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RESONANCE RELATIONS FOR SOLUTIONS OF THE ELLIPTIC QKZB EQUATIONS, FUSION RULES, AND EIGENVECTORS OF TRANSFER MATRICES OF RESTRICTED INTERACTION-ROUND-A-FACE MODELS

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ABSTRACT. Conformal blocks for the WZW model on tori can be represented by vector valued Weyl anti-symmetric theta functions on the Cartan subalgebra satisfying vanishing conditions on root hyperplanes. We introduce a quantum version of these vanishing conditions in the sl_2 case. They are compatible with the qKZB equations and are obeyed by the hypergeometric solutions as well as by their critical level counterpart, which are Bethe eigenfunctions of IRF row-to-row transfer matrices. In the language of IRF models the vanishing conditions turn out to be equivalent to the sl_2 fusion rules defining restricted models.

1. INTRODUCTION

This paper has two motivations: to understand quantization of vanishing conditions for conformal blocks on tori and to construct eigenvectors of row-to-row transfer matrices of restricted interaction-round-a-face models in statistical mechanics.

The Knizhnik-Zamolodchikov-Bernard equations are a system of differential equations arising in conformal field theory on Riemann surfaces. For each $g, n \in \mathbb{Z}_{\geq 0}$, a simple complex Lie algebra \mathfrak{g} , n highest weight \mathfrak{g} -modules V_i and a complex parameter κ , we have such a system of equations. In the case of genus g = 1, they have the form

(1)
$$\kappa \partial_{z_j} v = -\sum_{\nu} h_{\nu}^{(j)} \partial_{\lambda_{\nu}} v + \sum_{l, l \neq j} r(z_j - z_l, \lambda, \tau)^{(j,l)} v,$$

(2)
$$4\pi i\kappa \partial_{\tau} v = \Delta_{\lambda} v + \frac{1}{2} \sum_{i,j} s(z,\lambda,\tau)^{(i,j)} v.$$

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The unknown function v takes values in the zero weight space $V[0] = \bigcap_{x \in \mathfrak{h}} \operatorname{Ker}(x)$ of the tensor product $V = V_1 \otimes \cdots \otimes V_n$ with respect to the Cartan subalgebra \mathfrak{h} of \mathfrak{g} . It depends on variables $z_1, \ldots, z_n \in \mathbb{C}$, modulus τ of the elliptic curve and $\lambda = \sum \lambda_{\nu} h_{\nu} \in \mathfrak{h}$, where (h_{ν}) is an orthonormal basis of \mathfrak{h} with respect to a fixed invariant bilinear form. The notation $x^{(i)}$ for $x \in \operatorname{End}(V_i)$ or $x \in \mathfrak{g}$ means $1 \otimes \cdots \otimes x \otimes \cdots \otimes 1$. Similarly $x^{(i,j)}$ denotes the action on the *i*-th and *j*-th factors of $x \in \operatorname{End}(V_i \otimes V_j)$. In the equation, $r, s \in \mathfrak{g} \otimes \mathfrak{g}$ are suitable given tensor valued functions.

The KZB equations can be viewed as equations for horizontal sections of a connection with fiber over $z_1, ..., z_n, \tau$ being the space of functions of λ . If κ is an integer not less than the dual Coxeter number h^{\vee} of \mathfrak{g} , V_i are irreducible representations with highest weights Λ_i , $(\theta, \Lambda_i) \leq \kappa - h^{\vee}$ with θ being the highest root, then the connection has an invariant finite dimensional subbundle of conformal blocks coming from conformal field theory. The fiber over $(z_1, ..., z_n, \tau)$ of the subbundle of conformal blocks consists of holomorphic functions of λ obeying the following three conditions [FG], [FW], [EFK].

I. The functions $v(\lambda)$ are V[0]-valued theta functions. Namely, $v(\lambda)$ are periodic with respect to the coroot lattice Q^{\vee} and

$$v(\lambda + q\tau) = e^{-\pi i \kappa(q,q) - 2\pi i \kappa(q,\lambda) + 2\pi i \sum_{j} q^{(j)} z_j} v(\lambda), \qquad \forall q \in Q^{\vee},$$

- II. The functions $v(\lambda)$ are anti-symmetric with respect to the standard action of the Weyl group of \mathfrak{g} on V[0]-valued functions of $\lambda \in \mathfrak{h}$,
- III. The functions $v(\lambda)$ satisfy the vanishing conditions. Namely, for all roots α , integers $r, s, l, l \geq 0$, and $\xi \in \mathfrak{g}_{\alpha}$, where \mathfrak{g}_{α} is the root space corresponding to α , we have

$$\left(\sum_{j=1}^{n} e^{2\pi i s z_j} \xi^{(j)}\right)^l v = O((\alpha(\lambda) - r - s\tau)^{l+1}),$$

as $\alpha(\lambda) \to r + s\tau$.

In [F], an elliptic quantum group $E_{\tau,\eta}(\mathfrak{g})$ was introduced and a difference version of the KZB equations (1) was defined. Later a difference version of the KZB heat equation (2) was suggested in [FV3]. For the case $\mathfrak{g} = sl_2$ considered in this paper, the quantum Knizhnik-Zamolodchikov-Bernard equations are linear difference equations with step $p \in \mathbb{C}$ on a V[0]-valued function $v(z_1, ..., z_n, \lambda, \tau)$ of the form

(3)

$$v(z_1, ..., z_j + p, z_n, \lambda, \tau) = K_j(z_1, ..., z_n, \tau, p, \eta) v(z_1, ..., z_n, \lambda, \tau), \qquad j = 1, ..., n,$$

where the linear operators K_j are defined in terms of R-matrices of the elliptic quantum group $E_{\tau,\eta}(sl_2)$. More precisely, the space V is endowed with an $E_{\tau,\eta}(sl_2)$ -module structure depending on $(z_1, ..., z_n)$ and denoted $V_1(z_1) \otimes ... \otimes V_n(z_n)$, and the operators are defined in terms of this module structure. The difference version on the KZB heat equation (2) has the form

(4)
$$v(z_1, ..., z_n, \tau) = T(z_1, ..., z_n, \tau, p, \eta) v(z_1, ..., z_n, \lambda, \tau + p)$$

for a suitable linear operator T, but we do not consider the qKZB heat equation in this paper.

In the semiclassical limit $p, \eta \to 0$ with $p/\eta = -2\kappa$ fixed, the qKZB equations and the qKZB heat equation turn into the corresponding differential equations.

In [FTV2], [MV], hypergeometric solutions of the qKZB equations (3) we constructed. In [FV4], under certain conditions, it was shown that all solutions of the qKZB equations are sums of the hypergeometric solutions. Under the same conditions it was shown that the hypergeometric solutions also satisfy the qKZB heat equation (4).

The qKZB equations (3) can be viewed as equations for horizontal sections of a discrete connection with fiber over $(z_1, ..., z_n)$ being the space of functions of λ

The problem is to find difference analogs of Conditions I-III for conformal blocks in conformal field theory. The difference analog of Condition I is indicated in Section 6.5 of [FTV2], it is a theta function property analogous to I. It is not difficult to describe a difference analog of Condition II. In Section 9 we consider the case when all V_i are finite dimensional irreducible sl_2 modules. In this case we introduce an action of the Weyl group on V[0] and show that the qKZB operators commute with this action, thus allowing us to consider the Weyl anti-symmetric solutions.

Our first main motivation was to find an analog of vanishing conditions III for Weyl anti-symmetric solutions of the qKZB equations. It turns out that each coordinate of a Weyl anti-symmetric hypergeometric solution of the qKZB equations is equal to zero at a special set of values of λ depending on the \mathfrak{h} -weight of the coordinate. This set of special values of λ is described in terms of fusion rules of sl_2 , see Section 10. We show in Section 8 in the simplest nontrivial case that the semiclassical limit of the vanishing conditions for solutions of the qKZB equations is the vanishing condition III for conformal blocks.

Now we shall formulate the vanishing conditions in a special case. For i = 1, ...n, let V_i be the irreducible two dimensional representations of sl_2 with the standard weight basis e[1], e[-1]. For an even n, the zero weight space V[0] is nontrivial. Any V[0]-valued function has the form $u = \sum u_{m_1,...,m_n} e[m_1] \otimes ... \otimes e[m_n]$ where $m_1 + ... + m_n = 0$ and $u_{m_1,...,m_n}$ are scalar functions.

Theorem 1. Let $u(z_1, ..., z_n, \lambda)$ be a Weyl anti-symmetric V[0]-valued hypergeometric solution of the qKZB equations and $u_{m_1,...,m_n}(z_1, ..., z_n, \lambda)$ one of its coordinates. Form a sequence of integers $\Sigma^1, ..., \Sigma^n$ where $\Sigma^j = m_1 + ... + m_j$. Then for a non-negative integer k, we have $u_{m_1,...,m_n}(z_1, ..., z_n, 2\eta k) = 0$, if at least one of the integers $-\Sigma^1 + k, ..., -\Sigma^n + k$ is not positive, and $u_{m_1,...,m_n}(z_1, ..., z_n, -2\eta k) = 0$, if at least one of the integers $\Sigma^1 + k, ..., \Sigma^n + k$ is not positive.

Fix $(z_1, ..., z_n) \in \mathbb{C}^n$. Consider the $E_{\tau,\eta}(sl_2)$ -module $V_1(z_1) \otimes ... \otimes V_n(z_n)$, where all V_i are two dimensional irreducible sl_2 modules. For any complex number w, one defines a second order difference operator

$$T(w)f(\lambda) = A(w,\lambda)f(\lambda - 2\eta) + D(w,\lambda)f(\lambda + 2\eta)$$

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acting on the space of V[0]-valued meromorphic functions of λ and called the transfer matrix, see Section 11.6. Here $A, D : V[0] \to V[0]$ are linear operators meromorphically depending on w, λ . The transfer matrices commute for different values of w. In [FV2], we introduced an algebraic Bethe ansatz and constructed common eigenfunctions of the transfer matrices. The Bethe eigenfunctions are holomorphic functions of λ . The transfer matrices commute with the action of the Weyl group on V[0]. Thus antisymmetrization of a Bethe eigenfunction produces a holomorphic Weyl anti-symmetric common eigenfunction of the transfer matrices. In Section 11.6 we show that the Weyl anti-symmetric Bethe eigenfunctions obey the following vanishing conditions.

Theorem 2. Let $u(\lambda)$ be a Weyl anti-symmetric Bethe eigenfunction and $u_{m_1,...,m_n}(\lambda)$ one of its coordinates. Form a sequence of integers $\tilde{\Sigma}^1, ..., \tilde{\Sigma}^n$ where $\tilde{\Sigma}^j = m_j + ... + m_n$. Then for a non-negative integer k, we have $u_{m_1,...,m_n}(2\eta k) = 0$, if at least one of the integers $-\tilde{\Sigma}^1 + k, ..., -\tilde{\Sigma}^n + k$ is not positive, and $u_{m_1,...,m_n}(z_1, ..., z_n, -2\eta k) = 0$, if at least one of the integers $\tilde{\Sigma}^1 + k, ..., \tilde{\Sigma}^n + k$ is not positive.

For $\mu \in \mathbb{C}$, let $C_{\mu} = \{2\eta(\mu + j) \mid j \in \mathbb{Z}\}$ and let $\mathcal{F}_{\mu}(V[0])$ be the space of V[0]-valued functions on C_{μ} . For generic μ , the transfer matrices act on $\mathcal{F}_{\mu}(V[0])$ and are well defined. The restriction of Bethe eigenfunctions to C_{μ} give common eigenfunctions. The case $\mu = 0$ and $2\eta = 1/N$ for a natural number N is of special interest. Although the transfer matrices are not well defined on whole $\mathcal{F}_{\mu=0}(V[0])$, if one considers the finite dimensional subspace of V[0]-valued function on $C_{\mu=0}^{\{N\}} = \{2\eta j \mid j = 1, ..., N-1\}$, then the the restriction of the transfer matrices to this subspace is well defined and the transfer matrices commute for different values of w. The resulting commuting linear operators on this finite dimensional space are called the row-to-row transfer matrices of the restricted interaction-round-a-face model in statistical mechanics considered in [ABF].

Our second main motivation was to construct common eigenvectors of the row-to-row transfer matrices. In Section 12.5 we show that the restriction of a Weyl anti-symmetric Bethe eigenfunction to $C_{\mu=0}^{\{N\}}$ is a common eigenvector of the row-to-row transfer matrices. The vanishing conditions of Theorem 2 play the decisive role in the proof.

Thus, our two main results, the quantization of vanishing conditions for conformal block on tori and the construction of eigenvectors for restricted models, display another connection between conformal field theory and statistical mechanics.

The paper is organized as follows. We begin by introducing the notion of R-matrices and the qKZB equations in Section 2. In Section 3 we give a geometric construction of Rmatrices as transition matrices between special bases in suitable spaces of functions. The elements of those bases are called the weight functions. The weight functions are central objects of this paper. In Section 3 we prove resonance relations for weight functions, the relations which form the ground for all variants of vanishing conditions in this paper.

In Section 4 we study poles of the R-matrices and relations for their matrix coefficients. In Section 5 we describe the hypergeometric solutions of the qKZB equations. Section 6 is devoted to resonance relations for solutions of the qKZB equations with values in a tensor product of Verma modules. In Section 7 we show that the resonance relations are

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necessary and sufficient conditions for a holomorphic function of λ remain holomorphic under the action of the qKZB operators and the R-matrices permuting tensor factors.

In Section 8 we consider the semiclassical limit of the resonance relations in the simplest nontrivial case.

In Section 9 we introduce an action of the Weyl group and show that the action commutes with the qKZB operators. In Section 10 we prove the vanishing conditions for Weyl anti-symmetric hypergeometric solutions of the qKZB equations with values in a tensor product of finite dimensional irreducible representations and formulate the result in terms of sl_2 fusion rules. Notice that the KZB differential equations (1), (2) have singularities at the hyperplanes $\alpha(\lambda) = r + s\tau$, $r, s \in \mathbb{Z}$. In [FV5], it was shown that meromorphic solutions of the KZB equations obeying Condition II obey also Condition III. In the same spirit the vanishing conditions are, for Weyl anti-symmetric functions, consequences of resonance relations, which are necessary and sufficient conditions for a holomorphic function of λ to remain holomorphic after acting with the qKZB operators K_i and with the R-matrices permuting tensor factors.

In Section 11 we study vanishing conditions for Bethe ansatz eigenfunctions, the results are formulated in terms of sl_2 fusion rules and for special values of parameters in terms of $U_q(sl_2)$ fusion rules where q is a root of unity. In Section 12 we construct eigenvectors of restricted interaction-round-a-face models.

2. *R*-matrices, QKZB Equations

2.1. *R*-matrices. The qKZB equations are given in terms of *R*-matrices of elliptic quantum groups. In the sl_2 case, these *R*-matrices have the following properties. Let $\mathfrak{h} = \mathbb{C}h$ be a one-dimensional Lie algebra with generator h. For each $\Lambda \in \mathbb{C}$ consider the \mathfrak{h} -module $V_{\Lambda} = \bigoplus_{j=0}^{\infty} \mathbb{C}e_j$, with $he_j = (\Lambda - 2j)e_j$. For each pair Λ_1 , Λ_2 of complex numbers we have a meromorphic function, called the *R*-matrix, $R_{\Lambda_1,\Lambda_2}(z,\lambda)$ of two complex variables, with values in End $(V_{\Lambda_1} \otimes V_{\Lambda_2})$.

The main properties of the R-matrices are

I. The zero weight property: for any Λ_i , z, λ , $[R_{\Lambda_1,\Lambda_2}(z,\lambda), h^{(1)} + h^{(2)}] = 0$.

II. For any $\Lambda_1, \Lambda_2, \Lambda_3$, the dynamical Yang–Baxter equation

$$R_{\Lambda_1,\Lambda_2}(z,\lambda-2\eta h^{(3)})^{(12)}R_{\Lambda_1,\Lambda_3}(z+w,\lambda)^{(13)}R_{\Lambda_2,\Lambda_3}(w,\lambda-2\eta h^{(1)})^{(23)}$$

= $R_{\Lambda_2,\Lambda_3}(w,\lambda)^{(23)}R_{\Lambda_1,\Lambda_3}(z+w,\lambda-2\eta h^{(2)})^{(13)}R_{\Lambda_1,\Lambda_2}(z,\lambda)^{(12)},$

holds in End $(V_{\Lambda_1} \otimes V_{\Lambda_2} \otimes V_{\Lambda_3})$ for all z, w, λ .

III. For all Λ_1 , Λ_2 , z, λ , $R_{\Lambda_1,\Lambda_2}(z,\lambda)^{(12)}R_{\Lambda_2,\Lambda_1}(-z,\lambda)^{(21)} = \text{Id.}$ This property is called the "unitarity".

We use the following notation: if $X \in \text{End}(V_i)$, then we denote by $X^{(i)} \in \text{End}(V_1 \otimes \cdots \otimes V_n)$ the operator $\cdots \otimes \text{Id} \otimes X \otimes \text{Id} \otimes \cdots$, acting non-trivially on the *i*th factor of a tensor product of vector spaces, and if $X = \sum X_k \otimes Y_k \in \text{End}(V_i \otimes V_j)$, then we set $X^{(ij)} = \sum X_k^{(i)} Y_k^{(j)}$. If $X(\mu_1, \ldots, \mu_n)$ is a function with values in $\text{End}(V_1 \otimes \cdots \otimes V_n)$, then $X(h^{(1)}, \ldots, h^{(n)})v = X(\mu_1, \ldots, \mu_n)v$ if $h^{(i)}v = \mu_i v$, for all $i = 1, \ldots, n$.

For each τ in the upper half plane and generic $\eta \in \mathbb{C}$ ("Planck's constant") a system of R-matrices $R_{\Lambda_1,\Lambda_2}(z,\lambda)$ obeying I–III was constructed in [FV1]. They are characterized by an intertwining property with respect to the action of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ on tensor products of evaluation Verma modules. We recall a geometric construction of the R-matrices in Sec. 3.

2.2. The qKZB equations. Let n > 1 be a natural number. Fix the parameters τ, η . Fix also n complex numbers $\Lambda_1, \ldots, \Lambda_n$ and an additional parameter $p \in \mathbb{C}$. Let $V_{\vec{\Lambda}} = V_{\Lambda_1} \otimes \cdots \otimes V_{\Lambda_n}$. The kernel of $h^{(1)} + \cdots + h^{(n)}$ on $V_{\vec{\Lambda}}$ is called the zero-weight space and is denoted $V_{\vec{\Lambda}}[0]$.

Let $K_i(z, \tau, p)$ be the qKZB operators acting on the space $\mathcal{F}(V_{\vec{\Lambda}}[0])$ of meromorphic functions of $\lambda \in \mathbb{C}$ with values in the zero weight space $V_{\vec{\Lambda}}[0]$. They have the form:

$$K_{j}(z,\tau,p) = R_{j,j-1}(z_{j}-z_{j-1}+p,\tau)\cdots R_{j,1}(z_{j}-z_{1}+p,\tau)$$

$$\Gamma_{j}R_{j,n}(z_{j}-z_{n},\tau)\cdots R_{j,j+1}(z_{j}-z_{j+1},\tau).$$

The operators $R_{i,k}(z,\tau)$ are defined by the formula

$$R_{j,k}(z,\tau) v(\lambda) = R_{\Lambda_j,\Lambda_k}(z,\lambda - 2\eta \sum_{l=1,l\neq j}^{k-1} h^{(l)},\tau) v(\lambda),$$

and $(\Gamma_i v)(\lambda) = v(\lambda - 2\eta\mu)$ if $h^{(j)}v(\lambda) = \mu v(\lambda)$.

Let δ_j , $j = 1, \ldots, n$ be the standard basis of \mathbb{C}^n . The qKZB system of difference equations

(5)
$$v(z+p\delta_j) = K_j(z,\tau,p)v(z), \qquad j = 1,...n,$$

for a function v(z) on \mathbb{C}^n with values in $\mathcal{F}(V_{\vec{\Lambda}}[0])$ is compatible, i.e., we have

(6)
$$K_j(z+p\delta_l,\tau,p)K_l(z,\tau,p) = K_l(z+p\delta_j,\tau,p)K_j(z,\tau,p)$$

for all j, l, as a consequence of the dynamical Yang–Baxter equations satisfied by the R-matrices.

We also consider the *mirror* qKZB operators

$$K_{j}^{\vee}(z,\tau,p) = R_{j,j+1}^{\vee}(z_{j}-z_{j+1}+p,\tau)\cdots R_{j,n}^{\vee}(z_{j}-z_{n}+p,\tau)$$

$$\Gamma_{j}R_{j,1}^{\vee}(z_{j}-z_{1},\tau)\cdots R_{j,j-1}^{\vee}(z_{j}-z_{j-1},\tau),$$

with

$$R_{j,k}^{\vee}(z,\tau)\,v(\lambda) = R_{\Lambda_j,\Lambda_k}(z,\lambda-2\eta\sum_{l=k+1,l\neq j}^n h^{(l)},\tau)\,v(\lambda),$$

The corresponding system of qKZB equations

(7)
$$v(z+p\delta_j) = K_j^{\vee}(z,\tau,p) v(z), \qquad j = 1, \dots n,$$

is also compatible. In fact, if we write $x^{\vee} = (x_n, \ldots, x_1)$ for any $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and let $P: V_{\vec{\Lambda}} \to V_{\vec{\Lambda}^{\vee}}$ be the linear map sending $v_1 \otimes \cdots \otimes v_n$ to $v_n \otimes \cdots \otimes v_1$, then we have

$$K_i^{\vee}(z,\tau,p,\vec{\Lambda}) = P^{-1}K_{n+1-i}(z^{\vee},\tau,p,\vec{\Lambda}^{\vee})P.$$

2.3. Fundamental hypergeometric solutions. In [FTV2] we constructed a fundamental hypergeometric solution of the qKZB equations: it is constructed out of a universal hypergeometric function $u(z, \lambda, \mu, \tau, p)$ taking values in $V_{\vec{\Lambda}}[0] \otimes V_{\vec{\Lambda}}[0]$ and obeying the equations

(8)
$$u(z + \delta_j p, \tau, p) = K_j(z, \tau, p) \otimes D_j u(z, \tau, p),$$
$$u(z + \delta_j \tau, \tau, p) = D_j^{\vee} \otimes K_j^{\vee}(z, \tau, p) u(z, \tau, p),$$
$$u(z + \delta_j, \tau, p) = u(z, \tau, p).$$

Here we view u as taking values in the space of functions of λ and μ with values in $V_{\vec{\Lambda}}[0] \otimes V_{\vec{\Lambda}}[0]$. K_j acts on the variable λ and K_j^{\vee} on the variable μ . The operators D_j, D_j^{\vee} act by multiplication by diagonal matrices $D_j(\mu), D_j^{\vee}(\lambda)$, respectively. For our purpose, the most convenient description of these matrices is in terms of the function

$$\alpha(\lambda) = \exp(-\pi i \lambda^2 / 4\eta).$$

We have, for $j = 1, \ldots, n$,

$$D_{j}(\mu) = \frac{\alpha(\mu - 2\eta(h^{(j+1)} + \dots + h^{(n)}))}{\alpha(\mu - 2\eta(h^{(j)} + \dots + h^{(n)}))} e^{\pi i \eta \Lambda_{j}(\sum_{l=1}^{j-1} \Lambda_{l} - \sum_{l=j+1}^{n} \Lambda_{l})},$$

$$D_{j}^{\vee}(\lambda) = \frac{\alpha(\lambda - 2\eta(h^{(1)} + \dots + h^{(j-1)}))}{\alpha(\lambda - 2\eta(h^{(1)} + \dots + h^{(j)}))} e^{-\pi i \eta \Lambda_{j}(\sum_{l=1}^{j-1} \Lambda_{l} - \sum_{l=j+1}^{n} \Lambda_{l})}.$$

The fundamental hypergeometric solution is then

(9)
$$v = (1 \otimes \prod_{j=1}^{n} D_j^{-z_j/p}) u.$$

It obeys the qKZB equations in the first factor:

(10)
$$v(z+p\delta_j,\tau,p) = K_j(z,\tau,p) \otimes 1 v(z,\tau,p).$$

In other words, for every complex μ and every linear form on $V_{\vec{\Lambda}}[0]$, we have a solution of the qKZB equations. We call such solutions the elementary hypergeometric solutions. A solution $w(z, \lambda)$ of the qKZB equations is called a hypergeometric solution if it can be represented in the form

(11)
$$w(z,\lambda) = \sum_{j} a_{j}(z) v_{j}(z,\lambda)$$

where v_j are elementary hypergeometric solutions and a_j are scalar functions of $z_1, ..., z_n$ periodic with respect to *p*-shifts of variables.

We recall the construction of the universal hypergeometric function in Sec. 5.

The second system of equations in (8) gives the monodromy of these solutions, see [FTV2].

2.4. Finite-dimensional representations. If Λ is a non-negative integer, V_{Λ} contains the subspace $SV_{\Lambda} = \bigoplus_{j=\Lambda+1}^{\infty} \mathbb{C}e_j$ with the property that, for any M, $SV_{\Lambda} \otimes V_{M}$ and $V_{M} \otimes SV_{\Lambda}$ are preserved by the *R*-matrices $R_{\Lambda,M}(z,\lambda)$ and $R_{M,\Lambda}(z,\lambda)$, respectively, see [FV1] and Sec. 3. Let $L_{\Lambda} = V_{\Lambda}/SV_{\Lambda}$, $\Lambda \in \mathbb{Z}_{\geq 0}$. Then, in particular, for any nonnegative integers Λ and M, $R_{\Lambda,M}(z,\lambda)$ induces a map, also denoted by $R_{\Lambda,M}(z,\lambda)$, on the finite-dimensional space $L_{\Lambda} \otimes L_{M}$.

The simplest nontrivial case is $\Lambda = M = 1$. Then $R_{1,1}(z,\lambda)$ is defined on a fourdimensional vector space and coincides with the *fundamental* R-matrix, the matrix of structure constants of the elliptic quantum group $E_{\tau,\eta}(sl_2)$.

If $\Lambda_1, \ldots, \Lambda_n$ are non-negative integers, we can consider the qKZB equations (5), (7) on functions with values in the zero weight space of $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$.

3. Function spaces and R-matrices

In this section we geometrically construct R-matrices of the qKZB equations.

Let us fix complex parameters τ , η with $\text{Im } \tau > 0$, and complex numbers $\Lambda_1, \ldots, \Lambda_n$. We set $a_i = \eta \Lambda_i, i = 1, \ldots, n$.

3.1. A space of symmetric functions, [FTV1]. Introduce a space of functions with an action of the symmetric group. Recall that the Jacobi theta function

$$\theta(t) = \theta(t,\tau) = -\sum_{j \in \mathbb{Z}} e^{\pi i (j + \frac{1}{2})^2 \tau + 2\pi i (j + \frac{1}{2})(t + \frac{1}{2})},$$

has multipliers -1 and $-\exp(-2\pi i t - \pi i \tau)$ as $t \to t + 1$ and $t \to t + \tau$, respectively. It is an odd entire function whose zeros are simple and lie on the lattice $\mathbb{Z} + \tau \mathbb{Z}$. It has the product formula

$$\theta(t) = 2e^{\pi i\tau/4} \sin(\pi t) \prod_{j=1}^{\infty} (1-q^j)(1-q^j e^{2\pi it})(1-q^j e^{-2\pi it}), \qquad q = e^{2\pi i\tau}.$$

For complex numbers $a_1, \ldots, a_n, z_1, \ldots, z_n, \lambda$, let $\tilde{F}^m_{a_1,\ldots,a_n}(z_1, \ldots, z_n, \lambda)$ be the space of meromorphic functions $f(t_1, \ldots, t_m)$ of m complex variables such that

(i) $\prod_{i < j} \theta(t_i - t_j + 2\eta) \prod_{i=1}^m \prod_{k=1}^n \theta(t_i - z_k - a_k) f$ is a holomorphic function on \mathbb{C}^m . (ii) f is periodic with period 1 in each of its arguments and

$$f(\cdots, t_j + \tau, \cdots) = e^{-2\pi i(\lambda + 4\eta j - 2\eta)} f(\cdots, t_j, \cdots),$$

for all $j = 1, \ldots, m$.

The symmetric group S_m acts on $\tilde{F}^m_{a_1,\ldots,a_n}(z_1,\ldots,z_n,\lambda)$ so that the transposition of jand j+1 acts as

$$s_j f(t_1, \dots, t_m) = f(t_1, \dots, t_{j+1}, t_j, \dots, t_m) \frac{\theta(t_j - t_{j+1} - 2\eta)}{\theta(t_j - t_{j+1} + 2\eta)}$$

For any $m \in \mathbb{Z}_{>0}$, let $F_{a_1,\dots,a_n}^m(z_1,\dots,z_n,\lambda) = \tilde{F}_{a_1,\dots,a_n}^m(z_1,\dots,z_n,\lambda)^{S_m}$ be the space of S_m -invariant functions. If m = 0, then we set $F_{a_1,\dots,a_n}^0(z_1,\dots,z_n,\lambda) = \mathbb{C}$. We denote by Sym the symmetrization operator Sym $= \sum_{s \in S_m} s : \tilde{F}^m \to F^m$. Also, we set

$$F_{a_1,\ldots,a_n}(z_1,\ldots,z_n,\lambda)=\oplus_{m=0}^{\infty}F_{a_1,\ldots,a_n}^m(z_1,\ldots,z_n,\lambda),$$

and define an \mathfrak{h} -module structure on $F_{a_1,\ldots,a_n}(z_1,\ldots,z_n,\lambda)$ by letting h act by

$$h|_{F^m_{a_1,\dots,a_n}(z_1,\dots,z_n,\lambda)} = (\sum_{i=1}^n \Lambda_i - 2m) \mathrm{Id}, \qquad a_i = \eta \Lambda_i$$

Clearly, $F^m_{a_{\sigma(1)},\ldots,a_{\sigma(n)}}(z_{\sigma(1)},\ldots,z_{\sigma(n)}) = F^m_{a_1,\ldots,a_n}(z_1,\ldots,z_n,\lambda)$ for any permutation $\sigma \in S_n$.

 $F_{a_1,\ldots,a_n}^m(z_1,\ldots,z_n,\lambda)$ is a finite-dimensional vector space of dimension $\binom{n+m-1}{m}$. **Example**: Let n = 1. Then $F_a^m(z,\lambda)$ is a one-dimensional space spanned by

(12)
$$\omega_m(t_1,\ldots,t_m,\lambda,z) = \prod_{i< j} \frac{\theta(t_i-t_j)}{\theta(t_i-t_j+2\eta)} \prod_{j=1}^m \frac{\theta(\lambda+2\eta m+t_j-z-a)}{\theta(t_j-z-a)}$$

3.2. Tensor products, [FTV1]. Let n = n' + n'', m = m' + m'' be non-negative integers and $a_1, \ldots, a_n, z_1, \ldots, z_n$ be complex numbers. The formula

$$k(t_1, \dots, t_m) = \frac{1}{m'!m''!} \operatorname{Sym}\left(f(t_1, \dots, t_{m'})g(t_{m'+1}, \dots, t_m)\prod_{\substack{m' < j \le m \\ 1 \le l \le n'}} \frac{\theta(t_j - z_l + a_l)}{\theta(t_j - z_l - a_l)}\right)$$

correctly defines a linear map $\Phi : f \otimes g \mapsto k = \Phi(f \otimes g)$,

$$\bigoplus_{m'=0}^{m} F_{a_1,\dots,a_{n'}}^{m'}(z_1,\dots,z_{n'},\lambda) \otimes F_{a_{n'+1},\dots,a_n}^{m''}(z_{n'+1},\dots,z_n,\lambda-2\nu)$$

$$\to F_{a_1,\dots,a_n}^m(z_1,\dots,z_n,\lambda),$$

$$\oplus_{m'=0}^{m} F_{a_1,\dots,a_{n'}}^{m'}(z_1,\dots,z_{n'},\lambda) \otimes F_{a_{n'+1},\dots,a_n}^{m''}(z_{n'+1},\dots,z_n,\lambda-2\nu) \to F_{a_1,\dots,a_n}^m(z_1,\dots,z_n,\lambda),$$

where $\nu = a_1 + \cdots + a_{n'} - 2\eta m'$. For generic values of the parameters z_j , λ , the map Φ is an isomorphism. Moreover, Φ is associative in the sense that, for any three functions $f, g, h, \Phi(\Phi(f \otimes g) \otimes h) = \Phi(f \otimes \Phi(g \otimes h))$, whenever defined.

By iterating this construction, we get for all $n \ge 1$ a linear map Φ_n , defined recursively by $\Phi_1 = \text{Id}, \ \Phi_n = \Phi(\Phi_{n-1} \otimes \text{Id})$, from

$$\oplus_{m_1+\cdots+m_n=m} \otimes_{i=1}^n F_{a_i}^{m_i}(z_i, \lambda - 2\eta(\mu_1 + \cdots + \mu_{i-1}))$$

to $F_{a_1,...,a_n}^m(z_1,...,z_n,\lambda)$, with $\mu_j = a_j/\eta - 2m_j, \ j = 1,...,n$.

Let $V_{\Lambda}^* = \bigoplus_{j=0}^{\infty} \mathbb{C}e_j^*$ be the restricted dual of the module $V_{\Lambda} = \bigoplus_{j=0}^{\infty} \mathbb{C}e_j$. It is spanned by the basis (e_j^*) dual to the basis (e_j) . We let \mathfrak{h} act on V_{Λ}^* by $he_j^* = (\Lambda - 2j)e_j^*$. Then the map that sends e_j^* to ω_j (see (12)) defines an isomorphism of \mathfrak{h} -modules

$$\omega(z,\lambda): V_{\Lambda}^* \to F_a(z,\lambda), \qquad a = \eta \Lambda.$$

By composing this with the maps Φ , we obtain homomorphisms (of \mathfrak{h} -modules)

$$\omega(z_1,\ldots,z_n,\lambda):V^*_{\Lambda_1}\otimes\cdots\otimes V^*_{\Lambda_n}\to F_{a_1,\ldots,a_n}(z_1,\ldots,z_n,\lambda)$$

which are isomorphisms for generic values of $z_1, \ldots, z_n, \lambda$. The restriction of the map $\omega(z_1, \ldots, z_n, \lambda)$ to $e_{m_1}^* \otimes \cdots \otimes e_{m_n}^*$ is

$$\Phi_n(\omega(z_1,\lambda)e_{m_1}^*\otimes\omega(z_2,\lambda-2\eta\mu_1)e_{m_2}^*\otimes\cdots\otimes\omega(z_n,\lambda-2\eta(\mu_1+\cdots+\mu_{n-1}))e_{m_n}^*)$$

where $\mu_j = \Lambda_j - 2m_j, \ j = 1, \dots, n$.

For example, if n = 2, then $\omega(z_1, z_2, \lambda)$ sends $e_i^* \otimes e_k^*$ to

$$\frac{1}{j!k!} \operatorname{Sym}\left(\omega_j(t_1,\ldots,t_j,\lambda,z_1)\omega_k(t_{j+1},\ldots,t_{j+k},\lambda-2a_1+4\eta j,z_2)\prod_{i=j+1}^{j+k}\frac{\theta(t_i-z_1+a_1)}{\theta(t_i-z_1-a_1)}\right),$$

where $\{\omega_j(t_1,\ldots,t_j,\lambda,z)\}$ is the basis (12) of $F_a(z,\lambda)$.

More generally, we have an explicit formula for the image of $e_{m_1}^* \otimes \cdots \otimes e_{m_n}^*$, which we discuss next.

3.3. Bases of $F_{a_1,\ldots,a_n}(z_1,\ldots,z_n,\lambda)$, [FTV1]. The space V_{Λ} comes with a basis e_j . Thus we have the natural basis $e_{m_1}^* \otimes \cdots \otimes e_{m_n}^*$ of the tensor product of $V_{\Lambda_i}^*$ in terms of the dual bases of the factors. The map $\omega(z_1,\ldots,z_n,\lambda)$ maps, for generic z_i , this basis to a basis of $F_{a_1,\ldots,a_n}(z_1,\ldots,z_n,\lambda)$, which is an essential part of our formulae for integral representations for solutions of the qKZB equations.

We give here an explicit formula for the basis vectors. Let $m \in \mathbb{Z}_{\geq 0}$, $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n) \in \mathbb{C}^n$, and let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ be generic. Set $a_i = \eta \Lambda_i$. Let

$$u(t_1,\ldots,t_m) = \prod_{i < j} \frac{\theta(t_i - t_j + 2\eta)}{\theta(t_i - t_j)}$$

Then, for generic $\lambda \in \mathbb{C}$, the functions

 $\omega_{m_1,\dots,m_n}(t_1,\dots,t_m,\lambda,z) = \omega(z,\lambda) e_{m_1}^* \otimes \dots \otimes e_{m_n}^*$

labeled by $m_1, \ldots, m_n \in \mathbb{Z}$ with $\sum_k m_k = m$ form a basis of $F_a^m(z, \lambda)$ and are given by the explicit formula

(13)
$$\omega_{m_1,\dots,m_n}(t_1,\dots,t_m,\lambda,z,\tau) = u(t_1,\dots,t_m)^{-1} \sum_{I_1,\dots,I_n} \prod_{l=1}^n \prod_{i\in I_l} \prod_{k=1}^{l-1} \frac{\theta(t_i-z_k+a_k)}{\theta(t_i-z_k-a_k)}$$
$$\times \prod_{k$$

The summation is over all *n*-tuples I_1, \ldots, I_n of disjoint subsets of $\{1, \ldots, m\}$ such that I_j has m_j elements, $1 \le j \le n$.

We shall call the functions $\omega_{m_1,\ldots,m_n}(t_1,\ldots,t_m,\lambda,z,\tau)$ the weight functions.

For any permutation $\sigma \in S_n$ the space $F_{a_{\sigma(1)},...,a_{\sigma(n)}}(z_{\sigma(1)},...,z_{\sigma(n)},\lambda)$ coincides with $F_{a_1,...,a_n}(z_1,\ldots,z_n,\lambda)$. Thus, for any σ , we have a map

$$\omega^{\sigma}(z_1,\ldots,z_n,\lambda):V^*_{\Lambda_{\sigma(1)}}\otimes\cdots\otimes V^*_{\Lambda_{\sigma(n)}}\to F_{a_1,\ldots,a_n}(z_1,\ldots,z_n,\lambda)$$

and the corresponding basis in $F_{a_1,\ldots,a_n}(z_1,\ldots,z_n,\lambda)$. The basis corresponding to the permutation σ , such that $\sigma(j) = n - j + 1$ for all j, is denoted $\tilde{\omega}_{m_1,\ldots,m_n}(t_1,\ldots,t_m,\lambda,z,\tau)$. We have

$$(14) \quad \tilde{\omega}_{m_1,\dots,m_n}(t_1,\dots,t_m,\lambda,z,\tau) = u(t_1,\dots,t_m)^{-1} \sum_{I_1,\dots,I_n} \prod_{l=1}^n \prod_{i\in I_l} \prod_{k=l+1}^n \frac{\theta(t_i - z_k + a_k)}{\theta(t_i - z_k - a_k)} \\ \times \prod_{k>l} \prod_{i\in I_k, j\in I_l} \frac{\theta(t_i - t_j + 2\eta)}{\theta(t_i - t_j)} \prod_{k=1}^n \prod_{j\in I_k} \frac{\theta(\lambda + t_j - z_k - a_k + 2\eta m_k - 2\eta \sum_{l=k+1}^n (\Lambda_l - 2m_l))}{\theta(t_j - z_k - a_k)}.$$

We call these functions the mirror weight functions.

3.4. Geometric construction of *R*-matrices, [FTV1]. Let $a = \eta \Lambda$ and $b = \eta M$ be complex numbers. Since $F_{ab}(z, w, \lambda)$ coincides with $F_{ba}(w, z, \lambda)$, we obtain a family of isomorphisms between $V_{\Lambda}^* \otimes V_{M}^*$ and $V_{M}^* \otimes V_{\Lambda}^*$. The composition of this family with the flip $P : V_{M}^* \otimes V_{\Lambda}^* \to V_{\Lambda}^* \otimes V_{M}^*$, $Pv \otimes w = w \otimes v$ gives a family of automorphisms of $V_{\Lambda}^* \otimes V_{M}^*$:

Let z, w, λ be such that $\omega(z, w, \lambda, \Lambda, M) : V_{\Lambda}^* \otimes V_M^* \to F_{ab}(z, w, \lambda)$ is invertible. The *R*matrix $R_{\Lambda,M}(z, w, \lambda) \in \operatorname{End}_{\mathfrak{h}}(V_{\Lambda} \otimes V_M)$ is the dual map to the composition $R_{\Lambda,M}^*(z, w, \lambda)$:

$$V_{\Lambda}^* \otimes V_{\mathrm{M}}^* \xrightarrow{P} V_{\mathrm{M}}^* \otimes V_{\Lambda}^* \xrightarrow{\omega(w, z, \lambda, \mathrm{M}, \Lambda)} F_{ab}(z, w, \lambda) \xrightarrow{\omega(z, w, \lambda, \Lambda, \mathrm{M})^{-1}} V_{\Lambda}^* \otimes V_{\mathrm{M}}^*$$

where we identify canonically $V_{\Lambda}^* \otimes V_{\mathrm{M}}^*$ with $(V_{\Lambda} \otimes V_{\mathrm{M}})^*$.

Alternatively, the *R*-matrix $R_{\Lambda,M}(z, w, \lambda)$ can be thought of as the transition matrix expressing the basis of mirror weight functions $\tilde{\omega}_{ij} = \omega(w, z, \lambda, M, \Lambda) e_j^* \otimes e_i^*$ of the space $F_{ab}(z, w, \lambda)$ in terms of the basis of weight functions $\omega_{ij} = \omega(z, w, \lambda, \Lambda, M) e_i^* \otimes e_j^*$: if $R_{\Lambda,M}(z, w, \lambda) e_i \otimes e_j = \sum_{kl} R_{ij}^{kl} e_k \otimes e_l$, then

(15)
$$\tilde{\omega}_{kl} = \sum_{ij} R^{kl}_{ij} \omega_{ij}.$$

 $R_{\Lambda,\mathrm{M}}(z,w,\lambda)$ is a meromorphic function of $\Lambda, \mathrm{M}, z, w, \lambda$. If Λ is generic, then $R_{\Lambda,\Lambda}(z,w,\lambda)$ is regular at z = w and $\lim_{z\to w} R_{\Lambda,\Lambda}(z,w,\lambda) = P$, where P is the flip $u \otimes v \mapsto v \otimes u$. $R_{\Lambda,\mathrm{M}}(z,w,\lambda)$ depends only on the difference z-w. Accordingly, we write $R_{\Lambda,\mathrm{M}}(z-w,\lambda)$ instead of $R_{\Lambda,\mathrm{M}}(z,w,\lambda)$ in what follows.

The R-matrices $R_{\Lambda,M}(z,\lambda)$ satisfy the dynamical Yang-Baxter equation, they obey I–III of Sec. 2.

Consider the case of positive integer weights. In this case the *R*-matrices have invariant subspaces. If $\Lambda \in \mathbb{Z}_{\geq 0}$ we let SV_{Λ} be the subspace of V_{Λ} spanned by $e_{\Lambda+1}, e_{\Lambda+2}, \ldots$. The $\Lambda + 1$ -dimensional quotient V_{Λ}/SV_{Λ} will be denoted L_{Λ} , and will be often identified with $\bigoplus_{j=0}^{m} \mathbb{C}e_{j}$.

Let z, η, λ be generic and $\Lambda, M \in \mathbb{C}$.

- (i) If $\Lambda \in \mathbb{Z}_{\geq 0}$, then $R_{\Lambda,M}(z,\lambda)$ preserves $SV_{\Lambda} \otimes V_{M}$
- (ii) If $M \in \mathbb{Z}_{\geq 0}$, then $R_{\Lambda,M}(z,\lambda)$ preserves $V_{\Lambda} \otimes SV_{M}$
- (iii) If $\Lambda \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 0}$, then $R_{\Lambda,M}(z,\lambda)$ preserves $SV_{\Lambda} \otimes V_{M} + V_{\Lambda} \otimes SV_{M}$.

In particular, if Λ and/or M are non-negative integers, then $R_{\Lambda,M}(z,\lambda)$ induces operators, still denoted by $R_{\Lambda,M}(z,\lambda)$, on the quotients $L_{\Lambda} \otimes V_{M}$, $V_{\Lambda} \otimes L_{M}$ and/or $L_{\Lambda} \otimes L_{M}$. They obey the dynamical Yang–Baxter equation.

3.5. Evaluation Verma modules and their tensor products. Here we recall the relation between the geometric construction of tensor products and *R*-matrices and the representation theory of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ [FV1].

Recall the definition of a representation of $E_{\tau,\eta}(sl_2)$: let \mathfrak{h} act on \mathbb{C}^2 via h = diag(1, -1). A representation of $E_{\tau,\eta}(sl_2)$ is an \mathfrak{h} -module W with diagonalizable action of h and finite-dimensional eigenspaces, together with an operator $L(z,\lambda) \in \text{End}(\mathbb{C}^2 \otimes W)$ (the "L-operator"), commuting with $h^{(1)} + h^{(2)}$, and obeying the relations

$$R^{(12)}(z-w,\lambda-2\eta h^{(3)}) L^{(13)}(z,\lambda) L^{(23)}(w,\lambda-2\eta h^{(1)})$$

= $L^{(23)}(w,\lambda) L^{(13)}(z,\lambda-2\eta h^{(2)}) R^{(12)}(z-w,\lambda)$

in End ($\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes W$). The fundamental *R*-matrix $R(z, \lambda) \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is the following solution of the dynamical Yang–Baxter equation: let e_0 , e_1 be the standard basis of \mathbb{C}^2 , then with respect to the basis $e_0 \otimes e_0$, $e_0 \otimes e_1$, $e_1 \otimes e_0$, $e_1 \otimes e_1$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$R(z,\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha(z,\lambda) & \beta(z,\lambda) & 0 \\ 0 & \beta(z,-\lambda) & \alpha(z,-\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

where

$$\alpha(z,\lambda) = \frac{\theta(\lambda+2\eta)\theta(z)}{\theta(\lambda)\theta(z-2\eta)}, \qquad \beta(z,\lambda) = -\frac{\theta(\lambda+z)\theta(2\eta)}{\theta(\lambda)\theta(z-2\eta)}.$$

Theorem 3. [FTV1] Let us identify the two-dimensional space $L_1 = V_{\Lambda=1}/SV_{\Lambda=1}$ with \mathbb{C}^2 via the basis e_0, e_1 . Then the *R*-matrix $R_{1,1}(z, \lambda) \in \text{End}(L_1 \otimes L_1)$ coincides with the fundamental *R*-matrix.

Corollary 4. For any $w, M \in \mathbb{C}$, the \mathfrak{h} -module V_M together with the operator $L(z, \lambda) = R_{1,M}(z-w,\lambda) \in \text{End}(L_1 \otimes V_M)$ defines a representation of $E_{\tau,\eta}(sl_2)$.

This representation is called in [FV1] the evaluation Verma module with evaluation point w and highest weight M. It is denoted $V_{\rm M}(w)$.

The tensor product construction of 3.2 is related to the tensor product of representations of the elliptic quantum group. Recall that if W_1 , W_2 are representations of the elliptic quantum group with *L*-operators $L_1(z, \lambda)$, $L_2(z, \lambda)$, then their tensor product $W = W_1 \otimes W_2$ with *L*-operator

$$L(z,\lambda) = L_1(z,\lambda - 2\eta h^{(3)})^{(12)} L_2(z,\lambda)^{(13)} \in \operatorname{End} \left(\mathbb{C}^2 \otimes W\right)$$

is also a representation of the elliptic quantum group.

Theorem 5. [FTV1]

Let $\Lambda_1, \ldots, \Lambda_n \in \mathbb{C}$ and z_1, \ldots, z_n be generic complex numbers. Let $V = V_{\Lambda_1} \otimes \cdots \otimes V_{\Lambda_n}$ and $L(z, \lambda) \in \text{End}(V_{\Lambda=1} \otimes V)$ be defined by the relation

$$\omega(z, z_1, \dots, z_n, \lambda) L(z, \lambda)^* = \omega(z_1, \dots, z_n, z, \lambda) P$$

in End $((V_1 \otimes V)^*) =$ End $(V_1^* \otimes V^*)$, where $Pv_1 \otimes v = v \otimes v_1$, if $v_1 \in V_1^*$, $v \in V^*$. Then $L(z, \lambda)$ is well-defined as an endomorphism of the quotient $L_1 \otimes V = \mathbb{C}^2 \otimes V$, and defines a structure of a representation of $E_{\tau,\eta}(sl_2)$ on V. This representation is isomorphic to the tensor product of evaluation Verma modules

$$V_{\Lambda_n}(z_n) \otimes \cdots \otimes V_{\Lambda_1}(z_1),$$

with the isomorphism $u_1 \otimes \cdots \otimes u_n \mapsto u_n \otimes \cdots \otimes u_1$.

Finally, the dynamical Yang–Baxter equation in $L_1 \otimes V_\Lambda \otimes V_M$ can be stated as saying that $R_{\Lambda,M}(z - w, \lambda)P$ is an isomorphism from $V_M(w) \otimes V_\Lambda(z)$ to $V_\Lambda(z) \otimes V_M(w)$, see [FV1].

3.6. Weight functions. For a natural number m, define

(16)
$$\mathbb{Z}_m^2 = \{ (m_1, m_2) \in \mathbb{Z}_{\geq 0}^2 \mid m_1 + m_2 = m \}$$

Define a lexicographical order on \mathbb{Z}_m^2 : $(l_1, l_2) > (m_1, m_2)$ if $l_1 > m_1$. Let $z = (z_1, z_2) \in \mathbb{C}^2$. Define the points $T_M \in \mathbb{C}^m$, $M \in \mathbb{Z}_m^2$, by

(17)
$$T_M = (z_1 - \eta \Lambda_1 + 2\eta (m_1 - 1), z_1 - \eta \Lambda_1 + 2\eta (m_1 - 2), ..., z_1 - \eta \Lambda_1, z_2 - \eta \Lambda_2 + 2\eta (m_2 - 1), z_2 - \eta \Lambda_2 + 2\eta (m_2 - 2), ..., z_2 - \eta \Lambda_2).$$

Lemma 6. Let $\omega_M(t_1, ..., t_m, \lambda, z, \tau)$, $M \in \mathbb{Z}_m^2$, be the weight functions associated with $\vec{\Lambda} = (\Lambda_1, \Lambda_2)$ and defined in (13). Then for all $L, M \in \mathbb{Z}_m^2$, L > M, we have

$$\omega_M(T_L, \lambda, z, \tau) = 0.$$

Proof. According to (13),

(18)
$$\omega_{m_1,m_2}(t_1,...,t_m,\lambda,z,\tau) = u(t_1,...,t_m)^{-1} \sum_{I_1,I_2} \prod_{i\in I_1,j\in I_2} \frac{\theta(t_i-t_j+2\eta)}{\theta(t_i-t_j)} \times \prod_{i\in I_1} \frac{\theta(\lambda+t_i-z_1-\eta\Lambda_1+2\eta m_1)}{\theta(t_i-z_1-\eta\Lambda_1)} \prod_{j\in I_2} \frac{\theta(\lambda+t_j-z_2-\eta\Lambda_2+2\eta m_2-2\eta(\Lambda_1-2m_1))}{\theta(t_j-z_2-\eta\Lambda_2)} \frac{\theta(t_j-z_1+\eta\Lambda_1)}{\theta(t_j-z_1-\eta\Lambda_1)}$$

where the summation is over all pairs I_1, I_2 of disjoint subsets of $\{1, \ldots, m\}$ such that I_j has m_j elements, j = 1, 2.

It is convenient to set apart the term of the sum in (18) corresponding to the distinguished partition $I_1 = \{1, ..., m_1\}, I_2 = \{m_1 + 1, ..., m_2\},$

$$\prod_{1 \le i < j \le m_1} \frac{\theta(t_i - t_j)}{\theta(t_i - t_j + 2\eta)} \prod_{m_1 + 1 \le i < j \le m} \frac{\theta(t_i - t_j)}{\theta(t_i - t_j + 2\eta)} \prod_{i=1}^{m_1} \frac{\theta(\lambda + t_i - z_1 - \eta\Lambda_1 + 2\eta m_1)}{\theta(t_i - z_1 - \eta\Lambda_1)}$$
(19)
$$\times \prod_{i=m_1+1}^m \frac{\theta(\lambda + t_i - z_2 - \eta\Lambda_2 + 2\eta m_2 - 2\eta(\Lambda_1 - 2m_1))}{\theta(t_i - z_2 - \eta\Lambda_2)} \frac{\theta(t_i - z_1 + \eta\Lambda_1)}{\theta(t_i - z_1 - \eta\Lambda_1)}.$$

This term will be called *distinguished*.

Let us show that $\omega_M(T_L) = 0$ for L > M. Consider a term in the sum in (18) corresponding to a partition I_1, I_2 . Since $l_1 > m_1$, at least one of the numbers $1, 2, ..., l_1$ belong to I_2 . If $l_1 \in I_2$, then the factor $\theta(t_{l_1} - z_1 + \eta \Lambda_1)$ in (18) is zero. If l_1 does not belong to I_2 , then there is a pair of numbers $i, i + 1 < l_1$ such that $i \in I_2$ and $i + 1 \in I_1$. In that case the factor $\theta(t_{i+1} - t_i + 2\eta)$ of the first product in (18) equals zero. \Box

Lemma 7.

- I. Let $L, M \in \mathbb{Z}_m^2$, $L \leq M$. According to (18), decompose $\omega_M(T_L, \lambda, z, \tau)$ into the sum of the terms corresponding to partitions I_1, I_2 . Then all of the not distinguished terms are equal to zero.
- II. For any $M \in \mathbb{Z}_m^2$, we have

(20)
$$\omega_{M}(T_{M},\lambda,z,\tau) = \prod_{l=1}^{m_{1}} \left[\frac{\theta(2\eta)}{\theta(2\eta l)} \frac{\theta(\lambda - 2\eta(\Lambda_{1} - m_{1} - l + 1))}{\theta(-2\eta(\Lambda_{1} - l + 1))} \right] \\ \times \prod_{l=1}^{m_{2}} \left[\frac{\theta(2\eta)}{\theta(2\eta l)} \frac{\theta(\lambda - 2\eta(\Lambda_{1} + \Lambda_{2} - 2m_{1} - m_{2} - l + 1))}{\theta(-2\eta(\Lambda_{2} - l + 1))} \right] \\ \times \frac{\theta(-z_{1} + z_{2} - \eta\Lambda_{1} + \eta\Lambda_{2} + 2\eta(l - 1))}{\theta(-z_{1} + z_{2} - \eta\Lambda_{1} - \eta\Lambda_{2} + 2\eta(l - 1))} \right].$$

Proof. The values of all of the not distinguished summands in (18) equal zero for the same reason as in Lemma 6. Namely, if $l_1 \in I_2$, then the factor $\theta(t_{l_1} - z_1 + \eta \Lambda_1)$ in (18) is zero.

Now let $l_1 \in I_1$. If there is $j < l_1$ such that $j \in I_2$, then there is a pair of numbers $i, i+1 < l_1$ such that $i \in I_2$ and $i+1 \in I_1$. In that case the factor $\theta(t_{i+1} - t_i + 2\eta)$ of the first product in (18) equals zero.

So assume that $\{1, ..., l_1\} \in I_1$ and the partition I_1, I_2 is not distinguished, then there is a pair of numbers $l_1 < i, i + 1$ such that $i \in I_2$ and $i + 1 \in I_1$. In that case the factor $\theta(t_{i+1} - t_i + 2\eta)$ of the first product in (18) equals zero. This proves the first part of the Lemma.

The RHS in (20) is the value of the distinguished summand in (18). This proves the second part. $\hfill \Box$

Lemma 8. Let $\tilde{\omega}_M(t_1, ..., t_m, \lambda, z, \tau)$, $M \in \mathbb{Z}_m^2$, be the mirror weight functions associated with $\vec{\Lambda} = (\Lambda_1, \Lambda_2)$ and defined in (13). Then for all $L, M \in \mathbb{Z}_m^2$, L < M, we have

$$\tilde{\omega}_M(T_L,\lambda,z,\tau)=0$$

Moreover, for any $M \in \mathbb{Z}_m^2$, we have

(21)
$$\tilde{\omega}_{M}(T_{M},\lambda,z,\tau) = \prod_{l=1}^{m_{2}} \left[\frac{\theta(2\eta)}{\theta(2\eta l)} \frac{\theta(\lambda - 2\eta(\Lambda_{2} - m_{2} - l + 1))}{\theta(-2\eta(\Lambda_{2} - l + 1))} \right] \\ \times \prod_{l=1}^{m_{1}} \left[\frac{\theta(2\eta)}{\theta(2\eta l)} \frac{\theta(\lambda - 2\eta(\Lambda_{1} + \Lambda_{2} - m_{1} - 2m_{2} - l + 1))}{\theta(-2\eta(\Lambda_{1} - l + 1))} \right] \\ \times \frac{\theta(z_{1} - z_{2} + \eta\Lambda_{1} - \eta\Lambda_{2} + 2\eta(l - 1))}{\theta(z_{1} - z_{2} - \eta\Lambda_{1} - \eta\Lambda_{2} + 2\eta(l - 1))} \right].$$

The Lemmas have important corollaries. Namely, consider the functional space $F_{a_1,a_2}^m(z_1, z_2, \lambda)$ associated with $\vec{\Lambda}$. Consider the triangular matrices $A_{m_1,m_2}^{l_1,l_2} = \omega_{m_1,m_2}(T_{l_1,l_2}, \lambda, z_1, z_2)$ and $\tilde{A}_{m_1,m_2}^{l_1,l_2} = \tilde{\omega}_{m_1,m_2}(T_{l_1,l_2}, \lambda, z_1, z_2)$ for $(l_1, l_2), (m_1, m_2) \in \mathbb{Z}_m^2$. Assume that $z_1, z_2, \Lambda_1, \Lambda_2, \eta$ are such that all elements of these matrices are well defined. (Notice that these conditions can be easily written explicitly.)

Corollary 9. The weight functions $\omega_M(t_1, ..., t_m, \lambda, z, \tau)$, $M \in \mathbb{Z}_m^2$ form a basis in $F_{a_1,a_2}^m(z_1, z_2, \lambda)$ unless one of the diagonal elements (20) of the matrix A equals zero. Similarly, the mirror weight functions $\tilde{\omega}_M(t_1, ..., t_m, \lambda, z, \tau)$, $M \in \mathbb{Z}_m^2$, form a basis in $F_{a_1,a_2}^m(z_1, z_2, \lambda)$ unless one of the diagonal elements (21) of the matrix \tilde{A} equals zero.

Corollary 10. Let $B_{m_1,m_2}^{l_1,l_2}$ be the matrix inverse to A, $\sum_{l_1,l_2} A_{m_1,m_2}^{l_1,l_2} B_{l_1,l_2}^{k_1,k_2} = \delta_{k_1,m_1} \delta_{k_2,m_2}$. Consider the elliptic *R*-matrix $R_{\Lambda_1,\Lambda_2}(z_1 - z_2, \lambda)$ defined by (15). Then

(22)
$$R_{ij}^{kl} = \sum_{r_1, r_2} \tilde{A}_{kl}^{r_1 r_2} B_{r_1 r_2}^{ij}.$$

In particular, the R-matrix is well defined unless one the diagonal elements (20) of the matrix A equals zero. The determinant of the R-matrix $R_{\Lambda_1,\Lambda_2}(z_1 - z_2, \lambda)$ restricted to the weight space of weight $\Lambda_1 + \Lambda_2 - 2m$ is equal to the product of all of the diagonal

elements (21) of the matrix \hat{A} divided by the product of all of the diagonal elements (20) of the matrix A.

3.7. The Shapovalov operator. Let $Q^{\Lambda_j}(\lambda, \tau)$ be the diagonal operator on V_{Λ_j} with diagonal matrix elements

$$Q_k^{\Lambda_j}(\lambda,\tau) = \left(\frac{\theta'(0,\tau)}{\theta(2\eta,\tau)}\right)^k \prod_{l=1}^k \frac{\theta(2\eta(\Lambda_j+1-l),\tau)\theta(2\eta l,\tau)}{\theta(\lambda+2\eta(\Lambda_j+1-k-l),\tau)\theta(\lambda-2\eta l,\tau)} \,.$$

Set

(23)
$$Q(\lambda,\tau) = Q^{\Lambda_1}(\lambda,\tau) \otimes Q^{\Lambda_2}(\lambda+2\eta h^{(1)},\tau) \otimes \cdots \otimes Q^{\Lambda_n}(\lambda+2\eta \sum_{j=1}^{n-1} h^{(j)},\tau).$$

Q defines a diagonal operator on the tensor product $V_{\vec{\Lambda}}$ called the Shapovalov operator. The Shapovalov operator plays an important role in the study of the qKZB equations [FV3].

Let $\vec{\Lambda} \in \mathbb{C}^n$ be such that $\Lambda_1 + \ldots + \Lambda_n = 2m$ for some natural number m. We consider the Shapovalov operator on the zero weight subspace $V_{\vec{\Lambda}}[0]$ and in particular its poles as a function of λ .

Let $e_M = e_{m_1} \otimes \ldots \otimes e_{m_n} \in V_{\vec{\Lambda}}[0]$ be a basis vector, where $M = (m_1, \ldots, m_n), m_1 + \ldots + m_n = m$, is a vector of non-negative integers. The corresponding diagonal matrix element of the operator Q is

$$Q_M(\lambda,\tau) = \left(\frac{\theta'(0,\tau)}{\theta(2\eta,\tau)}\right)^m \prod_{j=1}^n \prod_{l=1}^{m_j} \frac{\theta(2\eta(\Lambda_j+1-l),\tau)\theta(2\eta l,\tau)}{\theta(\lambda+2\eta(m_j-l+1)+2\eta\sum_{k=1}^j(\Lambda_k-2m_k),\tau)} \times \frac{1}{\theta(\lambda-2\eta l+2\eta\sum_{k=1}^{j-1}(\Lambda_k-2m_k),\tau)}.$$
(24)

For generic $\vec{\Lambda}$, this coefficient has 2m simple pairwise distinct poles. It is convenient to parametrize these poles as follows.

For $j = 1, ..., n - 1, 0 \le l \le m_j + m_{j+1}, l \ne m_j$, set

(25)
$$\lambda_{M,j,l} = -2\eta(\Lambda_j - m_j - l + \sum_{k=1}^{j-1} (\Lambda_k - 2m_k)),$$

and for $0 \leq l \leq m_n + m_1$, $l \neq m_n$, set

(26)
$$\lambda_{M,n,l} = -2\eta(m_n - l).$$

Denote this set of 2m numbers S_M .

Lemma 11. The set S_M is the set of poles of the coefficient Q_M considered as a function of λ . \Box

Any $\lambda_{M,j,l}$ defines a new index L as follows. If j < n, then $L = (m_1, ..., m_{j-1}, l, m_j + m_{j+1} - l, m_{j+2}, ..., m_n)$. If j = n, then $L = (m_1 + m_n - l, m_2, ..., m_{n-1}, l)$. We call this index dual to M with respect to $\lambda_{M,j,l}$.

Let *L* be dual to *M* with respect to $\lambda_{M,j,l}$. If j < n, then the set S_L contains the number $\lambda_{M,j,l} = \lambda_{L,j,m_j}$ and moreover *M* is dual to *L* with respect to $\lambda_{M,j,l}$. If j = n, then the set S_L contains $-\lambda_{M,n,l} = \lambda_{L,n,m_n}$ and moreover *M* is dual to *L* with respect to $-\lambda_{M,n,l}$.

This construction defines a partition of the set of the pairs (M, λ) , $\lambda \in S_M$, into the dual pairs.

3.8. The poles of the Shapovalov operator and weight functions. Define elliptic numbers by

$$[k] = \frac{\theta(2\eta k, \tau)}{\theta(2\eta, \tau)}$$

and elliptic factorials by

$$[k]! = [1] [2] \dots [k].$$

Let $\vec{\Lambda} \in \mathbb{C}^n$, *m* a positive integer. Consider the weight functions associated with the pair $\vec{\Lambda}$, *m* and defined in (13).

For j < n, let $m_1, ..., m_{j-1}, k, m_{j+2}, ..., m_n$ be non-negative integers such that $m_1 + ... + m_{j-1} + k + m_{j+2} + ... + m_n = m$. Let a, b be integers such that $a \neq b, 0 \leq a, b \leq k$. Let $M = (m_1, ..., m_{j-1}, a, k - a, m_{j+2}, ..., m_n)$ and $L = (m_1, ..., m_{j-1}, b, k - b, m_{j+2}, ..., m_n)$. Consider the number $\lambda_0 = 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l))$. Notice that $-\lambda_0$ is a pole of the coefficients $Q_M(\lambda), Q_L(\lambda)$ of the Shapovalov operator on $V_{\vec{\Lambda}}$, and the pairs $M, -\lambda_0$ and $L, -\lambda_0$ are dual in the sense of Sec. 3.7.

Theorem 12. Under these conditions, for the weight functions defined in (13), we have

(27)
$$[a]! [k-a]! \omega_M(t_1, \dots, t_m, 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l)), z, \tau) =$$
$$[b]! [k-b]! \omega_L(t_1, \dots, t_m, 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l)), z, \tau) .$$

It is easy to see that formula (27) for arbitrary $n \ge 2$ and j < n follows from the next Proposition, cf. (13).

Proposition 13. Let $\overline{\Lambda} = (\Lambda_1, \Lambda_2)$. Let m, a, b be non-negative integers such that $a \neq b$, $0 \leq a, b \leq m$. Then for the weight functions corresponding to $V_{\overline{\Lambda}}$ we have

(28)
$$[a]! [m-a]! \omega_{a,m-a}(t_1, \dots, t_m, 2\eta(\Lambda_1 - a - b), z, \tau) = [b]! [m-b]! \omega_{b,m-b}(t_1, \dots, t_m, 2\eta(\Lambda_1 - a - b), z, \tau) .$$

3.9. Proof of Proposition 13.

According to Lemmas 6-8, it is easy to see that in order to prove the Proposition it is enough to check that for any $N = (c, m - c) \in \mathbb{Z}_m^2$,

(29)
$$[a]! [m-a]! \omega_{a,m-a}(T_N, 2\eta(\Lambda_1 - a - b), z, \tau) = [b]! [m-b]! \omega_{b,m-b}(T_N, 2\eta(\Lambda_1 - a - b), z, \tau) .$$

Let a > b. If c > a, then

$$\omega_{a,m-a}(T_N, 2\eta(\Lambda_1 - a - b), z, \tau) = \omega_{b,m-b}(T_N, 2\eta(\Lambda_1 - a - b), z, \tau) = 0$$

by Lemma 6. If $a \ge c > b$, then $\omega_{b,m-b}(T_N, 2\eta(\Lambda_1 - a - b), z, \tau) = 0$ by Lemma 6.

Lemma 14. If $a \ge c > b$, then $\omega_{a,m-a}(T_N, 2\eta(\Lambda_1 - a - b), z, \tau) = 0$.

Proof. According to (18), $\omega_{a,m-a}(T_N, 2\eta(\Lambda_1 - a - b), z, \tau)$ is the sum over all pairs I_1, I_2 of disjoint subsets of $\{1, \ldots, m\}$ such that I_1 has a elements, and I_2 has m - a elements. By Lemma 7, all terms in this sum but the distinguished one are equal to zero. The distinguished summand is zero since the factor $\theta(\lambda + t_{c-b} - z_1 - \eta\Lambda_1 + 2\eta a) = \theta(2\eta(\Lambda_1 - a - b) + z_1 - \eta\Lambda_1 + 2\eta b - z_1 - \eta\Lambda_1 + 2\eta a)$ is zero.

Lemma 15. If $a > b \ge c$, then equation (29) holds.

Proof. By Lemma 7, it is enough to show that the distinguished summands D_1 and D_2 of respectively $[a]! [m-a]! \omega_{a,m-a}(T_N, 2\eta(\Lambda_1-a-b), z, \tau)$ and $[b]! [m-b]! \omega_{b,m-b}(T_N, 2\eta(\Lambda_1-a-b), z, \tau)$ are equal. We have

$$D_{1} = \prod_{j=1}^{a} \frac{\theta(2\eta j)}{\theta(2\eta)} \prod_{j=1}^{m-a} \frac{\theta(2\eta j)}{\theta(2\eta)} \prod_{j=1}^{c} \frac{\theta(2\eta)}{\theta(2\eta)} \prod_{j=1}^{c} \frac{\theta(2\eta)}{\theta(2\eta j)} \times \prod_{j=1}^{c} \frac{\theta(2\eta)}{\theta(2\eta j)} \times \prod_{j=1}^{c} \frac{\theta(2\eta)}{\theta(2\eta j)} \prod_{j=1}^{m-a} \frac{\theta(2\eta)}{\theta(2\eta j)} \prod_{j=1}^{m-a} \frac{\theta(2\eta)}{\theta(2\eta j)} \times \prod_{j=1}^{c} \frac{\theta(2\eta)}{\theta(2\eta j)} \prod_{j=1}^{m-a} \frac{\theta(2\eta)}{\theta(2\eta j)} \prod_{j=1}^{a} \frac{\theta(2\eta)}{\theta(2\eta j)} \times \prod_{i=1}^{c} \frac{\theta(2\eta(c-i-b))}{\theta(-2\eta\Lambda_{1}+2\eta(c-i))} \prod_{i=c+1}^{a} \frac{\theta(-z_{1}+z_{2}+\eta\Lambda_{1}-\eta\Lambda_{2}+2\eta(m-i-b))}{\theta(-z_{1}+z_{2}-\eta\Lambda_{1}-\eta\Lambda_{2}+2\eta(m-i))} \times \prod_{i=a+1}^{m} \frac{\theta(-2\eta\Lambda_{2}+2\eta(2m-i-b))}{\theta(-2\eta\Lambda_{2}+2\eta(m-i))} \frac{\theta(-z_{1}+z_{2}+\eta\Lambda_{1}-\eta\Lambda_{2}+2\eta(m-i))}{\theta(-z_{1}+z_{2}-\eta\Lambda_{1}-\eta\Lambda_{2}+2\eta(m-i))},$$

$$D_{2} = \prod_{j=1}^{b} \frac{\theta(2\eta j)}{\theta(2\eta)} \prod_{j=1}^{m-b} \frac{\theta(2\eta j)}{\theta(2\eta)} \prod_{j=1}^{c} \frac{\theta(2\eta)}{\theta(2\eta)} \prod_{j=1}^{c} \frac{\theta(2\eta)}{\theta(2\eta j)} \\ \times \prod_{i=1}^{c} \prod_{j=c+1}^{b} \frac{\theta(z_{1}-z_{2}-\eta\Lambda_{1}+\eta\Lambda_{2}+2\eta(c-m+j-i))}{\theta(z_{1}-z_{2}-\eta\Lambda_{1}+\eta\Lambda_{2}+2\eta(c-m+j-i+1))} \prod_{j=1}^{b-c} \frac{\theta(2\eta)}{\theta(2\eta j)} \prod_{j=1}^{m-b} \frac{\theta(2\eta)}{\theta(2\eta j)} \\ \times \prod_{i=1}^{c} \frac{\theta(2\eta(c-i-a))}{\theta(-2\eta\Lambda_{1}+2\eta(c-i))} \prod_{i=c+1}^{b} \frac{\theta(-z_{1}+z_{2}+\eta\Lambda_{1}-\eta\Lambda_{2}+2\eta(m-i-a))}{\theta(-z_{1}+z_{2}-\eta\Lambda_{1}-\eta\Lambda_{2}+2\eta(m-i))} \\ \times \prod_{i=b+1}^{m} \frac{\theta(-2\eta\Lambda_{2}+2\eta(2m-i-a))}{\theta(-2\eta\Lambda_{2}+2\eta(m-i))} \frac{\theta(-z_{1}+z_{2}+\eta\Lambda_{1}-\eta\Lambda_{2}+2\eta(m-i))}{\theta(-z_{1}+z_{2}-\eta\Lambda_{1}-\eta\Lambda_{2}+2\eta(m-i))}.$$

 D_1 and D_2 contain factors of five types: $\theta(-2\eta\Lambda_1 + 2\eta \text{ integer})$, $\theta(-2\eta\Lambda_2 + 2\eta \text{ integer})$, $\theta(-z_1 + z_2 + \eta\Lambda_1 - \eta\Lambda_1 + 2\eta \text{ integer})$, $\theta(-z_1 + z_2 - \eta\Lambda_1 - \eta\Lambda_1 + 2\eta \text{ integer})$, $\theta(2\eta \text{ integer})$. Comparing the factors of each type in D_1 and D_2 we conclude that $D_1 = D_2$.

Lemma 15, Proposition 13 and Theorem 12 are proved.

Theorem 16. Let r, s be integers. Then under conditions of Theorem 12, for the weight functions defined in (13), we have

(30)

$$[a]! [k-a]! e^{2\pi i s a (z_{j+1}-z_j+\eta\Lambda_{j+1}+\eta\Lambda_j)} \times \omega_M(t_1, \dots, t_m, r+s\tau + 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l)), z, \tau) = [b]! [k-b]! e^{2\pi i s b (z_{j+1}-z_j+\eta\Lambda_{j+1}+\eta\Lambda_j)} \times \omega_L(t_1, \dots, t_m, r+s\tau + 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l)), z, \tau).$$

The Theorem follows from Theorem 12, the explicit formula for the weight functions and the transformation properties of the theta function $\theta(\lambda, \tau)$ under the shifts of λ by 1 and τ .

4. Properties of R-matrices

4.1. The λ -poles of the R-matrix. According to Corollary 10, the poles of the R-matrix

 $R_{\Lambda_1,\Lambda_2}(z,\lambda,\tau)$ considered as a function of z,λ,τ come from zeros of the functions $\theta(\lambda - 2\eta(\Lambda_1 + k)), \theta(\lambda - 2\eta(\Lambda_1 + \Lambda_2 + k)), \theta(z - \eta(\Lambda_1 + \Lambda_2 + 2k)), \theta(-z + \eta(-\Lambda_1 + \Lambda_2 + 2k))$ where k is an integer. Thus the poles could be divided into the λ -poles and z-poles. The following Theorem describes the λ -poles of the R-matrix.

Theorem 17. Let $\Lambda_1, \Lambda_2, \eta, z$ be generic. Consider the *R*-matrix $R_{\Lambda_1,\Lambda_2}(z,\lambda,\tau)$ restricted to the weight space $(V_{\Lambda_1} \otimes V_{\Lambda_2})_{\Lambda_1+\Lambda_2-2m}$ of weight $\Lambda_1 + \Lambda_2 - 2m$. Then

- I. All λ -poles of the *R*-matrix have the form $\lambda = 2\eta(\Lambda_1 k) + r + s\tau$ where k = 1, ..., 2m 1 and $r, s \in \mathbb{Z}$.
- II. All λ -poles of the *R*-matrix are simple.
- III. Let $\lambda_0 = 2\eta(\Lambda_1 k) + r + s\tau$ be a λ -pole of the R-matrix, where 0 < k < 2m and $r, s \in \mathbb{Z}$. Let $K(z, \tau) \in \text{End}\left((V_{\Lambda_1} \otimes V_{\Lambda_2})_{\Lambda_1 + \Lambda_2 2m}\right)$ be the residue of the R-matrix at λ_0 . Then the kernel of K is the subspace in $(V_{\Lambda_1} \otimes V_{\Lambda_2})_{\Lambda_1 + \Lambda_2 2m}$ consisting of vectors $\sum_{i+j=m} u_{i,j}e_i \otimes e_j$ satisfying the relations

$$[a]! [m - a]! e^{2\pi i s a (-z + \eta \Lambda_1 + \eta \Lambda_2)} u_{a,m-a} = [b]! [m - b]! e^{2\pi i s b (-z + \eta \Lambda_1 + \eta \Lambda_2)} u_{b,m-b}$$

where a, b run through the set of all pairs satisfying a + b = k.

The Theorem will be proved in Sections 4.2-4.4. More information about the λ -poles of the R-matrices see in Section 7.2

4.2. Proof of Part I of Theorem 17. For any $M = (m_1, m_2) \in \mathbb{Z}_m^2$ and any meromorphic function $f(t_1, \ldots, t_m)$ in *m* variables we let $\operatorname{res}_M f$ be the complex number

$$\operatorname{res}_{M} f = \operatorname{res}_{t_{1}=z_{1}+\eta\Lambda_{1}-2\eta(m_{1}-1)} \cdots \operatorname{res}_{t_{m_{1}-1}=z_{1}+\eta\Lambda_{1}-2\eta\operatorname{res}_{t_{m_{1}}=z_{1}+\eta\Lambda_{1}}$$
$$\operatorname{res}_{t_{m_{1}+1}=z_{2}+\eta\Lambda_{2}-2\eta(m_{2}-1)} \cdots \operatorname{res}_{t_{m_{1}+m_{2}-1}=z_{2}+\eta\Lambda_{2}-2\eta\operatorname{res}_{t_{m_{1}+m_{2}}=z_{2}+\eta\Lambda_{2}}.$$

According to Proposition 30 in [FTV1], the matrices $A = \{\operatorname{res}_L \omega_M\}$, $B = \{\operatorname{res}_L \tilde{\omega}_M\}$ are triangular. Namely, $\operatorname{res}_L \omega_M = 0$ if L > M and $\operatorname{res}_L \tilde{\omega}_M = 0$ if L < M. Moreover, Proposition 30 gives explicit formulas for the diagonal elements of these matrices as alternating products of theta functions. The formulas show that the diagonal elements do not have factors of the form $\theta(\lambda - 2\eta(\Lambda_1 + \Lambda_2 + k))$. Since $R = A^{-1}B$, this shows that the R-matrix does not have λ -poles of the form $\lambda = 2\eta(\Lambda_1 + \Lambda_2 + k) + r + s\tau, k, r, s \in \mathbb{Z}$.

Now it follows from Corollary 10 and formulas (20) and (21) that the λ -poles of the R-matrix have the form $\lambda = 2\eta(\Lambda_1 - k)$ where 0 < k < 2m and $r, s \in \mathbb{Z}$.

4.3. The poles of the Shapovalov operator and the R-matrix. The following relation of the R-matrix with the Shapovalov operator is proved in [FV3].

Lemma 18. [FV3] For all j, k, r, s, we have

(31)
$$Q_j^{\Lambda_1}(\lambda + 2\eta(\Lambda_2 - 2k), \tau)Q_k^{\Lambda_2}(\lambda, \tau)R_{rs}^{jk}(-\lambda, z, \tau)$$
$$= Q_r^{\Lambda_1}(\lambda, \tau)Q_s^{\Lambda_2}(\lambda + 2\eta(\Lambda_1 - 2r), \tau)R_{jk}^{rs}(\lambda + 2\eta(\Lambda_1 + \Lambda_2 - 2(r+s)), z\tau).$$

Lemma 19. If $\Lambda_1, \Lambda_2, \eta$ are generic, then $R_{\Lambda_1, \Lambda_2}(z, \lambda, \tau)$, as a function of λ , has only simple poles.

Proof. According to Part I of Theorem 17, the poles of the R-matrix are of the form $\lambda = 2\eta(\Lambda_1 + \text{integer}) \pmod{\mathbb{Z} + \tau \mathbb{Z}}$.

Consider formula (31). Modulo $\mathbb{Z} + \tau \mathbb{Z} + 2\eta \mathbb{Z}$, the poles of the R-matrix on the left are at $-2\eta \Lambda_1$, whereas the poles of the R-matrix on the right are at $-2\eta \Lambda_2$, i.e., elsewhere.

Thus the poles of the R-matrix on the right must come from the poles of the Q's on the left. It is therefore sufficient to show that the product of Q's on the left-hand side has only simple poles.

Now $Q_j^{\Lambda}(\lambda, \tau)$ has simple poles at $\lambda = 2\eta l$, and $\lambda = 2\eta (-\Lambda + j + l - 1), l = 1, \dots, j$. Hence the only way

$$Q_j^{\Lambda_1}(\lambda + 2\eta(\Lambda_2 - 2k), \tau)Q_k^{\Lambda_2}(\lambda, \tau)$$

can have higher order poles is that a pole of the first factor at $\lambda = 2\eta(-\Lambda_2 + 2k + l)$, $l = 1, \ldots, j$ coincides with a pole of the second factor at $\lambda = 2\eta(-\Lambda_2 + k + l' - 1)$, $l' = 1, \ldots, k$. But this never happens since 2k + l > k + l' - 1 for l > 0 and $l' \le k$. So the R-matrix has only simple poles.

The Q - R-relation in (31) is powerful enough to tell which matrix elements of the R-matrix could have which poles.

Part II of Theorem 17 is proved.

4.4. **Proof of Part III of Theorem 17.** According to Propositions 13 and 14 in [FTV2], we have $R_{\Lambda_1,\Lambda_2}(z, \lambda + 1, \tau) = R_{\Lambda_1,\Lambda_2}(z, \lambda, \tau)$ and

$$R_{\Lambda_1,\Lambda_2}(z-w,\lambda+\tau,\tau) = e^{\pi i (h^{(1)}(-z-\eta\Lambda_2)+h^{(2)}(-w+\eta\Lambda_1))} R_{\Lambda_1,\Lambda_2}(z-w,\lambda,\tau) e^{\pi i (h^{(1)}(z-\eta\Lambda_2)+h^{(2)}(w+\eta\Lambda_1))}.$$

These two formulas show that the statement of Part III for arbitrary integers r, s follows from the statement of Part III for r = s = 0.

Proposition 20. Let $\lambda_0 = 2\eta(\Lambda_1 - k)$ be a λ -pole of the *R*-matrix, where 0 < k < 2m. Let $K(z,\tau) \in \text{End}((V_{\Lambda_1} \otimes V_{\Lambda_2})_{\Lambda_1 + \Lambda_2 - 2m})$ be the residue of the *R*-matrix at λ_0 . Then the kernel of *K* is the subspace in $(V_{\Lambda_1} \otimes V_{\Lambda_2})_{\Lambda_1 + \Lambda_2 - 2m}$ consisting of vectors $\sum_{i+j=m} u_{i,j} e_i \otimes e_j$ satisfying the relations

$$[a]! [m-a]! u_{a,m-a} = [b]! [m-b]! u_{b,m-b}$$

where a, b run through the set of all pairs satisfying a + b = k.

The Proposition follows from Lemmas 6 - 8, Corollary 10, Proposition 13, and the following Lemma from linear algebra.

Lemma 21. Let l, n be natural numbers, $A(\alpha) = \{A_{ij}(\alpha)\}$ be an $n \times n$ matrix depending on a parameter α . Assume that det $A(\alpha) = c\alpha^l + O(\alpha^{l+1}), c \neq 0$, as $\alpha \to 0$, and $A_{ij} = O(\alpha)$ for all j and i = 1, ..., l. Then the inverse matrix $A^{-1}(\alpha)$ has a simple pole at $\alpha = 0$. Moreover, if K is the residue of $A^{-1}(\alpha)$ at $\alpha = 0$, then K has rank l and $K_{ij} = 0$ for all i and j = l + 1, ..., n. \Box .

4.5. Relations for matrix coefficients of R-matrices.

Theorem 22. Let $r, s \in \mathbb{Z}$. Let $z, w, \Lambda_1, \Lambda_2, \eta$ be generic complex numbers. Then

I. For all a, a', b, c, c', d we have

(32)
$$e^{2\pi i s (b(-w+\eta\Lambda_1)+d(z-\eta\Lambda_2))} \frac{[b]!}{[d]!} R_{\Lambda_1,\Lambda_2}(z-w,2\eta(b'-b)+r+s\tau)^{a,b}_{d,c} = e^{2\pi i s (b'(-w+\eta\Lambda_1)+d'(z-\eta\Lambda_2))} \frac{[b']!}{[d']!} R_{\Lambda_1,\Lambda_2}(z-w,2\eta(b-b')+r+s\tau)^{a,b'}_{d',c}.$$

II. For all a, b, b', c, d, d' we have

$$e^{2\pi i s (a(-z-\eta\Lambda_2)+c(w+\eta\Lambda_1))} \frac{[a]!}{[c]!} R_{\Lambda_1,\Lambda_2}(z-w,2\eta(\Lambda_1+\Lambda_2-2b-a-a')+r+s\tau)^{a,b}_{d,c} = e^{2\pi i s (a'(-z-\eta\Lambda_2)+c'(w+\eta\Lambda_1))} \frac{[a']!}{[c']!} R_{\Lambda_1,\Lambda_2}(z-w,2\eta(\Lambda_1+\Lambda_2-2b-a-a')+r+s\tau)^{a',b}_{d,c'}.$$

Example. We have $R_{\Lambda_1,\Lambda_2}(z-w,\lambda)_{d,c}^{0,0} = \delta_{0,c}\delta_{0,d}$. Then equation (32) says that $R_{\Lambda_1,\Lambda_2}(z-w,-2\eta k)_{k,0}^{0,k} = 1$ and $R_{\Lambda_1,\Lambda_2}(z-w,-2\eta k)_{i,j}^{0,k} = 0$ for $(i,j) \neq (k,0)$. Equation (33) says that $R_{\Lambda_1,\Lambda_2}(z-w,2\eta(\Lambda_1+\Lambda_2-k))_{0,k}^{k,0} = 1$ and $R_{\Lambda_1,\Lambda_2}(z-w,2\eta(\Lambda_1+\Lambda_2-k))_{i,j}^{k,0} = 0$ for $(i, j) \neq (0, k)$.

We shall prove Part I of the Theorem, Part II is proved similarly.

According to the transformation properties of the R-matrix, described in Propositions 13 and 14 in [FTV2], in order to prove Part I of the Theorem it is enough to prove Part I for r = s = 0, thus it is enough to prove the following Proposition.

Proposition 23. for all a, a', b, c, c', d we have

$$\frac{[b]!}{[d]!} R_{\Lambda_1,\Lambda_2}(z-w,2\eta(b'-b))^{a,b}_{d,c} = \frac{[b']!}{[d']!} R_{\Lambda_1,\Lambda_2}(z-w,2\eta(b-b'))^{a,b'}_{d',c}.$$

Proof. For a natural number m, define

(34)
$$\mathbb{Z}_m^3 = \{ (m_1, m_2, m_3) \in \mathbb{Z}_{\geq 0}^3 \mid m_1 + m_2 + m_3 = m \}.$$

Let $\omega_{m_1,m_2,m_3}^{\Lambda_1,\Lambda_2,\Lambda_3}(t_1,...,t_m,\lambda,z_1,z_2,z_3,\tau), (m_1,m_2,m_3) \in \mathbb{Z}_m^3$, be the weight functions associated in Section 3.3 with the space $F_{a_1,a_2,a_3}^m(z_1,z_2,z_3)$ for $a_j = \eta \Lambda_j, j = 1, 2, 3$. Define a $V_{\Lambda_1} \otimes V_{\Lambda_2} \otimes V_{\Lambda_3}$ -valued function $\omega^{\Lambda_1,\Lambda_2,\Lambda_3}(t,\lambda,z_1,z_2,z_3,\tau)$ by

$$\omega^{\Lambda_1,\Lambda_2,\Lambda_3}(t_1 \quad , \quad \dots, t_m, \lambda, z_1, z_2, z_3, \tau) = \sum_{(m_1,m_2,m_3) \in \mathbb{Z}_m^3} \omega^{\Lambda_1,\Lambda_2,\Lambda_3}_{m_1,m_2,m_3}(t_1, \dots, t_m, \lambda, z_1, z_2, z_3, \tau) e_{m_1} \otimes e_{m_2} \otimes e_{m_3} .$$

We have

$$\omega^{\Lambda_1,\Lambda_3,\Lambda_2}(t,\lambda,z_1,z_3,z_2,\tau) = P^{(23)}R^{(23)}_{\Lambda_2,\Lambda_3}(z_2-z_3,\lambda-2\eta h^{(1)})\,\omega^{\Lambda_1,\Lambda_2,\Lambda_3}(t,\lambda,z_1,z_2,z_3,\tau)\,,$$

i.e.

$$\omega_{m_1,l_3,l_2}^{\Lambda_1,\Lambda_3,\Lambda_2}(t,\lambda,z_1,z_3,z_2,\tau) = \sum_{m_2,m_3} R_{\Lambda_2,\Lambda_3}(z_2-z_3,\lambda) -2\eta(\Lambda_1-2m_1) |_{m_2,m_3}^{l_2,l_3} \omega_{m_1,m_2,m_3}^{\Lambda_1,\Lambda_2,\Lambda_3}(t,\lambda,z_1,z_2,z_3,\tau).$$

The coordinates of the functions $\omega^{\Lambda_1,\Lambda_2,\Lambda_3}(t,\lambda,z_1,z_2,z_3,\tau)$ and $\omega^{\Lambda_1,\Lambda_3,\Lambda_2}(t,\lambda,z_1,z_3,z_2,\tau)$ satisfy the resonance relations of Theorem 12. Let us write one of the relation for coordinates of the function $\omega^{\Lambda_1,\Lambda_3,\Lambda_2}(t,\lambda,z_1,z_3,z_2,\tau)$,

$$[m_1]! [l_3]! \ \omega_{m_1, l_3, a}^{\Lambda_1, \Lambda_3, \Lambda_2}(t, \lambda_0, z_1, z_3, z_2, \tau) = [\tilde{m}_1]! [\tilde{l}_3]! \ \omega_{\tilde{m}_1, \tilde{l}_3, a}^{\Lambda_1, \Lambda_3, \Lambda_2}(t, \lambda_0, z_1, z_3, z_2, \tau)$$

where $\lambda_0 = 2\eta(\Lambda_1 - m_1 - \tilde{m}_1)$ and $m_1 + l_3 = \tilde{m}_1 + \tilde{l}_3$. We assume that $m_1 < \tilde{m}_1$. We can write this relation as

$$[m_1]![l_3]! \sum_{m_2,m_3} R_{\Lambda_2,\Lambda_3}(z_2 - z_3, 2\eta(m_1 - \tilde{m}_1))^{a,l_3}_{m_2,m_3} \omega^{\Lambda_1,\Lambda_2,\Lambda_3}_{m_1,m_2,m_3}(t,\lambda_0,z_1,z_2,z_3,\tau) = \\ [\tilde{m}_1]![\tilde{l}_3]! \sum_{\tilde{m}_2,\tilde{m}_3} R_{\Lambda_2,\Lambda_3}(z_2 - z_3, 2\eta(\tilde{m}_1 - m_1))^{a,\tilde{l}_3}_{\tilde{m}_2,\tilde{m}_3} \omega^{\Lambda_1,\Lambda_2,\Lambda_3}_{\tilde{m}_1,\tilde{m}_2,\tilde{m}_3}(t,\lambda_0,z_1,z_2,z_3,\tau) .$$

Using the resonance relations for the function $\omega^{\Lambda_1,\Lambda_2,\Lambda_3}(t,\lambda,z_1,z_2,z_3,\tau)$,

$$\begin{split} [m_1]! \, [m_2]! \, \omega_{m_1,m_2,m_3}^{\Lambda_1,\Lambda_2,\Lambda_3}(t,\lambda_0,z_1,z_2,z_3, \ \tau \) \ = \\ [\tilde{m}_1]! \, [\tilde{m}_2]! \, \omega_{\tilde{m}_1,\tilde{m}_2,m_3}^{\Lambda_1,\Lambda_2,\Lambda_3}(t,\lambda_0,z_1,z_3,z_2,\tau) \, , \end{split}$$

we can rewrite the last equation as

(35)
$$\sum_{m_3} \left(\frac{[l_3]!}{[a+l_3-m_3]!} R_{\Lambda_2,\Lambda_3} (z_2-z_3, 2\eta(m_1-\tilde{m}_1))^{a,l_3}_{a+l_3-m_3,m_3} - \frac{[\tilde{l}_3]!}{[a+\tilde{l}_3-m_3]!} R_{\Lambda_2,\Lambda_3} (z_2-z_3, 2\eta(\tilde{m}_1-m_1))^{a,\tilde{l}_3}_{a+\tilde{l}_3-m_3,m_3} \right) \times \omega^{\Lambda_1,\Lambda_2,\Lambda_3}_{m_1,a+l_3-m_3,m_3} (t,\lambda_0, z_1, z_2, z_3, \tau) = 0.$$

In this equation the R-matrix elements with negative indices are considered to be zero and 1/[j]! is zero for negative j.

Lemma 24. Let $\lambda_0 = 2\eta(\Lambda_1 - m_1 - \tilde{m}_1)$ where m_1, \tilde{m}_1 are non-zero integers and $m_1 < \tilde{m}_1$. Consider the functions $\omega_{m_1,a+l_3-m_3,m_3}^{\Lambda_1,\Lambda_2,\Lambda_3}(t,\lambda_0,z_1,z_2,z_3,\tau), m_3 = 0, ..., a + l_3$, as functions of $t_1, ..., t_m$. Then these functions are linearly independent over \mathbb{C} .

Indeed, for any $(k_1, k_2, k_3) \in \mathbb{Z}_m^3$ define a point $T_{k_1, k_2, k_3} \in \mathbb{C}^m$ by

$$T_{k_1,k_2,k_3} = (z_1 - \eta \Lambda_1 + 2\eta(k_1 - 1), z_1 - \eta \Lambda_1 + 2\eta(k_1 - 2), ..., z_1 - \eta \Lambda_1, z_2 - \eta \Lambda_2 + 2\eta(k_2 - 1), z_2 - \eta \Lambda_2 + 2\eta(k_2 - 2), ..., z_2 - \eta \Lambda_2, z_3 - \eta \Lambda_3 + 2\eta(k_3 - 1), z_3 - \eta \Lambda_3 + 2\eta(k_3 - 2), ..., z_3 - \eta \Lambda_3).$$

It is easy to see that the matrix $\{A_{m_3,k_3} = \omega_{m_1,a+l_3-m_3,m_3}^{\Lambda_1,\Lambda_2,\Lambda_3}(T_{m_1,a+l_3-k_3,k_3},\lambda_0,z_1,z_2,z_3,\tau)\}$, with $0 \leq m_3, k_3 \leq a+l_3$, is triangular with nonzero diagonal elements, cf. Lemmas 6, 7. This proves the Lemma.

By the Lemma, the coefficients of the weight functions in equation (35) must be zero. Thus the Proposition is proved. $\hfill \Box$

5. Hypergeometric solutions

5.1. **Phase functions.** We assume that p has a positive imaginary part, and set $r = e^{2\pi i p}$, $q = e^{2\pi i \tau}$. Then the convergent infinite product

(36)
$$\Omega_a(t) := \Omega_a(t,\tau,p) = \prod_{j=0}^{\infty} \prod_{k=0}^{\infty} \frac{(1-r^j q^k e^{2\pi i(t-a)})(1-r^{j+1} q^{k+1} e^{-2\pi i(t+a)})}{(1-r^j q^k e^{2\pi i(t+a)})(1-r^{j+1} q^{k+1} e^{-2\pi i(t-a)})},$$

is called a (one variable) phase function. It obeys the identities

$$\Omega_a(z+p,\tau,p) = e^{2\pi i a} \frac{\theta(z+a;\tau)}{\theta(z-a;\tau)} \Omega_a(z,\tau,p),$$
$$\Omega_a(z+1,\tau,p) = \Omega_a(z,\tau,p),$$
$$\Omega_a(z,\tau,p) = \Omega_a(z,p,\tau),$$

see [FV3]. Notice that the phase function obeys many other remarkable identities that lead to an $SL(3,\mathbb{Z})$ symmetry, see [FV3, FV4].

Fix $a = (a_1, ..., a_n) \in \mathbb{C}^n$ and a natural number *m*. Define the *m*-variable phase function by

(37)
$$\Omega_a(t_1, \dots, t_m, z_1, \dots, z_n, \tau, p) = \prod_{j=1}^m \prod_{l=1}^n \Omega_{a_l}(t_j - z_l) \prod_{1 \le i < j \le m} \Omega_{-2\eta}(t_i - t_j).$$

5.2. The universal hypergeometric function, [FTV2]. Let $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ be such that $\Lambda_1 + \ldots + \Lambda_n = 2m$ for some positive integer m, and set $a_i = \eta \Lambda_i$.

Introduce a meromorphic $V_{\vec{\Lambda}}[0] \otimes V_{\vec{\Lambda}}[0]$ -valued function u of variables $z, \vec{\Lambda} \in \mathbb{C}^n, \lambda, \mu \in \mathbb{C}$ by

(38)
$$u(z,\lambda,\mu,\tau,p,\vec{\Lambda}) = \sum_{I,J,|I|=|J|=m} u_{IJ}(z,\lambda,\mu,\tau,p,\vec{\Lambda}) e_I \otimes e_J$$

where

(

$$39) \ u_{IJ}(z,\lambda,\mu,\tau,p,\vec{\Lambda}) = e^{-\pi i \frac{\mu\lambda}{2\eta}} \int_{T^m} \Omega(t,z,a,\tau,p,\eta) \,\omega_I(t,\lambda,z,a,\tau,\eta) \,\tilde{\omega}_J(t,\mu,z,a,p,\eta) \,dt \,.$$

Here, for every $M = (m_1, ..., m_n)$, e_M denotes $e_{m_1} \otimes ... \otimes e_{m_n}$, Ω is the *m*-variable phase function (37), $\omega_I(t, \lambda, z, a, \tau, \eta)$ and $\tilde{\omega}_J(t, \mu, z, a, p, \eta)$ are the weight functions (13),(14), $dt = dt_1 \wedge ... \wedge dt_m$. The integrand is 1-periodic with respect to shifts of variables $t = (t_1, ..., t_m)$ and therefore defines a meromorphic function of t on $\mathbb{C}^m/\mathbb{Z}^m$. $T^m = T^m(z, \tau, p, \vec{\Lambda})$ is an *m*-dimensional cycle in $\mathbb{C}^m/\mathbb{Z}^m$ which is a suitable deformation of the torus $\mathbb{R}^m/\mathbb{Z}^m \subset \mathbb{C}^m/\mathbb{Z}^m$ determined by $z, \tau, p, \vec{\Lambda}$, see [FTV2]. Namely, the integral is defined by analytic continuation from the region where Re $(\Lambda_i) < 0, z_i \in \mathbb{R}$ and Im $(\eta) < 0$. In this region, the integration cycle is just the torus $\mathbb{R}^m/\mathbb{Z}^m$.

The function u will be called the universal hypergeometric function associated with $V_{\vec{\Lambda}}$.

For a fixed generic $\vec{\Lambda}$, the function u is a meromorphic function of the remaining parameters and satisfies equations (8), see explicit conditions on $\vec{\Lambda}$ in [FTV2] and [MV].

Remark. In [FTV2] we used only weight functions and no mirror weight functions. Then the qKZB equations (8) only involve qKZB operators and no mirror qKZB operators. The choices of this paper and [FV3, FV4] make the qKZB heat equation of [FV3, FV4] more transparent. The proof that u obeys the relations (8) is the same as the proof of Theorem 31 in [FTV2]. Note however that the conventions in the definition of D_j are different there.

5.3. Finite dimensional representations, [MV]. Assume that $\Lambda_1^0, ..., \Lambda_n^0$ are natural numbers, and for some natural number $m, \Lambda_1^0 + ... + \Lambda_n^0 = 2m$. Set $\vec{\Lambda}^0 = (\Lambda_1^0, ..., \Lambda_n^0)$.

Let $M = (m_1, ..., m_n)$ be a vector of non-negative integers. Say that M is $\vec{\Lambda}^0$ - admissible, if $m_j \leq \Lambda_j$ for all j.

Theorem 25. [MV] The coordinates u_{IJ} of the universal hypergeometric function defined by (38) are meromorphic functions of its variables. If at least one of the indices I, J is $\vec{\Lambda}^0$ -admissible, then $u_{IJ}|_{\vec{\Lambda}=\vec{\Lambda}^0}$, defined by analytic continuation, is a well defined meromorphic function of the remaining variables. Thus, for every fixed $\vec{\Lambda}^0$ -admissible index M,

(40)
$$\sum_{I, |I|=m} u_{IM}|_{\vec{\Lambda}=\vec{\Lambda}^0} e_I \otimes e_M \quad and \quad \sum_{I, |I|=m} u_{MI}|_{\vec{\Lambda}=\vec{\Lambda}^0} e_M \otimes e_I,$$

are well defined meromorphic functions. Moreover, the first of the functions satisfies the first and the third system of equations in (8), and the second of the functions satisfies the second and the third system of equations in (8).

Consider the tensor product $L_{\vec{\Lambda}^0} = L_{\vec{\Lambda}^0_1} \otimes ... \otimes L_{\vec{\Lambda}^0_n}$ of the corresponding finite dimensional vector spaces and the $V_{\vec{\Lambda}^0}[0] \otimes L_{\vec{\Lambda}^0}[0]$ -valued function

$$\tilde{u}(z,\lambda,\mu,\tau,p,\vec{\Lambda}^{0}) = \sum_{\substack{I, |I|=m,\\ \text{adm} J, |J|=m}} u_{IJ}(z,\lambda,\mu,\tau,p,\vec{\Lambda}^{0}) e_{I} \otimes e_{J}.$$

The function

$$\tilde{v} = (1 \otimes \prod_{j=1}^{n} D_j^{-z_j/p}) \, \tilde{u}$$

obeys the qKZB equations in the first factor (10). It means that for every complex μ and every linear form on $L_{\vec{\Lambda}}[0]$, we have a solution of the qKZB equations. We call such solutions the elementary hypergeometric solutions with values in $V_{\vec{\Lambda}^0}[0]$. A solution $w(z, \lambda)$ of the qKZB equations with values in $V_{\vec{\Lambda}^0}[0]$ is called a hypergeometric solution if it can be represented in the form (11).

Define the universal hypergeometric function associated with $L_{\vec{\Lambda}^0}$ by

(41)
$$u(z,\lambda,\mu,\tau,p,\vec{\Lambda}^{0}) = \sum_{\text{adm } I,J,|I|=|J|=m} u_{IJ}(z,\lambda,\mu,\tau,p,\vec{\Lambda}^{0}) e_{I} \otimes e_{J}$$

where the sum is over all $\vec{\Lambda}^0$ -admissible indices I, J. This is a $L_{\vec{\Lambda}^0}[0] \otimes L_{\vec{\Lambda}^0}[0]$ -valued function. Let $\pi : V_{\vec{\Lambda}^0}[0] \to L_{\vec{\Lambda}^0}[0]$ be the canonical projection. Then $u = \pi \otimes 1$ \tilde{u} .

It follows from the properties of R-matrices formulated in Sec. 3.4 that the universal hypergeometric function satisfies equations (8). Define the fundamental hypergeometric solution v associated with $L_{\vec{\Lambda}^0}$ by formula (9). The $L_{\vec{\Lambda}^0}[0] \otimes L_{\vec{\Lambda}^0}[0]$ -valued function v obeys the qKZB equations in the first factor (10). Thus for every complex μ and every linear form on $L_{\vec{\Lambda}^0}[0]$, we have a solution of the qKZB equations. We call such solutions the elementary hypergeometric solutions with values in $L_{\vec{\Lambda}^0}[0]$. A solution $w(z, \lambda)$ of the qKZB equations with values in $L_{\vec{\Lambda}^0}[0]$ is called a hypergeometric solution if it can be represented in the form (11).

The second system of equations in (8) gives the monodromy of these solutions, see [MV].

6. Resonance relations for solutions with values in Verma modules

6.1. Resonance relations. Let $u = \sum u_{IJ}e_I \otimes e_J$ be the universal hypergeometric function associated with $V_{\vec{\Lambda}}$, $\Lambda_1 + \ldots + \Lambda_n = 2m$.

Theorem 26. Let r, s be integers.

I. Let n > 1. For j < n, let $m_1, ..., m_{j-1}, k, m_{j+2}, ..., m_n$ be non-negative integers such that $m_1 + ... + m_{j-1} + k + m_{j+2} + ... + m_n = m$. Let a, b be integers such that $a \neq b, 0 \leq a, b \leq k$. Let $M = (m_1, ..., m_{j-1}, a, k - a, m_{j+2}, ..., m_n)$ and $L = (m_1, ..., m_{j-1}, b, k - b, m_{j+2}, ..., m_n)$. Then for any J we have

(42)

$$[a]! [k - a]! e^{2\pi i s a (z_{j+1} - z_j + \eta \Lambda_{j+1} + \eta \Lambda_j)} \times u_{MJ}(z, r + s\tau + 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l)), \mu, \tau, p, \vec{\Lambda}) = [b]! [k - b]! e^{2\pi i s b (z_{j+1} - z_j + \eta \Lambda_{j+1} + \eta \Lambda_j)} \times u_{LJ}(z, r + s\tau + 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l)), \mu, \tau, p, \vec{\Lambda}).$$

II. Let n > 1. Let $m_2, ..., m_{n-1}, k$ be non-negative integers such that $m_2 + ... + m_{n-1} + k = m$. Let a, b be such that $a \neq b, 0 \leq a, b \leq k$. Let $M = (k - a, m_2, ..., m_{n-1}, a)$ and $L = (k - b, m_2, ..., m_{n-1}, b)$. Then for any J we have

(43)

$$[a]! [k - a]! e^{2\pi i s a (z_1 - z_n + \eta \Lambda_1 + \eta \Lambda_n - p)} \times u_{MJ}(z, r + s\tau + 2\eta(a - b), \mu, \tau, p, \vec{\Lambda}) = [b]! [k - b]! e^{2\pi i s b (z_1 - z_n + \eta \Lambda_1 + \eta \Lambda_n - p)} \times u_{LJ}(z, r + s\tau + 2\eta(b - a), \mu, \tau, p, \vec{\Lambda})$$

III. Let n = 1, $\vec{\Lambda} = 2m$. In this case $u = u_{mm}e_m \otimes e_m$ and u does not depend on z. We claim that

$$(44)e^{2\pi i \, s \, a \, (4\eta m - p)} \, u_{mm}(\, r + s\tau \, + \, 2\eta a, \mu, \tau, p, \vec{\Lambda}) \, = \, u_{mm}(\, r + s\tau \, - \, 2\eta a, \mu, \tau, p, \vec{\Lambda})$$

for $a = 1, ..., m$.

Corollary 27. Any hypergeometric solution $w(z, \lambda) = \sum_J w_J(z, \lambda) e_J$ defined in (11) obeys the resonance relations

(45)

$$[a]! [k - a]! e^{2\pi i s a (z_{j+1} - z_j + \eta \Lambda_{j+1} + \eta \Lambda_j)} \times w_M(z, r + s\tau + 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l))) = [b]! [k - b]! e^{2\pi i s b (z_{j+1} - z_j + \eta \Lambda_{j+1} + \eta \Lambda_j)} \times w_L(z, r + s\tau + 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l)))$$

for j, a, b, k, L, M defined in Part I of Theorem 26 and the resonance relations

(46)
$$[a]! [k-a]! e^{2\pi i \, s \, a \, (z_1-z_n+\eta\Lambda_1+\eta\Lambda_n-p)} w_M(z, \, r+s\tau \,+\, 2\eta(a-b)) = [b]! [k-b]! e^{2\pi i \, s \, b \, (z_1-z_n+\eta\Lambda_1+\eta\Lambda_n-p)} w_L(z, \, r+s\tau \,+\, 2\eta(b-a))$$

for a, b, k, L, M defined in Part II of Theorem 26.

Formula (42) of Theorem 26 follows from Theorem 16 and formula (39). Hence any hypergeometric solution $w = \sum_{J} w_{J} e_{J}$ defined in (11) obeys relations (45) for j, a, b, k, L, M defined in Part I of Theorem 26.

We deduce (43) from (45) in Sec. 6.3. Part III is proved in Sec. 6.4.

6.2. The monodromy with respect to permutations of variables. In order to prove Part II of Theorem 26 we recall some facts about monodromy properties of the qKZB equations.

Theorem 28. [FTV2] For $\vec{\Lambda} \in \mathbb{C}^n$ such that $\Lambda_1 + ... + \Lambda_n = 2m$, let $v(z_1, ..., z_n, \lambda)$ be a solution of the qKZB equations with values in $V_{\vec{\Lambda}}[0] = (V_{\Lambda_1} \otimes ... \otimes V_{\Lambda_n})[0]$, step p and modulus τ . Then for any j = 1, ..., n - 1, the function

$$P^{(j,j+1)} R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_{j+1}-z_j,\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)},\tau) v(z_1,\ldots,z_{j+1},z_j,\ldots,z_n,\lambda)$$

is a solution of the qKZB equations with values in $(V_{\Lambda_1} \otimes \ldots V_{\Lambda_{j+1}} \otimes V_{\Lambda_j} \ldots \otimes V_{\Lambda_n})[0]$, step p and modulus τ . Here $P^{(j,j+1)}$ is the permutation of the j-th and j + 1-th factors, and $R_{\Lambda_j,\Lambda_{j+1}}(z,\lambda,\tau) \in \text{End}(V_{\Lambda_j} \otimes V_{\Lambda_{j+1}})$ is the elliptic R-matrix with modulus τ .

Let v denote the fundamental hypergeometric solution associated with $\vec{\Lambda}$. Let $\vec{\Lambda}^{j}$ denote the vector $(\Lambda_{1}, ..., \Lambda_{j+1}, \Lambda_{j}, ..., \Lambda_{n})$ and v^{j} denote the fundamental hypergeometric solution associated with $\vec{\Lambda}^{j}$. According to Theorem 28, the $V_{\vec{\Lambda}^{j}}[0] \otimes V_{\vec{\Lambda}}[0]$ -valued function

$$u^{j} = P^{(j,j+1)} R^{(j,j+1)}_{\Lambda_{j},\Lambda_{j+1}}(z_{j+1}-z_{j},\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)},\tau) \otimes 1 \quad v(z_{1},\ldots,z_{j+1},z_{j},\ldots,z_{n},\lambda,\mu,\tau,p)$$

and the $V_{\vec{\Lambda}^{j}}[0] \otimes V_{\vec{\Lambda}^{j}}[0]$ -valued function $v^{j}(z, \lambda, \mu, \tau, p)$ satisfy the same qKZB equations in the first factor.

The next Theorem describes a relation between the two solutions and can be considered as a description of the monodromy of the hypergeometric solutions constructed in Sec. 2.3 with respect to permutation of variables.

Introduce a new *R*-matrix $\tilde{R}_{A,B}(z,\mu,p) \in \text{End}(V_A \otimes V_B)$ by

(47)
$$R_{A,B}(z,\mu,p) = e^{2\pi i ABz/p} \left(\frac{\alpha(\mu)}{\alpha(\mu-2\eta h^{(2)})}\right)^{z/p} R_{A,B}(z,\mu,p) \left(\frac{\alpha(\mu-2\eta(h^{(1)}+h^{(2)}))}{\alpha(\mu-2\eta h^{(1)})}\right)^{z/p}.$$

Theorem 29. [FTV2]

(48)

$$u^{j}(z,\lambda,\mu,\tau,p) = 1 \otimes P^{(j,j+1)}(\tilde{R}^{(j,j+1)}_{\Lambda_{j+1},\Lambda_{j}}(z_{j}-z_{j+1},\mu-2\eta\sum_{l=j+1}^{n}h^{(l)},p))^{-1} v^{j}(z,\lambda,\mu,\tau,p).$$

Remark. According to Proposition 12 in [FTV2], the matrix $\tilde{R}_{A,B}(z,\mu,p)$ is *p*-periodic,

$$\tilde{R}_{A,B}(z+p,\mu,p) = \tilde{R}_{A,B}(z,\mu,p).$$

Hence, formula (48) expresses the solution u^j as a linear combination of solutions v^j with *p*-periodic coefficients.

Remark. Although we used in [FTV2] only weight functions and no mirror weight functions, the proof of Theorem 29 is the same as the proof on Theorem 36 in [FTV2].

6.3. **Proof of Part II of Theorem 26.** For $\vec{\Lambda} \in \mathbb{C}^n$ such that $\Lambda_1 + ... + \Lambda_n = 2m$, let $v(z_1, \ldots, z_n, \lambda)$ be the fundamental hypergeometric solution associated with $V_{\vec{\Lambda}}$. Let $\vec{\Lambda}^{\vee} = (\Lambda_1^{\vee}, ..., \Lambda_n^{\vee})$ be defined by $(\Lambda_1^{\vee}, ..., \Lambda_n^{\vee}) = (\Lambda_n, \Lambda_1, \Lambda_2, ..., \Lambda_{n-1})$. Let $\Delta : V_{\vec{\Lambda}} \to V_{\vec{\Lambda}^{\vee}}$ be the linear operator defined by

(49)
$$\Delta = \Gamma_1 P^{(1,2)} P^{(2,3)} \dots P^{(n-1,n)}.$$

Consider the $V_{\vec{\Lambda}^{\vee}}[0] \otimes V_{\vec{\Lambda}}[0]$ -valued function

$$w(z_1,...,z_n,\lambda,\mu,\tau,p) = \Delta \otimes 1 v(z_2,z_3,...,z_n,z_1,\lambda,\mu,\tau,p).$$

Write w in coordinate form $w = \sum w_{IJ} e_I \otimes e_J$ where $\{e_I\}$ is the standard basis in $V_{\vec{\Lambda}^{\vee}}[0]$ and $\{e_J\}$ is the standard basis in $V_{\vec{\Lambda}}[0]$.

Theorem 30. Let n > 1. Let $M = (a, k - a, m_3, ..., m_n)$ and $L = (b, k - b, m_3, ..., m_n)$. Then

(50)

$$[a]! [k-a]! e^{2\pi i \, s \, a \, (z_2 - z_1 + \eta \Lambda_1 + \eta \Lambda_n - p)} w_{MJ}(z, \, r + s\tau \, + \, 2\eta(\Lambda_n - a - b), \mu, \tau, p) = \\[b]! [k-b]! e^{2\pi i \, s \, b \, (z_2 - z_1 + \eta \Lambda_1 + \eta \Lambda_n - p)} w_{LJ}(z, \, r + s\tau \, + \, 2\eta(\Lambda_n - a - b), \mu, \tau, p)$$

Corollary 31. For any a, b, k, L, M defined in Part II of Theorem 26, the coordinates of the hypergeometric solution $v = \sum v_{IJ} e_I \otimes e_J$ obey the resonance relation in (43).

This completes the proof of Part II of Theorem 26 since u_{LJ} is obtained from v_{LJ} by multiplying with a nonzero factor independent of L or λ , see (9).

Proof of the Corollary. We have

$$w_{(a,k-a,m_3,...,m_n)J}(z_1,...,z_n, r+s\tau + 2\eta(\Lambda_n - a - b), \mu, \tau, p) = v_{(k-a,m_3,...,m_n,a)J}(z_2,...,z_n, z_1, r+s\tau + 2\eta(a - b), \mu, \tau, p)$$

and similarly for $w_{LJ}(z, r + s\tau + 2\eta(\Lambda_n - a - b), \mu, \tau, p)$. \Box

Proof of Theorem 30. Apply to the function v the transformation of Theorem 28 for j = n - 1, then apply to the result the transformation of Theorem 28 for j = n - 2, then repeatedly apply the transformations for j = n - 3, ..., 1. The resulting $V_{\vec{\Lambda}^{\vee}}[0] \otimes V_{\vec{\Lambda}}[0]$ -valued function $w'(z, \lambda, \mu, \tau, p)$ has the form

$$w'(z_1, ..., z_n, \lambda, \mu, \tau, p) = R^{(2,1)}_{\Lambda_1, \Lambda_n}(z_2 - z_1, \lambda, \tau) R^{(3,1)}_{\Lambda_2, \Lambda_n}(z_3 - z_1, \lambda - 2\eta h^{(2)}, \tau) \dots$$
$$R^{(n,1)}_{\Lambda_{n-1}, \Lambda_n}(z_n - z_1, \lambda - 2\eta \sum_{l=2}^{n-2} h^{(l)}, \tau) P^{(1,2)} P^{(2,3)} \dots P^{(n-1,n)} \otimes 1 v(z_2, ..., z_n, z_1, \lambda, \mu, \tau, p).$$

By Theorem 28 the function w' satisfies the qKZB equations in the first factor. In particular, $w'(z_1 + p, z_2, ..., z_n) = K_1(z) \otimes 1 w'(z_1, z_2, ..., z_n)$. But

$$K_1(z) \otimes 1 \ w'(z_1, z_2, ..., z_n) = \Delta \ \otimes \ 1 \ v(z_2, z_3, ..., z_n, z_1, \lambda, \mu, \tau, p) \,.$$

Write the function w' in coordinate form, $w' = \sum w'_{IJ} e_I \otimes e_J$. By Theorem 29, for any J, the $V_{\vec{\Lambda}^{\vee}}[0]$ -valued function $\sum_{J} w'_{IJ} e_{I}$ is a hypergeometric solution of the qKZB equations. Hence this function obeys the resonance relations (45) of Corollary 27. \Box

6.4. Proof of Part III of Theorem 26. The function $u_{m,m}$ has the form

$$u_{m,m}(\lambda,\mu,\tau,p) = e^{-\pi i \frac{\mu\lambda}{2\eta}} \int_{T^m} \Omega(t,\tau,p,\eta) \,\omega_m(t,\lambda,\tau) \,\omega_m(t,\mu,p) \,dt$$

Here

$$\Omega(t,\tau,p) = \prod_{j=1}^{m} \Omega_{2\eta m}(t_j,\tau,p) \prod_{1 \le i < j \le m} \Omega_{-2\eta}(t_i - t_j,\tau,p)$$

and

$$\omega_m(t,\lambda,\tau) = \prod_{1 \le i < j \le m} \frac{\theta(t_i - t_j,\tau)}{\theta(t_i - t_j + 2\eta,\tau)} \prod_{j=1}^m \frac{\theta(\lambda + t_j,\tau)}{\theta(t_j - 2\eta m,\tau)}.$$

Notice that $\omega_m(t, \lambda, \tau)$ generates the one-dimensional space $F^m_{2\eta m}(0, \lambda)$, see Sec. 3.1. For any function $f(t_1, ..., t_m)$, let Sym $f(t_1, ..., t_m) = \sum_{s \in S_m} [f(t_1, ..., t_m)]_s$ be the symmetrization with respect to the action of the symmetric group S_m defined in Sec. 3.1. For any a = 0, 1, ..., m introduce functions $\Sigma_a = \text{Sym } \sigma_a, \Sigma'_a = \text{Sym } \sigma'_a$ where

$$\sigma_a(t,\lambda,\tau) = \prod_{1 \le i < j \le m-a} \frac{\theta(t_i - t_j)}{\theta(t_i - t_j + 2\eta)} \prod_{j=1}^{m-a} \frac{\theta(\lambda + 2\eta(m-a) + t_j - 2\eta m)}{\theta(t_j - 2\eta m)}$$
$$\times \prod_{m-a+1 \le i < j \le m} \frac{\theta(t_i - t_j)}{\theta(t_i - t_j + 2\eta)} \prod_{j=m-a+1}^{m} \frac{\theta(\lambda + 2\eta a + t_j - 2\eta m + 4\eta(m-a))}{\theta(t_j - 2\eta m)}$$

and

$$\begin{aligned} \sigma_a'(t,\lambda,\tau) &= \prod_{m-a+1 \le i < j \le m} \frac{\theta(t_i - t_j)}{\theta(t_i - t_j + 2\eta)} \prod_{j=m-a+1}^m \frac{\theta(\lambda + 2\eta a + t_j - 2\eta m)}{\theta(t_j - 2\eta m)} \\ &\times \prod_{1 \le i < j \le m-a} \frac{\theta(t_i - t_j)}{\theta(t_i - t_j + 2\eta)} \prod_{j=1}^{m-a} \frac{\theta(\lambda + 2\eta(m-a) + t_j - 2\eta m + 4\eta a)}{\theta(t_j - 2\eta m)} \\ &\times \prod_{i=1}^{m-a} \prod_{j=m-a+1}^m \frac{\theta(t_i - t_j - 2\eta)}{\theta(t_i - t_j + 2\eta)} \,. \end{aligned}$$

It is easy to see that the functions Σ_a, Σ'_a are elements of the space $F^m_{2nm}(0,\lambda)$ and hence are proportional.

Lemma 32. For any a = 0, 1, ..., m we have

(51)
$$[m]! \omega_m(t, \lambda, \tau) = [a]! [m-a]! \Sigma_a(t, \lambda, \tau) = [a]! [m-a]! \Sigma'_a(t, \lambda, \tau)$$

where $[l]!$ is the elliptic factorial.

Proof. To prove the formula it is enough to compare the residues of all functions at the point $(t_1, ..., t_m) = (2\eta, 4\eta, ..., 2m\eta)$.

To prove Part II of Theorem 26 it is enough to notice that for a = 1, ..., m and any $s \in S_m$ we have

$$e^{2\pi i \, s \, a \, (4\eta m - p)} \, e^{-\pi i \frac{\mu}{2\eta}(r + s\tau + 2\eta a)} \int_{T^m} \Omega(t, \tau, p, \eta) \left[\, \sigma_a(t, r + s\tau + 2\eta a, \tau) \, \right]_s \, \omega_m(t, \mu, p) \, dt = e^{-\pi i \frac{\mu}{2\eta}(r + s\tau - 2\eta a)} \int_{T^m} \Omega(t, \tau, p, \eta) \left[\, \sigma_a'(t, r + s\tau - 2\eta a, \tau) \, \right]_s \, \omega_m(t, \mu, p) \, dt \, .$$

For example, for s = id, set $F_a(t, \lambda, \mu, \tau, p) = e^{-\pi i \frac{\mu \lambda}{2\eta}} \Omega(t, \tau, p, \eta) \sigma_a(t, \lambda, \tau) \omega_m(t, \mu, p)$ and $F'_a(t, \lambda, \mu, \tau, p) = e^{-\pi i \frac{\mu \lambda}{2\eta}} \Omega(t, \tau, p, \eta) \sigma'_a(t, \lambda, \tau) \omega_m(t, \mu, p)$. Then we have

$$e^{2\pi i s a (4\eta m - p)} F_a(t_1, \dots, t_m, r + s\tau + 2\eta a, \mu, \tau, p) = F'_a(t_1, \dots, t_{m-a}, t_{m-a+1} + p, \dots, t_m + p, r + s\tau - 2\eta a, \mu, \tau, p, \eta)$$

This formula easily follows from the explicit formulas for $\sigma_a(t, \lambda, \tau)$, $\sigma'_a(t, \lambda, \tau)$ and transformation properties of $\Omega(t, \tau, p)$ and $\omega(t, \mu, p)$.

For an arbitrary permutation $s \in S_m$ the proof is similar.

6.5. Resonance relations for reduced coefficients. For $\vec{\Lambda} \in \mathbb{C}^n$, consider the standard basis $e_J = e_{j_1} \otimes ... \otimes e_{j_n}$ in $V_{\vec{\Lambda}}$. Introduce a new reduced basis by

$$E_J = \frac{1}{[j_1]! [j_2]! \dots [j_n]!} e_J$$

where [j]! is the elliptic factorial of j.

Consider a hypergeometric solution $w(z, \lambda) = \sum_J w_J(z, \lambda) e_J$ of the qKZB equations with values in $V_{\vec{\Lambda}}[0]$. Using the reduced basis, the solution can be written as $w(z, \lambda) = \sum_J W_J(z, \lambda) E_J$ where its *reduced* coefficients are defined by

$$W_J = [j_1]! [j_2]! \dots [j_n]! w_J$$

Corollary 27 can be reformulated in terms of the reduced coefficients.

; From now on we will be interested only in the special case of Corollary 27 when the integers r, s are equal to zero. We have

Corollary 33. The reduced coefficients of a hypergeometric solution obey the resonance relations

(52)
$$W_M(z, 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l))) = W_L(z, 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l)))$$

for j, a, b, k, L, M defined in Part I of Theorem 26 and the resonance relations

(53)
$$W_M(z, 2\eta(a-b)) = W_L(z, 2\eta(b-a))$$

for a, b, k, L, M defined in Part II of Theorem 26.

Consider a relation of Corollary 33 involving a coefficient $W_M(z, 2\eta k)$ where k is either $\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l)$ or a - b. Assume that the pair of indices M, L in a relation of Corollary 33 is ordered from M to L, then a relation will be called a transformation at level k from W_M to W_L . The transformation in (52) will be denoted $T_j(k), j = 1, ..., n-1$, and the transformation in (53) will be denoted $T_n(k)$.

If $M = (m_1, ..., m_n)$ and j < n, then the transformation $T_j(k)$ can be applied to $W_M(z, 2\eta k)$ if there exists an integer a such that

(54)
$$a \in [0, m_j + m_{j+1}] - \{m_j\},$$

(55)
$$\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + \Lambda_j - m_j - a = k.$$

Thus,

(56)
$$a = \sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + \Lambda_j - m_j - k.$$

The result of the transformation $T_i(k)$ is the index

(57)

$$L = (m_1, ..., m_{j-1}, \sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + \Lambda_j - m_j - k, \ m_{j+1} + k - \sum_{l=1}^{j} (\Lambda_l - 2m_l), \ m_{j+2}, ..., m_n).$$

and the value $W_L(z, 2\eta k)$.

If $M = (m_1, ..., m_n)$, then the transformation $T_n(k)$ can be applied to $W_M(z, 2\eta k)$ if there exists an integer *a* such that

(58)
$$a \in [0, m_n + m_1] - \{m_n\},$$

$$(59) m_n - a = k.$$

Note that equation (59) can be written as $\sum_{l=1}^{n-1} (\Lambda_l - 2m_l) + \Lambda_n - m_n - a = k$ similarly to (55).

The result of the transformation $T_n(k)$ is the index

(60)
$$L = (m_1 + k, m_2, ..., m_{n-1}, m_n - k)$$

and the value $W_L(z, -2\eta k)$.

RESONANCE RELATIONS

7. Regularity and Resonance Relations

7.1. Statement of results. Hypergeometric solutions are functions holomorphic with respect to variable λ . At the same time they satisfy the qKZB equations, $v(z_1, ..., z_j + p, ..., z_n) = K_j(z_1, ..., z_n, \tau, p)v(z_1, ..., z_n)$, where the qKZB operators K_j have poles with respect to λ . Also if $v(z_1, ..., z_n)$ is a hypergeometric solution with values in $V_{\Lambda_1} \otimes$ $... \otimes V_{\Lambda_n}$ [0], then for any j = 1, ..., n - 1, the function $P^{(j,j+1)} R^{(j,j+1)}_{\Lambda_j,\Lambda_{j+1}}(z_{j+1} - z_j, \lambda - 2\eta \sum_{l=1}^{j-1} h^{(l)}, \tau)v(z_1, ..., z_{j+1}, z_j, ..., z_n, \lambda)$ is a hypergeometric solution of the qKZB equations with values in $(V_{\Lambda_1} \otimes ... V_{\Lambda_{j+1}} \otimes V_{\Lambda_j} ... \otimes V_{\Lambda_n})$ [0]. The R-matrix has λ -poles while the new solution remains holomorphic with respect to λ . To preserve the λ holomorphy under the action of these transformations a function has to possess special properties. In this Section we show that these properties are the resonance relations discussed in Section 6.

Let n > 1 be a natural number. Let $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n) \in \mathbb{C}^n$ be such that $\Lambda_1 + \ldots + \Lambda_n = 2m$ for some natural number m. Let $u(\lambda) = \sum_{m_1 + \ldots + m_n = m} u_{m_1, \ldots, m_n}(\lambda) e_{m_1} \otimes \ldots \otimes e_{m_n}$ be a $V_{\vec{\Lambda}}[0]$ -valued holomorphic function of λ . Let j be a natural number, $1 \leq j \leq n-1$. Let $(z_1, \ldots, z_n) \in \mathbb{C}^n$. We say that the function $u(\lambda)$ obeys the resonance relation C_j with respect to the vector $\vec{\Lambda}$ at the point (z_1, \ldots, z_n) if the following holds. Let $m_1, \ldots, m_{j-1}, k, m_{j+2}, \ldots, m_n$ be any non-negative integers such that $m_1 + \ldots + m_{j-1} + k + m_{j+2} + \ldots + m_n = m$. Let a, b be any integers such that $a \neq b, 0 \leq a, b \leq k$. Let $M = (m_1, \ldots, m_{j-1}, a, k - a, m_{j+2}, \ldots, m_n)$ and $L = (m_1, \ldots, m_{j-1}, b, k - b, m_{j+2}, \ldots, m_n)$. Then we require that

(61)

$$[a]! [k-a]! e^{2\pi i \, s \, a \, (z_{j+1}-z_j+\eta\Lambda_{j+1}+\eta\Lambda_j)} \, u_M(r+s\tau + 2\eta(\Lambda_j-a-b+\sum_{l=1}^{j-1}(\Lambda_l-2m_l))) = [b]! [k-b]! e^{2\pi i \, s \, b \, (z_{j+1}-z_j+\eta\Lambda_{j+1}+\eta\Lambda_j)} \, u_L(r+s\tau + 2\eta(\Lambda_j-a-b+\sum_{l=1}^{j-1}(\Lambda_l-2m_l))).$$

We say that the function $u(\lambda)$ obeys the resonance relation C_n with respect to the vector $\vec{\Lambda}$ at the point $(z_1, ..., z_n)$ if the following holds. Let $m_2, ..., m_{n-1}, k$ be any non-negative integers such that $m_2 + ... + m_{n-1} + k = m$. Let a, b be any non-negative integers that $a \neq b, 0 \leq a, b \leq k$. Let $M = (k - a, m_2, ..., m_{n-1}, a)$ and $L = (k - b, m_2, ..., m_{n-1}, b)$. Then we require

(62)

$$[a]! [k-a]! e^{2\pi i s a (z_1 - z_n + \eta \Lambda_1 + \eta \Lambda_n - p)} u_M(r + s\tau + 2\eta(a-b)) = [b]! [k-b]! e^{2\pi i s b (z_1 - z_n + \eta \Lambda_1 + \eta \Lambda_n - p)} u_L(r + s\tau + 2\eta(b-a)).$$

For $z \in \mathbb{C}$ and $1 \leq j \leq n-1$, we introduce an operator $s_j(z) : \mathcal{F}(V_{\Lambda_1,\dots,\Lambda_n}) \to \mathcal{F}(V_{\Lambda_1,\dots,\Lambda_{j-1},\Lambda_{j+1},\Lambda_j,\Lambda_{j+2},\dots,\Lambda_n})$ by

$$s_j(z)u(\lambda) = P^{(j,j+1)}R_{\Lambda_j,\Lambda_{j+1}}(z,\lambda-2\eta\sum_{l=1}^{j-1}h^{(l)})^{(j,j+1)}u(\lambda).$$

Theorem 34. Let n > 1 be a natural number. Let $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ be a generic vector in \mathbb{C}^n with property $\Lambda_1 + \ldots + \Lambda_n = 2m$ for some natural number m. Let $(z_1, \ldots, z_n) \in \mathbb{C}^n$ be a generic point. Suppose that a $V_{\vec{\Lambda}}[0]$ -valued function $u(\lambda)$ is holomorphic and obeys the conditions $C_l, l = 1, \ldots, n$, with respect to the vector $(\Lambda_1, \ldots, \Lambda_n)$ at the point (z_1, \ldots, z_n) . Then

- I. For $1 \leq j \leq n-1$, the function $s_j(z_j z_{j+1})u(\lambda)$ is holomorphic and obeys the conditions C_l , l = 1, ..., n, with respect to the vector $(\Lambda_1, ..., \Lambda_{j+1}, \Lambda_j, ..., \Lambda_n)$ at the point $(z_1, ..., z_{j+1}, z_j, ..., z_n)$.
- II. For $1 \leq j \leq n$, the function $K_j(z_1, ..., z_n, \tau, p)u(\lambda)$ is holomorphic and obeys the conditions C_l , l = 1, ..., n, with respect to the vector $(\Lambda_1, ..., \Lambda_n)$ at the point $(z_1, ..., z_{j-1}, z_j + p, z_{j+1}, ..., z_n)$.

Conversely, suppose that for some generic vector $(\Lambda_1, ..., \Lambda_n)$ with $\Lambda_1 + ... + \Lambda_n = 2m$, generic point $(z_1, ..., z_n) \in \mathbb{C}^n$ and a holomorphic $V_{\vec{\Lambda}}[0]$ -valued function $u(\lambda)$ we have the following properties:

- a) For any $1 \leq j \leq n-1$, the function $s_j(z_j z_{j+1})u(\lambda)$ of λ is a holomorphic function of λ .
- b) The function $K_2(z_1, ..., z_n)u(\lambda)$ is a holomorphic function of λ .

Then $u(\lambda)$ obeys the conditions C_l , l = 1, ..., n, with respect to the vector $(\Lambda_1, ..., \Lambda_n)$ at the point $(z_1, ..., z_n)$.

Remark. The vector $(\Lambda_1, ..., \Lambda_n)$ is generic if all λ -poles of the R-matrices appearing in the proof below are simple. The point $(z_1, ..., z_n)$ is generic if all R-matrices $R_{\Lambda_i, \Lambda_j}(z_i - z_j(+p), \lambda)$ appearing in the proof below are well defined meromorphic functions of λ .

7.2. **Proof of Theorem 34.** Let $\vec{\Lambda}^{\vee} = (\Lambda_n, \Lambda_1, \Lambda_2, ..., \Lambda_{n-1})$. Let $\Delta : V_{\vec{\Lambda}} \to V_{\vec{\Lambda}^{\vee}}$ be the linear operator defined by $\Delta = \Gamma_1 P^{(1,2)} P^{(2,3)} ... P^{(n-1,n)}$.

We have $s_j(z)s_j(-z) = \text{Id}$, by the "unitarity" of the *R*-matrix.

Lemma 35. Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then the *j*-th qKZB operator can be written as:

$$K_{j}(z,\tau,p) = s_{j-1}(z_{j}-z_{j-1}+p)\cdots s_{2}(z_{j}-z_{2}+p)s_{1}(z_{j}-z_{1}+p)$$

$$\Delta s_{n-1}(z_{j}-z_{n})\cdots s_{j}(z_{j}-z_{j+1}).$$

Proof. Recall that if we set $R_{jk} = R(z_j - z_k(+p), \lambda - 2\eta \sum_{j < k, j \neq l} h^{(l)})$, where we add p only if k < j, and $\hat{R}_{jk} = P^{(j,k)}R_{jk}$, then

$$\begin{aligned} K_{j}(z,\tau,p) &= R_{jj-1}\cdots R_{j1}\Gamma_{j}R_{jn}\cdots R_{jj+1} \\ &= P^{(j,j-1)}\hat{R}_{jj-1}\cdots P^{(j,1)}\hat{R}_{j1}\Gamma_{j}P^{(j,n)}\hat{R}_{jn}\cdots P^{(j,j+1)}\hat{R}_{jj+1} \\ &= s_{j-1}(z_{j}-z_{j-1}+p)\cdots s_{1}(z_{j}-z_{1}+p) \\ P^{(j,j-1)}\cdots P^{(j,1)}\Gamma_{j}P^{(j,n)}\cdots P^{(j,j+1)}s_{n-1}(z_{j}-z_{n})\cdots s_{j}(z_{j}-z_{j+1}). \end{aligned}$$

We have moved the $P^{(j,k)}$ to the middle using the rule $P^{(i,k)}X^{(k)} = X^{(i)}P^{(i,k)}$. Using the same rule to move $P^{(j,j-1)}$ to the right, we obtain

 $P^{(j,j-1)}\cdots P^{(j,1)}\Gamma_j P^{(j,n)}\cdots P^{(j,j+1)} = P^{(j-1,j-2)}\cdots P^{(j-1,1)}\Gamma_{j-1}P^{(j-1,n)}\cdots P^{(j-1,j)}.$

This expression is therefore independent of j, and can thus be replaced by its value at j = 1, which is, by definition, Δ .

Lemma 36. Let $(\Lambda_1, ..., \Lambda_n)$ be a generic vector. Let $(z_1, ..., z_n) \in \mathbb{C}^n$ be a generic point. Let $u(\lambda)$ be a holomorphic function with values in $V_{\vec{\Lambda}}$. Let $1 \leq j \leq n-1$. Then, $s_j(z_j - z_{j+1})u(\lambda)$ is holomorphic if and only if $u(\lambda)$ obeys the condition C_j with respect to the vector $(\Lambda_1, ..., \Lambda_n)$ at the point $(z_1, ..., z_n)$.

Proof. For generic Λ_j, Λ_{j+1} , the R-matrix $R_{\Lambda_j,\Lambda_{j+1}}(z_j - z_{j+1}, \lambda - 2\eta \sum_{l=1}^{j-1} h^{(l)})$ has only simple λ -poles. The kernel of the residue of this matrix at a λ -pole is described in Theorem 17. It follows from Theorem 17 that a function $u(\lambda)$ obeys the condition C_j with respect to the vector $(\Lambda_1, ..., \Lambda_n)$ at a point $(z_1, ..., z_n)$ if and only if for any λ -pole λ_0 of $R_{\Lambda_j,\Lambda_{j+1}}(z_j - z_{j+1}, \lambda - 2\eta \sum_{l=1}^{j-1} h^{(l)})$ the vector $u(\lambda_0)$ belongs to the kernel of the residue of $R_{\Lambda_j,\Lambda_{j+1}}(z_j - z_{j+1}, \lambda - 2\eta \sum_{l=1}^{j-1} h^{(l)})$ at λ_0 . Thus the function $R_{\Lambda_j,\Lambda_{j+1}}(z_j - z_{j+1}, \lambda - 2\eta \sum_{l=1}^{j-1} h^{(l)})$ at λ_0 belongs the condition C_j . The Lemma is proved since $s_j = PR$ and the permutation P is regular.

Lemma 37. Let $1 \leq k \leq n-1$. If $u(\lambda)$ is a holomorphic function with values in $V_{\vec{\Lambda}}$ and obeys the condition C_k with respect to the vector $(\Lambda_1, ..., \Lambda_n)$ at a generic point $(z_1, ..., z_n)$, then for any j the function $s_j(z_j - z_{j+1})u(\lambda)$ obeys the condition C_k with respect to the vector $(\Lambda_1, ..., \Lambda_{j+1}, \Lambda_j, ..., \Lambda_n)$ at the point $(z_1, ..., z_{j+1}, z_j, ..., z_n)$.

Proof. The only non-trivial cases are when $|j - k| \leq 1$.

Let j = k. By Lemma 36, $s_j(z_j - z_{j+1})u(\lambda)$ is holomorphic and $s_j(z_{j+1} - z_j)s_j(z_j - z_{j+1})u(\lambda) = u(\lambda)$ is holomorphic. Hence by Lemma 36, $s_j(z_j - z_{j+1})u(\lambda)$ obeys C_j .

Let j = k + 1. The fact, that $s_j(z_j - z_{j+1})u(\lambda)$ satisfies the condition C_{j+1} if $u(\lambda)$ satisfies the condition C_{j+1} , follows from identities (32) and is similar to the proof of the identities (32).

The case j = k - 1 follows from identities (33).

Lemma 38. If $u(\lambda)$ is a holomorphic function with values in $V_{\vec{\Lambda}}$ and obeys the condition C_n with respect to the vector $(\Lambda_1, ..., \Lambda_n)$ at a generic point $(z_1, ..., z_n)$, then for any j the function $s_j(z_j - z_{j+1})u(\lambda)$ obeys the condition C_n with respect to the vector $(\Lambda_1, ..., \Lambda_{j+1}, \Lambda_j, ..., \Lambda_n)$ at the point $(z_1, ..., z_{j+1}, z_j, ..., z_n)$.

Proof. The statement is trivial if $2 \le j \le n-2$. Suppose that j = 1, and let

$$u(\lambda) = \sum_{m_1, m_2, m_n} e_{m_1} \otimes e_{m_2} \otimes u_{m_1, m_2, m_n}(\lambda) \otimes e_{m_n}$$

with $u_{m_1,m_2,m_n}(\lambda) \in V_{\Lambda_3} \otimes \ldots \otimes V_{\Lambda_{n-1}}$. Let

$$v(\lambda) = s_1(z_1 - z_2)u(\lambda) = \sum_{l_2, l_1, m_n} e_{l_2} \otimes e_{l_1} \otimes v_{l_2, l_1, m_n}(\lambda) \otimes e_{m_n},$$

where

(63)
$$v_{l_2,l_1,m_n}(\lambda) = \sum_{m_1,m_2} R_{\Lambda_1,\Lambda_2}(z_1 - z_2,\lambda)^{l_1,l_2}_{m_1,m_2} u_{m_1,m_2,m_n}(\lambda).$$

Our goal is to prove the identity

(64)
$$[l_2]![m_n]!e^{2\pi i s \tilde{m}_n(z_2-z_n+\eta\Lambda_2+\eta\Lambda_n-p)}v_{l_2,l_1,m_n}(r+s\tau+2\eta(m_n-\tilde{m}_n)) = [\tilde{l}_2]![\tilde{m}_n]!e^{2\pi i s \tilde{m}_n(z_2-z_n+\eta\Lambda_2+\eta\Lambda_n-p)}v_{\tilde{l}_2,l_1,\tilde{m}_n}(r+s\tau+2\eta(\tilde{m}_n-m_n))$$

for any $l_1, l_2, \tilde{l}_2, m_n \tilde{m}_n$. Using (63), we can rewrite (64) as

(65)
$$[l_{2}]![m_{n}]!e^{2\pi i s m_{n}(z_{2}-z_{n}+\eta\Lambda_{2}+\eta\Lambda_{n}-p)} \times \sum_{m_{2}} R_{\Lambda_{1},\Lambda_{2}}(z_{1}-z_{2},r+s\tau+2\eta(\tilde{l}_{2}-l_{2}))^{l_{1},l_{2}}_{l_{1}+l_{2}-m_{2},m_{2}} \times u_{l_{1}+l_{2}-m_{2},m_{2},m_{n}} (r+s\tau+2\eta(m_{n}-\tilde{m}_{n})) = [\tilde{l}_{2}]![\tilde{m}_{n}]!e^{2\pi i s \tilde{m}_{n}(z_{2}-z_{n}+\eta\Lambda_{2}+\eta\Lambda_{n}-p)} \times \sum_{m_{2}} R_{\Lambda_{1},\Lambda_{2}}(z_{1}-z_{2},r+s\tau+2\eta(l_{2}-\tilde{l}_{2}))^{l_{1},\tilde{l}_{2}}_{l_{1}+\tilde{l}_{2}-m_{2},m_{2}} \times u_{l_{1}+\tilde{l}_{2}-m_{2},m_{2},\tilde{m}_{n}} (r+s\tau+2\eta(\tilde{m}_{n}-m_{n})).$$

Now (65) follows from identities (32) and the condition C_n for the function $u(\lambda)$.

A similar consideration shows that the Lemma holds also for j = n - 1.

Lemma 39. Let $u(\lambda)$ be a meromorphic function with values in $V_{\vec{\Lambda}}$. Then for any j = 1, ..., n, the function $\Delta u(\lambda)$ is holomorphic and obeys C_j with respect to the vector $(\Lambda_n, \Lambda_1, \Lambda_2, ..., \Lambda_{n-1})$ at the point $(z_n + p, z_1, ..., z_{n-1})$ if and only if the function $u(\lambda)$ is holomorphic and obeys C_{j-1} with respect to the vector $(\Lambda_1, \Lambda_2, ..., \Lambda_n)$ at the point $(z_1, z_2, ..., z_n)$ (where $C_0 = C_n$).

The Lemma is proved by direct verification.

The proof of Theorem 34 can be concluded. It follows from Lemma 36 that if $u(\lambda)$ is holomorphic and obeys the conditions $C_1, ..., C_n$, then the functions $s_j(z_j - z_{j+1})u(\lambda)$ are holomorphic and obey $C_1, ..., C_n$. Using the fact that $K_j(z_1, ..., z_n)$ can be expressed through $s_k(z)$ and Δ (Lemma 35), we deduce that the functions $K_j(z_1, ..., z_n, \tau, p)u(\lambda)$ are holomorphic and obey $C_1, ..., C_n$.

Now we prove the "conversely" part. By Lemma 36, the function $u(\lambda)$ obey conditions $C_1, ..., C_{n-1}$ with respect to the vector $(\Lambda_1, ..., \Lambda_n)$ at the point $(z_1, ..., z_n)$. We have $K_2(z_1, ..., z_n, \tau, p) = s_1(z_2 - z_1 + p)\Delta s_{n-1}(z_2 - z_n)...s_2(z_2 - z_3)$. For j = 2, ..., n - 1, set $v_j(\lambda) = s_j(z_2 - z_{j+1})...s_2(z_2 - z_3)u(\lambda)$. By Lemmas 36 and 37, the functions $v_2(\lambda), ..., v_{n-1}(\lambda)$ are holomorphic and the function $v_{n-1}(\lambda)$ obeys $C_1, ..., C_{n-1}$ with respect to the vector $(\Lambda_1, \Lambda_3, ..., \Lambda_n, \Lambda_2)$ at the point $(z_1, z_3, ..., z_n, z_2)$. By Lemma 39, the function $\Delta v_{n-1}(\lambda)$ is holomorphic and obeys $C_2, ..., C_n$ with respect to the vector $(\Lambda_2, \Lambda_1, \Lambda_3, ..., \Lambda_n)$ at the point $(z_2+p, z_1, z_3, ..., z_n)$. The function $s_1(z_2-z_1+p)\Delta v_{n-1}(\lambda)$ is holomorphic. By Lemma 36, the function $\Delta v_{n-1}(\lambda)$ obeys C_1 with respect to the vector $(\Lambda_2, \Lambda_1, \Lambda_3, ..., \Lambda_n)$ at the point $(z_2+p, z_1, z_3, ..., z_n)$. By Lemma 39, the function $v_{n-1}(\lambda)$ obeys C_n with respect to the vector $(\Lambda_1, \Lambda_3, ..., \Lambda_n)$ at the point $(z_1, z_3, ..., z_n, \Lambda_2)$ at the point $(z_1, z_3, ..., z_n, z_2)$. By Lemma 39, the function $v_{n-1}(\lambda)$ obeys C_n with respect to the vector $(\Lambda_1, \Lambda_3, ..., \Lambda_n, \Lambda_2)$ at the point $(z_1, z_3, ..., z_n, z_2)$. By Lemma 39, the function $v_{n-2}(\lambda) = s_{n-1}(z_n - z_2)v_{n-1}(\lambda)$ obeys C_n with respect to the vector to the vector $(\Lambda_1, \Lambda_3, ..., \Lambda_{n-1}, \Lambda_2, \Lambda_n)$ at the point $(z_1, z_3, ..., z_{n-1}, z_2, z_n)$. Repeating this procedure we conclude that $u(\lambda)$ obeys C_n with respect to the vector $(\Lambda_1, \ldots, \Lambda_n)$ at the point (z_1, \ldots, z_n) . The "conversely" part is proved.

8. Remarks on Semiclassical Limit of Resonance Relations

We consider the semiclassical limit of the resonance relations in the simplest nontrivial case. Let n = 1, $\vec{\Lambda} = 2m$. The universal hypergeometric function u associated with $\vec{\Lambda}$ is a scalar function. The resonance relations are $u(2\eta a, \mu, \tau, p, \eta, \vec{\Lambda}) = u(-2\eta a, \mu, \tau, p, \eta, \vec{\Lambda})$ for a = 1, ..., m.

Lemma 40. Let $v(\lambda, \eta)$ be a smooth function such that for small η we have $v(2\eta a, \eta) = v(-2\eta a, \eta)$ for a = 1, ..., m. Then

$$\frac{\partial v}{\partial \lambda}(0,0) = \frac{\partial^3 v}{\partial \lambda^3}(0,0) = \dots = \frac{\partial^{2m-1} v}{\partial \lambda^{2m-1}}(0,0) = 0.$$

Moreover, for the anti-symmetric function $Av(\lambda) = v(\lambda) - v(-\lambda)$ we have $\frac{\partial^a Av}{\partial \lambda^a}(0,0) = 0$ for a = 0, 1, ..., 2m.

Proof.

$$2\frac{\partial^{2m-1}v}{\partial\lambda^{2m-1}}(0,0) = \lim_{\eta \to 0} \frac{1}{(2\eta)^{2m-1}} \left(\sum_{k=0}^{k=2m-1} (-1)^k \left(\sum_{k=0}^{2m-1} v(2\eta(m-k),\eta) + \sum_{k=0}^{k=2m-1} (-1)^k \binom{2m-1}{k} v(2\eta(m-k-1),\eta)\right) \right)$$

It is easy to see that the expression in the parentheses can be written as $\sum_{a=1}^{m} c_a(v(2\eta a, \eta) - v(-2\eta a, \eta))$ for suitable coefficients c_a . Thus the expression is zero. The other equalities to zero are proved similarly.

The semiclassical limit of the qKZB equations is the limit $\eta, p \to 0$ with $p/\eta = -2\kappa$ fixed. In this limit the qKZB equations after a suitable normalization turn into the KZB differential equations [FW, FV5]. In the classical case, in addition to the KZB equations, that are associated to the variation of the marked points on the elliptic curve, one also has an equation associated to the variation of the modulus τ of the elliptic curve. This additional differential equation, compatible with the KZB equations, is called the KZB *heat equation* (its difference version is suggested in [FV3]). For example, if n = 1 and $\vec{\Lambda} = 2m$, then the heat differential equation takes the form

(66)
$$2\pi i\kappa \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2} - m(m+1)\,\wp(\lambda,\tau)u$$

where \wp is the Weierstrass function. The Weierstrass function is an even function of λ with second order pole at $\lambda = 0$. The solutions of the heat equation in the neighborhood of $\lambda = 0$ are meromorphic functions. The Weyl reflection $\lambda \to -\lambda$ preserves the equation and acts on the space of solutions. If m is even, then a λ -odd solution has a λ -pole of order m at the origin and a λ -even solution has zero of order m+1. If m is odd, then a λ -even solution has a λ -pole of order m and a λ -odd solution has zero of order m+1. Change the variable in (66), $w = \theta(\lambda, \tau)^m u$. Then the Weyl reflection still acts on the space of solutions of the transformed heat equation. Any λ -meromorphic solution of the transformed heat equation is now regular at the origin. Its Taylor expansion has the form $w(\lambda, \tau) = \sum_{k=0}^{m-1} w_{2k}(\tau)\lambda^{2k} + \sum_{k>2m} w_k(\tau)\lambda^k$, and the Taylor expansion of a λ -odd solution of the transformed equation has zero of order 2m + 1.

It is expected that the semiclassical limit of the universal hypergeometric function $u(\lambda, \mu, \tau, p, \eta, \vec{\Lambda})$ after multiplication by a suitable function of τ will give solutions of the transformed heat equation (depending on μ as a parameter). Lemma 40 shows that the semiclassical limit of resonance relations turn into the regularity properties of the solutions of the heat equation.

Equation (66) can be considered as the equation for horizontal sections of a connection over the upper half plane whose fiber is the space of functions of λ . If κ is an integer, then the connection has an invariant finite dimensional subbundle of conformal blocks coming from conformal field theory. The subbundle consists of Weyl anti-symmetric theta functions satisfying the vanishing conditions that we find in the semiclassical limit of our resonance relations, see [FG], [FW], [EFK].

9. The Weyl reflection

9.1. The Weyl reflection, R-matrices and qKZB operators. For a positive integer Λ , let $L_{\Lambda}(z)$ be the $\Lambda + 1$ dimensional $E_{\tau,\eta}(sl_2)$ evaluation module with the standard basis e_j , $j = 1, ..., \Lambda$. Introduce a new basis E_j by $E_j = e_j/[j]!$ where [j]! is the elliptic

factorial. Define an involution

 $s_{\Lambda} : L_{\Lambda} \to L_{\Lambda}, \qquad E_j \mapsto E_{\Lambda-j}, \qquad \text{for} \qquad j = 0, ..., \Lambda.$

Theorem 41. For natural numbers Λ_1, Λ_2 , the *R*-matrix $R_{\Lambda_1,\Lambda_2}(z_1-z_2,\lambda) \in \text{End} (L_{\Lambda_1}(z_1) \otimes L_{\Lambda_1}(z_2))$ obeys

(67)
$$R_{\Lambda_1,\Lambda_2}(z_1-z_2,-\lambda) \ s_{\Lambda_1} \otimes s_{\Lambda_2} = s_{\Lambda_1} \otimes s_{\Lambda_2} \ R_{\Lambda_1,\Lambda_2}(z_1-z_2,\lambda) \,.$$

Let $\vec{\Lambda} = (\Lambda_1, ..., \Lambda_n)$ be a vector of natural numbers. Consider the qKZB equations with values in $L_{\vec{\Lambda}}[0]$ and the corresponding qKZB operators $K_i(z, \tau, p)$ acting on the space $\mathcal{F}(L_{\vec{\Lambda}}[0])$ of meromorphic functions of $\lambda \in \mathbb{C}$ with values in the zero weight space $L_{\vec{\Lambda}}[0]$.

Define the Weyl reflection $S : \mathcal{F}(L_{\vec{\Lambda}}[0]) \to \mathcal{F}(L_{\vec{\Lambda}}[0])$ by

(68)
$$(Sf)(\lambda) = s_{\Lambda_1} \otimes \ldots \otimes s_{\Lambda_n} f(-\lambda).$$

Theorem 42. For all *i*, the qKZB operators obey the relation

(69)
$$S K_i(z,\tau,p) = K_i(z,\tau,p) S.$$

The Theorem is a direct corollary of (67) and the obvious relation $S\Gamma_i = \Gamma_i S$.

Corollary 43. If u is a solution of the qKZB equations, then Su is.

9.2. Proof of Theorem 41. The Theorem is proved by induction on highest weights Λ_1, Λ_2 . First we prove some properties of R-matrices.

Let V, W be representations of $E_{\tau,\eta}(sl_2)$ with L-operators $L_V \in \text{End} (\mathbb{C}^2 \otimes V)$ and $L_W \in \text{End} (\mathbb{C}^2 \otimes W)$. An R-matrix $R_{V,W}(\lambda) \in \text{End} (V \otimes W)$ is an operator depending on λ and such that $R_{V,W}(\lambda)P_{V,W}$ is an $E_{\tau,\eta}(sl_2)$ intertwiner from $W \otimes V$ to $V \otimes W$, thus

$$R_{VW}(\lambda)^{(23)} P_{VW} L_W(z,\lambda-2\eta h^{(3)})^{(12)} L_V(z,\lambda)^{(13)} = L_V(z,\lambda-2\eta h^{(3)})^{(12)} L_W(z,\lambda)^{(13)} R_{VW}(\lambda-2\eta h^{(1)})^{(23)} P_{VW}.$$

So the relation determining R is

$$R_{VW}(\lambda)^{(23)}L_W(z,\lambda-2\eta h^{(2)})^{(13)}L_V(z,\lambda)^{(12)} = L_V(z,\lambda-2\eta h^{(3)})^{(12)}L_W(z,\lambda)^{(13)}R_{VW}(\lambda-2\eta h^{(1)})^{(23)}.$$

Lemma 44. Let V_1, V_2, V_3 be representations of $E_{\tau,\eta}(sl_2)$. Then

$$R_{V_1, V_2 \otimes V_3}(\lambda) = R_{V_1, V_2}(\lambda - 2\eta h^{(3)})^{(12)} R_{V_1, V_3}(\lambda)^{(13)}$$

is an *R*-matrix for representations V_1 , $V_2 \otimes V_3$ and

$$R_{V_1 \otimes V_2, V_3}(\lambda) = R_{V_2, V_3}(\lambda)^{(23)} R_{V_1, V_3}(\lambda - 2\eta h^{(2)})^{(13)}$$

is an *R*-matrix for representations $V_1 \otimes V_2$, V_3 .

Proof. We prove the first relation.

$$L_{V_{1}}(z,\lambda-2\eta(h^{(3)}+h^{(4)}))^{(12)}L_{V_{2}\otimes V_{3}}(z,\lambda)^{(1,34)}R_{V_{1},V_{2}\otimes V_{3}}(\lambda-2\eta h^{(1)})^{(2,34)}$$

$$=L_{V_{1}}(z,\lambda-2\eta(h^{(3)}+h^{(4)}))^{(12)}L_{V_{2}}(z,\lambda-2\eta h^{(4)})^{(13)}L_{V_{3}}(z,\lambda)^{(14)}$$

$$\times R_{V_{1},V_{2}}(\lambda-2\eta(h^{(1)}+h^{(4)})^{(23)}R_{V_{1},V_{3}}(\lambda-2\eta h^{(1)})^{(24)}$$

$$=L_{V_{1}}(z,\lambda-2\eta(h^{(3)}+h^{(4)}))^{(12)}L_{V_{2}}(z,\lambda-2\eta h^{(4)})^{(13)}$$

$$\times R_{V_{1},V_{2}}(\lambda-2\eta(h^{(1)}+h^{(4)})^{(23)}L_{V_{3}}(z,\lambda)^{(14)}R_{V_{1},V_{3}}(\lambda-2\eta h^{(1)})^{(24)}$$

At this point it is easy to move the *R*-matrices to the left, with the desired result. \Box

For a positive integer Λ , let $L_{\Lambda}(z)$ be the $\Lambda + 1$ dimensional $E_{\tau,\eta}(sl_2)$ evaluation module with the basis E_j , $j = 1, ..., \Lambda$. Then Theorem 10 in [FV1] says that the map

$$j_{\Lambda}: L_{\Lambda+1}(z) \to L_1(z-\eta\Lambda) \otimes L_{\Lambda}(z+\eta), \qquad E_k \mapsto \sum_l E_l \otimes E_{k-l}$$

define an embedding of $E_{\tau,\eta}(sl_2)$ modules.

Lemma 45. For natural numbers Λ_1, Λ_2 , let $R_{\Lambda_1,\Lambda_2}(z,\lambda) \in \text{End} (L_{\Lambda_1} \otimes L_{\Lambda_2})$ be the *R*-matrix constructed in Sec. 3. Then the *R*-matrix $R_{\Lambda_1,\Lambda_2}(z,\lambda)$ obeys the recursion relations

$$j_{\Lambda_1} \otimes 1 R_{\Lambda_1+1,\Lambda_2}(z,\lambda) = R_{\Lambda_1,\Lambda_2}(z+\eta,\lambda)^{(23)} R_{1,\Lambda_2}(z-\eta\Lambda_1,\lambda-2\eta h^{(2)})^{(13)} j_{\Lambda_1} \otimes 1, 1 \otimes j_{\Lambda_2} R_{\Lambda_1,\Lambda_2+1}(z,\lambda) = R_{\Lambda_1,1}(z+\eta\Lambda_2,\lambda-2\eta h^{(3)})^{(12)} R_{\Lambda_1,\Lambda_2}(z-\eta,\lambda)^{(13)} 1 \otimes j_{\Lambda_2}.$$

Proof. The Lemma follows from Lemma 44 and the normalization property $R_{\Lambda,M}(z,\lambda) E_0 \otimes E_0 = E_0 \otimes E_0$.

Now Theorem 41 is proved by induction in Λ_1 , Λ_2 . For $\Lambda_1 = \Lambda_2 = 1$ it is checked by direct calculation. The general case follows using

$$(s_1 \otimes s_\Lambda) \circ j_\Lambda = j_{\Lambda+1} \circ s_{\Lambda+1}$$
.

10. Vanishing conditions for hypergeometric solutions with values in finite dimensional modules

10.1. Fusion rules for sl_2 . For a non-negative integer a, let L_a denote the finite dimensional sl_2 -module with highest weight a. Let a, b, c be non-negative integers. We say that a, b, c obey the fusion rules for sl_2 if the sl_2 -module $L_a \otimes L_b$ contains L_c as a submodule. It means that $a - b \in \{-c, -c + 2, ..., c - 2, c\}$ and $c \leq a + b$. If the triple a, b, c obeys the fusion rules, then each of its its permutation does.

Let $\Lambda = (\Lambda_1, ..., \Lambda_n)$ be an *n*-tuple of positive integers. Let $a = (a_1, ..., a_n)$ be an *n*-tuple of integers. We say that *a* obeys the fusion rules for sl_2 with respect to $\vec{\Lambda}$, if $a_1, ..., a_n$ are non-negative and for every j = 1, ..., n, the triple a_{j-1}, a_j, Λ_j obeys the fusion rules. Here we assume that $a_0 = a_n$.

Define the weight vector, $w(a) = (w_1, ..., w_n)$, of an *n*-tuple *a* by $w_j = a_j - a_{j-1}$. The *n*-tuple *a* obeys the fusion rules if and only if for every *j*, we have $w_j \in \{-\Lambda_j, -\Lambda_j + 2, ..., \Lambda_j - 2, \Lambda_j\}$ and $\Lambda_j \leq a_j + a_{j-1}$.

If $(a_1, ..., a_n)$ obeys the fusion rules with respect to $\vec{\Lambda}$, then $(a_1 + 1, ..., a_n + 1)$ also obeys them.

Let $w = (w_1, ..., w_n)$ be an *n*-tuple of integers such that $w_1 + ... + w_n = 0$ and $w_j \in \{-\Lambda_j, -\Lambda_j + 2, ..., \Lambda_j - 2, \Lambda_j\}$ for all j = 1, ..., n. We call w a weight vector. The vector -w will be called the weight vector dual to w.

For a weight vector w, define an *n*-tuple $\Sigma(w) = (\Sigma^1, ..., \Sigma^n)$ by the rule $\Sigma^j = \sum_{l=1}^j w_j$. Note that $\Sigma^n = 0$.

It is clear that there exists a (non-negative) integer k(w) such that for every integer k the *n*-tuple $(\Sigma^1 + k, ..., \Sigma^n + k)$ obeys the fusion rules with respect to $\vec{\Lambda}$ if and only if $k \ge k(w)$. The integer k(w) will be called *the shift number* associated with a weight vector w. Every *n*-tuple obeying the fusion rules has the form $\Sigma(w) + (k, ..., k)$ for a suitable weight vector w and a non-negative integer k.

10.2. Vanishing conditions and fusion rules. Let $\vec{\Lambda} = (\Lambda_1, ..., \Lambda_n)$ be a vector of natural numbers, $\Lambda_1 + ... + \Lambda_n = 2m$, and let $v(z, \lambda)$ be a hypergeometric solution of the qKZB equations with values in $V_{\vec{\Lambda}}[0]$ in the sense of Sec. 5.3. Write the solution in coordinates,

$$v(z,\lambda) = \sum_{M, |M|=m} v_M(z,\lambda) E_M$$

where $\{E_M\}$ is the reduced basis in $V_{\vec{\Lambda}}[0]$. The following Theorem describes the resonance relations for admissible coordinates of the hypergeometric solution.

Let $M = (m_1, ..., m_n)$ be an admissible index, $m_j \leq \Lambda_j$ for all j. Let $w_M = (\Lambda_1 - 2m_1, ..., \Lambda_n - 2m_n)$ be the \mathfrak{h} -weight vector of the basis vector E_M and $-w_M$ the weight vector dual to w_M . The weight vector $-w_M$ is the \mathfrak{h} -weight of the basis vector $E_{s(M)}$, $s(M) = (\Lambda_1 - m_1, ..., \Lambda_n - m_n)$. Let $k(w_M)$ and $k(-w_M)$ be the shift numbers defined in Sec. 10.1.

Consider the resonance relations of Corollary 33 involving the coefficient $v_M(z, \lambda)$. Since $\vec{\Lambda}$ is a vector of natural numbers, in any such a relation we have $\lambda = 2\eta k$ for a suitable integer k.

Theorem 46.

Let $M = (m_1, ..., m_n)$ be an admissible index.

- I. If $k > k(-w_M)$ or $k < -k(w_M)$, then there is no resonance relation of Corollary 33 involving $v_M(z, 2\eta k)$.
- II. If $k \in [-k(w_M), k(-w_M)]$, then the resonance relations of Corollary 33 imply the relation

$$v_M(z, 2\eta k) = v_{s(M)}(z, -2\eta k).$$

Consider a hypergeometric solution $\tilde{v}(z,\lambda)$ of the qKZB equations with values in $L_{\vec{\Lambda}}[0]$ in the sense of Sec. 5.3. Any such a solution has the form $\tilde{v} = \pi v$, where v is a hypergeometric solution of the qKZB equations with values in $V_{\vec{\Lambda}}[0]$ and $\pi : V_{\vec{\Lambda}}[0] \to L_{\vec{\Lambda}}[0]$ is the canonical projection. Consider the anti-symmetrization,

 $A\tilde{v}(z,\lambda) = \tilde{v}(z,\lambda) - S\tilde{v}(z,\lambda) = \tilde{v}(z,\lambda) - s_{\Lambda_1} \otimes \ldots \otimes s_{\Lambda_n} \tilde{v}(z,-\lambda),$

of \tilde{v} with respect to the Weyl reflection. By Theorem 41, $A\tilde{v}$ is a solution of the qKZB equations. Write $A\tilde{v}$ in coordinates,

$$A\tilde{v}(z,\lambda) = \sum_{\operatorname{adm} M, |M|=m} A\tilde{v}_M(z,\lambda) E_M$$

Corollary 47. (Vanishing Conditions) For any (admissible) index M and any integer $k \in [-k(w_M), k(-w_M)]$, we have

$$A \tilde{v}_M(z, 2\eta k) = 0$$
.

In other words, let $M = (m_1, ..., m_n)$ be an admissible index. Let $w = (w_1, ..., w_n)$, $w_j = \Lambda_j - 2m_j$, be the \mathfrak{h} -weight of the basis vector E_M . Form the sums $\Sigma^j = \sum_{i=1}^j w_j$. Let k be a non-negative integer. Corollary 47 says that $A\tilde{v}_M(z, 2\eta k) = 0$ unless the vector

$$(-\Sigma^{1} + k - 1, -\Sigma^{2} + k - 1, ..., -\Sigma^{n} + k - 1)$$

satisfies the sl_2 fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$ and $A\tilde{v}_M(z, -2\eta k) = 0$ unless the vector

 $(\Sigma^1+k-1,\Sigma^2+k-1,...,\Sigma^n+k-1)$

satisfies the sl_2 fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$. In particular, $A\tilde{v}_M(z, 0) = 0$ since $\Sigma^n = 0$.

Remark. Corollary 47 has an analog for the case when n = 1, $\vec{\Lambda} = 2m$. Namely, let $u = u_{mm}(\lambda, \mu, \tau, p)e_m \otimes e_m \in V_{\vec{\Lambda}}[0] \otimes V_{\vec{\Lambda}}[0]$ be the corresponding universal hypergeometric function. Define the Weyl anti-symmetric function $Au(\lambda, \mu, \tau, p) = (u_{mm}(\lambda, \mu, \tau, p) - u_{mm}(-\lambda, \mu, \tau, p))e_m \otimes e_m$. Then for all k = -m, -m + 1, ..., m - 1, m we have

$$Au(2\eta k, \mu, \tau, p) = 0$$

see Part III of Theorem 26.

10.3. Proof of Theorem 46.

Lemma 48. If $k > k(-w_M)$ or $k < -k(w_M)$, then there is no resonance relation of Corollary 33 involving $v_M(z, 2\eta k)$.

Proof. For a given integer k, a relation involving $v_M(z, 2\eta k)$ exists, if for some $j \in [1, n]$, there is $a \in [0, m_j + m_{j+1}] - \{m_j\}$ such that

(70)
$$\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + \Lambda_j - m_j - a = k.$$

Let $k > k(-w_M)$. It means that for all j = 1, ..., n we have

$$\left(-\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + k\right) + \left(-\sum_{l=1}^{j} (\Lambda_l - 2m_l) + k\right) > \Lambda_j.$$

Here $m_{n+1} = m_1$. This is equivalent to

$$k > \sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + \Lambda_j - m_j$$

This inequality contradicts (70).

Let $k < -k(w_M)$. It means that for all j = 1, ..., n we have

$$\left(\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) - k\right) + \left(\sum_{l=1}^{j} (\Lambda_l - 2m_l) - k\right) > \Lambda_j.$$

This is equivalent to

$$k < \sum_{l=1}^{j-2} (\Lambda_l - 2m_l) + \Lambda_{j-1} - m_{j-1} - (m_{j-1} + m_j).$$

This inequality contradicts (70).

Lemma 49. If $k \in [-k(w_M), k(-w_M)]$, then either there is a resonance relation of Corollary 33 involving $v_M(z, 2\eta k)$ or k = 0, all Λ_j , j = 1, ..., n, are even, $M = (\Lambda_1/2, ..., \Lambda_n/2)$, s(M) = M and we have the statement of Theorem 46, $v_M(z, 0) = v_{s(M)}(z, 0)$.

Proof. There are only four cases. We prove the Lemma for each of them.

1. For all $j \in [1, n]$,

(71)
$$\sum_{l=1}^{j} (\Lambda_l - 2m_l) < k.$$

2. For all $j \in [1, n]$,

(72)
$$\sum_{l=1}^{j} (\Lambda_l - 2m_l) > k.$$

3. There is $j \in [1, n]$, such that

$$\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) = k.$$

4. There is $j \in [1, n]$, such that

$$\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) > k > \sum_{l=1}^{j} (\Lambda_l - 2m_l).$$

In case 1, we have $0 < k \leq k(-w_M)$. This means that there is $j \in [1, n]$ such that

$$\left(-\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + k\right) + \left(-\sum_{l=1}^{j} (\Lambda_l - 2m_l) + k\right) \le \Lambda_j,$$

which can be rewritten as

$$k \leq \sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + \Lambda_j - m_j.$$

Combining with (71) we get

$$\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + \Lambda_j - 2m_j < k \leq \sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + \Lambda_j - m_j.$$

Hence there is $a \in [0, m_j)$ such that

$$\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + \Lambda_j - m_j - a = k.$$

Then there is a relation involving $v_M(z, 2\eta k)$ corresponding to this equation.

In case 2, we have $-k(w_M) \leq k < 0$. This means that there is $j \in [1, n]$ such that

$$(\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) - k) + (\sum_{l=1}^{j} (\Lambda_l - 2m_l) - k) \le \Lambda_j,$$

which can be rewritten as

$$k \geq \sum_{l=1}^{j-2} (\Lambda_l - 2m_l) + \Lambda_{j-1} - m_{j-1} - (m_{j-1} + m_j).$$

Combining with (72) we get

$$\sum_{l=1}^{j-2} (\Lambda_l - 2m_l) + \Lambda_{j-1} - 2m_{j-1} > k \ge \sum_{l=1}^{j-2} (\Lambda_l - 2m_l) + \Lambda_{j-1} - m_{j-1} - (m_{j-1} + m_j).$$

Hence there is $a \in (m_{j-1}, m_{j-1} + m_j]$ such that

$$\sum_{l=1}^{j-2} (\Lambda_l - 2m_l) + \Lambda_{j-1} - m_{j-1} - a = k.$$

Then there is a relation involving $v_M(z, 2\eta k)$ corresponding to this equation.

In case 3, either k = 0 and $M = (\Lambda_1/2, ..., \Lambda_n/2)$ or there is $j \in [1, n]$ such that

$$\sum_{l=1}^{j} (\Lambda_l - 2m_l) = k > \sum_{l=1}^{j+1} (\Lambda_l - 2m_l).$$

In the last case, we have $0 \leq \Lambda_{j+1} - m_{j+1} < m_{j+1}$ and a relation of Corollary 33 corresponding to the equation

$$\sum_{l=1}^{j} (\Lambda_l - 2m_l) + \Lambda_{j+1} - m_{j+1} - (\Lambda_{j+1} - m_{j+1}) = k.$$

In case 4, there is $a \in [0, m_i)$ such that

$$\sum_{l=1}^{j-2} (\Lambda_l - 2m_l) + \Lambda_{j-1} - m_{j-1} - a = k,$$

and then there is a relation of Corollary 33 corresponding to this equation.

Lemma 50. Let $k \in [-k(w_M), k(-w_M)]$ and for all $j \in [1, n]$,

(73)
$$\sum_{l=1}^{j} (\Lambda_l - 2m_l) < k.$$

Then there is a sequence of relations of Corollary 33 implying $v_M(z, 2\eta k) = v_{s(M)}(z, -2\eta k)$.

Proof. By Lemma 49, there are $j \in [1, n]$ and $a \in [0, m_j + m_{j+1}] - \{m_j\}$ such that

(74)
$$\sum_{l=1}^{j-1} (\Lambda_l - 2m_l) + \Lambda_j - m_j - a = k.$$

Here $a = m_j + \sum_{l=1}^{j} (\Lambda_l - 2m_l) - k$. This means that the transformation $T_j(k)$ can be applied to $v_M(z, 2\eta k)$, see Sec. 6.5. We prove that the product of transformations $T_{j-1}(-k)...T_2(-k)$ $T_1(-k)$ $T_n(k)$... $T_{j+1}(k)T_j(k)$ can be applied to $v_M(z,2\eta k)$ and gives $v_{s(M)}(z, -2\eta k).$

We assume that $1 \leq j < n$, the proof for j = n is similar. For all *i*, define Σ^i by $\Sigma^i = \sum_{l=1}^i (\Lambda_l - 2m_l)$. Equation (74) implies a relation

$$v_M(z, 2\eta k) = v_{(m_1, \dots, m_{j-1}, m_j + \Sigma^j - k, m_{j+1} - \Sigma^j + k, m_{j+2}, \dots, m_n)}(z, 2\eta k).$$

We have an equation

$$\Sigma^{j-1} + \Lambda_j - 2(m_j + \Sigma^j - k) + \Lambda_{j+1} - (m_{j+1} - \Sigma^j + k) - (\Lambda_{j+1} - m_{j+1}) = k,$$

and assumption (73) implies

(76)
$$(\Lambda_{j+1} - m_{j+1}) < m_{j+1} - \Sigma^j + k .$$

Formulas (75) and (76) imply a relation

 $v_{(m_1,\dots,m_{j-1},m_j+\Sigma^j-k,m_{j+1}-\Sigma^j+k,m_{j+2},\dots,m_n)}(z, 2\eta k) = v_{(m_1,\dots,m_{j-1},m_j+\Sigma^j-k,\Lambda_{j+1}-m_{j+1},m_{j+2}-\Sigma^{j+1}+k,m_{j+3},\dots,m_n)}(z, 2\eta k).$

Now we continue the same way and get

$$v_M(z, 2\eta k) = v_{(m_1, \dots, m_{j-1}, m_j + \Sigma^j - k, \Lambda_{j+1} - m_{j+1}, \dots, \Lambda_{n-1} - m_{n-1}, m_n - \Sigma^{n-1} + k)}(z, 2\eta k).$$

We have an equation

(77)
$$(m_n - \Sigma^{n-1} + k) - (\Lambda_n - m_n) = k$$

and assumption (73) implies

(78)
$$(m_n - \Sigma^{n-1} + k) - (\Lambda_n - m_n) > 0.$$

Formulas (77) and (78) imply a relation

$$v_M(z, 2\eta k) = v_{(m_1+k, m_2, \dots, m_{j-1}, m_j + \Sigma^j - k, \Lambda_{j+1} - m_{j+1}, \dots, \Lambda_n - m_n)}(z, -2\eta k).$$

We have an equation

(79)
$$\Lambda_1 - (m_1 + k) - (\Lambda_1 - m_1) = -k,$$

and assumption (73) implies

(80)
$$(m_1 + k) - (\Lambda_1 - m_1) > 0.$$

Formulas (79) and (80) imply a relation

$$v_M(z, 2\eta k) = v_{(\Lambda_1 - m_1, m_2 + k - \Sigma^1, m_3, \dots, m_{j-1}, m_j + \Sigma^j - k, \Lambda_{j+1} - m_{j+1}, \dots, \Lambda_n - m_n)}(z, -2\eta k).$$

Continuing this way, we get

 $v_M(z, 2\eta k) = v_{(\Lambda_1 - m_1, \dots, \Lambda_{j-2} - m_{j-2}, m_{j-1} + k - \Sigma^{j-2}, m_j + \Sigma^j - k, \Lambda_{j+1} - m_{j+1}, \dots, \Lambda_n - m_n)}(z, -2\eta k),$ and finally

$$v_M(z, 2\eta k) = v_{s(M)}(z, -2\eta k).$$

Thus, starting from a relation, corresponding to a reconstruction of the pair of j-th and j + 1-st indices, then moving the total circle to the right and applying the relations, corresponding to reconstruction of pairs of indices (j + 1, j + 2), ..., (n, 1), ..., (j - 1, j), we proved the Lemma.

Lemma 51. Let $k \in [-k(w_M), k(-w_M)]$ and for all $j \in [1, n]$,

(81)
$$\sum_{l=1}^{j} (\Lambda_l - 2m_l) > k.$$

Then there is a sequence of relations of Corollary 33 implying $v_M(z, 2\eta k) = v_{s(M)}(z, -2\eta k)$.

Proof. The proof of this Lemma is similar to the proof of Lemma 50, but now we move over the total circle to the left.

Namely, by Lemma 49, there are $j \in [1, n]$ and $a \in [0, m_j + m_{j+1}] - \{m_j\}$ such that equation (74) is satisfied. This means that the transformation $T_j(k)$ can be applied to $v_M(z, 2\eta k)$.

We prove that the product of transformations $T_{j+1}(-k)...T_{n-1}(-k) T_n(k)T_1(k)T_2(k)$... $T_{j-1}(k)T_j(k)$ can be applied to $v_M(z, 2\eta k)$ and gives $v_{s(M)}(z, -2\eta k)$.

We assume that $1 < j \le n$, the proof for j = 1 is similar.

Equation (74) implies a relation

$$v_M(z, 2\eta k) = v_{(m_1, \dots, m_{j-1}, m_j + \Sigma^j - k, m_{j+1} - \Sigma^j + k, m_{j+2}, \dots, m_n)}(z, 2\eta k).$$

We also have a relation

(82)
$$v_{(m_1,...,m_{j-1},m_j+\Sigma^j-k,m_{j+1}-\Sigma^j+k,m_{j+2},...,m_n)}(z, 2\eta k) = v_{(m_1,...,m_{j-2},m_{j-1}+\Sigma^{j-1}-k,\Lambda_j-m_j,m_{j+1}-\Sigma^j+k,m_{j+2},...,m_n)}(z, 2\eta k),$$

since we have equations

$$\Sigma^{j-2} + \Lambda_{j-1} - m_{j-1} - (m_{j-1} + \Sigma^{j-1} - k) = k,$$

$$m_{j-1} + (m_j + \Sigma^j - k) = (m_{j-1} + \Sigma^{j-1} - k) + (\Lambda_j - m_j),$$

and an inequality

$$m_j + \Sigma^j - k > \Lambda_j - m_j$$

implied by assumption (81).

We continue the same way and get

$$v_M(z, 2\eta k) = v_{(m_1 + \Sigma^1 - k, \Lambda_2 - m_2, \dots, \Lambda_j - m_j, m_{j+1} - \Sigma^j + k, m_{j+2}, \dots, m_n)}(z, 2\eta k)$$

Similarly we have a relation

$$v_M(z, 2\eta k) = v_{(\Lambda_1 - m_1, \dots, \Lambda_j - m_j, m_{j+1} - \Sigma^j + k, m_{j+2}, \dots, m_{n-1}, m_n - k)}(z, -2\eta k),$$

then relations

$$v_M(z, 2\eta k) = v_{(\Lambda_1 - m_1, \dots, \Lambda_j - m_j, m_{j+1} - \Sigma^j + k, m_{j+2}, \dots, m_{n-2}, m_{n-1} + \Sigma^{n-1} - k, \Lambda_n - m_n)}(z, -2\eta k),$$

$$v_M(z, 2\eta k) = v_{(\Lambda_1 - m_1, \dots, \Lambda_j - m_j, m_{j+1} - \Sigma^j + k, m_{j+2} + \Sigma^{j+2} - k, \Lambda_{j+3} - m_{j+3}, \dots, \Lambda_n - m_n)}(z, -2\eta k),$$

and finally

$$v_M(z, 2\eta k) = v_{s(M)}(z, -2\eta k).$$

The Lemma is proved.

In order to finish the proof of Theorem 46 we assume that

$$\min\{\Sigma^1, ..., \Sigma^n\} \le k \le \max\{\Sigma^1, ..., \Sigma^n\}$$

and prove that

$$v_M(z, 2\eta k) = v_{s(M)}(z, -2\eta k).$$

Considering the resonance relations it is convenient to think of the factors of the tensor product $V_{\vec{\Lambda}}$ as positioned in the cyclic order so that the factor V_{Λ_1} follows the factor V_{Λ_n} . Therefore, considering the index of a factor we shall consider it modulo n so that the n+l-th factor is the same as the l-th factor. The sums Σ^j introduced above correspond to this convention since $\Sigma^n = 0$.

We call $j \in [1, n]$ a distinguished vertex if $\Sigma^j = k$.

We define top and bottom intervals.

A top interval is a sequence of vertices r, r+1, ..., s such that $\Sigma^r, \Sigma^{r+1}, ..., \Sigma^s > k$ and $\Sigma^{r-1}, \Sigma^{s+1} \leq k$. Here r can be equal to s.

A bottom interval is a sequence of vertices r, r+1, ..., s such that $\Sigma^r, \Sigma^{r+1}, ..., \Sigma^s < k$ and $\Sigma^{r-1}, \Sigma^{s+1} \geq k$. Here r can be equal to s.

In each of the two cases, the vertices r, r + 1, ..., s are called the vertices of the corresponding interval.

Each not distinguished vertex is a vertex of a unique top or bottom interval.

We define notions of the boundary and proper intervals. A top or bottom interval r, r+1, ..., s is called *the boundary interval* if $r \leq n \leq s$. All other top or bottom intervals are called *the proper intervals*.

There is no boundary interval if and only if k = 0.

Our goal is to transform the coordinates of the index $M = (m_1, ..., m_n)$ into the coordinates of the index $s(M) = (\Lambda_1 - m_1, ..., \Lambda_n - m_n)$. We construct a specially ordered product of transformations T_i , corresponding to not distinguished vertices, which can be applied to $v_M(z, 2\eta k)$ and makes the transformation.

First we associate a product T^{I} of transformations T_{i} to each top and bottom interval $I = \{r, r+1, ..., s\}.$

Namely, let I be a proper top interval, then $T^{I} = T_{r}(k)T_{r+1}(k)...T_{s}(k)$.

Let *I* be the boundary top interval, then $T^I = T_r(-k)T_{r+1}(-k)...T_{n-1}(-k)T_n(k)...T_s(k)$. Let *I* be a proper bottom interval, then $T^I = T_s(k)...T_{r+1}(k)T_r(k)$.

Let I be the boundary bottom interval, then $T^{I} = T_{s}(-k)...T_{n+1}(-k)T_{n}(k)...T_{r+1}(k)T_{r}(k)$. We order all top or bottom intervals as follows. If I is boundary and J is proper, then

I < J. If I and J are proper, then I < J if there exist $1 \le i < j \le n$ such that i is a vertex of I and j is a vertex of J.

Let $\{I_1, I_2, ..., I_r\}$ be the set of all top or bottom intervals and $I_1 < I_2 < ... < I_r$. Define the product of transformations T_i corresponding to $v_M(z, 2\eta k)$ by

$$P = T^{I_1} T^{I_2} ... T^{I_r}$$

Lemma 52. The product P of transformations T_i can be applied to $v_M(z, 2\eta k)$ and transforms $v_M(z, 2\eta k)$ to $v_{s(M)}(z, -2\eta k)$.

The Lemma finishes the proof of Theorem 46.

The proof of the Lemma is by direct verification similar to the proofs of Lemmas 50 and 51.

RESONANCE RELATIONS

11. Resonance relations for Bethe Ansatz Eigenfunctions

11.1. Bethe ansatz eigenfunctions of commuting difference operators. In this section we fix $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n) \in \mathbb{C}^n$ such that $\sum \Lambda_l = 2m$ for some positive integer m. Let $K_i(z, \tau, p)$ be the qKZB operators acting on the space $\mathcal{F}(V_{\vec{\Lambda}}[0])$ of meromorphic functions of $\lambda \in \mathbb{C}$ with values in the zero weight space $V_{\vec{\Lambda}}[0]$. The qKZB operators satisfy the compatibility conditions (6).

There is a closely related set of commuting difference operators $H_j(z)$ acting on $\mathcal{F}(V_{\vec{\lambda}}[0])$ and the corresponding eigenvalue problem

$$H_j(z)\psi = \varepsilon_j\psi, \qquad j = 1, \dots, n, \qquad \psi \in \mathcal{F}(V_{\vec{\Lambda}}[0]).$$

Here

(83)
$$H_j(z) = R_{j,j-1}(z_j - z_{j-1}) \cdots R_{j,1}(z_j - z_1) \Gamma_j R_{j,n}(z_j - z_n) \cdots R_{j,j+1}(z_j - z_{j+1}),$$

 $z = (z_1, \ldots, z_n)$ is a fixed generic point in \mathbb{C}^n , ψ is in $\mathcal{F}(V_{\bar{\Lambda}}[0])$ and the operators $R_{j,k}$ are defined in Sec. 2.2. The fact that the operators $H_j(z)$ commute with each other follows from the compatibility of the qKZB operators as $p \to 0$.

In [FTV1] we gave a formula for common quasiperiodic eigenfunctions of the operators H_j , i.e., functions ψ such that $H_j\psi = \varepsilon_j\psi$, $j = 1, \ldots, n$, for some $\varepsilon_j \in \mathbb{C}$. The quasiperiodicity assumption means that we require that $\psi(\lambda + 1) = \mu\psi(\lambda)$ for some multiplier $\mu \in \mathbb{C}^{\times}$.

The formula is given in terms of the weight functions $\omega_{j_1,...,j_n}(t_1,...,t_m,z,\lambda)$ associated with the tensor product $V_{\vec{\lambda}}$ and defined in Sec. 3.3.

Theorem 53. [FTV1] Let $c \in \mathbb{C}$. Suppose that t_1, \ldots, t_m obey the system of "Bethe ansatz" equations

(84)
$$\prod_{l=1}^{n} \frac{\theta(t_j - z_l + \eta \Lambda_l)}{\theta(t_j - z_l - \eta \Lambda_l)} \prod_{k: k \neq j} \frac{\theta(t_j - t_k - 2\eta)}{\theta(t_j - t_k + 2\eta)} = e^{-4\eta c}, \qquad j = 1, \dots, m.$$

Then

(85)
$$\psi(\lambda) = \sum_{J, |J|=m} e^{c\lambda} \omega_J(t_1, \dots, t_m, z, \lambda) e_J$$

is a common eigenfunction of the commuting operators H_j , j = 1, ..., n, with eigenvalues

$$\varepsilon_j = e^{-2c\eta\Lambda_j} \prod_{k=1}^m \frac{\theta(t_k - z_j - \eta\Lambda_j)}{\theta(t_k - z_j + \eta\Lambda_j)}$$

and multiplier $\mu = (-1)^m e^c$. Moreover, if t_1, \ldots, t_m are a solution of (84), and $\sigma \in S_m$ is any permutation, then $t_{\sigma(1)}, \ldots, t_{\sigma(m)}$ are also a solution. The eigenfunctions ψ corresponding to these two solutions are proportional.

11.2. **Resonance relations.** Let $\{E_J\}$ be the reduced basis in $V_{\vec{\Lambda}}[0]$. Let $\psi(\lambda)$ be an eigenfunction defined in Theorem 53 and corresponding to a solution (t_1, \ldots, t_m) of the Bethe ansatz equations. Write the eigenfunction in the reduced bases,

$$\psi(\lambda) = \sum_{J} \psi_{J}(t_{1},\ldots,t_{m},z,\lambda) E_{J},$$

where the reduced coefficients $\psi_J(t_1, \ldots, t_m, \lambda)$ are defined by

$$\psi_J(t_1,\ldots,t_m,z,\lambda) = [j_1]!\ldots[j_n]! e^{c\lambda} \omega_J(t_1,\ldots,t_m,z,\lambda).$$

Theorem 54.

I. Let n > 1. Then the reduced coefficients of an eigenfunction $\psi(\lambda)$ obey the resonance relations

(86)
$$\psi_M(t_1, \dots, t_m, z, 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l))) = \psi_L(t_1, \dots, t_m, z, 2\eta(\Lambda_j - a - b + \sum_{l=1}^{j-1} (\Lambda_l - 2m_l)))$$

for j, a, b, k, L, M defined in Part I of Theorem 26 and the resonance relations

(87)
$$\psi_M(t_1, \dots, t_m, z, 2\eta(a-b)) = \psi_L(t_1, \dots, t_m, z, 2\eta(b-a))$$

for a, b, k, L, M defined in Part II of Theorem 26.

II. Let n = 1, $\vec{\Lambda} = 2m$. In this case $\psi = \psi_m E_m$ and ψ does not depend on z. We claim that

(88)
$$\psi_m(t_1, ..., t_m, 2\eta a, \tau) = \psi_m(t_1, ..., t_m, -2\eta a, \tau)$$

for
$$a = 1, ..., m$$
.

11.3. **Proof of Theorem 54.** The proof of Part II is similar to the proof of Part III of Theorem 26.

The resonance relations in formula (86) follow from Theorem 12. The proof of the resonance relations in formula (87) is similar to the proof of Part II of Theorem 26.

Namely, let S_n act on \mathbb{C}^n by permutations of the coordinates, and let s_j be the transposition of the *j*-th and (j + 1)-st coordinates.

The equations (84) do not depend on the ordering of the parameters z_i, Λ_i . Let us fix a solution t^* of (84) with parameters $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ and $z = (z_1, \ldots, z_n)$. Then for each permutation $\sigma \in S_n$, the point t^* is still a solution of the equations (84) with parameters $\sigma \vec{\Lambda}, \sigma z$ and we have a corresponding function $\psi_{\sigma}(\lambda) \in \mathcal{F}(V_{\sigma \vec{\Lambda}}[0])$ given by the formula (85) with parameters $\sigma \vec{\Lambda}, \sigma z$.

It follows from the definition of R-matrices that

$$\psi_{s_j}(\lambda) = P^{(j,j+1)} R^{(j,j+1)}_{\Lambda_j \Lambda_{j+1}}(z_j - z_{j+1}, \lambda - 2\eta (h^{(1)} + \dots + h^{(j-1)})) \psi(\lambda).$$

Introduce a permutation $s^{\vee} \in S_n$ by $s^{\vee} = s_1 s_2 \dots s_{n-1}$. Then $s^{\vee} \vec{\Lambda} = \vec{\Lambda}^{\vee} = (\Lambda_n, \Lambda_1, \dots, \Lambda_{n-1})$. Introduce an operator $\Delta : V_{\vec{\Lambda}} \to V_{\vec{\Lambda}^{\vee}}$ by formula (49).

Lemma 55.

$$\Delta \psi(\lambda) = \varepsilon_1 \psi_{s^{\vee}}(\lambda).$$

The proof of the Lemma is similar to the argument in the proof of Theorem 30.

The Lemma finishes the proof of Theorem 54 since the resonance relations for the eigenfunction $\psi(\lambda)$ in formula (87) are the resonance relations for the eigenfunction $\psi_{s^{\vee}}(\lambda)$ in formula (86) for j = 1.

11.4. The fusion rules and the resonance relations for the Bethe ansatz eigenfunctions with values in finite dimensional modules. Let $\vec{\Lambda} = (\Lambda_1, ..., \Lambda_n)$ be a vector of natural numbers, $\Lambda_1 + ... + \Lambda_n = 2m$, and let $\psi(\lambda)$ be a Bethe ansatz eigenfunction with values in $V_{\vec{\Lambda}}[0]$ and corresponding to a solution of the Bethe ansatz equations. Write the eigenfunction in coordinates,

$$\psi(\lambda) = \sum_{M, |M|=m} \psi_M(\lambda) E_M$$

where $\{E_M\}$ is the reduced basis in $V_{\vec{\Lambda}}[0]$. The following Theorem describes the resonance relations for admissible coordinates of the eigenfunction.

Let $M = (m_1, ..., m_n)$ be an admissible index, $m_j \leq \Lambda_j$ for all j. Let $w_M = (\Lambda_1 - 2m_1, ..., \Lambda_n - 2m_n)$ be the \mathfrak{h} -weight vector of the basis vector E_M and $-w_M$ the weight vector dual to w_M . Let $k(w_M)$ and $k(-w_M)$ be the shift numbers defined in Sec. 10.1.

Consider the resonance relations of Theorem 54 involving the coefficient $\psi_M(\lambda)$. Since $\vec{\Lambda}$ is a vector of natural numbers, in any such a relation we have $\lambda = 2\eta k$ for a suitable integer k.

Theorem 56.

Let $M = (m_1, ..., m_n)$ be an admissible index.

- I. If $k > k(-w_M)$ or $k < -k(w_M)$, then there is no resonance relation of Theorem 54 involving $\psi_M(2\eta k)$.
- II. If $k \in [-k(w_M), k(-w_M)]$, then the resonance relations of Theorem 54 imply the relation

$$\psi_M(2\eta k) = \psi_{s(M)}(-2\eta k).$$

The proof of this Theorem coincides with the proof of Theorem 46.

Let $L_{\vec{\Lambda}} = L_{\Lambda_1} \otimes ... \otimes L_{\Lambda_n}$ be the corresponding product of the finite dimensional spaces and $\pi : V_{\vec{\Lambda}} \to L_{\vec{\Lambda}}$ the canonical projection. The operators $H_j(z)$ induce on $L_{\vec{\Lambda}}[0]$ a system of commuting difference operators which also will be denoted $H_j(z)$.

Let $\psi(\lambda)$ be the Bethe ansatz eigenfunction with values in $V_{\vec{\Lambda}}[0]$ and corresponding to a solution of the Bethe ansatz equations. Then the function

$$\widetilde{\psi}(\lambda) = \pi \psi(\lambda) = \sum_{\operatorname{adm} M, |M|=m} \psi_M(\lambda) E_M$$

is an eigenfunction of the commuting operators $H_j(z)$ acting on $\mathcal{F}(L_{\vec{\Lambda}}[0]), H_j(z)\tilde{\psi}(\lambda) = \varepsilon_j \tilde{\psi}(\lambda)$, where the eigenvalues ε_j are defined in Theorem 53. The values of $\tilde{\psi}(\lambda)$ obey the resonance relations of Theorem 56.

Let $S : \mathcal{F}(L_{\vec{\Lambda}}[0]) \to \mathcal{F}(L_{\vec{\Lambda}}[0])$ be the Weyl reflection. The Weyl reflection commutes with the operators $H_j(z)$, $S H_j(z) = H_j(z) S$. Hence $S\tilde{\psi}(\lambda)$ is also an eigenfunction of $H_j(z)$ with the same eigenvalues.

Consider the anti-symmetrization,

$$A\tilde{\psi}(\lambda) = \tilde{\psi}(\lambda) - S\tilde{\psi}(\lambda) = \tilde{\psi}(\lambda) - s_{\Lambda_1} \otimes ... \otimes s_{\Lambda_n} \tilde{\psi}(-\lambda),$$

of $\hat{\psi}(\lambda)$ with respect to the Weyl reflection. Write $A\hat{\psi}(\lambda)$ in coordinates,

$$A\tilde{\psi}(\lambda) = \sum_{\operatorname{adm} M, |M|=m} A\tilde{\psi}_M(\lambda) E_M.$$

Corollary 57. (Vanishing Conditions) For any (admissible) index M and any integer $k \in [-k(w_M), k(-w_M)]$, we have

$$A\psi_M(2\eta k) = 0.$$

In other words, let $M = (m_1, ..., m_n)$ be an admissible index and $w = (w_1, ..., w_n)$, $w_j = \Lambda_j - 2m_j$, the \mathfrak{h} -weight of the basis vector E_M . Form the sums $\Sigma^j = \sum_{i=1}^j w_j$. Let k be a non-negative integer. Corollary 57 says that $A\tilde{\psi}_M(2\eta k) = 0$ unless the vector

 $(-\Sigma^1+k-1,-\Sigma^2+k-1,...,-\Sigma^n+k-1)$

satisfies the sl_2 fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$ and $A\tilde{\psi}_M(-2\eta k) = 0$ unless the vector

$$(\Sigma^1 + k - 1, \Sigma^2 + k - 1, ..., \Sigma^n + k - 1)$$

satisfies the sl_2 fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$. In particular, $A\tilde{\psi}_M(0) = 0$ since $\Sigma^n = 0$.

Remark. The Corollary 57 has an analog for the case when n = 1, $\vec{\Lambda} = 2m$. Namely, let $\psi(\lambda) = \psi_m(\lambda) e_m$ be the Bethe ansatz eigenfunction with values in $V_{\vec{\Lambda}}[0]$ and corresponding to a solution of the Bethe ansatz equations. Consider the Weyl antisymmetric eigenfunction $A\psi(\lambda) = (\psi_m(\lambda) - \psi_m(-\lambda)) e_m$. Then for k = -m, -m + 1, ..., m - 1, m we have

$$A\psi(2\eta k) = 0.$$

11.5. The case $\eta = 1/2N$ and $e^{2c} = 1$; the resonance relations for the Bethe ansatz eigenfunctions and the fusion rules for the quantum group $U_{e^{2\pi i/N}}(sl_2)$. In this section we assume that the parameter η has the form $\eta = \frac{1}{2N}$, where N is a natural number, and assume that the parameter c in the Bethe ansatz equations obeys the relation $e^{2c} = 1$, see Theorem 53. Let $\vec{\Lambda} = (\Lambda_1, ..., \Lambda_n)$ be a vector of natural numbers, $\Lambda_1 + ... + \Lambda_n = 2m$, and let

$$\tilde{\psi}(\lambda) = \sum_{\operatorname{adm} M, |M|=m} \tilde{\psi}_M(\lambda) E_M$$

be the Bethe ansatz eigenfunction with values in $L_{\tilde{\Lambda}}[0]$ corresponding to a solution of the Bethe ansatz equations with parameter $c \in \mathbb{C}$. Let $\tilde{\psi}_M(2\eta k) = \tilde{\psi}_{s(M)}(-2\eta k)$ for $k \in [-k(w_M), k(-w_M)]$ be a resonance relation of Theorem 56 for the coefficients of the eigenfunction. Since the function $\tilde{\psi}(\lambda)$ is quasiperiodic, $\tilde{\psi}(\lambda + 1) = (-1)^m e^c \tilde{\psi}(\lambda)$, and since $2\eta = 1/N$, the resonance relation implies a relation $e^c \tilde{\psi}_M(2\eta(k+N)) =$ $e^{-c} \tilde{\psi}_{s(M)}(-2\eta(k+N))$. If $e^{2c} = 1$, then a resonance relation of Theorem 56, $\tilde{\psi}_M(2\eta k) =$ $\tilde{\psi}_{s(M)}(-2\eta k)$, implies a series of new resonance relations,

$$\tilde{\psi}_M(2\eta(k+lN)) = \tilde{\psi}_{s(M)}(-2\eta(k+lN)), \quad \text{for} \quad l \in \mathbb{Z}.$$

Therefore, under the above assumptions on η and c, the resonance relations become N-periodic with respect to k and to describe all the resonance relations it is enough to describe the relations for $k \in [0, N)$.

Corollary 58. Under assumptions $\eta = 1/2N$ and $e^{2c} = 1$, for all $k \in [0, N-1]$, we have

$$\tilde{\psi}_M(2\eta k) = \tilde{\psi}_{s(M)}(-2\eta k) \quad and \quad A\tilde{\psi}_M(2\eta k) = 0$$

unless

(89)
$$k(-w_M) < k < N - k(w_M).$$

The relation (89) can be interpreted in terms of the fusion rules for the quantum group $U_q(sl_2)$ with $q = e^{2\pi i/N}$.

First we remind the fusion rules for $U_q(sl_2)$ with $q = e^{2\pi i/N}$.

For an integer $a \in [0, N-2]$, let L_a denote the a+1-dimensional $U_q(sl_2)$ -module with highest weight q^a . We say that integers a, b, c obey the fusion rules for $U_q(sl_2), q^{2\pi i/N}$, if $a, b, c \in [0, N-2]$ and the $U_q(sl_2)$ -module $L_a \otimes L_b$ contains L_c as a submodule.

It means that $a - b \in \{-c, -c + 2, ..., c - 2, c\}$ and $c \leq a + b \leq 2N - c - 4$. If the triple a, b, c obeys the fusion rules, then each of its its permutation does.

Let $\Lambda = (\Lambda_1, ..., \Lambda_n)$, $a = (a_1, ..., a_n)$ be *n*-tuples of integers. We say that *a* obeys the fusion rules for $U_q(sl_2)$ with respect to Λ , if for every j = 1, ..., n, the triple a_{j-1}, a_j, Λ_j obeys the fusion rules for $U_q(sl_2)$. This means that for j = 1, ..., n we have $a_{j-1}, a_j, \Lambda_j \in [0, N-2]$ and $\Lambda_j \leq a_{j-1} + a_j \leq 2N - \Lambda_j - 4$. Here we assume that $a_0 = a_n$.

Lemma 59. An integer k satisfies (89), if and only if the vector

(90)
$$(-\Sigma^1 + k - 1, -\Sigma^2 + k - 1, ..., -\Sigma^n + k - 1)$$

obeys the fusion rules for $U_q(sl_2)$ with respect to $\vec{\Lambda} = (\Lambda_1, ..., \Lambda_n)$.

According to the Lemma, if k satisfies (89), then $-\Sigma^j + k - 1$, Λ_j belong to [0, N-2] for all j. The Theorem also implies that if there is j such that $\Lambda_j > N-2$, then we have

$$\tilde{\psi}_M(2\eta k) = \tilde{\psi}_{s(M)}(-2\eta k)$$
 and $A\tilde{\psi}_M(2\eta k) = 0$

for all M and k.

Proof. First we prove that (89) implies the $U_q(sl_2)$ fusion rules. Indeed, the inequality $k(-w_M) < k$ implies $\Lambda_j < -\Sigma^{j-1} + k - \Sigma^j + k$ for all j. Similarly, the inequality $k < N - k(w_M)$ implies $-\Sigma^{j-1} + k - \Sigma^j + k < 2N - \Lambda_j$ for all j. Hence,

$$\Lambda_j < -\Sigma^{j-1} + k - \Sigma^j + k < 2N - \Lambda_j.$$

Notice that the number $-\Sigma^{j-1} + k - \Sigma^j + k$ has the same parity as Λ_j . Therefore, $\Lambda_j \in [0, N-2]$ and

$$\Lambda_j \le \Sigma^{j-1} + k - 1 - \Sigma^j + k - 1 \le 2N - \Lambda_j - 4.$$

The vector $(-\Sigma^1 + k(-w_M), -\Sigma^2 + k(-w_M), ..., -\Sigma^n + k(-w_M))$ obeys the fusion rules for sl_2 . Hence, the inequality $k(-w_M) < k$ implies $-\Sigma^j + k > 0$ for all j.

The vector $(\Sigma^1 + k(w_M), \Sigma^2 + k(w_M), ..., \Sigma^n + k(w_M))$ obeys the fusion rules for sl_2 . Hence, the inequality $k(w_M) < N - k$ implies $N > -\Sigma^j + k$ for all j. Thus $-\Sigma^j + k - 1 \in [0, N-2]$. Thus the vector (90) obeys the $U_q(sl_2)$ fusion rules. The converse implication is proved similarly.

11.6. Resonance Relations for Eigenfunctions of Transfer Matrices. In this section we discuss the resonance relations for the eigenfunctions of the transfer matrix of highest weight representations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$, see [FV2, FTV1].

Let W be a representation of $E_{\tau,\eta}(sl_2)$ with L-operator $L(z,\lambda) \in \text{End} (\mathbb{C}^2 \otimes W)$. Let e_0, e_1 be the standard basis of \mathbb{C}^2 and E_{ij} the linear operators on \mathbb{C}^2 defined by $E_{ij}e_k = \delta_{jk}e_i$. Writing $L = E_{00} \otimes a + E_{01} \otimes b + E_{10} \otimes c + E_{11} \otimes d$ we get four operators, $a(z,\lambda), b(z,\lambda), c(z,\lambda), d(z,\lambda)$, the matrix elements of the L-operator, acting on W, and obeying the various relations of $E_{\tau,\eta}(sl_2)$. The transfer matrix $T_W(z) \in \text{End} (\mathcal{F}(W[0]))$ acts on functions by

$$T_W(z)f(\lambda) = a(z,\lambda)f(\lambda - 2\eta) + d(z,\lambda)f(\lambda + 2\eta)$$

The relations imply that $T_W(z)T_W(w) = T_W(w)T_W(z)$ for all $z, w \in \mathbb{C}$.

In [FV2], common eigenfunctions of $T_W(z), z \in \mathbb{C}$, are constructed in the form

$$b(t_1)\cdots b(t_m)v_c$$

where

$$v_c(\lambda) = e^{c\lambda} \prod_{j=1}^m \frac{\theta(\lambda - 2\eta j)}{\theta(2\eta)} v^0,$$

 v^0 is the highest weight vector of W and b(t) is the difference operator $(b(t)f)(\lambda) = b(t,\lambda)f(\lambda+2\eta), f \in \mathcal{F}(W)$ (both the transfer matrix and the difference operators b(t)

are part of the operator algebra of the elliptic quantum group, see [FV1, FV2]). The variables t_1, \ldots, t_m obey a set of Bethe ansatz equations, which are up to a shift the same as the ones in (84).

The eigenfunctions of the transfer matrix in a tensor product of Verma modules are described by the following two Theorems.

Theorem 60. [FTV1] Let $V_{\vec{\Lambda}} = V_{\Lambda_1}(z_1) \otimes \cdots \otimes V_{\Lambda_n}(z_n)$ be a tensor product of $E_{\tau,\eta}(sl_2)$ evaluation Verma modules with generic evaluation points z_1, \ldots, z_n and let

$$v_c(\lambda) = e^{c\lambda} \prod_{j=1}^m \frac{\theta(\lambda - 2\eta j)}{\theta(2\eta)} e_0 \otimes \dots \otimes e_0 \in \mathcal{F}(V_{\vec{\Lambda}}).$$

Then

$$\prod_{j=1}^{m} b(t_j + \eta) v_c = e^{c(\lambda + 2\eta m)} (-1)^m \prod_{i < j} \frac{\theta(t_i - t_j + 2\eta)}{\theta(t_i - t_j)} \times \sum_{j_1 + \dots + j_n = m} \tilde{\omega}_{j_1, \dots, j_n} (t_1, \dots, t_n, \lambda, z, \tau) e_{j_1} \otimes \dots \otimes e_{j_n},$$

where $\tilde{\omega}_{j_1,\ldots,j_n}(t_1,\ldots,t_n,\lambda,z,\tau)$ are the mirror weight functions associated with the tensor product $V_{\vec{\lambda}}$.

Theorem 61. [FV2] Let $V_{\vec{\Lambda}}$ and v_c be as in Theorem 60, and let $T_{V_{\vec{\Lambda}}}(w)$ be the corresponding transfer matrix acting on functions in $\mathcal{F}(V_{\vec{\Lambda}}[0])$. Then for any solution (t_1, \ldots, t_m) of the Bethe ansatz equations

(91)
$$\prod_{j:j\neq i} \frac{\theta(t_j - t_i - 2\eta)}{\theta(t_j - t_i + 2\eta)} \prod_{k=1}^n \frac{\theta(t_i - z_k - (1 + \Lambda_k)\eta)}{\theta(t_i - z_k - (1 - \Lambda_k)\eta)} = e^{4\eta c}, \qquad i = 1, \dots, m,$$

such that, for all i < j, $t_i \neq t_j \mod \mathbb{Z} + \tau \mathbb{Z}$, the vector $\psi = b(t_1) \cdots b(t_m) v_c \in \mathcal{F}(V_{\vec{\Lambda}}[0])$ is a common eigenfunction of all transfer matrices $T_{V_{\vec{\Lambda}}}(w)$ with eigenvalues

$$\varepsilon(w) = e^{-2\eta c} \prod_{j=1}^{m} \frac{\theta(t_j - w - 2\eta)}{\theta(t_j - w)} + e^{2\eta c} \prod_{j=1}^{m} \frac{\theta(t_j - w + 2\eta)}{\theta(t_j - w)} \prod_{k=1}^{n} \frac{\theta(w - z_k - (1 - \Lambda_k)\eta)}{\theta(w - z_k - (1 + \Lambda_k)\eta)}.$$

Moreover $\psi(\lambda + 1) = (-1)^m e^c \psi(\lambda)$.

Now we can reformulate the results of Sections 11.2, 11.4, 11.5 for eigenfunctions of the transfer matrices.

Let $\{E_J\}$ be the reduced basis in $V_{\vec{\Lambda}}[0]$. Let $\psi(\lambda)$ be the eigenfunction of the transfer matrix corresponding to a solution (t_1, \ldots, t_m) of the Bethe ansatz equations (91). Write the eigenfunction in the reduced bases, $\psi(\lambda) = \sum_J \psi_J(t_1, \ldots, t_m, z, \lambda) E_J$.

Theorem 62.

I. Let $n > 1, 1 < j \le n$. Let $m_1, ..., m_{j-2}, k, m_{j+1}, ..., m_n$ be non-negative integers such that $m_1 + ... + m_{j-2} + k + m_{j+1} + ... + m_n = m$. Let a, b be integers such that $a \ne b, 0 \le a, b \le k$. Let $M = (m_1, ..., m_{j-2}, k - a, a, m_{j+1}, ..., m_n)$ and $L = (m_1, ..., m_{j-2}, k - b, b, m_{j+1}, ..., m_n)$. Then

(92)
$$\psi_M(t_1, ..., t_m, z, 2\eta(\Lambda_j - a - b + \sum_{l=j+1}^n (\Lambda_l - 2m_l))) = \psi_L(t_1, ..., t_m, z, 2\eta(\Lambda_j - a - b + \sum_{l=j+1}^n (\Lambda_l - 2m_l)))$$

II. Let n > 1. Let $k, m_2, ..., m_n$ be non-negative integers such that $k+m_2+...+m_n = m$. Let a, b be such that $a \neq b, 0 \leq a, b \leq k$. Let $M = (a, m_2, ..., m_{n-1}, k - a)$ and $L = (b, m_2, ..., m_{n-1}, k - b)$. Then

(93)
$$\psi_M(t_1, ..., t_m, z, 2\eta(a-b)) = \\ \psi_L(t_1, ..., t_m, z, 2\eta(b-a)).$$

III. Let n = 1, $\vec{\Lambda} = 2m$. Then $\psi = \psi_m E_m$ and for all a = 1, ..., m we have

(94)
$$\psi(t_1, ..., t_m, z, 2\eta a) = \psi(t_1, ..., t_m, z, -2\eta a)$$

Notice that the resonance relations of this Theorem differ from the resonance relations of Theorem 54 because the coordinates of the eigenfunction in this Theorem are mirror weight functions while the coordinates of the eigenfunction in Theorem 54 are ordinary weight functions.

Now assume that $\vec{\Lambda} = (\Lambda_1, ..., \Lambda_n)$ is a vector of natural numbers, $\Lambda_1 + ... + \Lambda_n = 2m$. Let $L_{\vec{\Lambda}} = L_{\Lambda_1} \otimes ... \otimes L_{\Lambda_n}$ be the corresponding product of the finite dimensional spaces and $\pi : V_{\vec{\Lambda}} \to L_{\vec{\Lambda}}$ the canonical projection.

Let $\psi(\hat{\lambda})$ be the Bethe ansatz eigenfunction with values in $V_{\vec{\Lambda}}[0]$ and corresponding to a solution of the Bethe ansatz equations (91). Then the function

$$\widetilde{\psi}(\lambda) = \pi \psi(\lambda) = \sum_{\operatorname{adm} M, |M|=m} \psi_M(\lambda) E_M$$

is an eigenfunction of the transfer matrix $T_{L_{\vec{\Lambda}}}(w)$ acting on $\mathcal{F}(L_{\vec{\Lambda}}[0]), T_{L_{\vec{\Lambda}}}(w)\tilde{\psi}(\lambda) = \varepsilon(w)\tilde{\psi}(\lambda)$, where the eigenvalue $\varepsilon(w)$ are defined in Theorem 61.

Let $S : \mathcal{F}(L_{\vec{\Lambda}}[0]) \to \mathcal{F}(L_{\vec{\Lambda}}[0])$ be the Weyl reflection. The Weyl reflection commutes with the transfer matrix $T_{L_{\vec{\Lambda}}}(w)$. Hence $S\tilde{\psi}(\lambda)$ is also an eigenfunction of the transfer matrix with the same eigenvalue.

Consider the anti-symmetrization,

$$A\tilde{\psi}(\lambda) = \tilde{\psi}(\lambda) - S\tilde{\psi}(\lambda) = \tilde{\psi}(\lambda) - s_{\Lambda_1} \otimes ... \otimes s_{\Lambda_n} \tilde{\psi}(-\lambda).$$

Write $A\psi(\lambda)$ in coordinates,

$$A\tilde{\psi}(\lambda) = \sum_{\operatorname{adm} M, |M|=m} A\tilde{\psi}_M(\lambda) E_M.$$

The vanishing conditions for this anti-symmetric eigenfunction are described by the sl_2 modified fusion rules.

Recall that if $(a_1, ..., a_n)$ is a vector of integers, then we say that $(a_1, ..., a_n)$ satisfies the sl_2 fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$ if all a_j are non-negative and for every j the triple a_{j-1}, a_j, Λ_j obeys the sl_2 fusion rules (where $a_0 = a_n$). We shall say that $(a_1, ..., a_n)$ satisfies the sl_2 modified fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$ if all a_j are non-negative and for every j the triple a_j, a_{j+1}, Λ_j obeys the sl_2 fusion rules (where $a_{n+1} = a_1$). Notice that if $\Lambda_1 = \Lambda_2 = ... = \Lambda_n$, then the two fusion rules coincide.

Theorem 63. (Vanishing Conditions)

Let $M = (m_1, ..., m_n)$ be an admissible index and $w = (w_1, ..., w_n)$, $w_j = \Lambda_j - 2m_j$, the \mathfrak{h} -weight of the basis vector E_M . Form the sums $\tilde{\Sigma}^j = \sum_{i=j}^n w_j$. Let k be a non-negative integer. Then $A\tilde{\psi}_M(2\eta k) = 0$ unless the vector

$$(-\tilde{\Sigma}^1+k-1,-\tilde{\Sigma}^2+k-1,...,-\tilde{\Sigma}^n+k-1)$$

satisfies the sl_2 modified fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$ and $A\tilde{\psi}_M(-2\eta k) = 0$ unless the vector

$$(\tilde{\Sigma}^1 + k - 1, \tilde{\Sigma}^2 + k - 1, ..., \tilde{\Sigma}^n + k - 1)$$

satisfies the sl_2 modified fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$. In particular, $A\tilde{\psi}_M(0) = 0$ since $\tilde{\Sigma}^1 = 0$.

The fact that this Theorem involves the modified fusion rules while the corresponding Corollary 57 uses the ordinary fusion rules is due to the fact that the coordinates of the eigenfunction in this Theorem are given in terms of mirror weight functions while the coordinates of the eigenfunction in Corollary 57 are given in terms of ordinary weight functions.

Remark. If n = 1, $\vec{\Lambda} = 2m$ and $\psi(\lambda) = \psi_m(\lambda) e_m$ is the Bethe ansatz eigenfunction of the transfer matrix with values in $V_{\vec{\Lambda}}[0]$. Then the corresponding Weyl anti-symmetric eigenfunction $A\psi(\lambda) = (\psi_m(\lambda) - \psi_m(-\lambda)) e_m$ satisfies

$$A\psi(2\eta k) = 0$$

for all k = -m, -m + 1, ..., m - 1, m.

Now we shall assume that the parameter η has the form $\eta = 1/2N$ where N is a natural number and the parameter c in the Bethe ansatz equations (91) obeys the relation $e^{2c} = 1$. Under these assumptions the vanishing conditions for anti-symmetric Bethe eigenfunctions are described in terms of the $U_{e^{2\pi i/N}}(sl_2)$ modified fusion rules.

Recall that if $(a_1, ..., a_n)$ is a vector of integers, then we say that $(a_1, ..., a_n)$ satisfies the $U_{e^{2\pi i/N}}(sl_2)$ fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$ if all a_j are non-negative and for every j the triple a_{j-1}, a_j, Λ_j obeys the $U_{e^{2\pi i/N}}(sl_2)$ fusion rules (where $a_0 = a_n$). We shall say that $(a_1, ..., a_n)$ satisfies the $U_{e^{2\pi i/N}}(sl_2)$ modified fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$ if all a_j are non-negative and for every j the triple a_j, a_{j+1}, Λ_j obeys the $U_{e^{2\pi i/N}}(sl_2)$ fusion rules (where $a_{n+1} = a_1$). If $\Lambda_1 = \Lambda_2 = ... = \Lambda_n$, then the two fusion rules coincide.

Theorem 64. Let $\eta = 1/2N$ and $e^{2c} = 1$. Let $A\psi(\lambda)$ be the anti-symmetrization of a Bethe ansatz eigenfunction of the transfer matrix. Then

$$A\psi(2\eta k) = (-1)^m e^c A\psi(2\eta(k+N)).$$

Moreover, for any admissible index M and $k \in \{0, 1, 2, ..., N-1\}$ we have $A\psi_M(2\eta k) = 0$ unless the vector

$$[-\tilde{\Sigma}^1 + k - 1, -\tilde{\Sigma}^2 + k - 1, ..., -\tilde{\Sigma}^n + k - 1)]$$

satisfies the $U_{e^{2\pi i/N}}(sl_2)$ modified fusion rules with respect to $(\Lambda_1, ..., \Lambda_n)$.

12. Restricted Interaction-round-a-face Models

12.1. **Discrete models**, [FV2]. The construction of the transfer matrix admits the following variation. The transfer matrix $T_{V_{\vec{\Lambda}}}(w)$ and the difference operators b(w) shift the argument of functions by $\pm 2\eta$. Therefore we may replace $\mathcal{F}(V_{\vec{\Lambda}})$ by the space $\mathcal{F}_{\mu}(V_{\vec{\Lambda}})$ of all functions from the set $C_{\mu} = \{2\eta(\mu + j) \mid j \in \mathbb{Z}\}$ to $V_{\vec{\Lambda}}$. The transfer matrix and the difference operators are then well-defined on $\mathcal{F}_{\mu}(V_{\vec{\Lambda}})$ if μ is generic. Also, it follows from Theorem 60 that the restriction to C_{μ} of the Bethe ansatz eigenfunctions is well defined for all μ . We thus have:

Corollary 65. Suppose t_1, \ldots, t_m is a solution to the Bethe ansatz equations (91). Then, for generic μ , the restriction to C_{μ} of $b(t_1) \cdots b(t_m)v_c$ is a common eigenfunction of the operators $T_{V_{\overline{\lambda}}}(w) \in \text{End} (\mathcal{F}_{\mu}(V_{\overline{\lambda}}[0])).$

12.2. Unrestricted interaction-round-a-face models, [FV2]. In this section, we consider a special case of the above construction, and relate T(w) to the transfer matrix of the (unrestricted) interaction-round-a-face (IRF) (also called solid-on-solid) models of [Ba, ABF]. Therefore, our formulae give, in particular, eigenvectors of transfer matrices of IRF models, extending the results of [BR].

Let $V = \mathbb{C}^2$. Denote the vectors e_0, e_1 of the standard basis in \mathbb{C}^2 by e[1], e[-1], respectively. Let $V = V[1] \oplus V[-1] = \mathbb{C}e[1] \oplus \mathbb{C}e[-1]$ be the weight decomposition of V. Let $R(z, \lambda) \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ be the fundamental R-matrix defined in Section 3.5.

Let $W = V^{\otimes n}$ be the $E_{\tau,\eta}(sl_2)$ representation with the L-operator $L(w,\lambda) \in \text{End} (V \otimes W)$ given by

$$L(w,\lambda) = R^{(01)}(w-z_1,\lambda) \otimes R^{(02)}(w-z_2,\lambda-2\eta h^{(1)}) \otimes .. \otimes R^{(0,n)}(w-z_n,\lambda-2\eta \sum_{j=1}^{n-1} h^{(j)}).$$

This representation is isomorphic to the tensor product of the evaluation fundamental representations

$$V(z_n) \otimes \ldots \otimes V(z_1),$$

with the isomorphism $u_1 \otimes \cdots \otimes u_n \mapsto u_n \otimes \cdots \otimes u_1$.

We assume that n is even, and therefore W has a nontrivial zero weight subspace W[0].

The corresponding transfer matrix $T(w) \in \text{End}(\mathcal{F}(W[0]))$ has the form

(95)
$$T(w) = \sum_{\nu} \operatorname{Tr}_{V[\nu]^{(0)}} L(w, \lambda) \Gamma_{\nu}$$

where $(\Gamma_{\nu} f)(\lambda) = f(\lambda - 2\eta\nu)$.

We define "Boltzmann weights" w(a, b, c, d; z), depending on complex parameters a, b, c, d, z, such that $a - b, b - c, c - d, a - d \in \{1, -1\}$, by

$$R(z,2\eta d) e[d-c] \otimes e[c-b] = \sum_{a} w(a,b,c,d;z) e[a-b] \otimes e[d-a],$$

(the sum is over one or two allowed values of a). We set w(a, b, c, d; z) = 0 if a, b, c, d do not satisfy $a - b, b - c, c - d, a - d \in \{1, -1\}$.

The dynamical quantum Yang–Baxter equation translates into the star-triangle equation

$$\sum_{g} w(a, b, g, f; z_2 - z_3) w(b, c, d, g; z_1 - z_3) w(g, d, e, f; z_1 - z_2)$$

=
$$\sum_{g} w(b, c, g, a; z_1 - z_2) w(a, g, e, f; z_1 - z_3) w(g, c, d, e; z_2 - z_3).$$

Introduce a basis $|a_1, \ldots, a_n\rangle$ of $\mathcal{F}_{\mu}(W[0])$ labeled by $a_i \in \mu + \mathbb{Z}$ with $a_i - a_{i+1} \in \{1, -1\}, i = 1, \ldots, n-1$, and $a_n - a_1 \in \{1, -1\}$. We let $\delta(\lambda) = 1$ if $\lambda = 0$ and 0 otherwise. Then we define

$$|a_1,\ldots,a_n\rangle(\lambda) = \delta(\lambda - 2\eta a_1)e[a_1 - a_2] \otimes e[a_2 - a_3] \otimes \cdots \otimes e[a_n - a_1].$$

If Γ is the shift operator $\Gamma f(\lambda) = f(\lambda - 2\eta)$, then $\Gamma | a_1, \ldots, a_n \rangle = |a_1 + 1, \ldots, a_n + 1 \rangle$. Using this, and the fact that $h^{(j)} | a_1, \ldots, a_n \rangle = (a_{j+1} - a_j) | a_1, \ldots, a_n \rangle$, we get

(96)
$$T(w)|a_1,\ldots,a_n\rangle = \sum_{b_1,\ldots,b_n} \prod_{j=1}^n w(b_{j+1},a_{j+1},a_j,b_j;w-z_j)|b_1,\ldots,b_n\rangle,$$

with the understanding that $b_{n+1} = b_1$, $a_{n+1} = a_1$. The (finite) sum is over the values of the indices b_i for which the Boltzmann weights are nonzero. Comparing with [Ba], we see that T(w), in this basis, is the row-to-row transfer matrix of the (unrestricted) interaction-round-a-face model associated to the solution w(a, b, c, d; z) of the star-triangle equation, see [Ba]. The situation is best visualized by looking at the graphical representation of Fig. 1.

Applying results of Section 11.6 to the representation W, we get the following Corollary:

Corollary 66. Let c be a complex number. Consider the function

$$\psi(\lambda) = \sum e^{c\lambda} \omega_{j_1,\dots,j_n}(t_1,\dots,t_{n/2},z_1,\dots,z_n,\lambda) e_{j_1} \otimes \dots \otimes e_{j_n} \in \mathcal{F}(W[0])$$



FIGURE 1. Graphical representation of the row-to-row transfer matrix of an IRF model. Each crossing represents a Boltzmann weight w whose arguments are the labels of the adjoining regions and the difference of the parameters associated to the lines.

where the sum is over all $j_1, ..., j_n$ such that $j_k = 0, 1, j_1 + ... + j_n = n/2$ and $\omega_{j_1,...,j_n}$ $(t_1, ..., t_{n/2}, z_1, ..., z_n, \lambda)$ are the weight functions defined in (13) for $a_1 = ... = a_n = \eta$. Then for any solution $(t_1, ..., t_{n/2})$ of the Bethe ansatz equations

(97)
$$\prod_{l=1}^{n} \frac{\theta(t_j - z_l + \eta)}{\theta(t_j - z_l - \eta)} \prod_{k: k \neq j} \frac{\theta(t_j - t_k - 2\eta)}{\theta(t_j - t_k + 2\eta)} = e^{-4\eta c}, \qquad j = 1, \dots, n/2,$$

such that, for all i < j, $t_i \neq t_j \mod \mathbb{Z} + \tau \mathbb{Z}$, the function $\psi(\lambda)$ is a common eigenfunction of all transfer matrices T(w), $T(w)\psi(\lambda) = \varepsilon(w)\psi(\lambda)$, where

$$(98)\,\varepsilon(w) = e^{-2\eta c} \prod_{j=1}^{n/2} \frac{\theta(t_j - w - \eta)}{\theta(t_j - w + \eta)} + e^{2\eta c} \prod_{j=1}^{n/2} \frac{\theta(t_j - w + 3\eta)}{\theta(t_j - w + \eta)} \prod_{k=1}^n \frac{\theta(w - z_k)}{\theta(w - z_k - 2\eta)}.$$

Moreover $\psi(\lambda + 1) = (-1)^{n/2} e^c \psi(\lambda).$

The Weyl reflection $S : \mathcal{F}(W[0]) \to \mathcal{F}(W[0])$ commutes with the transfer matrix. Hence $S\psi(\lambda)$ is also an eigenfunction of the transfer matrix with the same eigenvalue. In particular the Weyl anti-symmetric function $A\psi(\lambda) = \psi(\lambda) - S\psi(\lambda)$ is an eigenfunction of the transfer matrix with the same eigenvalue.

The restriction of the eigenfunctions $\psi(\lambda)$ and $A\psi(\lambda)$ to the set C_{μ} is well defined for any μ . The λ -poles of the L-operator L(w, z) have the form $\lambda = 2\eta k \mod \mathbb{Z} + \tau \mathbb{Z}$, where k is an integer. Therefore the transfer matrix T(w) restricted to the space $\mathcal{F}_{\mu}(W[0])$ is well defined for $\mu \neq 0 \mod \mathbb{Z} + \frac{1}{2\eta}\mathbb{Z} + \frac{\tau}{2\eta}\mathbb{Z}$. Thus, for those μ , the restriction of the functions $\psi(\lambda), A\psi(\lambda)$ to C_{μ} defines common eigenfunctions of all transfer matrices acting in $\mathcal{F}_{\mu}(W[0])$ and hence common eigenfunctions of all row-to-row transfer matrices of the corresponding unrestricted interaction-round-a-face models. Notice that the eigenvalues of the restrictions do not depend on μ .

12.3. Infinite restricted interaction-round-a-face models. In this section we assume that η is a generic complex number. Let $|a_1, ..., a_n\rangle$ be a delta function such that the numbers $a_1, ..., a_n$ are integers and for all j = 1, ..., n we have $|a_j - a_{j+1}| = 1$ (with

 $a_{n+1} = a_1$). Such a delta function is an element of $\mathcal{F}_{\mu=0}(W[0])$. The delta function will be called *positive* (resp. *negative*) if $a_1, ..., a_n$ are positive (resp. negative). The delta function will be called *neutral* if the set $a_1, ..., a_n$ contains zero.

Introduce the subspace $\mathcal{F}^+(W[0]) \subset \mathcal{F}_{\mu=0}(W[0])$ as the subspace generated by (infinite) linear combinations of all positive delta functions.

The transfer matrix T(w) is not defined on $\mathcal{F}_{\mu=0}(W[0])$ since some of the Boltzmann weights w(a, b, c, d; z) are not defined when a, b, c, d belong to \mathbb{Z} . Nevertheless it follows from the formulae for the fundamental R-matrix in Section 3.5 that the Boltzmann weights are well defined if all a, b, c, d are positive integers (and |a - b| = |b - c| =|c - d| = |d - a| = 1). Moreover, the weights w(1, 2, 1, 0; z), w(0, 1, 2, 1; z) are well defined and w(0, 1, 2, 1; z) = 0.

Consider formula (96) for $T(w)|a_1, ..., a_n\rangle$. If $|a_1, ..., a_n\rangle$ is a positive delta function, then in this formula only $|b_1, ..., b_n\rangle$ with non-negative integer coordinates can appear. Moreover, if $b_j = 0$ for some j, then $b_{j-1} = b_{j+1} = a_j = 1$ and $a_{j-1} = a_{j+1} = 2$. If there is such b_j , then the coefficient of $|b_1, ..., b_n\rangle$ in $T(w)|a_1, ..., a_n\rangle$ contains the product $w(0, 1, 2, 1; w - z_{j-1})w(1, 2, 1, 0; z - z_j)$ which is well defined and equal to zero. Thus, if $|a_1, ..., a_n\rangle$ is a positive delta function, then all terms in the formula for $T(w)|a_1, ..., a_n\rangle$ are well defined and

$$T(w)|a_1,\ldots,a_n\rangle = \sum_{b_1,\ldots,b_n} \prod_{j=1}^n w(b_{j+1},a_{j+1},a_j,b_j;w-z_j)|b_1,\ldots,b_n\rangle,$$

where the (finite) sum is over only the positive delta functions $|b_1, \ldots, b_n\rangle$ for which the Boltzmann weights are nonzero. This formula induces a well defined operator $T^+(w)$ on $\mathcal{F}^+(W[0])$ which is the row-to-row transfer matrix of the infinite restricted model.

The restricted infinite model, like the unrestricted one, satisfies the star-triangular equation, see [ABF]. This equation holds for all positive integers a, b, c, d, e, f such that |a - b| = |b - c| = c - d| = |d - e| = |e - f| = |f - a| = 1 under the condition that the summations are over all positive integers g. This fact easily follows from the unrestricted star-triangular equation and the explicit formulae for the matrix coefficients of the fundamental R-matrix. Namely, it is enough to prove the restricted star-triangular equation under the assumption that the set a, b, c, d, e, f contains 1. If the set a, b, c, d, e, f contains 1 and 3, then the restricted star-triangular equation coincides with the unrestricted one. Thus the only remaining cases to check are the cases when the set a, b, c, d, e, f is the set 1,2,1,2,1,2 or 2,1,2,1,2,1. Under these boundary conditions, it is easy to check that if in the unrestricted equation we have g = 0, then the corresponding term is well defined and equal to zero. Thus, the restricted star-triangular equation coincides with the unrestricted one and hence holds. As a consequence of this result we conclude that the row-to-row transfer matrices $T^+(w)$ of the infinite restricted model commute for different values of w.

12.4. Eigenfunctions of the infinite restricted model. In this section we show that the Weyl anti-symmetrization of a Bethe eigenfunction of Corollary 66 gives a common eigenvector of all row-to-row transfer matrices of the infinite restricted model.

Let $T(w) \in \text{End}(\mathcal{F}(W[0]))$ be the transfer matrix of the $E_{\tau,\eta}(sl_2)$ module W defined in (95). Let ψ be its Bethe eigenfunction corresponding to a complex number c and a solution $(t_1, ..., t_{n/2})$ of the Bethe ansatz equations (97). Let $\varepsilon(w)$ be the eigenvalue defined in (98), $T(w)\psi = \varepsilon(w)\psi$. Let $A\psi = \psi - S\psi$ be the corresponding Weyl antisymmetric eigenfunction of T(w) (with the same eigenvalue). Let $A\psi_0 \in \mathcal{F}_{\mu=0}(W[0])$ be the restriction of $A\psi$ to the set $C_{\mu=0}$. As an element of $\mathcal{F}_{\mu=0}(W[0])$, it has the form

$$A\psi_0 = \sum_{|a_1,...,a_n\rangle} (A\psi)_{|a_1,...,a_n\rangle} |a_1,...,a_n\rangle,$$

where $(A\psi)_{|a_1,...,a_n\rangle}$ are suitable numbers and the sum is over all $|a_1,...,a_n\rangle$ such that $a_1,...,a_n$ are integers satisfying the condition $|a_1-a_2| = ... = |a_{n-1}-a_n| = |a_n-a_1| = 1$. Define an element $A\psi^+ \in \mathcal{F}^+(W[0])$ by

(99)
$$A\psi^{+} = \sum_{|a_{1},...,a_{n}\rangle > 0} (A\psi)_{|a_{1},...,a_{n}\rangle} |a_{1},...,a_{n}\rangle.$$

where the sum is over all positive delta functions.

Theorem 67. The vector $A\psi^+ \in \mathcal{F}^+(W[0])$ is a common eigenvector (with the same eigenvalue) of all row-to-row transfer matrices $T^+(w)$ of the infinite restricted model, $T^+(w)A\psi^+ = \varepsilon(w)A\psi^+$.

Proof. The Weyl anti-symmetry of $A\psi_0$ means

(100)
$$(A\psi)_{|a_1,...,a_n\rangle} = -(A\psi)_{|-a_1,...,-a_n\rangle}$$

for all $|a_1, ..., a_n\rangle$. The vanishing conditions of Corollary 57 imply

$$(A\psi)_{|a_1,\dots,a_n\rangle} = 0$$

for all neutral delta functions $|a_1, ..., a_n\rangle$.

For generic μ , let

$$A\psi_{\mu} = \sum_{|\mu+a_1,...,\mu+a_n\rangle} (A\psi)_{|\mu+a_1,...,\mu+a_n\rangle} |\mu+a_1,...,\mu+a_n\rangle$$

be the restriction of $A\psi$ to C_{μ} . Here the summation is over integers $a_1, ..., a_n$ such that $|a_1 - a_2| = ... = |a_{n-1} - a_n| = |a_n - a_1| = 1$. Let

$$T(w)A\psi_{\mu} = \sum_{|\mu+b_{1},...,\mu+b_{n}\rangle} (T(w)A\psi)_{|\mu+b_{1},...,\mu+b_{n}\rangle} |\mu+b_{1},...,\mu+b_{n}\rangle$$

be its image under the action of the transfer matrix. Then we have

(101)
$$(T(w)A\psi)_{|\mu+b_1,\dots,\mu+b_n\rangle} = \\ \sum_{a_1,\dots,a_n} \prod_{j=1}^n w(\mu+b_{j+1},\mu+a_{j+1},\mu+a_j,\mu+b_j;w-z_j) (A\psi)_{|\mu+a_1,\dots,\mu+a_n\rangle},$$

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as well as

(102)
$$(T(w)A\psi)_{|\mu+b_1,\dots,\mu+b_n\rangle} = \varepsilon(w)(A\psi)_{|\mu+b_1,\dots,\mu+b_n\rangle}.$$

If $|b_1, ..., b_n\rangle$ is a positive delta function, then in formula (101) only $|\mu + a_1, ..., \mu + a_n\rangle$ with nonzero $a_1, ..., a_n$ could appear. If $a_j = 0$ for some j, then $a_{j-1} = a_{j+1} = b_j = 1$ and $b_{j-1} = b_{j+1} = 2$. If there is such a_j , then the coefficient of $(A\psi)_{|\mu+a_1,...,\mu+a_n\rangle}$ in $(T(w)A\psi)_{|\mu+b_1,...,\mu+b_n\rangle}$ contains the product $w(\mu + 1, \mu + 0, \mu + 1, \mu + 2; w - z_{j-1})w(\mu + 2, \mu + 1, \mu + 0, \mu + 1; z - z_j)$.

For a positive $|b_1, ..., b_n\rangle$, consider the limit of $(T(w)A\psi)_{|\mu+b_1,...,\mu+b_n\rangle}$ if μ tends to zero. According to explicit formulae, the factors $w(\mu + b_{j+1}, \mu + a_{j+1}, \mu + a_j, \mu + b_j; w - z_j)$ with positive $b_{j+1}, a_{j+1}, a_j, b_j$ are well defined for $\mu = 0$ as well as the factors $w(\mu + 1, \mu + 0, \mu + 1, \mu + 2; w - z_{j-1})$, $w(\mu + 2, \mu + 1, \mu + 0, \mu + 1; z - z_j)$. At the same time the limit of the coefficients $(A\psi)_{|\mu+a_1,...,\mu+a_n\rangle}$ is zero if $a_1, ..., a_n$ are non-negative and at least one of them is zero. Thus, taking the limit of formulae (101) and (102), we get $T^+(w)A\psi^+ = \varepsilon(w)A\psi^+$.

12.5. Finite interaction-round-a-face models. In this section we assume that $\eta = 1/2N$ where N is a natural number. A delta function $|a_1, ..., a_n\rangle$ with integers $a_1, ..., a_n$ obeying conditions $|a_1 - a_2| = ... = |a_{n-1} - a_n| = |a_n - a_1| = 1$ and $0 < a_k < N$ for all k will be called *admissible*. A delta function $|a_1, ..., a_n\rangle$ with at least one of the integers $a_1, ..., a_n$ divisible by N will be called *bad*. Introduce the finite dimensional subspace $\mathcal{F}^{\{N\}}(W[0])$ of $\mathcal{F}_{\mu=0}(W[0])$ as the subspace generated by the admissible delta functions.

The linear operator $T^{\{N\}}(w) \in \operatorname{End} (\mathcal{F}^{\{N\}}(W[0]))$ defined by

$$T^{\{N\}}(w)|a_1,\ldots,a_n\rangle = \sum_{b_1,\ldots,b_n} \prod_{j=1}^n w(b_{j+1},a_{j+1},a_j,b_j;w-z_j)|b_1,\ldots,b_n\rangle,$$

where the sum is over admissible delta functions, is called the row-to-row transfer matrix of the (finite) restricted interaction-round-a-face model. Considerations, similar to those in Section 12.3, show that the transfer matrix is well defined, and the transfer matrices commute for different values of w, see [ABF].

Let $T(w) \in \text{End}(\mathcal{F}(W[0]))$ be the transfer matrix of the $E_{\tau,\eta}(sl_2)$ module W defined in (95). Let ψ be its Bethe eigenfunction corresponding to a complex number c and a solution $(t_1, ..., t_{n/2})$ of the Bethe ansatz equations (97). In this section we always assume that $e^{2c} = 1$. Let $\varepsilon(w)$ be the eigenvalue defined in (98), $T(w)\psi = \varepsilon(w)\psi$. Let $A\psi = \psi - S\psi$ be the corresponding Weyl anti-symmetric eigenfunction of T(w). Let $A\psi_0 \in \mathcal{F}_{\mu=0}(W[0])$ be the restriction of $A\psi$ to the set $C_{\mu=0}$, $A\psi_0 = \sum_{|a_1,...,a_n\rangle} (A\psi)_{|a_1,...,a_n\rangle} |a_1,...,a_n\rangle$. Define an element $A\psi^{\{N\}} \in \mathcal{F}^{\{N\}}(W[0])$ by

(103)
$$A\psi^{\{N\}} = \sum_{\operatorname{adm}|a_1,\dots,a_n\rangle} (A\psi)_{|a_1,\dots,a_n\rangle} |a_1,\dots,a_n\rangle.$$

where the sum is over all admissible delta functions.

Theorem 68. The vector $A\psi^{\{N\}} \in \mathcal{F}^{\{N\}}(W[0])$ is a common eigenvector (with the same eigenvalue) of all row-to-row transfer matrices $T^{\{N\}}(w)$ of the (finite) restricted model, $T^{\{N\}}(w)A\psi^{\{N\}} = \varepsilon(w)A\psi^{\{N\}}$.

Proof. The Weyl anti-symmetry of $A\psi_0$ gives the relation (100). The condition $e^{2c} = 1$ implies

$$(A\psi)_{|a_1+N,...,a_n+N\rangle} = (-1)^{n/2} e^c (A\psi)_{|a_1,...,a_n\rangle}$$

for all $|a_1, ..., a_n\rangle$. The vanishing conditions of Theorem 64 imply

$$(A\psi)_{|a_1,\dots,a_n\rangle} = 0$$

for all bad $|a_1, ..., a_n\rangle$.

Now the Theorem is proved similarly to the proof of Theorem 67 considering the function $A\psi_{\mu}$ and taking the limit of the equation $T(w)A\psi_{\mu} = \varepsilon(w)A\psi_{\mu}$ as μ tends to zero.

Remark. The eigenvector $A\psi^{\{N\}}$ has an additional symmetry

$$(A\psi)_{|a_1,...,a_n\rangle} = (-1)^{n/2+1} e^c (A\psi)_{|N-a_1,...,N-a_n\rangle}$$

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