_q(j))}.$$

We will denote $l_{\Lambda'}(\mathbf{u},\mathbf{v})$ simply by $l(\mathbf{u},\mathbf{v})$, or $l(\mathbf{t})$. We will consider the complex

$$\Omega^\cdot_{alg}:\ 0 \longrightarrow \Omega^0_{alg} \longrightarrow \ldots \longrightarrow \Omega^N_{alg} \longrightarrow 0$$

where Ω^i_{alg} is the vector space consisting of expressions

$$\phi(\mathbf{t}) l(\mathbf{t}) dt_{p_1} \wedge \ldots \wedge dt_{p_i},$$

$\phi(\mathbf{t})$ being an algebraic rational function of $t_1, \ldots, t_N$. The differential is defined in the same way as for $\Omega^\cdot$, (27). The complex $\Omega^\cdot_{alg}$ is naturally a subcomplex of $\Omega^\cdot$; it inherits the action of the symmetric group $\Sigma = \Sigma_\pi$.

We denote by $\mathcal{A} : \Omega^\cdot_{alg} \longrightarrow \Omega^\cdot_{alg}$ the operator of skew symmetrization:

$$\mathcal{A}(\omega) = \sum_{\sigma \in \Sigma} (-1)^{|\sigma|} \sigma(\omega)$$

where $|\sigma|$ denotes the parity of a permutation.

5.6. Let us fix n disjoint sequences $\vec{I} = (\vec{I}_1, \ldots, \vec{I}_n)$ such that $I_1 \cup \ldots \cup I_n = [M]$. Set $\vec{J}_s = \pi_1(\vec{I}_s)$. We have the corresponding monomial

$$x = x_{\vec{J}_1,\ldots,\vec{J}_n} = f_{\vec{J}_1} v_1 \otimes \ldots \otimes f_{\vec{J}_n} v_n \in L_{\Lambda''} \tag{51}$$

Let us define a differential form

$$\tilde{\Omega}_m = B_{\vec{I}_1,\ldots,\vec{I}_n}(\mathbf{u};\mathbf{z}) \cdot l_{\Lambda''}(\mathbf{u}) \cdot d\mathbf{u} \wedge \tilde{\Omega}(1) \wedge \ldots \wedge \tilde{\Omega}(m) \in \Omega^N_{alg} \tag{52}$$

where $\mathbf{u} := u_1 \wedge \ldots \wedge u_M$, $B_{\vec{I}_1,\ldots,\vec{I}_n}(\mathbf{u};\mathbf{z})$ is as in (20), and

(53)
$$\tilde{\Omega}(i) = c_\theta d\{l_\theta(\mathbf{v}(i)) \prod(\mathbf{v}(i),\mathbf{v}(i-1)) \cdot \ldots \cdot \prod(\mathbf{v}(i),\mathbf{v}(1)) \cdot \prod(\mathbf{v}(i),\mathbf{u})) \cdot A_{\vec{J}(\theta)}(\mathbf{v}(i)) \cdot \nu(\mathbf{v}(i))\}.$$

Set

$$\Omega_m = \mathcal{A}(\tilde{\Omega}_m) \tag{54}$$

5.7. Let us consider a free Lie algebra $\tilde{\mathfrak{n}}$ with N generators $\tilde{f}_i,\ i \in [M]$ and ${}^p\tilde{f}_a,\ p \in [m], a \in [A]$. We have elements

$$\tilde{x} := \tilde{f}_{\vec{I}_1} \otimes \ldots \otimes \tilde{f}_{\vec{I}_n} \in U(\tilde{\mathfrak{n}})^{\otimes n}$$

and

$${}^p\tilde{f}_\theta := c_\theta[{}^p\tilde{f}_{\vec{I}(\theta)}] \in \tilde{\mathfrak{n}}$$

The construction of Section 3 gives us a map

$$\tilde{B} : U(\mathfrak{n})_I^{\otimes n} \longrightarrow R_I(\mathbf{z})$$

for each $I \subset [N]$. Note that

$$\tilde{B}(\tilde{x}) = B_{\vec{I}_1,\dots,\vec{I}_n}(x)$$

Recall the notation (21). Pick an i-tuple of mutually distinct numbers $(p_1, \dots, p_i) \in [m]^i$. By Lemma 3.10.1 we have

$$\tilde{B}\{(\sum_{s=1}^n z_s\, {}^{.p_1}\, \tilde{f}_\theta^{(s)}) \cdot \ldots \cdot (\sum_{s=1}^n z_s\, {}^{.p_i}\, \tilde{f}_\theta^{(s)}) \cdot \tilde{x}\} = W^{(p_1,\dots,p_i)}(\mathbf{u},\mathbf{v}) \cdot \tilde{B}(\tilde{x}) \cdot A_{\vec{J}(\theta)}(\mathbf{v}(p_1)) \cdot \ldots \cdot A_{\vec{J}(\theta)}(\mathbf{v}(p_i))$$

for a certain rational function $W^{p_1,\dots,p_i}(\mathbf{u},\mathbf{v})$.

Let us describe this function more explicitely.

Let us consider all rational functions of the form

$$X_{s_1,\dots,s_i;p_1,\dots,p_i}(q_1,\dots,q_i;\tilde{t}_1,\dots,\tilde{t}_i) = z_{s_1} \cdot \ldots \cdot z_{s_i} \frac{a(v_{q_1}(p_1),\tilde{t}_1)}{v_{q_1}(p_1) - \tilde{t}_1} \cdot \ldots \cdot \frac{a(v_{q_i}(p_i),\tilde{t}_i)}{v_{q_i}(p_i) - \tilde{t}_i} \tag{55}$$

where $s_1, \dots, s_i \in [n],\ q_1, \dots, q_i \in [A]$ and

$$\tilde{t}_j \in \{z_{s_j}\} \cup \mathbf{u}_{I_{s_j}} \bigcup_{j' \in [j-1] | s_{j'} = s_j} \mathbf{v}(p_{j'}),\ j = 1, \dots, i$$

(the last union may be empty). Let us call such functions *admissible terms.*

5.7.1. It follows from Lemma 3.10.1 that $W^{p_1,\dots,p_i}(\mathbf{u},\mathbf{v})$ is equal to the sum of all admissible terms with fixed $(p_1, \dots, p_i)$.

5.7.2. **Definition.** For any $i \in [m]$, the rational function $W_i = W_i(\mathbf{u},\mathbf{v})$ is defined as a sum

$$W_i(\mathbf{u},\mathbf{v}) = \sum W^{p_1,\dots,p_i}(\mathbf{u},\mathbf{v})$$

over all i-tuples of pairwise distinct numbers $(p_1, \dots, p_i) \in [m]^i$. We set $W_0 := 1$.

5.8. Theorem. *We have an equality*

(56) $$\Omega_m = c_\theta^m l(\boldsymbol{u},\boldsymbol{v})d\boldsymbol{u}d\boldsymbol{v}\cdot \mathrm{Sym}\{B_{\vec{I}_1,\dots,\vec{I}_n}(\boldsymbol{u};\boldsymbol{z})A_{\vec{J}(\theta)}(\boldsymbol{v}(1))\cdot\ldots\cdot A_{\vec{J}(\theta)}(\boldsymbol{v}(m))\cdot \sum_{i=0}^{m}(\prod_{j=i+1}^{m}C(j))W_i(\boldsymbol{u},\boldsymbol{v})\}$$

Here

$$C(j) := j + ja + jb + \frac{j(j-1)}{2}c + d$$

where $a,b,c,d\in\mathbb{C}$ are defined in (41).

Equality (56) will be called **Resonance identity**.

5.9. Let us deduce Theorem 4.3.1 from 5.8. From the definition (52) follows immediatedly that the form $\tilde{\Omega}_m$, and hence Ω_m, is exact. Let us rewrite (56) in the form

(57) $$\Omega_m = \sum_{i=0}^{m}(\prod_{j=i+1}^{m}C(j))\cdot\mathcal{W}_i(\mathbf{u},\mathbf{v})$$

where

(58) $$\mathcal{W}_i(\mathbf{u},\mathbf{v}) = c_\theta^m l(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v}\cdot \mathrm{Sym}\{B_{\vec{I}_1,\dots,\vec{I}_n}(\mathbf{u};\mathbf{z})A_{\vec{J}(\theta)}(\mathbf{v}(1))\cdot\ldots\cdot A_{\vec{J}(\theta)}(\mathbf{v}(m))\cdot W_i(\mathbf{u},\mathbf{v})\}$$

Now set $m=m_0$. By Lemma 4.4.3, $C(m_0)=0$. Thus

$$\Omega_{m_0} = \mathcal{W}_{m_0}$$

On the other hand, it follows from Lemma 3.10.1 that

$$\mathcal{W}_{m_0} = \omega((\mathbf{z}\cdot f_\theta)^{m_0}x)$$

where x is as in (51). This proves that the map ω takes the image of the operator $(\mathbf{z}\cdot f_\theta)^{m_0}$ to the subspace of exact forms, thus proving Theorem4.3.1. □

Theorem 5.8 will be proved in the next Section.

6. Proof of Resonance identity

We keep all the notations of the previous Section.

6.1. For each $i\in[m]$, let us consider the operators "partial differentials"

$$d^{(i)} := \sum_{q=1}^{A}\frac{\partial}{\partial v_q(i)}dv_q(i)$$

acting on our functions or forms. Note that in the expression (52) we can replace all forms $\tilde{\Omega}(i)$ by the forms

(59) $$\tilde{\Omega}'(i) = c_\theta\cdot d^{(i)}\{l_\theta(\mathbf{v}(i))\cdot\prod(\mathbf{v}(i),\mathbf{v}(i-1))\cdot\ldots\cdot\prod(\mathbf{v}(i),\mathbf{v}(1))\cdot\prod(\mathbf{v}(i),\mathbf{u}))\cdot \cdot A_{\vec{J}(\theta)}(\mathbf{v}(i))\cdot\nu(\mathbf{v}(i))\}$$

In fact, in the product of the first factor of $\tilde{\Omega}_m$ and the first $i-1$ forms $\tilde{\Omega}(j)$, we have already differentiated the variables $\mathbf{u}$ and $v_q(j)$ with $j<i$.

We can apply (45) to (59), and get

$$\tilde{\Omega}'(i) = c_\theta l_\theta(\mathbf{v}(i)) \cdot \prod(\mathbf{v}(i), \mathbf{v}(i-1)) \cdot \ldots \cdot \prod(\mathbf{v}(i), \mathbf{v}(1)) \cdot \prod(\mathbf{v}(i), \mathbf{u}) \cdot \tag{60}$$
$$\cdot A_{\vec{J}(\theta)}(\mathbf{v}(i)) \cdot (1 + b + \sum_{q\in[A];s\in[n]} \frac{a(v_q(i), z_s)v_q(i)}{v_q(i)-z_s} + \sum_{q\in[A];t\in\mathbf{U}\cup\mathbf{V}(1)\cup\ldots\cup\mathbf{V}(i-1)} \frac{a(v_q(i), t)v_q(i)}{v_q(i)-t})$$

In fact, the expression in brackets in (59) is the sum of expressions of the form (44) with $v_q(i)$ playing the role of t's in (44), and t's playing the role of z's.

Thus, we have

$$\Omega_m = c_\theta^m l(\mathbf{u},\mathbf{v}) d\mathbf{u} d\mathbf{v}\, \mathrm{Sym}\{B_{\vec{I}_1,\ldots,\vec{I}_n}(\mathbf{u};\mathbf{z}) A_{\vec{J}(\theta)}(\mathbf{v}(1)) \cdot \ldots \cdot A_{\vec{J}(\theta)}(\mathbf{v}(m)) \cdot T(1) \cdot \ldots \cdot T(m)\} \tag{61}$$

where

$$T(i) = 1 + b + \sum_{q\in[A];s\in[n]} \frac{a(v_q(i), z_s)v_q(i)}{v_q(i)-z_s} + \sum_{q\in[A];t\in\mathbf{U}\cup\mathbf{V}(1)\cup\ldots\cup\mathbf{V}(i-1)} \frac{a(v_q(i), t)v_q(i)}{v_q(i)-t} \tag{62}$$

6.2. Let us consider the function $T(m)$. Using the identity

$$\frac{X}{X-Y} = 1 + \frac{Y}{X-Y} \tag{63}$$

many times, one sees that $T(m)$ may be rewritten as

$$T(m) = 1 + a + b + (m-1)c + d + \sum_{s\in[n],q\in[A]} \frac{z_s a(v_q(m), z_s)}{v_q(m)-z_s} + \tag{64}$$
$$+ \sum_{q\in[A];t\in\mathbf{U}\cup\mathbf{V}(1)\cup\ldots\cup\mathbf{V}(m-1)} \frac{t\cdot a(v_q(m), t)}{v_q(m)-t}$$

Let us denote

$$\tilde{C}(m) = 1 + a + b + (m-1)c + d$$

Note that $\tilde{C}(1) = C(1)$.

Let us consider the expression $\frac{u_j}{v_q(m)-u_j}$ for some $j\in I_s\subset[M]$, and replace it by

$$\frac{u_j - z_s}{v_q(m)-u_j} + \frac{z_s}{v_q(m)-u_j}$$

The next Lemma is our main technical statement. In its proof we use in an essential way that θ is the highest root.

6.3. Highest root lemma. *We have*

$$\mathrm{Sym}\{B_{\vec{I}_s}(\boldsymbol{u}_{I_s};z_s)\cdot A_{\vec{J}(\theta)}(\boldsymbol{v}(m))\cdot\sum_{j\in I_s;\ q\in[A]}\frac{u_j-z_s}{v_q(m)-u_j}\}=0$$

Proof. Suppose $\vec{I}_s=(i_1,\dots,i_P)$. According to Key Lemma 3.5.1 we have

$$B_{\vec{I}_s}(\mathbf{u}_{I_s};z_s)=\sum_{\sigma\in\Sigma_P}S(\vec{I}_s,\sigma\vec{I}_s)\cdot E_{\sigma\vec{I}_s}(\mathbf{u}_{I_s};z_s)$$

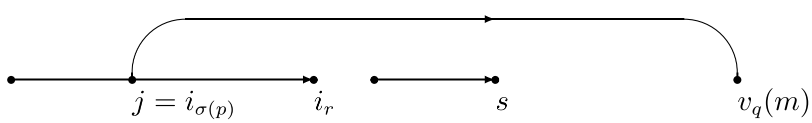
where $\sigma\vec{I}_s:=(i_{\sigma(1)},\dots,i_{\sigma(P)})$. Let us consider one product

$$E_{\sigma\vec{I}_s}(\mathbf{u}_{I_s};z_s)\cdot\frac{u_j-z_s}{v_q(m)-u_j}\tag{65}$$

for some $j\in I_s$, say $j=i_{\sigma(p)}$ for some $p\in[P]$, and rewrite u_j-z_s as a sum

$$(u_{i_{\sigma(p)}}-u_{i_{\sigma(p-1)}})+\ldots+(u_{i_{\sigma(1)}}-z_s)$$

The term (65) becomes a sum of terms corresponding - in diagrammatic notations of Section 3 - to graphs



Now let us fix a decomposition $I_s=I'\cup I''$ with $I'\cap I''=\emptyset$ and $I''\neq\emptyset$, and a sequence $\vec{I}'\in Seq(I')$. Let $\vec{I}''\in Seq(I'')$ be the total order induced from $\vec{I}_s$. Consider the sum of terms (65) (with their coefficients) which produce the above graphs with the right interval from i_r to s equal to $\vec{I}'$, the left interval varying.

One sees from 3.5.1 that the factors corresponding to the left interval give a multiple of $A_{\vec{I}''}(\mathbf{u}_{I''})$. After multiplication by $A_{\vec{J}(\theta)}(\mathbf{v}(m))$ and summing up over all connections between the left interval and the group $\mathbf{v}(m)$ and symmetrisation, we get a multiple of

$$A([f_{\vec{I}''},f_\theta]),$$

by Lemma 3.12. But $[f_{\vec{I}''},f_\theta]$ is zero since θ is a highest root. $\square$

6.4. Corollary. *Resonance identity is valid for $m=1$.*

Proof. In fact, making the above substitution in the expression for $T(1)$, and taking into account the Highest root lemma, we are left (after the symmetrisation) with the expression $C(1)+W_1(\mathbf{u},\mathbf{v}(1))$ which gives the Resonance identity for $m=1$. $\square$

6.5. We will proceed with the proof of Resonance identity by induction on m. Suppose we know it for $m-1$. Let us set

$$D_i^{(m)} = \prod_{j=i+1}^{m} C(j)$$

We also denote by $W_i^{(m)} = W_i^{(m)}(\mathbf{u}, \mathbf{v}(1), \dots, \mathbf{v}(m))$ the function denoted earlier $W_i(\mathbf{u}, \mathbf{v})$.

By (61), we have to show that

$$\mathrm{Sym}\{B_{\vec{I}_1,\dots,\vec{I}_n}(\mathbf{u};\mathbf{z})A_{\vec{J}(\theta)}(\mathbf{v}(1))\cdot\ldots\cdot A_{\vec{J}(\theta)}(\mathbf{v}(m))\cdot T(1)\cdot\ldots\cdot T(m)\} =$$
$$\mathrm{Sym}\{B_{\vec{I}_1,\dots,\vec{I}_n}(\mathbf{u};\mathbf{z})A_{\vec{J}(\theta)}(\mathbf{v}(1))\cdot\ldots\cdot A_{\vec{J}(\theta)}(\mathbf{v}(m))\cdot\sum_{i=0}^{m} D_i^{(m)}W_i^{(m)}\}$$

By induction hypothesis, one is reduced to proving that

$$\text{(66)}\quad \mathrm{Sym}\{B_{\vec{I}_1,\dots,\vec{I}_n}(\mathbf{u};\mathbf{z})A_{\vec{J}(\theta)}(\mathbf{v}(1))\cdot\ldots\cdot A_{\vec{J}(\theta)}(\mathbf{v}(m))\cdot(\sum_{i=0}^{m-1} D_i^{(m-1)}W_i^{(m-1)})\cdot T(m)\} =$$
$$\mathrm{Sym}\{B_{\vec{I}_1,\dots,\vec{I}_n}(\mathbf{u};\mathbf{z})A_{\vec{J}(\theta)}(\mathbf{v}(1))\cdot\ldots\cdot A_{\vec{J}(\theta)}(\mathbf{v}(m))\cdot\sum_{i=0}^{m} D_i^{(m)}W_i^{(m)}\}$$

By definition,

$$W_i^{(m-1)} = \sum_{p_1,\dots,p_i} W^{(m-1)p_1\dots,p_i}$$

the sum over all $(p_1,\dots,p_i)\in[m-1]^i$, p_j mutually distinct. Let us pick such $(p_1,\dots,p_i)$. Consider a product of an admissible term from $W^{(m-1)p_1\dots,p_i}$ (see (55)) and a summand from $T(m)$ (see (64)):

$$\text{(67)}\qquad X_{s_1,\dots,s_i;p_1,\dots,p_i}(q_1,\dots,q_i;\tilde{t}_1,\dots,\tilde{t}_i)\cdot\frac{a(v_q(m),v_{q'}(j))v_{q'}(j)}{v_q(m)-v_{q'}(j)}$$

$(j<m)$. These products occur in the left hand side of (66).

Two cases may occur.

1st Case. $j\notin\{p_1,\dots,p_i\}$. Then we can replace the factor

$$\text{(68)}\qquad \frac{v_{q'}(j)}{v_q(m)-v_{q'}(j)}$$

in the term (67) in the lhs of (66) by $-\frac{1}{2}$. In fact, we are doing the symmetrisation which permutes j and m, and we have

$$\frac{v_{q'}(j)}{v_q(m)-v_{q'}(j)}+\frac{v_q(m)}{v_{q'}(j)-v_q(m)}=-1.$$

2nd Case. $j=p_r$ for some r.

Claim. *In this case we can replace (68) by*

$$\text{(69)}\qquad \frac{z_{s_r}}{v_q(m)-v_{q'}(j)}.$$

In other words, if we substitute

$$\frac{v_{q'}(j) - z_{s_r}}{v_q(m) - v_{q'}(j)}$$

into the lhs of (66), we get 0 after symmetrisation. This claim is proved by the argument identical to the argument in the proof of Highest root lemma 6.3.

Let us denote for brevity

$$Y(\mathbf{t};\mathbf{z}) = B_{\vec{I}_1,\dots,\vec{I}_n}(\mathbf{u};\mathbf{z})A_{\vec{J}(\theta)}(\mathbf{v}(1)) \cdot \ldots \cdot A_{\vec{J}(\theta)}(\mathbf{v}(m))$$

Using Highest root lemma, we can rewrite the lhs of (66) as

$$\text{Sym}\{Y(\mathbf{t};\mathbf{z}) \cdot \sum_{i=0}^{m-1} D_i^{(m-1)} \sum_{p_1,\dots,p_i} W^{(m-1)p_1,\dots,p_i} \cdot (\tilde{C}(m) + \sum_{s,q}(\frac{a(v_q(m),z_s)z_s}{v_q(m)-z_s} + \sum_{j\in I_s} \frac{a(v_q(m),u_j)z_s}{v_q(m)-u_j}) + \sum_{j\in[m-1]-\{p_1,\dots,p_i\}} \sum_{q,q'}(-\frac{a(v_q(m),v_{q'}(j))}{2}) + \sum_{r=1}^{i}\sum_{q,q'} \frac{a(v_q(m),v_{q'}(p_r))z_{s_r}}{v_q(m)-v_{q'}(p_r)})\} = \tag{70}$$

$$\text{Sym}\{Y(\mathbf{t};\mathbf{z}) \cdot \sum_{i=0}^{m-1} D_i^{(m-1)} \sum_{p_1,\dots,p_i} W^{(m-1)p_1,\dots,p_i} \cdot (\tilde{C}(m) + \sum_{s,q}(\frac{a(v_q(m),z_s)z_s}{v_q(m)-z_s} + \sum_{j\in I_s} \frac{a(v_q(m),u_j)z_s}{v_q(m)-u_j}) - \frac{m-i-1}{2}c + \sum_{r=1}^{i}\sum_{q,q'} \frac{a(v_q(m),v_{q'}(p_r))z_{s_r}}{v_q(m)-v_{q'}(p_r)})\} =$$

We have to prove that (70) is equal to

$$\text{Sym}\{Y(\mathbf{t};\mathbf{z}) \cdot \sum_{i=0}^{m} D_i^{(m)} \sum_{p'_1,\dots,p'_i} W^{(m)p'_1,\dots,p'_i}\} \tag{71}$$

6.6. Let us consider more attentively the nature of symmetrisation. Let us denote by

$$\pi^{(m)} : [N] = [M + mA] \longrightarrow [r]$$

the map (49), and by

$$\pi^{(i)} : [M + iA] \longrightarrow [r]$$

the analogous map with m replaced by $i \in [m]$. Denote

$$\Sigma^{(i)} = \Sigma_{\pi^{(i)}}$$

The symmetrisation in (70), (71) is done over the group $\Sigma = \Sigma^{(m)}$.

Note that $\Sigma^{(m)}$ is equal to a disjoint union

$$\Sigma^{(m)} = \bigcup_{i=1}^{m} \Sigma^{(m-1)} \cdot (im)$$

where $(im) \in \Sigma^{(m)}$ denotes the transposition of the whole group $\mathbf{v}(i)$ with $\mathbf{v}(m)$.

More generally, the symmetric group Σ_m is naturally embedded in $\Sigma^{(m)}$ - it acts by permutations of groups $\mathbf{v}(i)$. This subgroup evidently fixes $Y(\mathbf{t};\mathbf{z})$.

Let us denote

$$Q = \{(1m), \dots, (mm)\} \subset \Sigma_m \subset \Sigma^{(m)}$$

We have

$$\mathrm{Sym}_{\Sigma(m)} = \mathrm{Sym}_{\Sigma(m-1)} \circ \mathrm{Sym}_Q$$

Let us pick mutually distinct $(p_1, \dots , p_i) \in [m-1]^i$, and consider the partial symmetrisation of the corresponding summand in (70):

$$\mathrm{Sym}_Q\{Y(\mathbf{t};\mathbf{z}) \cdot \sum_{i=0}^{m-1} D_i^{(m-1)} W^{(m-1)p_1,\dots,p_i} \cdot (\tilde{C}(m) + \sum_{s,q}(\frac{a(v_q(m), z_s)z_s}{v_q(m) - z_s} + \sum_{j\in I_s} \frac{a(v_q(m), u_j)z_s}{v_q(m) - u_j}) - \frac{m-i-1}{2}c + \sum_{r=1}^{i}\sum_{q,q'} \frac{a(v_q(m), v_{q'}(p_r))z_{s_r}}{v_q(m) - v_{q'}(p_r)})\} = \sum_{j=1}^{m} S_j \tag{72}$$

where

$$S_j = Y(\mathbf{t};\mathbf{z})\{\cdot \sum_{i=0}^{m-1} D_i^{(m-1)} W^{(m-1)p_1,\dots,p_i} \cdot (\tilde{C}(m) + \sum_{s,q}(\frac{a(v_q(m), z_s)z_s}{v_q(m) - z_s} + \sum_{j\in I_s} \frac{a(v_q(m), u_j)z_s}{v_q(m) - u_j}) - \frac{m-i-1}{2}c + \sum_{r=1}^{i}\sum_{q,q'} \frac{a(v_q(m), v_{q'}(p_r))z_{s_r}}{v_q(m) - v_{q'}(p_r)})\}^{(jm)} \tag{73}$$

Consider the j-th summand S_j. Two possibilities may occur:

(i) $j \in [m] - \{p_1, \dots , p_i\}$. In this case

$$S_j = Y(\mathbf{t};\mathbf{z}) \cdot (D_i^{(m-1)} \cdot (\tilde{C}(m) - \frac{m-i-1}{2} \cdot c) \cdot W^{(m)p_1,\dots,p_i} + D_i^{(m-1)} \cdot W^{(m)p_1,\dots,p_i,j})$$

(ii) $j = p_r$ for some r. Then

$$S_j = Y(\mathbf{t}; bz) \cdot (D_i^{(m-1)} \cdot (\tilde{C}(m) - \frac{m-i-1}{2} \cdot c) \cdot W^{(m)p_1,\dots,p_{r-1},m,p_{r+1},\dots,p_i} + D_i^{m-1} \cdot W^{(m)p_1,\dots,p_i,m}),$$

as one sees from definitions.

Now, if we pick mutually distinct $(p'_1, \dots , p'_i) \in [m]^i$, we see that the contribution into $W^{(m)p'_1,\dots,p'_i}$ from (72) comes with a coefficient

$$D_i^{(m-1)} \cdot (\tilde{C}(m) - \frac{m-i-1}{2} \cdot c) \cdot (m-i) + D_{i-1}^{(m-1)} = \tag{74}$$

$$D_i^{(m-1)} \cdot ((m-i) \cdot (\tilde{C}(m) - \frac{m-i-1}{2} \cdot c) + C(i))$$

(75)

6.7. Lemma. *We have*

$$C(m) = (m-i) \cdot (\tilde{C}(m) - \frac{m-i-1}{2} \cdot c) + C(i)$$

Proof. Immediate. □

It follows that (74) is equal to $D_i^{(m-1)}C(m) = D_i^{(m)}$. It follows that (70)=(71) which in turn implies m-th Resonance identity. □

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