HITCHIN'S CONJECTURE FOR SIMPLY-LACED LIE ALGEBRAS IMPLIES THAT FOR ANY SIMPLE LIE ALGEBRA

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Abstract. Let $\mathfrak{g}$ be any simple Lie algebra over $\mathbb{C}$. Recall that there exists an embedding of $\mathfrak{sl}_2$ into $\mathfrak{g}$, called a principal TDS, passing through a principal nilpotent element of $\mathfrak{g}$ and uniquely determined up to conjugation. Moreover, $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ is freely generated (in the super-graded sense) by primitive elements $\omega_1, \dots, \omega_\ell$, where ℓ is the rank of $\mathfrak{g}$. N. Hitchin conjectured that for any primitive element $\omega \in \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$, there exists an irreducible $\mathfrak{sl}_2$-submodule $V_\omega \subset \mathfrak{g}$ of dimension d such that ω is non-zero on the line $\wedge^d(V_\omega)$. We prove that the validity of this conjecture for simple simply-laced Lie algebras implies its validity for any simple Lie algebra.

Let G be a connected, simply-connected, simple, simply-laced algebraic group and let σ be a diagram automorphism of G with fixed subgroup K. Then, we show that the restriction map $R(G) \to R(K)$ is surjective, where R denotes the representation ring over $\mathbb{Z}$. As a corollary, we show that the restriction map in the singular cohomology $H^*(G) \to H^*(K)$ is surjective. Our proof of the reduction of Hitchin's conjecture to the simply-laced case relies on this cohomological surjectivity.

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over the complex numbers $\mathbb{C}$ with the associated connected, simply-connected complex algebraic group G. Recall that there is a unique (up to conjugation) embedding of $\mathfrak{sl}_2$ into $\mathfrak{g}$, called a principal TDS, such that the image passes through a principal nilpotent element of $\mathfrak{g}$. Under the adjoint action of a principal TDS, the Lie algebra $\mathfrak{g}$ decomposes as a direct sum of exactly ℓ irreducible $\mathfrak{sl}_2$-submodules $V_1, \dots, V_\ell$ of dimensions $2m_1 + 1, \dots, 2m_\ell + 1$ respectively, where ℓ is the rank of $\mathfrak{g}$ and $m_1, \dots, m_\ell$ are the exponents of $\mathfrak{g}$.

Further, the singular cohomology $H^*(G) = H^*(G, \mathbb{C})$ with complex coefficients is a Hopf algebra. Let $P(\mathfrak{g}) \subset H^*(G)$ be the graded subspace of primitive elements. Then, $P(\mathfrak{g})$ has a basis in degrees $2m_1 + 1, \dots, 2m_\ell + 1$. We identify $H^*(G)$ with $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ and consider $P(\mathfrak{g})$ as a subspace of $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$.

Now, N. Hitchin made the following conjecture [Hi]

Conjecture 1.1. *Let $\mathfrak{g}$ be any simple Lie algebra. For any primitive element $\omega \in P_d \subset \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$, there exists an irreducible subspace $V_\omega \subset \mathfrak{g}$ of dimension d with respect to the principal TDS action such that*

$$\omega|_{\wedge^d(V_\omega)} \neq 0.$$

The main motivation for Hitchin behind the above conjecture lies in its connection with the study of polyvector fields on the moduli space $M_G(\Sigma)$ of semistable principal G-bundles on a smooth projective curve Σ of any genus $g > 2$. Specifically, observe that the cotangent space

Date: June 8, 2021.

at a smooth point E of $M_G(\Sigma)$ is isomorphic with $H^0(\Sigma, \mathfrak{g}(E) \otimes \Omega)$, where $\mathfrak{g}(E)$ denotes the associated adjoint bundle and Ω is the canonical bundle of the curve Σ. Given a biinvariant differential form ω of degree k on G, i.e., $\omega \in \wedge^k(\mathfrak{g}^*)^{\mathfrak{g}}$, and elements $\Phi_j \in H^0(\Sigma, \mathfrak{g}(E)\otimes\Omega), 1 \leq j \leq k, \omega(\Phi_1, \ldots, \Phi_k)$ defines a skew form with values in the line bundle Ω^k. Dually, it defines a homomorphism

$$\Theta_\omega : H^1(\Sigma, \Omega^{1-k}) \to H^0(M_G(\Sigma), \wedge^k T),$$

where T is the tangent bundle of $M_G(\Sigma)$.

Now, as shown by Hitchin, the validity of the above conjecture would imply that the map Θ_ω is injective for any invariant form $\omega \in \wedge^k(\mathfrak{g}^*)^{\mathfrak{g}}$ (cf. [Hi]).

Any simple Lie algebra $\mathfrak{k}$ can be realized as the fixed point subalgebra of a diagram automorphism of an appropriate simple simply-laced Lie algebra $\mathfrak{g}$. We prove that the validity of the conjecture for $\mathfrak{g}$ implies the validity for $\mathfrak{k}$. Thus, one needs to verify the conjecture only for the simple Lie algebras of types A, D and E. Specifically, we have the following result (cf. Theorem 2.5).

Theorem 1.2. *If Hitchin's conjecture is valid for any simply-laced simple Lie algebra $\mathfrak{g}$, then it is valid for any simple Lie algebra.*

More precisely, if Hitchin's conjecture is valid for $\mathfrak{g}$ of type $(A_{2\ell-1}; A_{2\ell}; D_4; E_6)$, then it is valid for $\mathfrak{g}$ of type $(C_\ell; B_\ell; G_2; F_4)$ respectively.

The proof relies on constructing a principal TDS in $\mathfrak{k}$ which remains a principal TDS in $\mathfrak{g}$. Moreover, we need to use the surjectivity of the space of primitive elements $P(\mathfrak{g}) \to P(\mathfrak{k})$, which allows us to lift primitive elements $\omega_d \in \wedge^d(\mathfrak{k}^*)^{\mathfrak{k}}$ to primitive elements $\widetilde{\omega}_d \in \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$.

Let K be the algebraic subgroup of G with Lie algebra $\mathfrak{k}$, where $\mathfrak{k}$ is the fixed subalgebra under a diagram automorphism of a simple simply-laced Lie algebra $\mathfrak{g}$. Our next main result of the paper (cf. Theorem 3.1) asserts the following.

Theorem 1.3. *The canonical map $\phi : R(G) \to R(K)$ is surjective, where $R(G)$ denotes the representation ring of G (over $\mathbb{Z}$).*

In particular, the canonical restriction map $\psi : S^\bullet(\mathfrak{g}^*)^{\mathfrak{g}} \to S^\bullet(\mathfrak{k}^*)^{\mathfrak{k}}$ is surjective.

Finally, we use H. Cartan's transgression map and the surjectivity of ψ to obtain the desired surjectivity of $\gamma_o : P(\mathfrak{g}) \to P(\mathfrak{k})$ and thereby the surjectivity of $\gamma : H^*(G) \to H^*(K)$ (cf. Theorem 3.5). In our view, the surjectivity of ϕ, γ and γ_o is of independent interest.

Acknowledgements: The second author is grateful to Nigel Hitchin for explaining his conjecture and to Michel Brion for asking the question answered in Theorem 3.1. The first author would like to thank Swarnava Mukhopadhyay for many helpful discussions. Both the authors were supported by the NSF grant number DMS-1201310.

2. Reduction of Hitchin's conjecture to simply-laced Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with the associated connected simply-connected complex algebraic group G (with Lie algebra $\mathfrak{g}$).

Definition 2.1. A Lie algebra embedding $\varphi : \mathfrak{sl}_2 \to \mathfrak{g}$ (or its image) is called a principal TDS if $\varphi(X)$ is a principal nilpotent element of $\mathfrak{g}$, i.e., $\mathrm{Ad}G \cdot \varphi(X)$ is the open orbit in the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$.

Here, $\mathfrak{sl}_2$ is the Lie algebra of traceless 2×2 matrices over $\mathbb{C}$ with the standard basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\varphi' : \mathfrak{sl}_2 \to \mathfrak{g}$ be another principal TDS. Then, by a result of Kostant [Ko], Corollary 3.7, φ' is conjugate to φ, i.e., there exists a $g \in G$ such that

$$\varphi' = \mathrm{Ad} g \cdot \varphi. \tag{1}$$

Decompose the adjoint representation of $\mathfrak{g}$ with respect to a principal TDS φ into irreducible components:

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell,$$

labeling them so that

$$n_1 \leq \cdots \leq n_\ell, \quad \text{where} \quad n_i = \dim V_i. \tag{2}$$

Then, it is known (cf. [Ko], Corollary 8.7) that

(a) $\ell =$ rank of $\mathfrak{g}$.

(b) Each n_i is an odd integer $2m_i + 1$. Moreover,

$$m_1 \leq m_2 \leq \cdots \leq m_\ell$$

are the exponents of $\mathfrak{g}$. (The list of exponents for any $\mathfrak{g}$ can be found in [Bo], Planche I - IX.)

(c) Except when $\mathfrak{g}$ is of type D_ℓ (with ℓ even), each V_i is an isotypical component (in particular, uniquely determined) for the principal TDS φ, i.e., $m_1 < m_2 < \cdots < m_\ell$.

When $\mathfrak{g}$ is of type D_ℓ (with ℓ even), the exponents are:

$$1, 3, 5, \cdots, \ell - 3, \ell - 1, \ell - 1, \ell + 1, \cdots, 2\ell - 3.$$

Hence, the isotypical component for the highest weight $2\ell - 2$ is a direct sum of two copies of the irreducible module $V_{\mathfrak{sl}_2}(2\ell - 2)$ with highest weight $2\ell - 2$.

By the identity (1), we see that the decomposition of $\mathfrak{g}$ with respect to another principal TDS φ' looks like

$$\mathfrak{g} = \big(\mathrm{Ad} g \cdot V_1\big) \oplus \big(\mathrm{Ad} g \cdot V_2\big) \oplus \cdots \oplus \big(\mathrm{Ad} g \cdot V_\ell\big). \tag{3}$$

Definition 2.2. Recall that the singular cohomology with complex coefficients $H^*(G) = H^*(G, \mathbb{C})$ is a Hopf algebra, where the product of course comes from the cup product, and the coproduct $\Delta : H^*(G) \to H^*(G) \otimes H^*(G)$ is induced from the multiplication map $\mu : G \times G \to G$.

Let $P = P(\mathfrak{g}) \subset H^*(G)$ be the subspace of primitive elements, i.e.,

$$P = \{x \in H^*(G) \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

(Observe that $H^*(G)$ does not depend upon the isogeny class of G and hence the notation $P(\mathfrak{g})$ is justified.)

Since Δ is a graded homomorphism, $P \subset H^*(G)$ is a graded linear subspace. It is well-known that, by a result of Hopf-Koszul-Samelson, P is concentrated in odd degrees and, moreover, the canonical map, induced from the product,

$$\theta : \wedge^\bullet(P) \to H^*(G)$$

is a graded algebra isomorphism. In particular, P generates $H^*(G)$ as an algebra over $\mathbb{C}$.

We can think of $\wedge(\mathfrak{g}^*)$ as the algebra of left invariant $\mathbb{C}$-valued forms on a maximal compact subgroup G_o of G. By a result of Koszul ([K], Théorème 9.2, Chapitre IV), any $\omega \in \wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ is a closed form and, moreover, the induced map (identifying $H^*(G_o)$ with the de Rham cohomology $H^*_{dR}(G_o, \mathbb{C})$)

$$\eta : \wedge(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{\sim} H^*(G_o) \cong H^*(G)$$

is a graded algebra isomorphism, where the restriction map $H^*(G) \to H^*(G_o)$ is an isomorphism since G_o is a deformation retract of G.

Via the isomorphism η, we identify the graded subspace $P \subset H^*(G)$ of primitive elements with a graded subspace (still denoted by) $P \subset \wedge(\mathfrak{g}^*)^{\mathfrak{g}}$.

For any $d \geq 1$, let P_d be the subspace of P of (homogeneous) degree d elements. Then, by [Ko], Corollary 8.7,

$$\dim P_d = \#\{1 \leq i \leq \ell \mid n_i = d\}, \tag{4}$$

where n_i's (given by (2)) are the dimensions of irreducible components of $\mathfrak{g}$ under the principal TDS action.

In particular, if $\mathfrak{g}$ is not of type D_ℓ (with ℓ even), then

$$\dim P_d \leq 1 \tag{5}$$

and P_d is of dimension 1 if and only if d is equal to one of the $n_i's$. If $\mathfrak{g}$ is of type D_ℓ (with ℓ even),

$$\dim P_d \leq 1 \ \text{ if } \ d \neq 2\ell - 1, \ \text{ and } \ \dim P_{2\ell-1} = 2. \tag{6}$$

Fix a principal TDS. Hitchin made the following conjecture (cf. [Hi]).

Conjecture 2.3. *Let $\mathfrak{g}$ be any simple Lie algebra. For any primitive element $\omega \in P_d \subset \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$, there exists an irreducible subspace $V_\omega \subset \mathfrak{g}$ of dimension d with respect to the principal TDS action such that*

$$\omega|_{\wedge^d(V_\omega)} \neq 0.$$

Remark 2.4. (a) Unless $\mathfrak{g}$ is of type D_ℓ (with ℓ even), given $\omega \in P_d$, there exists a unique irreducible submodule V of dimension d in $\mathfrak{g}$ with respect to the principal TDS. Thus, V_ω is uniquely determined.

If $\mathfrak{g}$ is of type D_ℓ (with ℓ even), unless $d = 2\ell - 1$, given $\omega \in P_d$, there is a unique irreducible submodule V of dimension d in $\mathfrak{g}$. Thus, again V_ω is uniquely determined (for $d \neq 2\ell - 1$).

(b) A different choice of principal TDS results in the irreducible submodules being equal to $\mathrm{Ad}g \cdot V$, for some $g \in G$, and some irreducible submodule V for the original principal TDS. But, since we are only considering forms $\omega \in \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$ (which are, by definition, $\mathrm{Ad}\, G$-invariant), $\omega|_{\wedge^d(\mathrm{Ad}g \cdot V)} \neq 0$ if and only if $\omega|_{\wedge^d(V)} \neq 0$.

Now, we come to the main result of this section.

Theorem 2.5. *If Hitchin's conjecture is valid for any simply-laced simple Lie algebra $\mathfrak{g}$, then it is valid for any simple Lie algebra.*

More precisely, if Hitchin's conjecture is valid for $\mathfrak{g}$ of type $(A_{2\ell-1}; A_{2\ell}; D_4; E_6)$, then it is valid for $\mathfrak{g}$ of type $(C_\ell; B_\ell; G_2; F_4)$ respectively.

Proof: Let $\mathfrak{k}$ be a non simply-laced simple Lie algebra. Then, there exists a simply-laced simple Lie algebra $\mathfrak{g}$ together with a diagram automorphism σ (i.e., an automorphism σ of $\mathfrak{g}$ induced from a diagram automorphism of its Dynkin diagram) such that $\mathfrak{k}$ is the σ-fixed point $\mathfrak{g}^\sigma$ of $\mathfrak{g}$. Moreover, given $\mathfrak{k}$, we can choose $\mathfrak{g}$ to be of type given in the statement of the theorem. (For more details, see Section 3.1 on diagram folding.) In particular, we never need to take $\mathfrak{g}$ of type D_ℓ except D_4.

Choose a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ such that they both are stable under σ. let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{t}^*$ be the set of simple roots of $\mathfrak{g}$, where ℓ is the rank of $\mathfrak{g}$. Since σ keeps $\mathfrak{b}$ and $\mathfrak{t}$ stable, σ permutes the simple roots. Let $\{\widetilde{\beta}_1, \dots, \widetilde{\beta}_{\ell_\mathfrak{k}}\}$ be a set of simple roots taken exactly one simple root from each orbit of σ in Δ. Then, the fixed subalgebra $\mathfrak{b}_\mathfrak{k} := \mathfrak{b}^\sigma$ is a Borel subalgebra of $\mathfrak{k}$, $\mathfrak{t}_\mathfrak{k} := \mathfrak{t}^\sigma$ is a Cartan subalgebra of $\mathfrak{k}$ and $\{\beta_1, \dots, \beta_{\ell_\mathfrak{k}}\}$ is the set of simple roots of $\mathfrak{k}$, where $\beta_i := \widetilde{\beta}_i|_{\mathfrak{t}_\mathfrak{k}}$ (cf. [S1]). In particular, $\ell_\mathfrak{k}$ is the rank of $\mathfrak{k}$.

For any $1 \le n \le \ell_\mathfrak{k}$, choose a nonzero element $x_n \in \mathfrak{g}_{\widetilde{\beta}_n}$, where $\mathfrak{g}_{\widetilde{\beta}_n}$ is the root space of $\mathfrak{g}$ corresponding to the root $\widetilde{\beta}_n$. Define

$$y_n = \sum_{i=1}^{\mathrm{ord}(\sigma)} \sigma^i(x_n),$$

where $\mathrm{ord}(\sigma)$ is the order of σ (which is 2 except when $\mathfrak{g}$ is of type D_4 and $\mathfrak{k}$ is of type G_2, in which case it is 3). If $\widetilde{\beta}_n$ is fixed by σ, then σ acts trivially on $\mathfrak{g}_{\widetilde{\beta}_n}$ (cf. [S1]), hence y_n is never zero. Of course, $y_n \in \mathfrak{k}$ and, in fact, $y_n \in \mathfrak{k}_{\beta_n}$. Define the element $y \in \mathfrak{k}$ by

$$y = \sum_{n=1}^{\ell_\mathfrak{k}} y_n.$$

By [Ko], Theorem 5.3, y is a principal nilpotent element of $\mathfrak{k}$ and hence there exists a principal TDS in $\mathfrak{k}$:

$$\varphi : \mathfrak{sl}_2 \to \mathfrak{k} \quad \text{such that} \quad \varphi(X) = y.$$

Moreover, since

$$y = \sum_{n=1}^{\ell_\mathfrak{k}} \sum_{i=1}^{\mathrm{ord}(\sigma)} \sigma^i(x_n),$$

again using [Ko], Theorem 5.3, we get that y is a principal nilpotent of $\mathfrak{g}$ as well. Hence, φ is a principal TDS of $\mathfrak{g}$ also. Decompose $\mathfrak{g}$ under the adjoint action of $\mathfrak{sl}_2$ via φ:

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_{\ell_\mathfrak{k}} \oplus V_{\ell_\mathfrak{k}+1} \oplus \cdots \oplus V_\ell,$$

where $V_1 \oplus \cdots \oplus V_{\ell_\mathfrak{k}}$ is a decomposition of $\mathfrak{k}$.

Take a primitive element $\omega_d \in P_d(\mathfrak{k}) \subset \wedge^d(\mathfrak{k}^*)^\mathfrak{k}$, where $P_d(\mathfrak{k})$ is the space of primitive elements for $\mathfrak{k}$. By (subsequent) Theorem 3.5 , the canonical restriction map $\wedge^d(\mathfrak{g}^*) \to \wedge^d(\mathfrak{k}^*)$ induces a surjection

$$P_d(\mathfrak{g}) \to P_d(\mathfrak{k}), \quad \text{for any } d > 0.$$

Take a preimage $\widetilde{\omega}_d \in P_d(\mathfrak{g})$ of ω_d. By (4)-(5), there exists a unique irreducible $\mathfrak{sl}_2$-submodule V_{ω_d} of $\mathfrak{k}$ of dimension d. Further, by (4)-(6), there exists a unique irreducible $\mathfrak{sl}_2$-submodule $V_{\widetilde{\omega}_d} \subset \mathfrak{g}$ of dimension d. (For any $\mathfrak{k}$ not of type G_2, the uniqueness of $V_{\widetilde{\omega}_d}$ follows since we have chosen $\mathfrak{g}$ not of type D_ℓ; for $\mathfrak{k}$ of type G_2, $P_d(\mathfrak{k})$ is nonzero if and only if $d = 3, 11$ (cf. §2.1). Again, for these values of d, $\dim P_d(D_4) = 1$.) Hence, $V_{\omega_d} = V_{\widetilde{\omega}_d}$. Assuming the validity of Hitchin's conjecture for $\mathfrak{g}$, we get that $\widetilde{\omega}_d|_{\wedge^d(V_{\widetilde{\omega}_d})} \neq 0$. Hence,

$$\omega_d|_{\wedge^d(V_{\omega_d})} = \widetilde{\omega}_d|_{\wedge^d(V_{\widetilde{\omega}_d})} \neq 0.$$

This proves the theorem. ∎

3. GIT quotient $G//Ad\,G$ and diagram automorphisms

Let $\mathfrak{g}$ be a simple, simply-laced Lie algebra over $\mathbb{C}$ and let G be the connected, simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$. Let σ be a diagram automorphism of $\mathfrak{g}$ and let $\mathfrak{k} = \mathfrak{g}^\sigma$ be the fixed subalgebra. Then, $\mathfrak{k}$ is a simple Lie algebra again. Let K be the connected subgroup of G with Lie algebra $\mathfrak{k}$. In fact, $K = G^\sigma$ (cf. [S1]). For the connection of the root datum of K with that of G, we refer, e.g., to [S1].

With this notation, we have the following main result of this section.

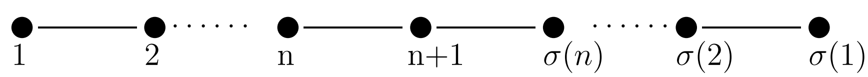
Theorem 3.1. *The canonical map $\phi : R(G) \to R(K)$ is surjective, where $R(G)$ denotes the representation ring of G (over $\mathbb{Z}$).*

In particular, the canonical map $K//Ad\,K \to G//Ad\,G$, between the GIT quotients, is a closed embedding.

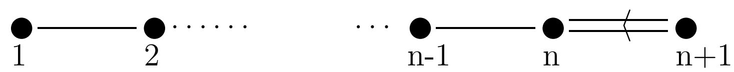
Before we come to the proof of the theorem, we need some notational preliminaries on diagram automorphisms and 'diagram folding' (i.e., the process of getting $\mathfrak{k}$ from $\mathfrak{g}$). As in Section 2, fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ of $\mathfrak{g}$ stable under σ. Then, $\mathfrak{b}_\mathfrak{k} := \mathfrak{b}^\sigma$ (resp. $\mathfrak{t}_\mathfrak{k} := \mathfrak{t}^\sigma$) is a Borel (resp. Cartan) subalgebra of $\mathfrak{k}$. Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset \mathfrak{t}^*$ be the simple roots of $\mathfrak{g}$ and let $\{\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{\ell_\mathfrak{k}}\}$ be a set of simple roots taken exactly one simple root from each orbit of σ in Δ. Then, $\Delta_\mathfrak{k} := \{\beta_1, \ldots, \beta_{\ell_\mathfrak{k}}\} \subset \mathfrak{t}_\mathfrak{k}^*$ is the set of simple roots of $\mathfrak{k}$, where $\beta_i := \widetilde{\beta}_i|_{\mathfrak{t}_\mathfrak{k}}$. In the following diagrams, we will make a specific choice of indexing convention in each case of diagram folding.

3.1. Diagram Folding: Dynkin diagrams of $(\mathfrak{g}, \mathfrak{k})$.

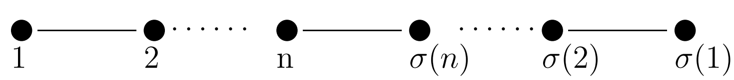
(A_{2n+1}, C_{n+1}) :



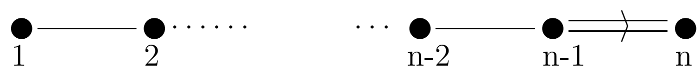
$\beta_i := \alpha_{i|\mathfrak{t}_\mathfrak{k}}$ for $i \leq n+1$ and β_{n+1} is a long root.



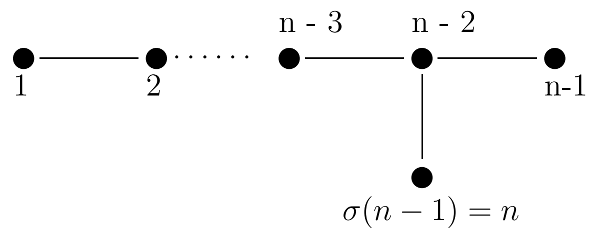
(A_{2n}, B_n) :



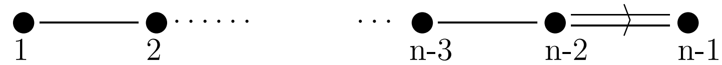
$\beta_i = \alpha_{i|\mathfrak{t}_\mathfrak{k}}$ for $1 \leq i \leq n$ and β_n is a short root.



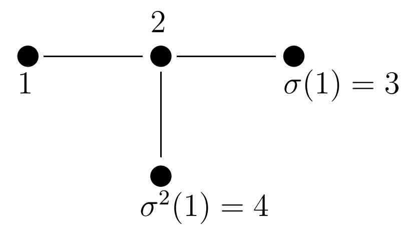
(D_n, B_{n-1}) :



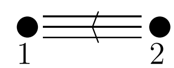
$\beta_i := \alpha_{i|\mathfrak{t}_\mathfrak{k}}$ for $1 \le i \le n-1$ and β_{n-1} is a short root.



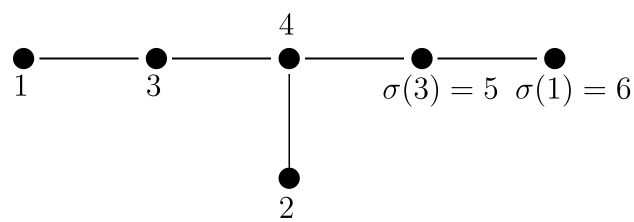
<u>(D_4, G_2) :</u>



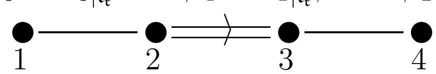
$\beta_1 := \alpha_{1|\mathfrak{t}_\mathfrak{k}}$, $\beta_2 := \alpha_{2|\mathfrak{t}_\mathfrak{k}}$, and β_2 is a long root.



<u>(E_6, F_4) :</u>



$\beta_1 = \alpha_{2|\mathfrak{t}_\mathfrak{k}}$, $\beta_2 = \alpha_{4|\mathfrak{t}_\mathfrak{k}}$, $\beta_3 = \alpha_{3|\mathfrak{t}_\mathfrak{k}}$ and $\beta_4 = \alpha_{1|\mathfrak{t}_\mathfrak{k}}$, with β_2 a long root.



Let $\{\varpi_1, \dots, \varpi_\ell\}$ (resp. $\{\nu_1, \dots, \nu_{\ell_\mathfrak{k}}\}$) be the fundamental weights for the root system of $\mathfrak{g}$ (resp. $\mathfrak{k}$). We next prove two facts unique to our context. For any simple root α, we denote the corresponding coroot by $\alpha^\vee$. We follow the indexing convention as in Subsection 3.1.

Lemma 3.2. *(a) If G is not of type A_{2n} or E_6, then $\rho(\varpi_i) = \nu_i$ for $1 \le i \le \ell_\mathfrak{k} :=$ rank$(\mathfrak{k})$.*

(b) If G is of type A_{2n}, then $\rho(\varpi_i) = \rho(\varpi_{2n-i+1}) = \nu_i$ for $1 \le i \le n-1$, and $\rho(\varpi_n) = \rho(\varpi_{n+1}) = 2\nu_n$.

(c) If G is of type E_6, $\rho(\varpi_1) = \rho(\varpi_6) = \nu_4$; $\rho(\varpi_2) = \nu_1$; $\rho(\varpi_3) = \rho(\varpi_5) = \nu_3$; $\rho(\varpi_4) = \nu_2$.

Proof: (a) It suffices to show

$$\langle \rho(\varpi_i), \beta_j^\vee \rangle = \delta_{i,j}, \quad \text{for } 1 \le i, j \le \ell_\mathfrak{k}. \tag{7}$$

In this case, we have ([S1])

$$\beta_j^\vee = \sum \alpha_k^\vee,$$

where the summation runs over the orbit of α_j under σ. For $1 \leq j \leq \ell_{\mathfrak{k}}$, no α_k is in the σ-orbit of α_j for any $1 \leq k \leq \ell_{\mathfrak{k}}$. Thus, the equation (7) follows.

(b) When G is of type A_{2n}, by [S1],

$$\beta_j^\vee = \begin{cases} \alpha_j^\vee + \alpha_{2n-j+1}^\vee, & \text{for } j \leq n-1, \\ \\ 2\alpha_n^\vee + 2\alpha_{n+1}^\vee, & \text{for } j = n. \end{cases}$$

So, for $1 \leq i \leq 2n$,

$$\begin{aligned} \langle \rho(\varpi_i), \beta_j^\vee \rangle &= \begin{cases} \langle \varpi_i, \alpha_j^\vee \rangle + \langle \varpi_i, \alpha_{2n-j+1}^\vee \rangle, & \text{for } j \leq n-1, \\ \\ 2\langle \varpi_i, \alpha_n^\vee \rangle + 2\langle \varpi_i, \alpha_{n+1}^\vee \rangle, & \text{for } j = n. \end{cases} \\ &= \begin{cases} \delta_{i,j} + \delta_{i,2n-j+1}, & \text{for } j \leq n-1, \\ \\ 2\delta_{i,n} + 2\delta_{i,n+1}, & \text{for } j = n. \end{cases} \end{aligned}$$

From this (b) follows.

(c) By [S1], following the indexing convention as in Subsection 3.1, we get that

$$\beta_1^\vee = \alpha_2^\vee,\ \beta_2^\vee = \alpha_4^\vee,\ \beta_3^\vee = \alpha_3^\vee + \alpha_5^\vee,\ \beta_4^\vee = \alpha_1^\vee + \alpha_6^\vee.$$

Thus,

$$\begin{aligned} &\rho(\varpi_1) = \rho(\varpi_6) = \nu_4, \\ &\rho(\varpi_2) = \nu_1, \\ &\rho(\varpi_3) = \rho(\varpi_5) = \nu_3, \\ &\rho(\varpi_4) = \nu_2. \end{aligned}$$

■

Let $\Lambda^+(\mathfrak{g}) \subset \mathfrak{t}^*$ (resp. $\Lambda^+(\mathfrak{k}) \subset \mathfrak{t}_{\mathfrak{k}}^*$) be the set of dominant integral weights for the root system of $\mathfrak{g}$ (resp. $\mathfrak{k}$) and let $\Lambda^+(K) \subset \Lambda^+(\mathfrak{k})$ be the submonoid of dominant characters for the group K, i.e., $\Lambda^+(K)$ is the set of characters of the maximal torus T_K (with Lie algebra $\mathfrak{t}_{\mathfrak{k}}$) of K which are dominant with respect to the group K. Observe that since G is simply-connected, $\Lambda^+(G) = \Lambda^+(\mathfrak{g})$. Moreover, under the restriction map $\rho : \mathfrak{t}^* \to \mathfrak{t}_{\mathfrak{k}}^*$,

$$\rho(\Lambda^+(\mathfrak{g})) = \Lambda^+(K). \tag{8}$$

To see this, let $\Lambda(K)$ be the character lattice of K (similarly for $\Lambda(G) = \Lambda(\mathfrak{g})$). Then, by Springer's original construction of $\Lambda(K)$ [S1], the restriction $\rho : \Lambda(\mathfrak{g}) \to \Lambda(K)$ is surjective. Further, from the description of the coroots of $\mathfrak{k}$ as in [S1], $\rho(\Lambda^+(\mathfrak{g})) \subset \Lambda^+(\mathfrak{k})$. Thus, we have

$$\rho(\Lambda^+(\mathfrak{g})) \subset \Lambda^+(\mathfrak{k}) \cap \Lambda(K) = \Lambda^+(K).$$

Conversely, in all cases except for $\mathfrak{g}$ of type A_{2n}, by Lemma 3.2, $\rho(\Lambda^+(\mathfrak{g})) = \Lambda^+(\mathfrak{k}) \supset \Lambda^+(K)$, so equation (8) holds in these cases. When $\mathfrak{g}$ is of type A_{2n}, again by Lemma 3.2,

$$\rho(\Lambda^+(\mathfrak{g})) = \left(\oplus_{i=1}^{n-1} \mathbb{Z}_+\nu_i\right) \oplus 2\mathbb{Z}_+\nu_n,$$

and

$$\Lambda(K) = \rho(\Lambda(\mathfrak{g})) = \left(\oplus_{i=1}^{n-1} \mathbb{Z}\nu_i\right) \oplus 2\mathbb{Z}\nu_n.$$

From this again, we see that (8) is satisfied. This proves (8) in all cases.

For any $\lambda \in \Lambda^+(\mathfrak{g})$, let $V(\lambda)$ be the irreducible G-module with highest weight λ. Similarly, for $\mu \in \Lambda^+(K)$, let $W(\mu)$ be the irreducible K-module with highest weight μ. We denote the fundamental representations $V(\varpi_i)$ of $\mathfrak{g}$ by V_i and $W(\nu_j)$ of $\mathfrak{k}$ by W_j.

Lemma 3.3. *For any $\lambda \in \Lambda^+(\mathfrak{g})$, $W(\rho(\lambda))$ has multiplicity one in $V(\lambda)$ as a $\mathfrak{k}$-module. (Observe that by* (8)*, $\rho(\lambda) \in \Lambda^+(K)$.)*

Proof: Note that the Borel subalgebra $\mathfrak{b}_{\mathfrak{k}}$ of $\mathfrak{k}$ is contained in the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. So, if v_λ is the highest weight vector of $V(\lambda)$ (of weight λ), then v_λ remains a highest weight vector of weight $\rho(\lambda)$ in $V(\lambda)$ for the action of $\mathfrak{k}$. Hence, $W(\rho(\lambda)) \subset V(\lambda)$.

Multiplicity one is clear from the weight consideration. ■

3.2. **Proof of Theorem 3.1.** Let $\{\mu_1, \ldots, \mu_N\} \subset \Lambda^+(K)$ be a set of semigroup generators of $\Lambda^+(K)$. Then, the classes $\{[W(\mu_j)]\}_{1\le j\le N}$ generate the $\mathbb{Z}$-algebra $R(K)$, where $[W(\mu_j)] \in R(K)$ denotes the class of the irreducible K-module $W(\mu_j)$ (cf. [P], Theorem 3.12).

We proceed separately for each of the five cases depending on the type of $(\mathfrak{g}, \mathfrak{k})$.

Case I (A_{2n+1}, C_{n+1}): By Lemmas 3.2 and 3.3, for $1 \le j \le n+1$, $W_j \subset V_j$ (as $\mathfrak{k}$-modules). Recall that $V_1 \simeq W_1 \simeq \mathbb{C}^{2n+2}$ (so $W_1 = V_1$) and $V_j = \wedge^j V_1$ for all $1 \le j \le 2n+1$. Also, for $2 \le j \le n+1$, W_j is given as the kernel of the surjective $\mathfrak{k}$-equivariant contraction map $\wedge^j W_1 \to \wedge^{j-2} W_1$. Hence, for $2 \le j \le n+1$, in $R(\mathfrak{k})$ (where $R(\mathfrak{k})$ is the representation ring of $\mathfrak{k}$), by [FH], Theorem 17.5,

$$[W_j] + [\wedge^{j-2} W_1] = [\wedge^j W_1].$$

Thus,

$$\phi([V_1]) = [W_1], \text{ and } \phi([V_j]) - \phi([V_{j-2}]) = [W_j], \text{ for } 2 \le j \le n+1,$$

where V_0 is interpreted as the trivial one dimensional module $\mathbb{C}$. Thus, the class $[W_j]$ of each fundamental representation lies in the image of ϕ, and hence ϕ is surjective.

Case II. (A_{2n}, B_n): By Lemmas 3.2 and 3.3, for $1 \le j \le n-1$, $W_j \subset V_j$ and $W(2\nu_n) \subset V_n$ (as $\mathfrak{k}$-modules). Recall that $V_1 \simeq W_1 \simeq \mathbb{C}^{2n+1}$ (so $W_1 = V_1$), and $V_j = \wedge^j V_1$ for all $1 \le j \le 2n$. Also, $W_j = \wedge^j W_1$ for $1 \le j \le n-1$ and $W(2\nu_n) = \wedge^n W_1$ (see, e.g., [FH], Theorem 19.14). Thus, as $\mathfrak{k}$-modules,

$$W_j = V_j, \; j \le n-1; \quad W(2\nu_n) = V_n.$$

Thus,

$$[W_1], \ldots, [W_{n-1}], [W(2\nu_n)] \in \text{ Image } \phi.$$

By Lemma 3.2 (b) and the identity (8), $\Lambda^+(K)$ is generated (as a semigroup) by $\{\nu_1, \ldots, \nu_{n-1}, 2\nu_n\}$. Hence, ϕ is surjective in this case.

Case III. (D_n, B_{n-1})**:** Recall that $V_1 \simeq \mathbb{C}^{2n}$ and $W_1 \simeq \mathbb{C}^{2n-1}$. By Lemmas 3.2 and 3.3, for $1 \le j \le n-1$, $W_j \subset V_j$ (as $\mathfrak{k}$-modules). Since $W_1 \subset V_1$ (as $\mathfrak{k}$-modules), we get (as $\mathfrak{k}$-modules):

$$V_1 = W_1 \oplus \mathbb{C}.$$

Thus, for $1 \le k \le n-2$, as $\mathfrak{k}$-modules,

$$V_k = \wedge^k V_1 = \wedge^k (W_1 \oplus \mathbb{C}) \simeq (\wedge^k W_1) \oplus (\wedge^{k-1} W_1) = W_k \oplus W_{k-1},$$

where the first equality is by [FH], Theorem 19.2; W_0 is interpreted as the one dimensional trivial module and the last equality is from the proof of Case II.

Since $W_{n-1} \subset V_{n-1}$ as $\mathfrak{k}$-modules, and both being spin representations have the same dimension 2^{n-1} (see, e.g., [GW], Section 6.2.2), we get $V_{n-1} = W_{n-1}$. Therefore,

$$\phi([V_k]) = [W_k] + [W_{k-1}] \text{ for } 1 \leq k \leq n-2, \text{ and } \phi([V_{n-1}]) = [W_{n-1}].$$

In particular, each of $[W_1], \dots, [W_{n-1}]$ lies in the image of ϕ, proving the surjectivity of ϕ in this case.

Case IV. (D_4, G_2): The two fundamental representations W_1 and W_2 have respective dimensions 7 and 14 ([FH], Section 22.3). On the other hand, V_1 is eight dimensional and $V_2 = \wedge^2 V_1$. Since $\rho(\varpi_1) = \nu_1$ (by Lemma 3.2), by Lemma 3.3 we get $W_1 \subset V_1$ (as $\mathfrak{k}$-modules). So, we have the decomposition (as $\mathfrak{k}$-modules):

$$V_1 = W_1 \oplus \mathbb{C}.$$

Thus, as $\mathfrak{k}$-modules,

$$V_2 = \wedge^2 V_1 = \wedge^2 (W_1 \oplus \mathbb{C}) \simeq \big(\wedge^2 W_1\big) \oplus W_1.$$

But, $\wedge^2 W_1 \simeq W_2 \oplus W_1$ ([FH], Section 22.3). Hence, as $\mathfrak{k}$-modules,

$$V_2 = W_2 \oplus W_1^{\oplus 2}.$$

This gives

$$\phi([V_1]) = [W_1] + 1 \text{ and } \phi([V_2]) = [W_2] + 2[W_1],$$

which proves the surjectivity of ϕ in this case.

Case V. (E_6, F_4): By Lemma 3.2(c), we see that ρ is surjective with kernel given by $\{a\varpi_1 + b\varpi_3 - b\varpi_5 - a\varpi_6 \mid a, b \in \mathbb{Z}\}$. Considering the images of ϖ_i under ρ, we have as $\mathfrak{k}$-modules (by Lemmas 3.2(c) and 3.3),

$$\begin{aligned} &W_1 \subset V_2, \\ &W_2 \subset V_4, \\ &W_3 \subset V_3, V_5, \\ &W_4 \subset V_1, V_6. \end{aligned}$$

Using [Sl], Tables 44 and 47 or [LiE], we obtain

$$\begin{array}{ll} \dim(W_1) = 52, & \dim(V_2) = 78, \\ \dim(W_2) = 1274, & \dim(V_4) = 2925, \\ \dim(W_3) = 273, & \dim(V_3) = \dim(V_5) = 351, \\ \dim(W_4) = 26, & \dim(V_1) = \dim(V_6) = 27. \end{array}$$

Along with the fundamental $\mathfrak{k}$-modules, there are only three other irreducible $\mathfrak{k}$-modules of dimensions at most 1651 ([Sl], Table 44, or [LiE]). These are $\dim(W(2\nu_4)) = 324$, $\dim(W(\nu_1 + \nu_4)) = 1053$, and $\dim(W(2\nu_1)) = 1053$.

Let U^k denote an arbitrary $\mathfrak{k}$-module of dimension k. Considering the dimensions, we get (as $\mathfrak{k}$-modules):

$$\begin{aligned} &V_1 = V_6 = W_4 \oplus \mathbb{C}, \\ &V_2 = W_1 \oplus U^{26}, \\ &V_3 = V_5 = W_3 \oplus U^{78}, \\ &V_4 = W_2 \oplus U^{1651}. \end{aligned}$$

Now, U^{26} must be either W_4 or the trivial module $\mathbb{C}^{26}$, and U^{78} must be some combination of W_4, W_1 and $\mathbb{C}$. Since $\phi([V_1]) - 1 = [W_4]$, this implies that $[W_4]$, $[W_1]$ and $[W_3]$ are in the image of ϕ. (We remark that [Sl] gives $F_4 \subset E_6$ branching, but we continue without these results for clarity and completeness.)

Using appropriate tensor product decompositions in [LiE], we get

$$[W(2\nu_4)] = [W_4]^2 - [W_3] - [W_1] - [W_4] - 1, \tag{9}$$

$$[W(\nu_1 + \nu_4)] = [W_1][W_4] - [W_3] - [W_4], \tag{10}$$

$$[W(2\nu_1)] = [W_1]^2 - [W_2] - [W(2\nu_4)] - [W_1] - 1. \tag{11}$$

Since W_2 appears in V_4 as a $\mathfrak{k}$-submodule exactly once by Lemma 3.3, from the above identities, we get that $[W_2]$ lies in the image of ϕ if $W(2\nu_1)$ is not a component of V_4. In fact, we prove below that $2\nu_1$ is not a $\mathfrak{k}$-weight of V_4 at all.

In order that $2\nu_1$ be a $\mathfrak{k}$-weight of V_4, we should have $2\nu_1 = \mu|_{\mathfrak{t}_\mathfrak{k}}$, where μ is a weight of V_4. This is only possible if there exists a weight of V_4 of the form $\mu = a\varpi_1 + 2\varpi_2 + b\varpi_3 - b\varpi_5 - a\varpi_6$, for some $a, b \in \mathbb{Z}$. We claim this is impossible. Indeed, all weights of V_4 are of the form $\varpi_4 - \sum_{i=1}^{6} d_i\alpha_i$, where $d_i \in \mathbb{Z}^+$. If such μ existed, then by [Bo], Planche V,

$$\begin{aligned}
&\textstyle\sum_{i=1}^{6} d_i\alpha_i = \varpi_4 - \mu \\
&= \varpi_4 + a(\varpi_6 - \varpi_1) - 2\varpi_2 + b(\varpi_5 - \varpi_3) \\
&= (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) + (a/3)(-2\alpha_1 - \alpha_3 + \alpha_5 + 2\alpha_6) \\
&-2(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6) + (b/3)(-\alpha_1 - 2\alpha_3 + 2\alpha_5 + \alpha_6),
\end{aligned}$$

from which we immediately see a contradiction since the α_2 coefficient is -1.

This completes the proof in this last case and hence the proof of the first part of Theorem 3.1 is completed.

To prove that $\eta : K//\mathrm{Ad}\ K \to G//\mathrm{Ad}\ G$ is a closed embedding, it suffices to show that the induced map between the affine coordinate rings $\eta^* : \mathbb{C}[G//\mathrm{Ad}\ G] \to \mathbb{C}[K//\mathrm{Ad}\ K]$ is surjective. But, by [P], Theorem 3.5, there is a functorial isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} R(G) \to \mathbb{C}[G//\mathrm{Ad}\ G],$$

and similarly we have an isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} R(K) \to \mathbb{C}[K//\mathrm{Ad}\ K].$$

From this the surjectivity of η^* follows from the surjectivity of $R(G) \to R(K)$. This proves the theorem. □

We give the following Lie algebra analogue as a corollary.

Corollary 3.4. *The canonical restriction map*

$$S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{k}^*)^{\mathfrak{k}}$$

is surjective.

Proof: By [St], §6.4, for any connected semisimple algebraic group H over $\mathbb{C}$, the restriction map

$$r : \mathbb{C}[H//\mathrm{Ad}\ H] \simeq \mathbb{C}[H]^H \to \mathbb{C}[T_H]^{W_H} \tag{12}$$

is an isomorphism of $\mathbb{C}$-algebras, where $T_H \subset H$ is a maximal torus and W_H is the Weyl group of H.

Similarly, the restriction map

$$r_o : \mathbb{C}[\mathfrak{h}]^H \to \mathbb{C}[\mathfrak{t}_\mathfrak{h}]^{W_H} \tag{13}$$

is a graded algebra isomorphism, where $\mathfrak{h}$ (resp. $\mathfrak{t}_\mathfrak{h}$) is the Lie algebra of H (resp. T_H). Thus, to prove the corollary, it suffices to show that the canonical restriction map

$$\beta_o^* : \mathbb{C}[\mathfrak{t}]^W \to \mathbb{C}[\mathfrak{t}_\mathfrak{k}]^{W_K}$$

is surjective,where W (resp. W_K) is the Weyl group of G (resp. K). Since β_o^* is a graded algebra homomorphism induced from the $\mathbb{C}^*$-equivariant map $\beta_o : \mathfrak{t}_\mathfrak{k}/W_K \to \mathfrak{t}/W$ (where the $\mathbb{C}^*$-action is the standard homothety action), it suffices to show that the tangent map between the Zariski tangent spaces at 0:

$$(d\beta_o)_0 : T_0(\mathfrak{t}_\mathfrak{k}/W_K) \to T_0(\mathfrak{t}/W)$$

is injective. Let T^{anal} denote the analytic tangent space. Then, the canonical map

$$T_x^{anal}(X) \to T_x(X)$$

is an isomorphism for any algebraic variety X and any point $x \in X$.

Consider the commutative diagram:

$$\begin{tikzcd} \mathfrak{t}_\mathfrak{k}/W_K \arrow[r, "\beta_o"] \arrow[d, "\mathrm{Exp}"] & \mathfrak{t}/W \arrow[d, "\mathrm{Exp}"] \\ T_K/W_K \arrow[r, "\beta"'] & T/W, \end{tikzcd}$$

where $T_K \subset K$ is the maximal torus with Lie algebra $\mathfrak{t}_\mathfrak{k}$ and $\beta : T_K/W_K \to T/W$ is the canonical map. Since T_K, T are tori, Exp is a local isomorphism in the analytic category. In particular, there exist open subsets (in the analytic topology) $0 \in U_\mathfrak{k} \subset \mathfrak{t}_\mathfrak{k}/W_K, 0 \in U \subset \mathfrak{t}/W, 1 \in V_K \subset T_K/W_K$ and $1 \in V \subset T/W$ such that $\beta_o(U_K) \subset U$ and $\mathrm{Exp}_{|U_\mathfrak{k}} : U_\mathfrak{k} \to V_K$ is an analytic isomorphism and so is $\mathrm{Exp}_{|U} : U \to V$. Since, by Theorem 3.1 and the isomorphism (12), β is a closed embedding,

$$(d\beta)_1 : T_1^{anal}(T_K/W_K) \simeq T_1(T_K/W_K) \to T_1^{anal}(T/W) \simeq T_1(T/W)$$

is injective and hence so is $T_0(\mathfrak{t}_\mathfrak{k}/W_K) \to T_0(\mathfrak{t}/W)$. This proves the corollary. ■

As a consequence of Corollary 3.4, we get the following.

Theorem 3.5. *With the notation and assumptions as in Theorem 3.1, the canonical restriction map $\gamma : H^*(G) \to H^*(K)$ is surjective. Moreover, this induces a surjective (graded) map*

$$\gamma_o : P(\mathfrak{g}) \to P(\mathfrak{k}),$$

where $P(\mathfrak{g}) \subset H^*(G)$ is the subspace of primitive elements.

Proof: From the definition of coproduct, it is easy to see that the following diagram is commutative:

$$\begin{tikzcd} H^*(G) \arrow[r, "\Delta_G"] \arrow[d, "\gamma"] & H^*(G) \otimes H^*(G) \arrow[d, "\gamma\otimes\gamma"'] \\ H^*(K) \arrow[r, "\Delta_K"'] & H^*(K) \otimes H^*(K). \end{tikzcd}$$

Thus, γ takes $P(\mathfrak{g})$ to $P(\mathfrak{k})$.

Let $\mathfrak{h}$ be a reductive Lie algebra. For any $v \in \mathfrak{h}$, define the derivation $i(v) : S(\mathfrak{h}^*) \to S(\mathfrak{h}^*)$ given by $i(v)(f) = f(v)$, for $f \in \mathfrak{h}^*$. Further, define an algebra homomorphism $\lambda : S(\mathfrak{h}^*) \to \wedge^{\text{even}}(\mathfrak{h}^*)$ by $\lambda(f) = df$, for $f \in \mathfrak{h}^* = S^1(\mathfrak{h}^*)$, where $d : \wedge^1(\mathfrak{h}^*) = \mathfrak{h}^* \to \wedge^2(\mathfrak{h}^*)$ is the standard differential in the Lie algebra cochain complex $\wedge^\bullet(\mathfrak{h}^*)$. Now, define the *transgression map*

$$\tau = \tau_{\mathfrak{h}} : S^+(\mathfrak{h}^*)^{\mathfrak{h}} \to \wedge^+(\mathfrak{h}^*)^{\mathfrak{h}}, \;\; \tau(p) = \sum_j e_j^* \wedge \lambda(i(e_j)p),$$

for $p \in S^+(\mathfrak{h}^*)^{\mathfrak{h}}$, where $\{e_j\}$ is a basis of $\mathfrak{h}$ and $\{e_j^*\}$ is the dual basis of $\mathfrak{h}^*$.

By a result of Cartan (cf. [Ca], Théorème 2; also see [L]), τ factors through

$$S^+(\mathfrak{h}^*)^{\mathfrak{h}}/(S^+(\mathfrak{h}^*)^{\mathfrak{h}}) \cdot (S^+(\mathfrak{h}^*)^{\mathfrak{h}})$$

to give an injective map

$$\bar{\tau} : S^+(\mathfrak{h}^*)^{\mathfrak{h}}/(S^+(\mathfrak{h}^*)^{\mathfrak{h}}) \cdot (S^+(\mathfrak{h}^*)^{\mathfrak{h}}) \to \wedge^+(\mathfrak{h}^*)^{\mathfrak{h}}$$

with image precisely equal to the space of primitive elements $P(\mathfrak{h})$. From the definition of τ, it is easy to see that the following diagram is commutative:

$$\begin{tikzcd} S^+(\mathfrak{g}^*)^{\mathfrak{g}} \arrow[r, "\tau_{\mathfrak{g}}"] \arrow[d] & \wedge^+(\mathfrak{g}^*)^{\mathfrak{g}} \arrow[d] \\ S^+(\mathfrak{k}^*)^{\mathfrak{k}} \arrow[r, "\tau_{\mathfrak{k}}"] & \wedge^+(\mathfrak{k}^*)^{\mathfrak{k}}, \end{tikzcd}$$

where the vertical maps are the canonical restriction maps. By using Corollary 3.4, this proves that $P(\mathfrak{g})$ surjects onto $P(\mathfrak{k})$. Since $P(\mathfrak{k})$ generates $\wedge^*(\mathfrak{k}^*)^{\mathfrak{k}} \simeq H^*(K)$ as an algebra, we get that γ is surjective. This proves the theorem. ■

Remark 3.6. As a consequence of the above theorem, we see that the Leray-Serre homology (or cohomology) spectral sequence with coefficients in $\mathbb{C}$ for the fibration

$$K \to G \to G/K$$

degenerates at the E^2-term.

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