BOUNDED-SIZE RULES: THE BARELY SUBCRITICAL REGIME

SHANKAR BHAMIDI, AMARJIT BUDHIRAJA, AND XUAN WANG

ABSTRACT. Bounded-size rules(BSR) are dynamic random graph processes which incorporate limited choice along with randomness in the evolution of the system. Typically one starts with the empty graph and at each stage two edges are chosen uniformly at random. One of the two edges is then placed into the system according to a decision rule based on the sizes of the components containing the four vertices. For bounded-size rules, all components of size greater than some fixed $K \geq 1$ are accorded the same treatment. Writing $\mathbf{BSR}(t)$ for the state of the system with $\lfloor nt/2 \rfloor$ edges, Spencer and Wormald [24] proved that for such rules, there exists a (rule dependent) critical time t_c such that when $t < t_c$ the size of the largest component is of order $\log n$ while for $t > t_c$, the size of the largest component is of order n. In this work we obtain upper bounds (that hold with high probability) of order $n^{2\gamma} \log^4 n$, on the size of the largest component, at time instants $t_n = t_c - n^{-\gamma}$, where $\gamma \in (0, 1/4)$. This result for the barely subcritical regime forms a key ingredient in the study undertaken in [4], of the asymptotic dynamic behavior of the process describing the vector of component sizes and associated complexity of the components for such random graph models in the critical scaling window. The proof uses a coupling of BSR processes with a certain family of inhomogeneous random graphs with vertices in the type space $\mathbb{R}_+ \times \mathcal{D}([0, \infty) : \mathbb{N}_0)$ where $\mathcal{D}([0, \infty) : \mathbb{N}_0)$ is the Skorohod D-space of functions that are right continuous and have left limits, with values in the space of nonnegative integers $\mathbb{N}_0$, equipped with the usual Skorohod topology. The coupling construction also gives an alternative characterization (than the usual explosion time of the susceptibility function) of the critical time t_c for the emergence of the giant component in terms of the operator norm of integral operators on certain L^2 spaces.

1. INTRODUCTION

The classical Erdős-Rényi random graph can be thought of as a dynamic random graph process on the vertex set $[n] := \{1, 2, \ldots, n\}$, where one starts with the empty graph $\mathbf{0}_n$ (the graph with n vertices and no edges) and at each discrete time step chooses an edge uniformly at random and places it in the configuration. Denote by $\mathbf{ER}^{(n)}(t)$ for the state of the graph obtained after $\lfloor nt/2 \rfloor$ steps. For any graph $\mathbf{G}$, denote $\mathcal{C}_i(\mathbf{G})$ for the i-th largest component, and $|\mathcal{C}_i(\mathbf{G})|$ for its size (number of vertices). Classical results ([7, 11, 12]) say that for fixed $t < 1$, the size of the largest component $|\mathcal{C}_1(\mathbf{ER}^{(n)}(t))|$ is $O(\log n)$ while for $t > 1$, $|\mathcal{C}_1(\mathbf{ER}^{(n)}(t))| \sim f(t)n$. Here $f(t) > 0$ is the survival probability of an associated supercritical branching process. For $t > 1$, the size of the second largest component $|\mathcal{C}_2(\mathbf{ER}^{(n)}(t))| = O(\log n)$. The largest component is often referred to as as the giant component.

There have been several works aimed at understanding the nature and emergence of this giant as t transitions from below to above $t_c = 1$ ([2, 13]). In recent years, motivated by a question of Achlioptas, there has been a significant interest in investigating more general dynamical random graph models. The driving theme has been to understand the role of limited choice along with randomness in the evolution of the network, in particular the time and nature of emergence of the giant component. The simplest such model that has been rigorously analyzed, referred to as the Bohman-Frieze process, can be described as follows. Start with the empty graph at time $t = 0$. At each discrete time step, choose two edges e_1, e_2 uniformly at random amongst all pairs of ordered edges. Place edge $e_1 = (v_1, v_2)$ if both end points v_1, v_2 are isolated vertices (components of size one), otherwise use edge e_2.

Despite this conceptually simple modification of the standard Erdős-Rényi random graph process, a rigorous understanding of this process turns out to be non-trivial. Write $\mathbf{BF}^{(n)}(t)$ for the state of the system when we have placed $\lfloor nt/2 \rfloor$ edges. Bohman and Frieze ([5, 6]) showed that there exists a time $t > 1$ and $\varepsilon \in (0, 1)$ such that the size of the largest component $|\mathcal{C}_1(\mathbf{BF}^{(n)}(t))| = o(n^\varepsilon)$. Thus this simple modification delays the time of emergence of a giant component.

In [24], these results were substantially refined and extended to the context of all *bounded-size rules* (BSR) which we now describe.

The bounded-size K-rule process $\{\mathbf{BSR}^{(n)}(t)\}_{t \geq 0}$. Fix $K \geq 0$, this will be a parameter in the construction of the process. Bounded-size rules treat all components of size greater than K in an identical fashion. Let $\Omega_K = \{1, 2, \ldots, K, \varpi\}$. Conceptually ϖ represents components of size greater than K. Given a graph $\mathbf{G}$ and a vertex $v \in \mathbf{G}$, write $\mathcal{C}_v(\mathbf{G})$ for the component that contains v. Define

$$c_{\mathbf{G}}(v) := \begin{cases} |\mathcal{C}_v(\mathbf{G})| & \text{if } |\mathcal{C}_v(\mathbf{G})| \leq K \\ \varpi & \text{if } |\mathcal{C}_v(\mathbf{G})| > K \end{cases} \qquad (1.1)$$

For a quadruple of (not necessarily distinct) vertices v_1, v_2, v_3, v_4, write $\vec{v}$ for the ordered quadruple $\vec{v} = (v_1, v_2, v_3, v_4)$. Let $c_{\mathbf{G}}(\vec{v}) = (c_{\mathbf{G}}(v_1), c_{\mathbf{G}}(v_2), c_{\mathbf{G}}(v_3), c_{\mathbf{G}}(v_4))$. Fix $F \subseteq \Omega_K^4$. The set F will be another parameter in the construction of the process. The F-bounded-size rule(F-BSR) is defined as follows:

(a) At time $k = 0$ start with the empty graph $\mathbf{BSR}_0^{(n)} := \mathbf{0}_n$ on $[n]$ vertices.
(b) For $k \geq 0$, having constructed the graph $\mathbf{BSR}_k^{(n)}$, construct $\mathbf{BSR}_{k+1}^{(n)}$ as follows. Choose 4 vertices $\vec{v} = (v_1, v_2, v_3, v_4)$ uniformly at random amongst all n^4 possible quadruples and let $c_k(\vec{v}) = c_{\mathbf{BSR}_k}(\vec{v})$. If $c_k(\vec{v}) \in F$ then $\mathbf{BSR}_{k+1}^{(n)} = \mathbf{BSR}_k^{(n)} \cup (v_1, v_2)$ else $\mathbf{BSR}_{k+1}^{(n)} = \mathbf{BSR}_k^{(n)} \cup (v_3, v_4)$.

We are interested in the dynamics of the rescaled process $\mathbf{BSR}^{(n)}(t) = \mathbf{BSR}_{\lfloor nt/2 \rfloor}^{(n)}$. Mathematically it is more convenient to work with a formulation in which edges are added at Poissonian time instants rather than at fixed discrete times. More precisely, we will consider the following random graph process (denoted once more as $\mathbf{BSR}^{(n)}(t)$). For every quadruple of vertices $\vec{v} = (v_1, v_2, v_3, v_4) \in [n]^4$, let $\mathcal{P}_{\vec{v}}$ be a Poisson process with rate $1/2n^3$, independent between quadruples. Note that this implies that the rate of creation of edges is $n^4 \times 1/2n^3 = n/2$. Thus we have sped up time by a factor $n/2$ as in the above discrete time construction. Start with $\mathbf{BSR}^{(n)}(0) = \mathbf{0}_n$. For any $t \geq 0$, at which there is a point in $\mathcal{P}_{\vec{v}}$ for a quadruple $\vec{v} \in [n]^4$, define

$$\mathbf{BSR}^{(n)}(t) = \begin{cases} \mathbf{BSR}^{(n)}(t-) \cup (v_1, v_2) & \text{if } c_{t-}(\vec{v}) \in F \\ \mathbf{BSR}^{(n)}(t-) \cup (v_3, v_4) & \text{otherwise,} \end{cases} \qquad (1.2)$$

where $c_{t-}(\vec{v}) = c_{\mathbf{BSR}(t-)}(\vec{v})$.

Two examples of such processes are Erdős-Rényi process (here $K = 0$, $\Omega_K = \{\varpi\}$ and $F = \{(\varpi, \varpi, \varpi, \varpi)\}$) and Bohman-Frieze process (here $K = 1$, $\Omega_K = \{1, \varpi\}$ and $F = \{(1, 1, j_3, j_4) : j_3, j_4 \in \Omega_K\}$). Spencer and Wormald[24] showed that every bounded-size rules exhibits a phase transition similar to the Erdős-Rényi random graph process. More precisely, write $\mathcal{C}_i^{(n)}(t)$ for the i-th largest component in $\mathbf{BSR}^{(n)}(t)$, and $|\mathcal{C}_i^{(n)}(t)|$ for the size of this component. Define the *susceptibility* function

$$\mathcal{S}_2(t) = \sum_{i=1}^{\infty} |\mathcal{C}_i^{(n)}(t)|^2. \tag{1.3}$$

Then [24] proves the following result.

Theorem 1.1 (Theorem 1.1, [24]). *Fix $F \in \Omega_K^4$. Then for the random graph process associated with the F-BSR, there exists deterministic monotonically increasing function $s_2(t)$ and a critical time t_c such that $\lim_{t \uparrow t_c} s_2(t) = \infty$ and*

$$\frac{\mathcal{S}_2(t)}{n} \xrightarrow{\mathbb{P}} s_2(t) \text{ as } n \to \infty, \text{ for all } t \in [0, t_c).$$

For fixed $t < t_c$, $|\mathcal{C}_1^{(n)}(t)| = O(\log n)$ while for $t > t_c$, $|\mathcal{C}_1^{(n)}(t)| = \Theta_P(n)$.

Here we use o, O, Θ in the usual manner. Given a sequence of random variables $\{\xi_n\}_{n \geq 1}$ and a function $f(n)$, we say $\xi_n = O(f)$ if there is a constant C such that $\xi_n \leq Cf(n)$ with high probability (whp), and we say $\xi_n = \Omega(f)$ if there is a constant C such that $\xi_n \geq Cf(n)$ whp. Say that $\xi_n = \Theta(f)$ if $\xi_n = O(f)$ **and** $\xi_n = \Omega(f)$. In addition, we say $\xi_n = o(f)$ if $\xi_n/f(n) \xrightarrow{\mathbb{P}} 0$.

Thus as t transitions from less than t_c to greater than t_c, the size of the largest component jumps from size $O(\log n)$ to a giant component $\Theta(n)$. The aim of this work is to study the barely subcritical regime, i.e. to analyze the behavior of the size of the largest component at times $t = t_c - \varepsilon_n$ where $\varepsilon_n \to 0$. The following is the main result.

Theorem 1.2 (**Barely subcritical regime**). *Fix $F \subset \Omega_K^4$ and $\gamma \in (0, 1/4)$. Then there exists $B \in (0, \infty)$ such that,*

$$\mathbb{P}\left\{|\mathcal{C}_1^{(n)}(t)| \leq B\frac{(\log n)^4}{(t_c - t)^2}, \ \forall t \leq t_c - n^{-\gamma}\right\} \to 1,$$

as $n \to \infty$.

As another consequence of our proofs, we obtain an alternative characterization of the critical time for a bounded-size rule given in Theorem 1.3 below. Let $\mathcal{X} = [0, \infty) \times \mathcal{D}([0, \infty) : \mathbb{N}_0)$ where $\mathcal{D}([0, \infty) : \mathbb{N}_0)$ is the Skorohod D-space of functions that are right continuous and have left limits with values in the space of nonnegative integers, equipped with the usual Skorohod topology. Given a finite measure μ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and a measurable map $\kappa : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ satisfying $\int_{\mathcal{X} \times \mathcal{X}} \kappa^2(\mathbf{x}, \mathbf{y})\mu(d\mathbf{x})\mu(d\mathbf{y}) < \infty$, define the integral operator $\mathcal{K} : L^2(\mathcal{X}, \mu) \to L^2(\mathcal{X}, \mu)$ as

$$\mathcal{K}f(\mathbf{x}) = \int_{\mathcal{X}} \kappa(\mathbf{x}, \mathbf{y})f(\mathbf{y})\mu(d\mathbf{y}), f \in L^2(\mathcal{X}, \mu), \ \mathbf{x} \in \mathcal{X}.$$

We refer to κ as a kernel on $\mathcal{X} \times \mathcal{X}$ and $\mathcal{K}$ as the integral operator associated with (κ, μ). We will show the following result.

Theorem 1.3 (**Characterization of the critical time**). *Fix $F \subset \Omega_K^4$. Then there exists a collection of kernels $\{\kappa_t\}_{t \geq 0}$ on $\mathcal{X} \times \mathcal{X}$ and finite measures $\{\mu_t\}_{t \geq 0}$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ such that*

the integral operators $\mathcal{K}_t$ associated with (κ_t, μ_t), $t > 0$, have the property that the operator norms $\rho(t) = \|\mathcal{K}_t\|$ are continuous and strictly increasing in t. Furthermore, t_c is the unique time instant such that $\rho(t_c) = 1$.

See Section 4.3 for a precise definition of κ_t and μ_t. We postpone the discussion of the connection between the integral operators in Theorem 1.3 and the BSR processes to Section 2. Using arguments similar to [17] for the Bohman-Frieze model, one can show that for any fixed $\varepsilon > 0$, the size of the largest component at time $t = t_c - \varepsilon$ can be lower bounded as $|\mathcal{C}_1^{(n)}(t)| \geq A \log n / (t_c - t)^2$ where A is a constant independent of ε. Thus up to logarithmic factors one expects the upper bound in Theorem 1.2 to be tight.

Theorem 1.2 plays a central role in the study of the asymptotic dynamic behavior of the process describing the vector of component sizes and associated surpluses for BSR processes in the critical scaling window and its connections with the augmented multiplicative coalescent process. This study is the subject of [4] to which we refer the reader for details.

1.1. **Organization of the paper.** The paper is organized as follows. In Section 2 we give a discussion of the main result. Section 3 collects some notation used in this work. In Section 4 we introduce and analyze certain inhomogeneous random graph processes associated with the BSR process. Finally, in Section 5 we complete the proofs of Theorems 1.2 and 1.3.

2. DISCUSSION

We now give some background, open problems and general discussion of the results in this work.

2.1. **Subcritical and supercritical random graphs:** There has been a considerable interest in understanding various properties of random graph models in the barely subcritical and supercritical regime, see for example ([15, 16, 21]) for various results on complex network models such as the configuration model in the subcritical regime, [10, 18] for structural properties of such graphs including mixing times of the nearly supercritical Erdős-Rényi random graph, [25] for an analysis of the Hamming cube near the critical regime and [8] for an extensive analysis of a general class of the inhomogeneous random graphs. In recent years there has been a significant effort in understanding a special (non-bounded-size) rule called the Product rule ([1]), where one uses the edge that minimizes the product of the components at the end points. Simulations in [1] suggest that the nature of the emergence of the giant component is different from that observed for rules such as the Erdős-Rényi or Bohman-Frieze process. Conceptually such rules tend to be harder to analyze since one needs to keep track not just of vertices in components upto size K for $K < \infty$ but for all K. There has been recent progress in understanding such rules [22]. The subcritical regime for such processes has been studied in [23] where it is shown that there exists a critical time t_c such that for $t > t_c$, the susceptibility function defined as in (1.3) satisfies $\mathcal{S}_2(t)/n \xrightarrow{\mathbb{P}} \infty$ as $n \to \infty$. Furthermore there exist functions $\{f_k(\cdot)\}_{k \geq 1}$ such that for $t < t_c$ the proportion of vertices in components of size k remains closely concentrated about $f_k(t)$ as $n \to \infty$.

For bounded-size rules, the only known results in the barely subcritical regime are in the context of the Bohman-Frieze process in [17] and [3] (also see [14] where scaling exponents for susceptibility functions in the Bohman-Frieze process were derived). In [17] it was shown that for fixed ε, the largest component in the Bohman-Frieze process at time $t_c - \varepsilon$ satisfies $|\mathcal{C}_1(\mathbf{BF}^{(n)}(t_c - \varepsilon))| = \Theta(\log n / \varepsilon^2)$. In [3], upper bounds when $\varepsilon = \varepsilon_n$, where $\varepsilon_n \to 0$, were obtained and a result analogous to Theorem 1.2 was shown using the special structure of certain differential equations associated with the Bohman-Frieze process.

2.2. **Optimal scaling and open questions:** Note that for a fixed $\varepsilon > 0$, Theorem 1.2 gives an upper bound of $B(\log n)^4/\varepsilon^2$. One would expect, similar to the special case of the Bohman-Frieze process treated in [17], that for fixed ε, $|\mathcal{C}_1(\mathbf{BF}^{(n)}(t_c - \varepsilon))| = \Theta(\log n/\varepsilon^2)$. It would be interesting to see if the results in the paper can be refined to prove this result for general bounded-size rules. Also note that Theorem 1.2 considers time instants $t_n = t_c - n^{-\gamma}$ for $\gamma \in (0, 1/4)$. A separate analysis that uses this result as a starting point is carried out in [4] to treat the component sizes in the critical scaling window i.e. when $\gamma = 1/3$. It would be interesting to extend the analysis of the current paper to get refined estimates on the size of the largest component when $\gamma \in [1/4, 1/3)$.

2.3. **Connection to the discrete time system:** We use the continuous time construction given in terms of Poisson processes, as opposed to the discrete time construction, for mathematical convenience (see e.g. the various martingale estimates in Section 4). It is easy to show that in the continuous time construction, by time $t_n = t_c - n^{-\gamma}$, the number of edges in the system is of the order $nt_c - n^{1-\gamma} + O(\sqrt{n})$. Using this and the monotonicity of the process it is easy to check that Theorem 1.2 holds for the discrete time version as well.

3. Notation

We collect some notation used through the rest of the paper. All unspecified limits are taken as $n \to \infty$. We use $\xrightarrow{\mathbb{P}}$ and $\xrightarrow{d}$ to denote convergence in probability and in distribution respectively. Given a sequence of events $\{E_n\}_{n\geq 1}$, we say E_n occurs with high probability (whp) if $\mathbb{P}\{E_n\} \to 1$.

For a set S and a function $\mathbf{g} : S \to \mathbb{R}^k$, we write $||\mathbf{g}||_\infty = \sum_{i=1}^k \sup_{s\in S} |g_i(s)|$, where $\mathbf{g} = (g_1, \cdots g_k)$. For a Polish space S, we denote by $\mathrm{BM}(S)$, the space of bounded measurable functions on S (equipped with the Borel sigma-field $\mathcal{B}(S)$.) For a finite set S, $|S|$ denotes the number of elements in the set. $\mathbb{N}_0$ is the set of nonnegative integers. For ease of notation, we shall often suppress the dependence on n and shall write for example $\mathbf{BSR}(t) = \mathbf{BSR}^{(n)}(t)$. Recall the Poisson processes $\mathcal{P}_{\vec{v}}$ used to construct $\mathbf{BSR}(\cdot)$ in Section 1. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the associated filtration: $\mathcal{F}_t = \sigma\left\{\mathcal{P}_{\vec{v}}(s) : s \leq t, \vec{v} \in [n]^4\right\}$. We shall often deal with $\{\mathcal{F}_t\}$-semimartingales $\{J(t)\}_{t\geq 0}$ of the form

$$dJ(t) := \alpha(t)dt + dM(t), \tag{3.1}$$

where M is a $\{\mathcal{F}_t\}$ local martingale. We shall denote $\alpha = \mathbf{d}(J)$ and $M = \mathbf{M}(J)$. For a local martingale $M(t)$, we shall write $\langle M, M \rangle(t)$ for the predictable quadratic variation process namely the predictable process of bounded variation such that $M(t)^2 - \langle M, M \rangle(t)$ is a local martingale.

4. Inhomogeneous random graphs

Fix $K \geq 0$ and a general bounded-size rule $F \subseteq \Omega_K^4$ and recall that $\{\mathbf{BSR}(t)\}_{t\geq 0}$ denotes the continuous time bounded-size rule process started with the empty graph at $t = 0$. Note that $K = 0$ case corresponds to the Erdős-Rényi random graph process for which results such as Theorem 1.2 are well known. Thus, henceforth we shall assume $K \geq 1$. We begin in Section 4.1 by analyzing the proportion of vertices in components of size i for $i \leq K$. As shown in [24], these converge to a set of deterministic functions which can be characterized as the unique solution of a set of differential equations. We will need precise rates of convergence for these proportions which we establish in Lemma 4.2. We then study the evolution of components of size larger than K in Section 4.2. Finally, we relate the evolution of these components to an inhomogeneous random graph (IRG) model in Section 4.3.

4.1. **Density of vertices in components of size bounded by K.** Recall from Section 1, (1.1) that $c_t(v) = c_{\mathbf{BSR}(t)}(v)$, for $v \in [n]$. For $t \geq 0$ and $i \in \Omega_K$, define

$$X_i(t) = |\{v \in [n] : c_t(v) = i\}| \text{ and } \bar{x}_i(t) = X_i(t)/n. \tag{4.1}$$

Following [24], the first step in analyzing bounded-size rules is understanding the evolution of $\bar{x}_i(\cdot)$ as functions of time as $n \to \infty$. Although [24] proves the convergence of $\bar{x}_i(t)$ as $n \to \infty$, we give here a self contained proof of this convergence with precise rates of convergence that will be needed in the proof of Theorem 1.2.

Note that the BSR process changes values at the occurrence of points in the Poisson processes $\mathcal{P}_{\vec{v}}$, $\vec{v} \in [n]^4$. We call each such occurrence as a 'round' and call a round **redundant** if the added edge in that round joins two vertices in the same component. Note that such rounds do not have any effect on component sizes or on the vector $\bar{\mathbf{x}}(t) = (\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_K(t), \bar{x}_\varpi(t))$. We will in fact observe that such rounds are quite rare. We now describe the effect of non-redundant rounds on $\bar{\mathbf{x}}(\cdot)$. For $\vec{j} \in \Omega_K^4$ and $i \in \Omega_K$, write $\Delta(\vec{j}; i)$ for the change $\Delta X_i(t) := X_i(t) - X_i(t-)$ at an occurrence time t if the chosen quadruple $\vec{v} \in [n]^4$ satisfies $c_{t-}(\vec{v}) = \vec{j}$ and the round is not redundant. It is easy to check (see Section 2.1, [24]) that, when $\vec{j} = (j_1, j_2, j_3, j_4) \in F$,

$$\Delta(\vec{j}; i) = i \cdot (\mathbf{1}_{\{j_1+j_2=i\}} - \mathbf{1}_{\{j_1=i\}} - \mathbf{1}_{\{j_2=i\}}), \text{ for } 1 \leq i \leq K,$$

$$\Delta(\vec{j}; \varpi) = \mathbf{1}_{\{j_1+j_2=\varpi\}}(j_1 \mathbf{1}_{\{j_1 \leq K\}} + j_2 \mathbf{1}_{\{j_2 \leq K\}}),$$

with the convention $j_1 + j_2 = \varpi$ when the sum of j_1, j_2 is greater than K, and $j_1 + \varpi = \varpi + j_1 = \varpi$ for all $j_1 \in \Omega_K$. For $\vec{j} = (j_1, j_2, j_3, j_4) \in F^c$ one uses the second edge $\{v_3, v_4\}$ and the expressions for $\Delta(\vec{j}; i)$ are identical to the above, with (j_3, j_4) replacing (j_1, j_2). Note that the corresponding change in the density $\bar{x}_i(t) = X_i(t)/n$ is given by $\Delta(\vec{j}; i)/n$. For $\vec{j} \in \Omega_K^4$ and $t > 0$, write

$$\mathcal{Q}(t; \vec{j}) := \left\{\vec{v} \in [n]^4 : c_t(\vec{v}) = \vec{j}\right\}.$$

Since each quadruple $\vec{v} \in [n]^4$ is selected according to the Poisson process $\mathcal{P}_{\vec{v}}$ with rate $1/2n^3$, the above description of the jumps of $X_i(\cdot)$ leads to a semi-martingale decomposition of $\bar{x}_i$ of the form (3.1) with

$$\mathbf{d}(\bar{x}_i)(t) = \sum_{\vec{j} \in F} \sum_{\vec{v} \in \mathcal{Q}(t; \vec{j})} \frac{\Delta(\vec{j}; i)}{2n^4} \mathbf{1}\{\mathcal{C}_{v_1}(t) \neq \mathcal{C}_{v_2}(t)\} + \sum_{\vec{j} \in F^c} \sum_{\vec{v} \in \mathcal{Q}(t; \vec{j})} \frac{\Delta(\vec{j}; i)}{2n^4} \mathbf{1}\{\mathcal{C}_{v_3}(t) \neq \mathcal{C}_{v_4}(t)\},$$

$$\tag{4.2}$$

where $\mathcal{C}_v(t) := \mathcal{C}_v(\mathbf{BSR}(t))$ denotes the component containing v in $\mathbf{BSR}(t)$.

Define for $i \in \Omega_K$, the functions $F_i : [0,1]^{K+1} \to \mathbb{R}$ mapping the vector $\mathbf{x} = (x_1, x_2, \ldots, x_K, x_\varpi) \in \mathbb{R}^{K+1}$ to

$$F_i^x(\mathbf{x}) = \frac{1}{2} \sum_{\vec{j} \in F} \Delta(\vec{j}; i) x_{j_1} x_{j_2} x_{j_3} x_{j_4} + \frac{1}{2} \sum_{\vec{j} \in F^c} \Delta(\vec{j}; i) x_{j_1} x_{j_2} x_{j_3} x_{j_4}. \tag{4.3}$$

Note that $|\Delta(\vec{j}; i)| \leq 2K$ for all $\vec{j} \in \Omega_K^4$. Also,

$$\max\left\{\sum_{\vec{j} \in F} \sum_{\vec{v} \in \mathcal{Q}(t; \vec{j})} \mathbf{1}\{\mathcal{C}_{v_1}(t) = \mathcal{C}_{v_2}(t)\}, \sum_{\vec{j} \in F^c} \sum_{\vec{v} \in \mathcal{Q}(t; \vec{j})} \mathbf{1}\{\mathcal{C}_{v_3}(t) = \mathcal{C}_{v_4}(t)\}\right\} \leq n^3 K.$$

Thus we have

$$|\mathbf{d}(\bar{x}_i)(t) - F_i^x(\bar{\mathbf{x}}(t))| \leq \frac{2K}{2n^4} \cdot 2Kn^3 = \frac{2K^2}{n}. \tag{4.4}$$

Note that $\bar{x}_1(0) = 1$ while $\bar{x}_i(0) = 0$ for other $i \in \Omega_K$. Guided by equations (4.2) − (4.4), [24] considered the system of differential equations for $\mathbf{x}(t) := (x_j(t) : j \in \Omega_K)$

$$x_i'(t) = F_i^x(\mathbf{x}(t)), \quad i \in \Omega_K, \quad t \geq 0, \quad \mathbf{x}(0) = (1, 0, ..., 0), \tag{4.5}$$

and showed the following result.

Theorem 4.1 (Theorem 2.1, [24]). *Equation* (4.5) *has a unique solution. For all $i \in \Omega_K$ and $t > 0$, $x_i(t) > 0$. Furthermore $\sum_{i \in \Omega_K} x_i(t) = 1$ and $\lim_{t \to \infty} x_\varpi(t) = 1$.*

The paper [24] also showed that the functions $\bar{x}_i(t) \xrightarrow{\mathbb{P}} x_i(t)$ for each fixed $t \geq 0$. We will need precise rates of convergence for our proofs for which we establish the following result.

Lemma 4.2. *Fix $\delta \in (0, 1/2)$ and $T > 0$. There exist $C_1, C_2 \in (0, \infty)$ such that for all n,*

$$\mathbb{P}\left(\sup_{i \in \Omega_K} \sup_{s \in [0,T]} |\bar{x}_i(t) - x_i(t)| > n^{-\delta} \right) < C_1 \exp\left(-C_2 n^{1-2\delta} \right).$$

Proof. Note that $F_i^x(\cdot)$ is a Lipchitz function, indeed for $\mathbf{x}, \tilde{\mathbf{x}} \in [0,1]^{K+1}$,

$$|F_i^x(\mathbf{x}) - F_i^x(\tilde{\mathbf{x}})| \leq 4K(K+1)^4 \sum_{i \in \Omega_K} |x_i - \tilde{x}_i| \leq 4K(K+1)^5 \sup_{i \in \Omega_K} |x_i - \tilde{x}_i|.$$

Write $D(t) := \sup_{i \in \Omega_K} |\bar{x}_i(t) - x_i(t)|$ and $M_i(t) := \mathbf{M}(\bar{x}_i)(t)$. Using (4.4), we get for all $i \in \Omega_K$ and $t \in [0, T]$,

$$|\bar{x}_i(t) - x_i(t)| \leq \int_0^t |F_i^x(\bar{\mathbf{x}}(s)) - F_i^x(\mathbf{x}(s))| ds + T \cdot \frac{2K^2}{n} + |M_i(t)|$$

$$\leq 4K(K+1)^5 \int_0^t D(s) ds + T \cdot \frac{2K^2}{n} + |M_i(t)|.$$

Taking $\sup_{i \in \Omega_K}$ on both sides and using Gronwall's lemma we have

$$\sup_{t \in [0,T]} D(t) \leq \left(\sup_{i \in \Omega_K} \sup_{t \in [0,T]} |M_i(t)| + \frac{2TK^2}{n} \right) e^{4K(K+1)^5 T}.$$

Thus, for a suitable $d_1 \in (0, \infty)$,

$$\mathbb{P}\left\{ \sup_{t \in [0,T]} D(t) > n^{-\delta} \right\} \leq \sum_{i \in \Omega_K} \mathbb{P}\left\{ \sup_{t \in [0,T]} |M_i(t)| > d_1 n^{-\delta} \right\}. \tag{4.6}$$

To complete the proof we will use exponential tail bounds for martingales. From Theorem 5 in Section 4.13 of [19] we have that, for a square integrable martingale M with $M(0) = 0$, $|\Delta M(t)| \leq c$ for all t, and $\langle M, M \rangle(T) \leq Q$, a.s., for some $c, Q \in (0, \infty)$,

$$\mathbb{P}\left\{ \sup_{0 \leq t \leq T} |M(t)| > \alpha \right\} \leq 2 \exp\left\{ -\sup_{\lambda > 0} [\alpha\lambda - Q\psi(\lambda)] \right\}, \quad \text{for all } \alpha > 0,$$

where $\psi(\lambda) = \frac{e^{\lambda c} - 1 - \lambda c}{c^2}$. Optimizing over λ, we get the bound

$$\mathbb{P}\left\{ \sup_{0 \leq t \leq T} |M(t)| > \alpha \right\} \leq 2 \exp\left\{ -\frac{\alpha}{c} \log\left(1 + \frac{\alpha c}{Q} \right) + \left[\frac{\alpha}{c} - \frac{Q}{c^2} \log\left(1 + \frac{\alpha c}{Q} \right) \right] \right\}. \tag{4.7}$$

In our context, note that for any $i \in \Omega_K$, $|\Delta M_i(t)| = |\Delta \bar{x}_i(t)| \leq 2K/n$. Also, the total rate of jumps is bounded by $n^4 \cdot \frac{1}{2n^3}$. Thus for all $i \in \Omega_K$, the quadratic variation process

$$\langle M_i, M_i \rangle(T) \leq \int_0^T \left(\frac{2K}{n} \right)^2 \times \frac{n^4}{2n^3} dt = \frac{2K^2 T}{n}.$$

Taking $\alpha = d_1 n^{-\delta}, Q = 2K^2 T/n$ and $c = 2K/n$ in (4.7) completes the proof. $\qquad\square$

4.2. **Evolution of components of size larger than K.** Let $\mathbf{BSR}^*(t)$ denote the subgraph of $\mathbf{BSR}(t)$ consisting of components of size greater than K. In this section, we will focus on the dynamics and evolution of $\mathbf{BSR}^*(t)$. Note that $\mathbf{BSR}^*(0) = \emptyset$, i.e. a graph with no vertices or edges. As time progresses components of size less than K merge and components of size greater than K emerge. Three distinct types of events affect the evolution $\mathbf{BSR}^*(t)$:

1. **Immigration:** This occurs when two components of size $\leq K$ merge into a single component of size $> K$. We view the resulting component as a new immigrant into $\mathbf{BSR}^*(t)$. Note that the first component to appear in $\mathbf{BSR}^*(t)$ is an immigrant.
2. **Attachments:** This occurs when a component of size $\leq K$ gets linked to a component of size larger than K. The former component enters $\mathbf{BSR}^*(t)$ via attaching itself to a component of size larger than K.
3. **Edge formation:** Two distinct components of size larger than K merge into a single component via formation of an edge between these components. In this case, the vertex set of $\mathbf{BSR}^*(t)$ remains unchanged.

We now introduce some functions that describe the rate of occurrence for each of the three types of events. For $i_1, i_2 \in \Omega_K$, define $F_{i_1,i_2}^x : [0,1]^{K+1} \to \mathbb{R}$ as

$$F_{i_1,i_2}^x(\mathbf{x}) = \frac{1}{2} \left(\sum_{\vec{j} \in F : \{j_1,j_2\}=\{i_1,i_2\}} x_{j_1} x_{j_2} x_{j_3} x_{j_4} + \sum_{\vec{j} \in F^c : \{j_3,j_4\}=\{i_1,i_2\}} x_{j_1} x_{j_2} x_{j_3} x_{j_4} \right). \tag{4.8}$$

For $i_1, i_2 \leq K$, denote $n \cdot R_{i_1,i_2}(t)$ as the rate at which two components of size i_1, i_2 merge. When $i_1 \neq i_2$, this rate is precisely

$$(2n^3)^{-1} \Big[\sum_{\substack{\vec{j} \in F \\ \{j_1,j_2\}=\{i_1,i_2\}}} X_{j_1}(t) X_{j_2}(t) X_{j_3}(t) X_{j_4}(t) + \sum_{\substack{\vec{j} \in F \\ \{j_3,j_4\}=\{i_1,i_2\}}} X_{j_1}(t) X_{j_2}(t) X_{j_3}(t) X_{j_4}(t) \Big] := n \cdot F_{i_1,i_2}^x(\bar{\mathbf{x}}(t)).$$

Thus $R_{i_1,i_2}(t) = F_{i_1,i_2}^x(\bar{\mathbf{x}}(t))$. The case $i_1 = i_2 \leq K$ is more subtle due to redundant rounds linking vertices in the same component. The rate of redundant rounds can be bounded by $\frac{1}{2n^3} \cdot K n^3 \cdot 2 = K$, from which it follows that

$$|R_{i,i}(t) - F_{i,i}^x(\bar{\mathbf{x}}(t))| \leq \frac{K}{n}.$$

The case $i_1 = i_2 = \varpi$ corresponds to creation of edges in $\mathbf{BSR}^*(t)$ and $n \cdot F_{\varpi,\varpi}^x(\bar{\mathbf{x}}(t))$ is the rate of creation of such edges.

We now give expressions for the rates for the three types of events that govern the evolution of $\mathbf{BSR}^*(t)$. The convention followed for the rest of this section is that for $i_1, i_2 \in \Omega_K$, $i_1 + i_2 = \varpi$ when the sum of is greater than K, and $\varpi + i_1 = i_1 + \varpi = \varpi$ for all $i_1 \in \Omega_K$.

I. Immigrating vertices: For $1 \leq i \leq K$, write $\mathfrak{I}_i(t) := n \cdot a_i^*(t)$ for the rate at which components of size $K + i$ immigrate into $\mathbf{BSR}^*(t)$ at time t. Using the above expressions for

the rate of merger of components of various sizes we have

$$\left| a_i^*(t) - \sum_{1 \le i_i, i_2 \le K: i_1 + i_2 = K + i} F_{i_1, i_2}^x(\bar{\mathbf{x}}(t)) \right| \le \frac{K}{n}. \tag{4.9}$$

As before the error is due to redundant rounds which can only occur for $i_1 = i_2 = (K+i)/2$ (and when $(K+i)/2$ is an integer). Now define functions $F_i^a : [0,1]^{K+1} \to \mathbb{R}_+$, and $a_i(\cdot) : [0, \infty) \to [0, \infty)$ by

$$F_i^a(\mathbf{x}) = \sum_{\substack{1 \le i_i, i_2 \le K, \\ i_1 + i_2 = K + i}} F_{i_1, i_2}^x(\mathbf{x}), \qquad a_i(t) = F_i^a(\mathbf{x}(t)), \tag{4.10}$$

where $\mathbf{x}(t)$ is as in (4.5). Then (4.9) says that

$$\sup_{t \in [0, \infty)} |a_i^*(t) - F_i^a(\bar{\mathbf{x}}(t))| \le K/n. \tag{4.11}$$

Note that for any $\delta < 1$, the error term in (4.11) is $o(n^{-\delta})$. Using this observation along with the Lipschitz property of F_{i_1, i_2}, we have from Lemma 4.2 that for any fixed $T > 0$ and $\delta < 1/2$,

$$\mathbb{P}(\sup_{1 \le i \le K} \sup_{s \in [0,T]} |a_i^*(t) - a_i(t)| > n^{-\delta}) \le C_1 \exp(-C_2 n^{1-2\delta}). \tag{4.12}$$

The constants C_1, C_2 here may be different from those in Lemma 4.2, however for notational ease we use the same symbols.

II. Attachments: Fix $1 \le i \le K$ and a vertex v contained in a component in $\mathbf{BSR}^*(t)$. Let, for $i \le K$, $c_i^*(t)$ denote the rate at which a component of size i attaches itself to the component of v through an edge connecting the former component to v. This rate can be calculated as follows. First note that the total rate of merger between a component of size i with a component in $\mathbf{BSR}^*(t)$ is $n \cdot F_{i,\varpi}^x(\bar{\mathbf{x}}(t))$. Since there are $X_\varpi(t)$ vertices in $\mathbf{BSR}^*(t)$ each of which has the same probability of being the vertex through which this attachment event happens, the rate at which a component of size i attaches to v is given by $n F_{i,\varpi}^x(\bar{\mathbf{x}}(t))/X_\varpi(t) = F_{i,\varpi}^x(\bar{\mathbf{x}}(t))/\bar{x}_\varpi(t)$. Since x_ϖ is a factor of $F_{i,\varpi}^x(\mathbf{x})$, $c_i^*(t)$ is a polynomial in $\bar{\mathbf{x}}(t)$. Define the functions $F_i^c : [0,1]^{K+1} \to \mathbb{R}$ and $c_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$F_i^c(\mathbf{x}) = F_{i,\varpi}^x(\mathbf{x})/x_\varpi, \qquad c_i(t) = F_i^c(\mathbf{x}(t)). \tag{4.13}$$

Then $c_i^*(t) = F_i^c(\bar{\mathbf{x}}(t))$. Once again using Lemma 4.2 we get for any $\delta < 1/2$ and $T > 0$,

$$\mathbb{P}(\sup_{1 \le i \le K} \sup_{s \in [0,T]} |c_i^*(t) - c_i(t)| > n^{-\delta}) \le C_1 \exp(-C_2 n^{1-2\delta}). \tag{4.14}$$

III. Edge formation: Note that the rate of creation of an edge between vertices in $\mathbf{BSR}^*(t)$ is $n F_{\varpi,\varpi}^x(\bar{\mathbf{x}}(t))$. Since such an edge is equally likely to be between any of the $X_\varpi^2(t)$ pairs of vertices in $\mathbf{BSR}^*(t)$, we have that the rate of creation of an edge between specified vertices $\{v_1, v_2\}$ with $v_1, v_2 \in \mathbf{BSR}^*(t)$ is $b^*(t)/n$ where $b^*(t) = F_{\varpi,\varpi}^x(\bar{\mathbf{x}}(t))/x_\varpi^2(t)$. Define $F^b : [0,1]^{K+1} \to \mathbb{R}$ and $b(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$F^b(\mathbf{x}) = F_{\varpi,\varpi}^x(\mathbf{x})/x_\varpi^2 \text{ and } b(t) = F^b(\mathbf{x}(t)). \tag{4.15}$$

Once more it is clear that $F^b(\mathbf{x})$ is a polynomial and furthermore $b^*(t) = F^b(\bar{\mathbf{x}}(t))$, so by Lemma 4.2, for any $\delta < 1/2$ and $T > 0$,

$$\mathbb{P}(\sup_{s \in [0,T]} |b^*(t) - b(t)| > n^{-\delta}) \le C_1 \exp(-C_2 n^{1-2\delta}). \tag{4.16}$$

Write $\mathbf{a}(t) := \{a_i(t)\}_{1 \leq i \leq K}$ and $\mathbf{c}(t) := \{c_i(t)\}_{1 \leq i \leq K}$. We refer to $(\mathbf{a}, b, \mathbf{c})$ as rate functions. In the proposition below we collect some properties of these rate functions. These properties are easy consequences of Theorem 4.1.

Proposition 4.3. (a) *For all* $1 \leq i \leq K$ *and* $t > 0$, $b(t), a_i(t), c_i(t) > 0$.
(b) *We have*

$$\|\mathbf{a}\|_\infty := \sup_{t \geq 0} \sum_{i=1}^{K} a_i(t) \leq 1/2, \quad \|\mathbf{c}\|_\infty := \sup_{t \geq 0} \sum_{i=1}^{K} c_i(t) \leq 1/2, \quad \|b\|_\infty := \sup_{t \geq 0} b(t) \leq 1/2.$$

(c) $\lim_{t \to \infty} b(t) = 1/2$.

Proof: Part(a) follows from Theorem 4.1 and the definition of the functions. For (b) observe that

$$\sum_{i=1}^{K} a_i(t) = \sum_{i=1}^{K} F_i^a(\mathbf{x}(t)) \leq \frac{1}{2} \sum_{\vec{j} \in \Omega_K} x_{j_1}(t) x_{j_2}(t) x_{j_3}(t) x_{j_4}(t) = \frac{1}{2} \left[\sum_{i \in \Omega_K} x_i(t) \right]^4 = \frac{1}{2}.$$

Statements on $\|c\|_\infty, \|b\|_\infty$ follow similarly.

For (c), note that $F_{\varpi,\varpi}^x(\mathbf{x}) \geq x_\varpi^4/2$ since when all the four vertices selected are from components of size greater than K, two components of size greater than K will surely be linked. From Theorem 4.1 $\lim_{t \to \infty} x_\varpi(t) = 1$ and thus $\limsup_{t \to \infty} b(t) \geq x_\varpi^2(t)/2$. The result now follows on combining this with (b). $\qquad\square$

4.3. **Connection to inhomogeneous random graphs.** In this section, we describe the inhomogeneous random graph (IRG) models that have been studied extensively in [8], and then approximate $\mathbf{BSR}^*(t)$ by a special class of such models. We will in fact use a variation of the models in [8] which uses a suitable weight function to measure the volume of a component. We begin by defining the basic ingredients in this model. Let $\mathcal{X}$ be a Polish space, equipped with the Borel σ-field $\mathcal{B}(\mathcal{X})$. We shall sometimes refer to this as the **type space**. Let μ be a non-atomic finite measure on $\mathcal{X}$ which we shall call the **type measure** on $\mathcal{X}$. A **kernel** will be a symmetric non-negative product measurable function $\kappa : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and a **weight function** $\phi : \mathcal{X} \to \mathbb{R}$ will be a non-negative measurable function on $\mathcal{X}$. We call such a quadruple $\{\mathcal{X}, \mu, \kappa, \phi\}$ a **basic structure**.

The **inhomogeneous random graph with weight function (IRG):** Associated with a basic structure $\{\mathcal{X}, \mu, \kappa, \phi\}$, the IRG model $\mathbf{RG}^{(n)}(\mathcal{X}, \mu, \kappa, \phi)$ is a random graph described as follows:

(a) **Vertices:** the vertex set $\mathcal{V}$ of this random graph is a Poisson point process on the space $\mathcal{X}$ with intensity measure $n\mu$.
(b) **Edges:** an edge is added between vertices $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ with probability $1 \wedge \frac{\kappa(\mathbf{x}, \mathbf{y})}{n}$, independent across different pairs. This defines the random graph.
(c) **Volume:** The volume of a component $\mathcal{C}$ of $\mathbf{RG}^{(n)}(\mathcal{X}, \mu, \kappa, \phi)$ is defined as

$$\text{vol}_\phi(\mathcal{C}) := \sum_{x \in \mathcal{C}} \phi(x). \tag{4.17}$$

For the rest of this section we take

$$\mathcal{X} := [0, \infty) \times \mathcal{W} \text{ where } \mathcal{W} := \mathcal{D}([0, \infty) : \mathbb{N}_0). \tag{4.18}$$

We first describe how, for each $t > 0$, $\mathbf{BSR}^*(t)$ can be identified with a random graph with vertex set in $\mathcal{X}$. Recall the three types of events governing the evolution of $\mathbf{BSR}^*(t)$, described in Section 4.2. Each component in $\mathbf{BSR}^*(t)$ contains at least one group of $K + i$

vertices, $i = 1, \cdots, K$ which appeared at instant $s \leq t$ in $\mathbf{BSR}^*(\cdot)$, as an immigrant. We denote the collection of all such groups as $\mathrm{Imm}(t)$. For $\mathcal{C} \in \mathrm{Imm}(t)$, we denote by $\tau_{\mathcal{C}} \in (0, t]$ the instant this immigrant appears. Also, to each $\mathcal{C} \in \mathrm{Imm}(t)$, we associate a path in $\mathcal{D}([0, \infty) : \mathbb{N}_0)$, denoted as $w_{\mathcal{C}}$, such that $w_{\mathcal{C}}(s) = 0$ for all $s < \tau_{\mathcal{C}}$; $w_{\mathcal{C}}(s) = w_{\mathcal{C}}(t)$ for all $s \in [t, \infty)$; and for $s \in [\tau_{\mathcal{C}}, t]$, $w_{\mathcal{C}}(s) = |\mathcal{A}_{\mathcal{C}}(s)|$, where $\mathcal{A}_{\mathcal{C}}(s)$ denotes the component that is formed by $\mathcal{C}$ and all the attachment components that link to $\mathcal{C}$ over the time interval $[\tau_{\mathcal{C}}, s]$. Then $\{(\tau_{\mathcal{C}}, w_{\mathcal{C}}) : \mathcal{C} \in \mathrm{Imm}(t)\}$ is a point process on $\mathcal{X}$ and forms the vertex set of a random graph which we denote by $\Gamma(t)$. We form edges between any two vertices $(\tau_{\mathcal{C}}, w_{\mathcal{C}})$, $(\tau'_{\mathcal{C}}, w'_{\mathcal{C}})$ in $\Gamma(t)$ if the components $\mathcal{A}_{\mathcal{C}}(t)$ and $\mathcal{A}_{\mathcal{C}'}(t)$ are directly linked by some edge in $\mathbf{BSR}^*(t)$.

Define, for $t > 0$, the weight function $\phi_t : \mathcal{X} \to [0, \infty)$ as

$$\phi_t(\mathbf{x}) = \phi_t(s, w) = w(t), \qquad \mathbf{x} = (s, w) \in \mathcal{X}. \tag{4.19}$$

Note that by construction there is a one to one correspondence between the components in $\mathbf{BSR}^*(t)$ and the components in $\Gamma(t)$. For a component $\mathcal{C}_0$ in $\mathbf{BSR}^*(t)$, denote by $I_{\mathcal{C}_0}$ the corresponding component in $\Gamma(t)$. Then note that

$$|\mathcal{C}_0| = \mathrm{vol}_{\phi_t}(I_{\mathcal{C}_0}). \tag{4.20}$$

We will now describe inhomogeneous random graph models that approximate $\Gamma(t)$ (and hence $\mathbf{BSR}^*(t)$). Given a set of nonnegative continuous bounded functions $\boldsymbol{\alpha} = \{\alpha_i\}_{1 \leq K}$, β and $\boldsymbol{\gamma} = \{\gamma_i\}_{1 \leq i \leq K}$ on $[0, \infty)$ we construct, for each $t > 0$, type measures $\mu_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ and kernels $\kappa_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ on $\mathcal{X}$ as follows. For $i = 1, \cdots K$ and $s \in [0, \infty)$, denote by $\tilde{\nu}_{s,i}$ the probability law on $\mathcal{D}([s, \infty) : \mathbb{N}_0)$ of the Markov process $\{\tilde{w}(r)\}_{r \in [s, \infty)}$ with infinitesimal generator

$$(\mathcal{A}(u)f)(k) = \sum_{j=1}^{K} k\gamma_j(u)(f(k+j) - f(k)), \ f \in \mathrm{BM}(\mathbb{N}_0) \tag{4.21}$$

and initial condition $\tilde{w}(s) = K + i$. In words, this is a pure jump Markov process which starts at time s at state $K + i$ and then at any time instant $u > s$, has jumps of size j at rate $\gamma_j(u)$. Denote by $\nu_{s,i}$ the probability law on $\mathcal{D}([0, \infty) : \mathbb{N}_0)$ of the stochastic process $\{w(r)\}_{r \in [0, \infty)}$, defined as

$$w(r) = \tilde{w}(r) \quad \text{for } r \geq s , \qquad w(r) = 0 \text{ otherwise.} \tag{4.22}$$

Now define the finite measure $\mu_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) \equiv \mu_t$ as

$$\int_{\mathcal{X}} f(\mathbf{x}) d\mu_t(\mathbf{x}) = \sum_{i=1}^{K} \int_0^t \alpha_i(u) \left(\int_{\mathcal{W}} f(u, w) \nu_{u,i}(dw) \right) du, \ f \in \mathrm{BM}(\mathcal{X}). \tag{4.23}$$

Next, define the kernel $\kappa_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) \equiv \kappa_t$ on $\mathcal{X} \times \mathcal{X}$ as

$$\kappa_t(\mathbf{x}, \mathbf{y}) = \kappa_t((s, w), (r, \tilde{w})) = \int_0^t w(u)\tilde{w}(u)\beta(u)du, \ \mathbf{x} = (s, w), \mathbf{y} = (r, \tilde{w}) \in \mathcal{X}. \tag{4.24}$$

With the above choice of μ_t, κ_t and with weight function ϕ_t as in (4.19) we now construct the random graph $\mathbf{RG}^{(n)}(\mathcal{X}, \mu_t, \kappa_t, \phi_t)$ which we denote by $\mathbf{RG}^{(n)}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})(t)$. We will refer to the set of functions $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ as above, as **rate functions**. These rate functions will typically arise as small perturbations of the functions $(\mathbf{a}, b, \mathbf{c})$, thus in view of Proposition 4.3(b) it will suffice to consider $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ such that $\max\{||\boldsymbol{\alpha}||_\infty, ||\beta||_\infty, ||\boldsymbol{\gamma}||_\infty\} < 1$. Throughout this work we will assume that all rate functions (and their perturbations) satisfy this bound.

The following key result says that for large n, $\Gamma(t)$ is suitably close to $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})(t)$, where $(\mathbf{a}, b, \mathbf{c})$ are the rate functions introduced below (4.16). In order to state the result precisely, we extend the notion of a "subgraph" to the setting with type space $\mathcal{X}$ and weight function ϕ. For $i = 1, 2$, consider graphs $\mathbf{G}_i$, with finite vertex set $\mathcal{V}_i \subset \mathcal{X}$ and edge set $\mathcal{E}_i$. We say $\mathbf{G}_1$ is a subgraph of $\mathbf{G}_2$, and write $\mathbf{G}_1 \subset \mathbf{G}_2$, if there exists a one to one mapping $\Psi : \mathcal{V}_1 \to \mathcal{V}_2$ such that

(i) $\phi(\mathbf{x}) \leq \phi(\Psi(\mathbf{x}))$, for all $\mathbf{x} \in \mathcal{V}_1$.
(ii) $\{\mathbf{x}_1, \mathbf{x}_2\} \in \mathcal{E}_1$ implies $\{\Psi(\mathbf{x}_1), \Psi(\mathbf{x}_2)\} \in \mathcal{E}_2$.

Lemma 4.4. *Fix $\delta \in (0, 1/2)$ and let $\varepsilon_n = n^{-\delta}$, $n \geq 1$. Define, for $t > 0$, the set of functions $\mathbf{a}^-(t) := \{(a_i(t) - \varepsilon_n) \vee 0\}_{1 \leq i \leq K}$, $\mathbf{a}^+(t) := \{a_i(t) + \varepsilon_n\}_{1 \leq i \leq K}$ and similarly $\mathbf{c}^-(t), \mathbf{c}^+(t)$ and $b^-(t), b^+(t)$. Define the inhomogeneous random graphs (IRG) with the above rate functions by*

$$\mathbf{RG}^-(t) := \mathbf{RG}(\mathbf{a}^-, b^-, \mathbf{c}^-)(t), \quad \mathbf{RG}^+(t) := \mathbf{RG}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t).$$

Then for every $T > 0$ there exist $C_3, C_4 \in (0, \infty)$, such that for all $t \in [0, T]$, there is a coupling of $\mathbf{RG}^-(t)$, $\mathbf{RG}^+(t)$ and $\Gamma(t)$ such that,

$$\mathbb{P}\left\{\mathbf{RG}^-(t) \subset \Gamma(t) \subset \mathbf{RG}^+(t)\right\} > 1 - C_3 \exp\left\{-C_4 n^{1-2\delta}\right\}.$$

Proof: The coupling between the three graphs is done in a manner such that $\Gamma(t)$ is obtained by a suitable thinning of vertices and edges in $\mathbf{RG}^+(t)$ and $\mathbf{RG}^-(t)$ is obtained by a thinning of $\Gamma(t)$. We will only provide details of the first thinning step. We first construct the vertex sets $\mathcal{V}^+$ and $\mathcal{V}^*$ in $\mathbf{RG}^+(t)$ and $\Gamma(t)$ respectively.

Let $\mathcal{V}^+$ be a Poisson point process on $\mathcal{X}$ with intensity $n\mu_t^+$, where $\mu_t^+ := \mu_t(\mathbf{a}^+, b^+, \mathbf{c}^+)$. For a fixed realization of $\mathcal{V}^+$, denote by $(x_1^+, \cdots, x_N^+)$, the points in $\mathcal{V}^+$, with $x_i^+ = (s_i^+, w_i^+)$ and $0 < s_1^+ < s_2^+ \cdots s_N^+ < t$. Write $\mathbf{w}^+ = (w_1^+, \cdots w_N^+)$. We now construct vertices in the corresponding realization of $\Gamma(t)$ (denoted as $\{x_1, \cdots x_{N_0}\}$), along with the realizations of $\bar{x}_i(s)$, $i \in \Omega_K$, $0 \leq s \leq t$, which then defines $(a_j^*(s), b^*(s), c_j^*(s))$, $0 \leq s \leq t$, $j = 1, \cdots K$, as functions of $\bar{\mathbf{x}}(s) = (\bar{x}_i(s))_{i \in \Omega_K}$ as in Section 4.2. For that, we will construct functions $w_j : [0, t] \to \mathbb{N}_0$, $1 \leq j \leq N$ and $\bar{x}_i : [0, t] \to [0, 1]$, $i \in \Omega_K$. We will only describe the construction of $w_j, \bar{x}_i$ until the first time instant $s \in (0, t]$, when the property

$$a_j^*(s) \leq a_j^+(s), \; b^*(s) \leq b^+(s), \; c_j^*(s) \leq c_j^+(s) \text{ for all } 1 \leq k \leq K \tag{4.25}$$

is violated. Denote σ for the first time that (4.25) is violated with σ taken to be t if the property holds for all $s \in [0, t]$. Subsequent to that time instant the construction can be done in any fashion that yields the correct probability law for $\Gamma(t)$. For simplicity, we assume henceforth that $\sigma = t$. After obtaining the functions $w_j, \bar{x}_i$, we set $x_i^* = (\tau_i^*, w_i^*)$, where τ_i^* is the first jump instant of w_i (taken to be $+\infty$ if there are no jumps) and $w_i^* \in \mathcal{D}([0, \infty) : \mathbb{N}_0)$ is defined as $w_i^*(s) = w_i(s) 1_{[0,t]}(s) + w_i(t) 1_{(t,\infty)}(s)$. The vertex set $\mathcal{V}^*$ is then defined as

$$\mathcal{V}^* = \{x_1, \cdots x_{N_0}\} = \{x_i^* : \tau_i^* < t, i = 1, \cdots N\}.$$

We now give the construction of $\mathbf{w}(s) = (w_1(s), \cdots w_N(s))$ and $\bar{\mathbf{x}}(s)$ for $s \leq t$. Denote by $\{t_i\}_{i=1}^M$, $0 = t_0 < t_1 < t_2 < ... t_M < t$, the collection of all time instants of jumps of $\{w_i^+\}_{i=1}^N$ before time t. Denote by i_k the coordinate of $\mathbf{w}^+$ that has a jump at time t_k, and denote the corresponding jump size by j_k. We set $\mathbf{w}(0) = 0$, $\bar{x}_i(0) = 0$ for $i \neq 1$ and $\bar{x}_1(0) = 1$. The construction proceeds recursively over the time intervals $(t_{k-1}, t_k]$, $k = 1, \cdots M + 1$, where $t_{M+1} = t$. Suppose that $(\mathbf{w}(s), \bar{\mathbf{x}}(s))$ have been defined for $s \in [0, t_{k-1}]$, for some $k \geq 1$. We now define these functions over the interval $(t_{k-1}, t_k]$.

Step 1: $s \in (t_{k-1}, t_k)$. Set $\mathbf{w}(s) = \mathbf{w}(t_{k-1})$. The values of $\bar{\mathbf{x}}(s)$ over the interval will be given as a realization of a jump process, for which jumps at time instant s occur at rates $n \cdot R_{i_1,i_2}(s)$, $i_1, i_2 \in \{1, \cdots K\}$, $i_1 + i_2 \leq K$, where the function $R_{i_1,i_2}(s)$, given as a function of $\bar{\mathbf{x}}(s)$ is defined as in Section 4.2. A jump at time instant s, corresponding to the pair (i_1, i_2) as above, changes the values of $\bar{\mathbf{x}}$ as:

$$\bar{x}_{i_1}(s) = \bar{x}_{i_1}(s-) - \frac{i_1}{n}, \ \bar{x}_{i_2}(s) = \bar{x}_{i_2}(s-) - \frac{i_2}{n}, \ \bar{x}_{i_1+i_2}(s) = \bar{x}_{i_1+i_2}(s-) + \frac{i_1 + i_2}{n}, \ \text{if } i_1 \neq i_2,$$

$$\bar{x}_{i_1}(s) = \bar{x}_{i_1}(s-) - \frac{2i_1}{n}, \ \bar{x}_{i_1+i_2}(s) = \bar{x}_{2i_1}(s-) + \frac{2i_1}{n}, \ \text{if } i_1 = i_2.$$

Remaining $\bar{x}_i$ stay unchanged. The values of $a_i^*(s)$, $b^*(s)$, $c_i^*(s)$ are determined accordingly.

Step 2: $s = t_k$. Recall that $w_{i_k}^+(t_k) - w_{i_k}^+(t_k-) = j_k$. We define $w_i(t_k) = w_i(t_k-)$ for all $i \neq i_k$. The values of $w_{i_k}(t_k)$ and $\bar{\mathbf{x}}(t_k)$ are determined as follows.

Case 1: $w_{i_k}^+(t_k-) = 0$. In this case $K + 1 \leq j_k \leq 2K$ and t_k is the first jump of $w_{i_k}^+$. Define for $1 \leq l \leq K$, $Q_k(l) := \sum_{i=1}^{l} R_{i,j_k-i}(t_k-)$, where by definition $R_{i,i'} = 0$ if $i' > K$. Note that $Q_k(K) = a_{j_k}^*(t_k-)$. We set $Q_k(0) = 0$. The values of $w_{i_k}(t_k)$ and $\bar{\mathbf{x}}(t_k)$ are now determined according to the realization of an independent Uniform $[0, 1]$ random variable u_k as follows.

- If $u_k > Q_k(K)/a_{j_k}(t_k-)$, define $(w_{i_k}(t_k), \bar{\mathbf{x}}(t_k)) = (w_{i_k}(t_k-), \bar{\mathbf{x}}(t_k-))$.
- Otherwise, suppose $1 \leq l_k \leq K$ is such that $Q_k(l_k - 1) < u_k \leq Q_k(l_k)$. Then define $w_{i_k}(t_k) = j_k$, $\bar{x}_\varpi(t_k) = \bar{x}_\varpi(t_k-) + \frac{j_k}{n}$ and

$$\bar{x}_{l_k}(t_k) = \bar{x}_{l_k}(t_k-) - \frac{l_k}{n}, \ \ \bar{x}_{j_k-l_k}(t_k) = \bar{x}_{j_k-l_k}(t_k-) - \frac{j_k - l_k}{n}, \ \ \text{if } l_k \neq j_k - l_k,$$

$$\bar{x}_{l_k}(t_k) = \bar{x}_{l_k}(t_k-) - \frac{2l_k}{n}, \ \ \ \text{if } l_k = j_k - l_k.$$

The value of all other x_i processes at t_k stay the same as their values at t_k-.

Case 2: $w_{i_k}^+(t_k-) \neq 0$. In this case $1 \leq j_k \leq K$. Once again, the values of $w_{i_k}(t_k)$ and $\bar{\mathbf{x}}(t_k)$ are determined according to the realization of an independent Uniform $[0, 1]$ random variable u_k as follows.

- If $u_k > \frac{w_{i_k}(t_k-)c_{j_k}^*(t_k-)}{w_{i_k}^+(t_k-)c_{j_k}^+(t_k-)}$, define $(w_{i_k}(t_k), \bar{\mathbf{x}}(t_k)) = (w_{i_k}(t_k-), \bar{\mathbf{x}}(t_k-))$.
- Otherwise,

$$w_{i_k}(t_k) = w_{i_k}(t_k-) + j_k, \ \ \bar{x}_{j_k}(t_k) = \bar{x}_{j_k}(t_k-) - \frac{j_k}{n}, \ \ \bar{x}_\varpi(t_k) = \bar{x}_\varpi(t_k-) + \frac{j_k}{n},$$

and the value of all other x_i processes stay the same as their value at t_k-.

This completes the construction of $(\mathbf{w}(s), \bar{\mathbf{x}}(s))$ for $s \in (t_{k-1}, t_k]$ and thus by this recursive procedure and our earlier discussion we obtain the vertex set

$$\mathcal{V}^* = \{x_1, \cdots x_{N_0}\} = \{x_i^* : \tau_i^* < t, i = 1, \cdots N\},$$

which will be used to construct $\Gamma(t)$.

Having constructed vertex sets $\mathcal{V}^+$ and $\mathcal{V}^*$, we now construct edges. For this we take realizations of independent Uniform $[0, 1]$ random variables $\{u_{i,j}\}_{1 \leq i < j < \infty}$ and construct edge between vertices x_i^+ and x_j^+ in $\mathcal{V}^+$ if

$$u_{i,j} \leq \frac{1}{n} \int_0^t b^+(s) w_i^+(s) w_j^+(s) ds.$$

This completes the construction of $\mathbf{RG}^+(t)$. Finally construct an edge between x_i^* and x_j^* if both vertices are in $\mathcal{V}^*$ and

$$u_{i,j} \leq 1 - \exp\left\{-\frac{1}{n}\int_0^t b^*(s)w_i(s)w_j(s)ds\right\}.$$

This completes the construction of $\Gamma(t)$. By construction $\Gamma(t) \subset \mathbf{RG}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)$ on the set $\sigma = t$. Also, from (4.12), (4.16) and (4.14) it follows that $\mathbb{P}(\sigma < t) \leq C_3 \exp\left\{-C_4 n^{1-2\delta}\right\}$ for suitable constant C_3, C_4. The result follows. $\qquad\square$

The following is an immediate corollary of Lemma 4.4.

Corollary 4.5. Fix $T > 0$. Then with $C_3, C_4 \in (0,\infty)$ and , for $t \in [0,T]$, a coupling of $\mathbf{RG}^-(t)$, $\mathbf{RG}^+(t)$ and $\Gamma(t)$ as in Lemma 4.4:

$$\mathbb{P}\left\{\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^-}(t)) \leq \mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\Gamma}(t)) \leq \mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^+}(t))\right\}$$

$$\geq 1 - C_3 \exp(-C_4 n^{1-2\delta}), \tag{4.26}$$

where $\mathcal{I}_1^{\Gamma}(t)$ denotes the component in $\Gamma(t)$ with the largest volume with respect to the weight function ϕ_t, and $\mathcal{I}_1^{\mathbf{RG}^-}(t)$, $\mathcal{I}_1^{\mathbf{RG}^+}(t)$ are defined similarly.

5. Proof of the main results

In this section, we will complete the proof of Theorems 1.2 and 1.3. Proof of Theorem 1.3 is given in Section 5.4 while proof of Theorem 1.2 is given in Section 5.5. Recall that Lemma 4.4 says that $\mathbf{BSR}^*$ can be approximated by $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})$. Sections 5.2 and 5.3 analyze properties of integral operators associated with $\mathbf{RG}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ for a general family of rate functions $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$. We begin in Section 5.1 by presenting some results about an IRG model $\mathbf{RG}^{(n)}(\mathcal{X}, \mu, \kappa, \phi)$ on a general type space $\mathcal{X}$.

5.1. Preliminary lemmas.
In this section, we collect some results about the general IRG model $\mathbf{RG}^{(n)}(\mathcal{X}, \mu, \kappa, \phi)$. Let $\mathcal{K}$ be the integral operator associated with (κ, μ), as defined in Section 1. Recall that the operator norm of $\mathcal{K}$, denoted as $\|\mathcal{K}\|$, is defined as

$$\|\mathcal{K}\| = \sup_{f \in L^2(\mathcal{X}, \mu), f \neq 0} \frac{\|\mathcal{K}f\|_2}{\|f\|_2}, \tag{5.1}$$

where for $f \in L^2(\mathcal{X}, \mu)$, $\|f\|_2 = \left(\int_{\mathcal{X}} |f(\mathbf{x})|^2 \mu(d\mathbf{x})\right)^{1/2}$.

Lemma 5.1. *Fix $(\mathcal{X}, \mu, \kappa, \phi)$. Denote the vertex set of $\mathbf{RG}^{(n)}(\mathcal{X}, \mu, \kappa, \phi) \equiv \mathbf{RG}^{(n)}$ by $\mathcal{P}_n$ which is a rate $n\mu$ Poisson point process on $\mathcal{X}$. Let $\mathcal{K}$ be the integral operator associated with (κ, μ). Suppose that $\|\mathcal{K}\| < 1$ and let $\Delta = 1 - \|\mathcal{K}\|$. Denote by $\mathcal{I}_1^{\mathbf{RG}}$ the component in $\mathbf{RG}^{(n)}$ with the largest volume (with respect to the weight function ϕ). Then the following hold.*

(i) *If $\|\phi\|_\infty < \infty$ and $\|\kappa\|_\infty < \infty$, then for all $m \in \mathbb{N}$ and $D \in (0,\infty)$*

$$\mathbb{P}\{vol_\phi(\mathcal{I}_1^{\mathbf{RG}}) > m\} \leq 2nD \exp\{-C\Delta^2 m\} + \mathbb{P}(|\mathcal{P}_n| \geq nD), \tag{5.2}$$

   where $C = (8\|\phi\|_\infty(1 + 3\|\kappa\|_\infty\mu(\mathcal{X})))^{-1}$.

(ii) *Let for $n \geq 1$, $\Lambda_n \in \mathcal{B}(\mathcal{X})$ be such that*

$$g(n) := 8 \sup_{\mathbf{x} \in \Lambda_n} |\phi(\mathbf{x})| \left(1 + 3\mu(\mathcal{X}) \sup_{(\mathbf{x},\mathbf{y}) \in \Lambda_n \times \Lambda_n} |\kappa(\mathbf{x},\mathbf{y})|\right) < \infty.$$

   Then for all $m \in \mathbb{N}$,

$$\mathbb{P}\{vol_\phi(\mathcal{I}_1^{\mathbf{RG}}) > m\} \leq n\mu(\Lambda_n^c) + 2nD \exp(-\Delta^2 m/g(n)) + \mathbb{P}(|\mathcal{P}_n| \geq nD).$$

Proof: Part (i) has been proved in [3] (see Lemmas 6.12 and 6.13 therein). We now prove (ii). Consider the truncated version of $\mathbf{RG}^{(n)}$ constructed using the basic structure $\{\mathcal{X}, \bar{\mu}, \bar{\kappa}, \bar{\phi}\}$ where $\bar{\mu} := \mu|_{\Lambda_n}$ (i.e. the restriction of μ to Λ_n), $\bar{\kappa}(\mathbf{x}, \mathbf{y}) = \kappa(\mathbf{x}, \mathbf{y})\mathbb{1}_{\Lambda_n}(\mathbf{x})\mathbb{1}_{\Lambda_n}(\mathbf{y})$ and $\bar{\phi}(\mathbf{x}) = \phi(\mathbf{x})\mathbb{1}_{\Lambda_n}(\mathbf{x})$. Note that $||\bar{\kappa}||_\infty < \infty$ and $||\bar{\phi}||_\infty < \infty$. Denote by $\bar{\mathcal{K}}$ the integral operator associated with $(\bar{\kappa}, \bar{\mu})$. Clearly $\|\bar{\mathcal{K}}\| \leq \|\mathcal{K}\|$ and thus $\bar{\Delta} = 1 - \|\bar{\mathcal{K}}\| \geq \Delta$. Consider the natural coupling between the truncated and original model by using the vertex set $\bar{\mathcal{P}}_n := \mathcal{P}_n \cap \Lambda_n$. Write $\bar{\mathcal{I}}_1^{\mathbf{RG}}$ for the component with the largest volume in the truncated model. Then we have

$$
\begin{aligned}
\mathbb{P}\{\mathrm{vol}_\phi(\mathcal{I}_1^{\mathbf{RG}}) > m\} &\leq \mathbb{P}\{\mathcal{P}_n \cap \Lambda_n^c \neq \emptyset\} + \mathbb{P}\{\mathcal{P}_n \subset \Lambda_n, \mathrm{vol}_\phi(\mathcal{I}_1^{\mathbf{RG}}) > m\} \\
&= \mathbb{P}\{\mathcal{P}_n \cap \Lambda_n^c \neq \emptyset\} + \mathbb{P}\{\mathrm{vol}_\phi(\bar{\mathcal{I}}_1^{\mathbf{RG}}) > m\} \\
&\leq (1 - \exp\{-n\mu(\Lambda_n^c)\}) + 2nD\exp\{-\Delta^2 m/g(n)\} + \mathbb{P}(|\mathcal{P}_n| \geq nD),
\end{aligned}
$$

where the last inequality follows from part (i) and the fact that $\Delta \leq \bar{\Delta}$. $\qquad\square$

For the proof of the following elementary lemma we refer the reader to Lemma 6.5 in [3].

Lemma 5.2. *Let κ, κ' be kernels on a common finite measure space $(\mathcal{X}, \mu)$, with the associated integral operators $\mathcal{K}, \mathcal{K}'$ respectively. Then*

(a) $\|\mathcal{K}\| \leq \|\kappa\|_{2,\mu} := \left(\int_{\mathcal{X} \times \mathcal{X}} \kappa^2(\mathbf{x}, \mathbf{y})\mu(d\mathbf{x})\mu(d\mathbf{y})\right)^{1/2}$.
(b) *If $\kappa \leq \kappa'$, then $||\mathcal{K}|| \leq ||\mathcal{K}'||$.*
(c) $||\mathcal{K} - \mathcal{K}'|| \leq ||\bar{\mathcal{K}}||$*, where $\bar{\mathcal{K}}$ is the integral operator associated with $(|\kappa - \kappa'|, \mu)$.*

For the proof of the following lemma we refer the reader to Lemma 6.18 of [3].

Lemma 5.3. *Let $\tilde{\mu}, \mu$ be two finite measures on the space $\mathcal{X}$. Assume $\tilde{\mu} \ll \mu$ and let $g = d\tilde{\mu}/d\mu$ be the Radon-Nikodym derivative. Let $\tilde{\kappa}$ be a kernel on $\mathcal{X} \times \mathcal{X}$, and define κ as*

$$
\kappa(\mathbf{x}, \mathbf{y}) := \sqrt{g(\mathbf{x})g(\mathbf{y})}\tilde{\kappa}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathcal{X}.
$$

Denote by $\mathcal{K}$ [resp. $\tilde{\mathcal{K}}$] the integral operator on $L^2(\mathcal{X}, \mu)$ [resp. $L^2(\mathcal{X}, \tilde{\mu})$] associated with (κ, μ) [resp. $(\tilde{\kappa}, \tilde{\mu})$]. Then $\|\mathcal{K}\|_{L^2(\mu)} = \|\tilde{\mathcal{K}}\|_{L^2(\tilde{\mu})}$, where $\|\mathcal{K}\|_{L^2(\mu)}$ [resp. $\|\tilde{\mathcal{K}}\|_{L^2(\tilde{\mu})}$] is the norm of the operator $\mathcal{K}$ [resp. $\tilde{\mathcal{K}}$] on $L^2(\mu)$ [resp. $L^2(\tilde{\mu})$].

We end this section with a lemma drawing a connection between the Yule process and the pure jump Markov processes with distribution $\nu_{s,i}$ that arose in the construction of the inhomogeneous random graphs $\mathbf{RG}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$, see (4.22). Fix $j \geq 1$ and recall that a rate one Yule process started at time $t = 0$ with j individuals is a pure birth Markov process $Y(t)$ with $Y(0) = j$ and the rate of going from state i to $i + 1$ given by $\lambda(i) := i$. Also recall from (4.18) that $\mathcal{W}$ denotes the Skorohod space $\mathcal{W} := \mathcal{D}([0, \infty) : \mathbb{N}_0)$.

Lemma 5.4. *Fix $1 \leq i \leq K$ and $s \geq 0$ and rate functions $\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}$. Let $\{w(t)\}_{t \geq 0}$ be a pure jump Markov process with law $\nu_{s,i} := \nu_{s,i}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ as in (4.22). Then*

(i) *The process $w^*(t) := w(t/K\|\boldsymbol{\gamma}\|_\infty)/K$ can be stochastically dominated by a Yule processes $Y(\cdot)$ starting with two particles (i.e. $Y(0) = 2$).*
(ii) *Fix $t > 0$, $s \in [0, t]$ and $1 \leq i \leq K$. Then we have*

$$
\int_{\mathcal{W}} [w(t)]^2 \nu_{s,i}(dw) \leq 6K^2 e^{2tK\|\boldsymbol{\gamma}\|_\infty},
$$

and for any $A > 0$ we have

$$
\nu_{s,i}(w(t) > A) \leq 2(1 - e^{-tK\|\boldsymbol{\gamma}\|_\infty})^{A/2K}.
$$

Proof: Let us first prove (i). Note that under the law $\nu_{s,i}$, the process w satisfies $w(u) = 0$ for $u < s$ and $w(s) = K + i \leq 2K$. Further for times $t > s$, by (4.21), the jumps of the w can be bounded as $\Delta w(t) := w(t) - w(t-) \leq K$ at rate at most $w(t) \|\boldsymbol{\gamma}\|_\infty$. The process $w^*(\cdot)$ is obtained by rescaling time and space for the process $w(\cdot)$. This is once again a pure jump Markov process with jump sizes $\Delta w^*(t) \leq 1$ which happen at rate at most one. Further $w^*(0) \leq 2$. This immediately implies that this process is stochastically dominated by a Yule process with $Y(0) = 2$. This completes the proof.

We now consider (ii). We will use the result in part (i). Note that a Yule process started with two individuals at time $t = 0$ has the same distribution as the sum of two independent Yule processes $\{Y_1(t)\}_{t \geq 0}$ and $\{Y_2(t)\}_{t \geq 0}$ with $Y_1(0) = Y_2(0) = 1$. Now fix $t > 0$, $s \leq t$ and $1 \leq i \leq K$. Let $w(\cdot)$ have distribution $\nu_{s,i}$. From (i) we have

$$w(t) \leq_d K \cdot (Y_1(tK\|\boldsymbol{\gamma}\|_\infty) + Y_2(tK\|\boldsymbol{\gamma}\|_\infty)). \tag{5.3}$$

For simplicity write $X_1 = Y_1(tK\|\boldsymbol{\gamma}\|_\infty)$ and $X_2 = Y_2(tK\|\boldsymbol{\gamma}\|_\infty)$. Well known results about Yule processes ([20, Chapter 2]) say that the random variables X_1 and X_2 have a Geometric distribution with $p := e^{-tK\|\boldsymbol{\gamma}\|_\infty}$. The first bound in (ii) follows from the Geometric distribution and the fact

$$\int_{\mathcal{W}} [w(t)]^2 \nu_{s,i}(dw) \leq K^2 \mathbb{E}[(X_1 + X_2)^2].$$

The second bound in (ii) follows from

$$\nu_{s,i}(\{w(t) > A\}) \leq 2\mathbb{P}\{X_1 > A/2K\}.$$

This completes the proof. □

5.2. **Some perturbation estimates for $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})$.** Recall that Lemma 4.4 coupled the evolution of $\Gamma(t)$ (equivalently $\mathbf{BSR}^*(t)$) with two inhomogeneous random graphs $\mathbf{RG}(\mathbf{a}^+, b, \mathbf{c}^+)(t)$ and $\mathbf{RG}(\mathbf{a}^-, b, \mathbf{c}^-)(t)$ which can be considered as perturbations of $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})(t)$. The aim of this section is to understand the effect of such perturbations on the associated operator norms. Throughout this section $\mathcal{X}$ and ϕ_t are as in (4.18) and (4.19), respectively. Given the basic structure $\{\mathcal{X}, \mu_t, \kappa_t, \phi_t\}$, $t > 0$, associated with rate functions $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$, we denote the norm of the integral operator $\mathcal{K}_t$ associated with (κ_t, ϕ_t) as $\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$.

The following proposition which is the main result of this section studies the effect of perturbations of $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ on this norm. For a K-dimensional vector $\mathbf{v} = (v_1, \cdots v_K)$ and a scalar θ, $\mathbf{v} + \theta$ denotes the vector $(v_1 + \theta, \cdots v_K + \theta)$ and $(\mathbf{v} + \theta)^+$ denotes the vector $((v_1 + \theta)^+, \cdots (v_K + \theta)^+)$.

Proposition 5.5. *For $\varepsilon > 0$ let $\rho_t^+ = \rho_t(\boldsymbol{\alpha} + \varepsilon, \beta + \varepsilon, \boldsymbol{\gamma} + \varepsilon)$ and $\rho_t^- = \rho_t((\boldsymbol{\alpha} - \varepsilon)^+, (\beta - \varepsilon)^+, (\boldsymbol{\gamma} - \varepsilon)^+)$, where $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ are rate functions. Assume that $\max\{\|\boldsymbol{\alpha} + \varepsilon\|_\infty, \|\beta + \varepsilon\|_\infty, \|\boldsymbol{\gamma} + \varepsilon\|_\infty\} < 1$. For every $T > 0$, there is a $C_5 \in (0, \infty)$ such that for all $\varepsilon > 0$ and $t \in [0, T]$,*

$$\max\{|\rho_t - \rho_t^+|, |\rho_t - \rho_t^-|\} \leq C_5 \sqrt{\varepsilon} \cdot (-\log \varepsilon)^2.$$

Proof of Proposition 5.5 relies on Lemmas 5.6 – 5.10 below, and is given at the end of the section. We analyze the effect of perturbation of β, $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ separately in Lemmas 5.6, 5.8) and 5.10, respectively.

Lemma 5.6 (Perturbations of β). *Let $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ be rate functions and β^ε be be a nonnegative function on $[0, \infty)$ with $\sup_{0 \leq s < \infty} |\beta^\varepsilon(s) - \beta(s)| \leq \varepsilon$. Then*

$$|\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_t(\boldsymbol{\alpha}, \beta^\varepsilon, \boldsymbol{\gamma})| \leq C\varepsilon,$$

where $C = 6t^2 K^3 \|\boldsymbol{\alpha}\|_\infty e^{2t\|\boldsymbol{\gamma}\|_\infty}$.

Proof: Let (μ_t, κ_t) be the type measure and kernel associated with $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ Note that a perturbation in β only affects the kernel κ_t and not μ_t. Recall the representation of μ_t in terms of probability measures $\{\nu_{u,i}, u \in [0,t], i = 1, \cdots, K\}$. From Lemma 5.4(ii)

$$\int_{\mathcal{W}} [w(t)]^2 \nu_{u,i}(dw) \leq 6K^2 e^{2tK\|\boldsymbol{\gamma}\|_\infty}, \text{ for all } u \in [0,t], i = 1, \cdots K. \tag{5.4}$$

Denote the kernel obtained by replacing β by β^ε in (4.24) by κ_t^ε. Since $\|\beta - \beta^\varepsilon\|_\infty < \varepsilon$, we have from (4.24) that

$$|\kappa_t(\mathbf{x}, \mathbf{y}) - \kappa_t^\varepsilon(\mathbf{x}, \mathbf{y})| \leq \varepsilon \int_0^t w(u)\tilde{w}(u)du \leq \varepsilon t w(t)\tilde{w}(t),$$

$\mu_t \otimes \mu_t$ a.e. $(\mathbf{x}, \mathbf{y}) = ((s, w), (r, \tilde{w}))$.

By Lemma 5.2 (a) and (c) we now have

$$|\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_t(\boldsymbol{\alpha}, \beta^\varepsilon, \boldsymbol{\gamma})| \leq \left(\int_{\mathcal{X} \times \mathcal{X}} |\kappa_t(\mathbf{x}, \mathbf{y}) - \kappa_t^\varepsilon(\mathbf{x}, \mathbf{y})|^2 d\mu_t(\mathbf{x}) d\mu_t(\mathbf{y}) \right)^{1/2}$$

$$\leq \left(\int_{\mathcal{X} \times \mathcal{X}} (\varepsilon t w(t)\tilde{w}(t))^2 d\mu_t(\mathbf{x}) d\mu_t(\mathbf{y}) \right)^{1/2}$$

$$= \varepsilon t \sum_{i=1}^K \int_0^t \alpha_i(s) \left[\int_{\mathcal{W}} [w(t)]^2 \nu_{s,i}(dw) \right] ds$$

$$\leq \varepsilon t \cdot t \|\boldsymbol{\alpha}\|_\infty \cdot K \cdot 6K^2 e^{2tK\|\boldsymbol{\gamma}\|_\infty},$$

where the last inequality follows from (5.4). The result follows. $\qquad \square$

When $\boldsymbol{\alpha}$ or $\boldsymbol{\gamma}$ is perturbed, the underlying measure μ_t changes as well and thus one needs to analyze the corresponding Radon-Nikodym derivatives. This is done in the following two lemmas. We denote by $\{\mathcal{G}_s\}_{0 \leq s < \infty}$ the canonical filtration on $\mathcal{D}([0,\infty) : \mathbb{N}_0)$. In the following we follow the convention that $0/0 = 1$.

Lemma 5.7. *Fix $\varepsilon > 0$ and let $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$, $(\tilde{\boldsymbol{\alpha}}, \tilde{\beta}, \tilde{\boldsymbol{\gamma}})$ be two sets of rate functions such that for all $1 \leq i \leq K$ and $s \geq 0$,*

$$\alpha_i(s) - \varepsilon \leq \tilde{\alpha}_i(s) \leq \alpha_i(s), \text{ and } \gamma_i(s) - \varepsilon \leq \tilde{\gamma}_i(s) \leq \gamma_i(s).$$

Fix $t \geq 0$ and let μ_t and $\tilde{\mu}_t$ be the corresponding type measures on $\mathcal{X}$. For $(s, w) \in \mathcal{X}$ and $j \geq 1$ let τ_j^s for the time of the j-th jump of $w(\cdot)$ after time s (μ_t a.s.). Also write $\Delta(u) = w(u) - w(u-)$ for $u \geq 0$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\tilde{\mu} \ll \mu$ and

$$\frac{d\tilde{\mu}_t}{d\mu_t}(s, w) = \frac{\tilde{\alpha}_{\Delta(s) - K}(s)}{\alpha_{\Delta(s) - K}(s)} \times \Pi_{j:\tau_j^s \leq t} \frac{\tilde{\gamma}_{\Delta(\tau_j^s)}(\tau_j^s)}{\gamma_{\Delta(\tau_j^s)}(\tau_j^s)} \times \exp \left\{ -\int_s^t w(u) \left[\sum_{i=1}^K \tilde{\gamma}_i(u) - \sum_{i=1}^K \gamma_i(u) \right] du \right\}.$$

Proof: For $i = 1, \cdots K$, define finite measures μ_t^i, $\tilde{\mu}_t^i$ on $\mathcal{X}$ as

$$\mu_t^i(du\, dw) = \alpha_i(u)\nu_{u,i}(dw)1_{[0,t]}(u)du, \ \tilde{\mu}_t^i(du\, dw) = \tilde{\alpha}_i(u)\tilde{\nu}_{u,i}(dw)1_{[0,t]}(u)du,$$

where $\nu_{u,i}$ is defined above (4.23) and $\tilde{\nu}_{u,i}$ is defined similarly on replacing γ_i with $\tilde{\gamma}_i$. We will show that

$$\text{for all } 1 \leq k \leq K \text{ and } s \in [0, T], \tilde{\nu}_{s,k} \ll \nu_{s,k} \text{ and } \frac{d\tilde{\nu}_{s,k}}{d\nu_{s,k}}(w) = L_s^t(w), \tag{5.5}$$

where

$$L_s^t := \Pi_{j \geq 1} \left(\frac{\tilde{\gamma}_{\Delta(\tau_j^s)}(\tau_j^s)}{\gamma_{\Delta(\tau_j^s)}(\tau_j^s)} \mathbb{1}_{\{\tau_j^s \leq t\}} \right) \times \exp\left\{ -\int_s^t w(u) \left[\sum_{i=1}^K \tilde{\gamma}_i(u) - \sum_{i=1}^K \gamma_i(u) \right] du \right\}. \quad (5.6)$$

The lemma is an immediate consequence of (5.5) on observing that μ_t^i and μ_t^j are mutually singular when $i \neq j$, and the relation $\mu_t = \sum_{i=1}^K \mu_t^i$, $\tilde{\mu}_t = \sum_{i=1}^K \tilde{\mu}_t^i$.

We now show (5.5). From the construction of $\nu_{s,k}$ it follows that, there are counting processes $\{N_i(u)\}_{u \in [s,t]}$, $i = 1, \cdots K$, on $\mathcal{W}$ such that

$$w(u) = w(s) + \sum_{i=1}^K i N_i(u), \text{ for } u \in [s,t], \text{ a.s. } \nu_{s,k} \quad (5.7)$$

and

$$M_i(u) := N_i(u) - \int_s^u w(r)\gamma_i(r)dr \quad \text{under } \nu_{s,k} \text{ is a } \{\mathcal{G}_u\}_{u \in [s,t]} \text{ local martingale for } u \in [s,t]. \quad (5.8)$$

From standard results it follows that L_s^t is a local-martingale and super-martingale (see Theorem VI.T2, p.165 of [9]). In order to show (5.5), it suffices to show that $\{L_s^u\}_{u \in [s,t]}$ is a martingale. By a change of variable formula it follows that (see e.g. Theorem A4.T4, p. 337 of [9])

$$L_s^v = 1 + \sum_{i=1}^K \int_s^v L_s^{u-} \cdot \left(\frac{\tilde{\gamma}_i(u)}{\gamma_i(u)} - 1 \right) dM_i(u), \ v \in [s,t]. \quad (5.9)$$

In order to show L_s^t is a martingale, it then suffices, in view of (5.8), to show that (see e.g. Theorem II.T8 in [9]) for all $1 \leq i \leq K$,

$$\int_{\mathcal{W}} \left[\int_s^t L_s^u \cdot |\tilde{\gamma}_i(u) - \gamma_i(u)| w(u) du \right] d\nu_{s,k}(w) < \infty.$$

Finally note that $L_s^u \leq e^{\varepsilon t w(t)}$. Using Lemma 5.4(i) and standard estimates for Yule processes, it follows that for ε sufficiently small $\sup_{s \in [0,t]} \sup_{1 \leq k \leq K} \int_{\mathcal{W}} w(t) e^{\varepsilon t w(t)} d\nu_{s,k}(w) < \infty$.

The result follows. $\qquad \square$

We will now use the above lemma to study the effect of perturbations in $\boldsymbol{\alpha}$ on $\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$.

Lemma 5.8 (Perturbations of $\boldsymbol{\alpha}$). *Fix $\varepsilon > 0$. Let $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ be rate functions and let $\boldsymbol{\alpha}^\varepsilon = (\alpha_1^\varepsilon, \cdots, \alpha_K^\varepsilon)$, where α_i^ε are continuous nonnegative bounded functions on $[0, \infty)$ such that for all $1 \leq i \leq K$ and $s \in [0, T]$*

$$\alpha_i(s) - \varepsilon \leq \alpha_i^\varepsilon(s) \leq \alpha_i(s).$$

Then for every $t > 0$,

$$|\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_t(\boldsymbol{\alpha}^\varepsilon, \beta, \boldsymbol{\gamma})| \leq C\sqrt{\varepsilon},$$

where $C = t\|\beta\|_\infty \cdot 6K^2 e^{2tK\|\boldsymbol{\gamma}\|_\infty} \cdot 4tK\sqrt{\|\boldsymbol{\alpha}\|_\infty}$.

Proof: Let (μ_t, κ_t) be the type measure and kernel associated with $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$. Also, let μ_t^ε be the type measure associated with $(\boldsymbol{\alpha}^\varepsilon, \beta, \boldsymbol{\gamma})$. By Lemma 5.7,

$$g(s,w) := \frac{d\mu_t^\varepsilon}{d\mu_t}(s,w) = \frac{\alpha_{\Delta(s)-K}^\varepsilon(s)}{\alpha_{\Delta(s)-K}(s)} \text{ for } (s,w) \in [0,t] \times \mathcal{W}.$$

Using Lemma 5.2 (c), (a), Lemma 5.3, and the fact that $|\kappa_t(\mathbf{x}, \mathbf{y})| \leq t\|\beta\|_\infty w(t)\tilde{w}(t)$, $\mu_t \otimes \mu_t$ a.e. $(\mathbf{x}, \mathbf{y}) = ((s, w), (\tilde{s}, \tilde{w}))$, we have

$$
|\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_t(\boldsymbol{\alpha}^\varepsilon, \beta, \boldsymbol{\gamma})| \leq \left(\int_{\mathcal{X} \times \mathcal{X}} |\sqrt{g(\mathbf{x})g(\mathbf{y})} - 1|^2 |\kappa_t(\mathbf{x}, \mathbf{y})|^2 d\mu_t(\mathbf{x}) d\mu_t(\mathbf{y}) \right)^{1/2}
$$
$$
\leq t\|\beta\|_\infty \left(\int_{\mathcal{X} \times \mathcal{X}} |\sqrt{g(\mathbf{x})g(\mathbf{y})} - 1|^2 w^2(t) \tilde{w}^2(t) d\mu_t(\mathbf{x}) d\mu_t(\mathbf{y}) \right)^{1/2}
$$
$$
\leq t\|\beta\|_\infty d_1 \left(\sum_{i,j=1}^{K} \int_{[0,t]^2} \left(\sqrt{\frac{\alpha_i^\varepsilon(s)\alpha_j^\varepsilon(u)}{\alpha_i(s)\alpha_j(u)}} - 1 \right)^2 \alpha_i(s)\alpha_j(u) ds du \right)^{1/2},
$$

$$(5.10)$$

where

$$
d_1 = \sup_{s \in [0,T]} \sup_{1 \leq i \leq K} \int_{\mathcal{W}} |w(t)|^2 \nu_{s,i}(dw) \leq 6K^2 e^{2tK\|\gamma\|_\infty},
$$

and the last inequality follows from (5.4). In order to bound (5.10), note that:

$$
\left| \sqrt{\alpha_i(s)\alpha_j(u)} - \sqrt{\alpha_i^\varepsilon(s)\alpha_j^\varepsilon(u)} \right| = \left| \sqrt{\alpha_i(s)} \left(\sqrt{\alpha_j(u)} - \sqrt{\alpha_j^\varepsilon(u)} \right) + \left(\sqrt{\alpha_i(s)} - \sqrt{\alpha_i^\varepsilon(s)} \right) \sqrt{\alpha_j^\varepsilon(u)} \right|
$$
$$
\leq 2\sqrt{\varepsilon} \left(\sqrt{\alpha_i(s)} + \sqrt{\alpha_j(u)} \right).
$$

Plugging the above bound in (5.10) gives the desired result. □

We will now analyze the effect of perturbations in $\boldsymbol{\gamma}$ on $\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$. We need the following preliminary truncation lemma.

Lemma 5.9. *For every $T > 0$, there exist $C_6, C_7, A_0 \in (0, \infty)$ such that for any $t \in [0, T]$ and rate functions $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ the following holds: Let μ_t, κ_t be the type measure and kernel associated with $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$. Define, for $A \in (0, \infty)$, the kernel $\kappa_{A,t}$ as*

$$
\kappa_{A,t}(\mathbf{x}, \mathbf{y}) = \kappa_t(\mathbf{x}, \mathbf{y}) \mathbb{1}_{\{w(t) \leq A, \tilde{w}(t) \leq A\}} \text{ where } \mathbf{x} = (s, w), \mathbf{y} = (r, \tilde{w}). \tag{5.11}
$$

Then for all $A > A_0$,

$$
\rho(\kappa_t) - C_6 e^{-C_7 A} \leq \rho(\kappa_{A,t}) \leq \rho(\kappa_t),
$$

where $\rho(\kappa_t)$ [resp. $\rho(\kappa_{A,t})$] denotes the norm of the operator associated with (κ_t, μ_t) [resp. $(\kappa_{A,t}, \mu_t)$].

Proof: The upper bound in the lemma is immediate from Lemma 5.2 (b). We now consider the lower bound. For the rest of the proof, we suppress the dependence of $\kappa_t, \kappa_{A,t}, \mu_t$ on t.

Note that, from Lemma 5.2 (a,c)

$$\rho(\kappa) - \rho(\kappa_A) \leq \left(\int_{\mathcal{X} \times \mathcal{X}} (\kappa(\mathbf{x}, \mathbf{y}) - \kappa_A(\mathbf{x}, \mathbf{y}))^2 d\mu(\mathbf{x}) d\mu(\mathbf{y}) \right)^{1/2}$$

$$\leq 2 \left(\int_{\mathcal{X} \times \mathcal{X}} (t\|\beta\|_\infty w(t) \tilde{w}(t) \mathbb{1}\{\tilde{w}(t) > A\})^2 d\mu(\mathbf{x}) d\mu(\mathbf{y}) \right)^{1/2}$$

$$\leq 2t\|\beta\|_\infty \left(d_1 \sum_{i=1}^K \sum_{j=1}^K \int_{[0,t] \times [0,t]} \alpha_i(s) \alpha_j(u) ds du \right)^{1/2}$$

$$\leq 2t\|\beta\|_\infty \cdot t\|\boldsymbol{\alpha}\|_\infty \cdot \sqrt{d_1}, \tag{5.12}$$

where

$$d_1 = \int_{\mathcal{W}} [w(t)]^2 \nu_{s,i}(dw) \int_{\mathcal{W}} [w(t)]^2 \mathbb{1}_{\{w(t) > A\}} \nu_{s,i}(dw). \tag{5.13}$$

By (5.3), $w(t) \leq_d K(X_1 + X_2)$ where X_1, X_2 are independent and identically distributed with Geometric $p = e^{-tK\|\boldsymbol{\gamma}\|_\infty}$ distribution.

$$\int_{\mathcal{W}} [w(t)]^2 \mathbb{1}_{\{w(t) > A\}} \nu_{s,i}(dw) \leq K^2 \mathbb{E} \left[(X_1 + X_2)^2 \mathbb{1}_{\{X_1 + X_2 > A/K\}} \right]$$

$$= K^2 \mathbb{E} \left[(X_1 + X_2)^2 (\mathbb{1}_{\{X_1 + X_2 > C, X_1 \geq X_2\}} + \mathbb{1}_{\{X_1 + X_2 > C, X_1 < X_2\}}) \right]$$

$$\leq 4K^2 \mathbb{E} \left[X_1^2 \mathbb{1}_{\{X_1 > A/2K\}} + X_2^2 \mathbb{1}_{\{X_2 > A/2K\}} \right],$$

The above quantity can be bounded by

$$d_2 (1 - e^{-2TK\|\boldsymbol{\gamma}\|_\infty})^{A/2K} \leq d_2 \exp\left\{ -\frac{e^{-2TK\|\boldsymbol{\gamma}\|_\infty}}{K} A \right\}$$

for some constant d_2. The result now follows on using the above bound and (5.4) in (5.13) and (5.12). $\qquad\square$

Lemma 5.10 (Perturbations of $\boldsymbol{\gamma}$). *For every $T > 0$, there exists $C_8 \in (0, \infty)$ and $\varepsilon_0 \in (0, 1)$ such that for all $t \in [0, T]$ and rate functions $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ the following holds: Suppose $\varepsilon \in (0, \varepsilon_0)$ and $\boldsymbol{\gamma}^\varepsilon = (\gamma_1^\varepsilon, \cdots \gamma_K^\varepsilon)$, where γ_i^ε are continuous, nonnegative maps on $[0, T]$ such that for all $1 \leq i \leq K$*

$$\gamma_i(s) - \varepsilon \leq \gamma_i^\varepsilon(s) \leq \gamma_i(s), \text{ for all } s \in [0, T].$$

Then

$$|\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)| \leq C_8 \sqrt{\varepsilon} \cdot (-\log \varepsilon)^2.$$

Proof: Let (μ_t, κ_t) [resp. $(\mu_t^\varepsilon, \kappa_t^\varepsilon)$] be the type measure and kernel associated with $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ [resp. $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)$]. By Lemma 5.7, for $(s, w) \in [0, t] \times \mathcal{W}$

$$\frac{d\mu_t^\varepsilon}{d\mu_t}(s, w) = \Pi_{j \geq 1} \left(\frac{\gamma_{\Delta(\tau_j^s)}^\varepsilon(\tau_j^s)}{\gamma_{\Delta(\tau_j^s)}(\tau_j^s)} \mathbb{1}_{\{\tau_j^s \leq t\}} \right) \times \exp\left\{ -\int_s^t w(u) \left[\sum_{i=1}^K \gamma_i^\varepsilon(u) - \sum_{i=1}^K \gamma_i(u) \right] du \right\}.$$

Denote the right side as $L_s^t(w)$. Then, as in the proof of Lemma 5.7, it follows that $\{L_s^u(w)\}_{u \in [s,t]}$ is a $\{\mathcal{G}_u\}_{u \in [s,t]}$ martingale under $\nu_{s,k}$ for every $k = 1, \cdots K$. Fix $A \in (A_0, \infty)$, where A_0 is as in Lemma 5.9, and let $\kappa_{A,t}$ be defined by (5.11). Similarly define $\kappa_{A,t}^\varepsilon$ by replacing κ_t with κ_t^ε in (5.11). Denote the operator norm of the integral operators associated

with $(\kappa_{A,t}, \mu_t)$ and $(\kappa_{A,t}^\varepsilon, \mu_t^\varepsilon)$ by $\rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ and $\rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)$, respectively. Then, by Lemma 5.3 and Lemma 5.2 (a,c),

$$|\rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)|$$

$$\leq \left(\int_{\mathcal{X} \times \mathcal{X}} \left| \sqrt{\frac{d\mu_t^\varepsilon}{d\mu_t}(s, w) \frac{d\mu_t^\varepsilon}{d\mu_t}(u, \tilde{w})} - 1 \right|^2 (\kappa_{A,t}(\mathbf{x}, \mathbf{y}))^2 d\mu_t(\mathbf{x}) d\mu_t(\mathbf{y}) \right)^{1/2}$$

$$\leq tA^2 \|\beta\|_\infty \left(\sum_{i=1}^K \sum_{j=1}^K \int_{[0,t] \times [0,t]} \alpha_i(s) \alpha_j(u) \int_{\mathcal{W} \times \mathcal{W}} \left| \sqrt{L_s^t(w) L_u^t(w)} - 1 \right|^2 \nu_{s,i}(dw) \nu_{u,j}(d\tilde{w}) \right)^{1/2}.$$

$$(5.14)$$

Next, using the martingale property of L_s^t, we have

$$\int_{\mathcal{W} \times \mathcal{W}} \left| \sqrt{L_s^t(w) L_u^t(w)} - 1 \right|^2 \nu_{s,i}(dw) \nu_{u,j}(d\tilde{w})$$

$$= 2 - 2 \int_{\mathcal{W}} \sqrt{L_s^t(w)} \nu_{s,i}(dw) \int_{\mathcal{W}} \sqrt{L_u^t(w)} \nu_{u,j}(dw)$$

$$\leq 4 - 2 \int_{\mathcal{W}} \sqrt{L_s^t(w)} \nu_{s,i}(dw) - 2 \int_{\mathcal{W}} \sqrt{L_u^t(w)} \nu_{u,j}(dw), \qquad (5.15)$$

where the inequality on the last line follows on observing that from Jensen's inequality the two integrals on the second line are bounded by 1 and using the elementary inequality $a_1 + a_2 \leq a_1 a_2 + 1$, for $a_1, a_2 \in [0, 1]$. We will now estimate the two integrals on the last line of (5.15) by using the martingale properties of $\{L_s^u\}_{u \in [s,t]}$ and the representations (5.7) and (5.9) in the proof of Lemma 5.7. For the rest of the proof we write L_s^u as $L_s(u)$. By an application of Ito's formula, we have that for every $k = 1, \cdots K$, $\nu_{s,k}$ a.s.

$$\sqrt{L_s(t)} - 1 - \sum_{i=1}^K \int_s^t \frac{\sqrt{L_s(u-)}}{2} \left(\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)} - 1 \right) dM_i(u)$$

$$= \sum_{s < u \leq t} \left(\sqrt{L_s(u)} - \sqrt{L_s(u-)} \right) - \sum_{i=1}^K \int_s^t \frac{\sqrt{L_s(u-)}}{2} \left(\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)} - 1 \right) dN_i(u)$$

$$= \sum_{i=1}^K \int_s^t \sqrt{L_s(u-)} \left(\sqrt{\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)}} - 1 \right) dN_i(u) - \sum_{i=1}^K \int_s^t \frac{\sqrt{L_s(u-)}}{2} \left(\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)} - 1 \right) dN_i(u)$$

$$= -\frac{1}{2} \sum_{i=1}^K \int_s^t \sqrt{L_s(u-)} \left(\sqrt{\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)}} - 1 \right)^2 dN_i(u),$$

$$(5.16)$$

where the second equality follows on observing that for $u \in (s, t]$ $\sqrt{L_s(u)} = \sum_{i=1}^K \sqrt{L_s(u-)} \sqrt{\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)}} \Delta N_i(u)$.

As in the proof of Lemma 5.7, we can check that for all i, k,

$$\int_{\mathcal{W}} \left[\int_s^t \sqrt{L_s(u)} \cdot |\gamma_i^\varepsilon(u) - \gamma_i(u)| w(u) du \right] d\nu_{s,k}(w) < \infty,$$

and consequently the last term on the left side of (5.16) is a martingale. Denoting the expectation operator corresponding to the probability measure $\nu_{s,k}$ on $\mathcal{W}$ by $\mathbb{E}_{s,k}$, we have

$$
\begin{aligned}
1 - \mathbb{E}_{s,k}[\sqrt{L_s(t)}] =& \frac{1}{2} \sum_{i=1}^{K} \mathbb{E}_{s,k} \left[\int_s^t \sqrt{L_s(u-)} \left(\sqrt{\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)}} - 1 \right)^2 dN_i(u) \right] \\
=& \frac{1}{2} \sum_{i=1}^{K} \mathbb{E}_{s,k} \left[\int_s^t \sqrt{L_s(u)} \left(\sqrt{\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)}} - 1 \right)^2 w(u) \gamma_i(u) du \right] \\
=& \frac{1}{2} \sum_{i=1}^{K} \int_s^t \mathbb{E}_{s,k} \left[\sqrt{L_s(u)} w(u) \left(\sqrt{\gamma_i^\varepsilon(u)} - \sqrt{\gamma_i(u)} \right)^2 \right] du \\
\leq& \frac{1}{2} \int_s^t K\varepsilon \cdot \mathbb{E}_{s,k}[\sqrt{L_s(u)} w(u)] du \\
\leq& \frac{K\varepsilon}{2} \int_s^t \left(\mathbb{E}_{s,k}[L_s(u)] \mathbb{E}_{s,k}[w^2(u)] \right)^{1/2} du \\
\leq& \frac{K\varepsilon}{2} \cdot t \cdot (6K^2 e^{2TK\|\boldsymbol{\gamma}\|_\infty})^{1/2},
\end{aligned}
$$

where the last inequality follows from (5.4). Using the above bound in (5.14) we now have

$$
|\rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)| \leq tA^2 \|\beta\|_\infty \cdot t \|\boldsymbol{\alpha}\|_\infty \cdot \left[2K\varepsilon t (6K^2 e^{2TK\|\boldsymbol{\gamma}\|_\infty})^{1/2} \right]^{1/2}.
$$

Finally, by Lemma 5.9, we have

$$
\begin{aligned}
|\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)| \leq& |\rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)| + 2C_6 e^{-C_7 A} \\
<& d_1 A^2 \varepsilon^{1/2} + 2C_6 e^{-C_7 A},
\end{aligned}
$$

where $d_1 = tA^2 \|\beta\|_\infty \cdot t \|\boldsymbol{\alpha}\|_\infty \cdot \left[2Kt(6K^2 e^{2TK\|\boldsymbol{\gamma}\|_\infty})^{1/2} \right]^{1/2}$. The result now follows on taking $A = -\log \varepsilon$ in the above display and taking ε_0 sufficiently small (in particular such that $-\log(\varepsilon_0) > A_0$). $\square$

Now we combine all the above ingredients to complete the proof of Proposition 5.5.

Proof of Proposition 5.5: Using Lemma 5.10, 5.6 and 5.8, we get

$$
\begin{aligned}
|\rho_t^+ - \rho_t| \leq& |\rho_t(\boldsymbol{\alpha} + \varepsilon, \beta + \varepsilon, \boldsymbol{\gamma} + \varepsilon) - \rho_t(\boldsymbol{\alpha} + \varepsilon, \beta + \varepsilon, \boldsymbol{\gamma})| \\
& + |\rho_t(\boldsymbol{\alpha} + \varepsilon, \beta + \varepsilon, \boldsymbol{\gamma}) - \rho_t(\boldsymbol{\alpha} + \varepsilon, \beta, \boldsymbol{\gamma})| + |\rho_t(\boldsymbol{\alpha} + \varepsilon, \beta, \boldsymbol{\gamma}) - \rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})| \\
\leq& C_8 \varepsilon^{1/2} (-\log \varepsilon)^2 + d_1 \varepsilon + d_2 \varepsilon^{1/2},
\end{aligned}
$$

where $d_1 = 6T^2 K^3 e^{2TK}$ and $d_2 = 24K^3 T^2 e^{2TK}$. A similar bound holds for $|\rho_t^- - \rho_t|$. The result follows. $\square$

5.3. **Effect of time perturbation on ρ_t.** Throughout this section we fix rate functions $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$. The aim of this section is to understand the evolution of the operator norm $\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ as t changes. The main result of the section is Proposition 5.11 which studies continuity and differentiability properties of the function $\rho(t) := \rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$, $t \geq 0$.

Proposition 5.11. *Suppose that $\beta(t) > 0$ for $t > 0$ and $\liminf_{t\to\infty} \beta(t) > 0$. Then*

(i) *ρ is a continuous strictly increasing function on $\mathbb{R}_+$ with $\rho(0) = 0$ and $\lim_{t\to\infty} \rho(t) = \infty$.*

(ii) *There is a unique value $t_c' = t_c'(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ such that $\rho(t_c') = 1$.*

The proof of the proposition relies on the following lemma and is given after the proof of the lemma.

Lemma 5.12. *Let $0 < t_1 \le t_2 < \infty$. Then*

$$|t_2 - t_1| \cdot \frac{\inf_{t_1 \le u \le t_2} \beta(u)}{t_1 \|\beta\|_\infty} \cdot \rho(t_1) \le \rho(t_2) - \rho(t_1) \le |t_2 - t_1| \cdot 6t_2 K^2 \|\beta\|_\infty \|\boldsymbol{\alpha}\|_\infty e^{2t_2 K \|\boldsymbol{\gamma}\|_\infty}.$$

Proof: Letting $\mu := \mu_{t_2}$ we have

$$|\rho(t_2) - \rho(t_1)| \le \left(\int_{\mathcal{X} \times \mathcal{X}} (\kappa_{t_2}(\mathbf{x}, \mathbf{y}) - \kappa_{t_1}(\mathbf{x}, \mathbf{y}))^2 d\mu(\mathbf{x}) d\mu(\mathbf{y}) \right)^{1/2}$$

$$\le \left(\int_{\mathcal{X} \times \mathcal{X}} (\|\beta\|_\infty w(t_2) \tilde{w}(t_2) |t_2 - t_1|)^2 d\mu(\mathbf{x}) d\mu(\mathbf{y}) \right)^{1/2}$$

$$\le |t_2 - t_1| \cdot \|\beta\|_\infty \cdot t_2 \|\boldsymbol{\alpha}\|_\infty \cdot 6K^2 e^{2t_2 K \|\boldsymbol{\gamma}\|_\infty},$$

where the last inequality once again follows from (5.4). This proves the upper bound.

Next note that, for $\mu \otimes \mu$ a.e. $(\mathbf{x}, \mathbf{y})$ such that $\kappa_{t_1}(\mathbf{x}, \mathbf{y}) \ne 0$, we have

$$\frac{\kappa_{t_2}(\mathbf{x}, \mathbf{y})}{\kappa_{t_1}(\mathbf{x}, \mathbf{y})} = 1 + \frac{\int_{t_1}^{t_2} w(u) \tilde{w}(u) \beta(u) du}{\int_0^{t_1} w(u) \tilde{w}(u) \beta(u) du}$$

$$\ge 1 + \frac{w(t_1) \tilde{w}(t_1) \inf_{t_1 \le u \le t_2} \beta(u) \cdot (t_2 - t_1)}{w(t_1) \tilde{w}(t_1) \|\beta\|_\infty t_1}.$$

Thus $\kappa_{t_2}(\mathbf{x}, \mathbf{y}) \ge \left(1 + |t_2 - t_1| \cdot \frac{\inf_{t_1 \le u \le t_2} \beta(u)}{t_1 \|\beta\|_\infty} \right) \kappa_{t_1}(\mathbf{x}, \mathbf{y})$ which from Lemma 5.2 (b) implies

$$\rho(t_2) - \rho(t_1) \ge |t_2 - t_1| \cdot \frac{\inf_{t_1 \le u \le t_2} \beta(u)}{t_1 \|\beta\|_\infty} \cdot \rho(t_1).$$

This completes the proof of the lower bound. $\square$

Proof of Proposition 5.11: Since $\kappa_0 = 0$, the property $\rho(0) = 0$ is immediate. Also Lemma 5.12 shows that ρ is continuous and strictly increasing. Finally since $\inf_{t \to \infty} \beta(t) > 0$, there exists $\delta > 0$ and a $t^* \in (0, \infty)$ such that for all $t \ge t^*$, $\beta(t) \ge \delta$. From Lemma 5.12 we then have, for $t \ge t^*$, $\rho(t) - \rho(t^*) \ge \frac{(t - t^*)\delta}{t^* \|\beta\|_\infty}$. This proves that $\rho(t) \to \infty$ as $t \to \infty$ and completes the proof of (i). Part (ii) is immediate from (i). $\square$

5.4. **Operator norm of $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})$ and critical time of BSR.** In this section we will prove Theorem 1.3. Recall that by Lemma 4.4, for any fixed time t, $\mathbf{BSR}^*(t)$ (more precisely, $\Gamma(t)$) can be approximated by perturbations of $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})(t)$. To estimate the volume of the largest component in $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})(t)$ we will use Lemma 5.1. In order to identify suitable Λ_n as in part (ii) of the lemma, we start with the following lemma.

Lemma 5.13. *Let $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ be rate functions and let μ_t be the associated type measure. Fix $T > 0$. Define $\Lambda \in \mathcal{B}(\mathcal{X})$ as $\Lambda = \{(s, w) \in \mathcal{X} : w(T) \le l\}$ for $l \in \mathbb{R}_+$. Then, for every $l \in \mathbb{R}_+$*

$$\mu_t(\Lambda^c) < 2T \|\boldsymbol{\alpha}\|_\infty \cdot \exp\left(-l \frac{e^{-TK\|\boldsymbol{\gamma}\|_\infty}}{2K} \right).$$

Proof: Note that

$$\mu_t(\Lambda^c) = \sum_{i=1}^{K} \int_0^t \alpha_i(u)\nu_{u,i}(\Lambda^c) \le ||\boldsymbol{\alpha}||_\infty T \sup_{u\in[0,T]} \sup_{1\le i\le K} \nu_{u,i}(\Lambda^c). \tag{5.17}$$

By (5.3),

$$\nu_{u,i}(\Lambda^c) = \nu_{u,i}(\{w : w(T) \ge l\}) \le \mathbb{P}(X_1 + X_2 \ge l/K) \le 2(1 - e^{-TK||\boldsymbol{\gamma}||_\infty})^{l/2K}.$$

where X_i are iid with $\text{Geom}(e^{-T||\boldsymbol{\gamma}||_\infty})$ distribution. Using this estimate in (5.17), we have

$$\mu_t(\Lambda^c) \le ||\boldsymbol{\alpha}||_\infty T \cdot 2(1 - e^{-TK||\boldsymbol{\gamma}||_\infty})^{l/2K}.$$

The result follows. $\square$

We will now use the above lemma along with Lemma 5.1 to estimate the largest component in $\mathbf{RG}^{(n)}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})(t)$. Recall the notation $\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ from Section 5.2.

Lemma 5.14. *Let $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ be rate functions and denote by $\mathcal{I}_1^{\mathbf{RG}}(t)$ the component with the largest volume, with respect to the weight function ϕ_t, in $\mathbf{RG}^{(n)}(t) := \mathbf{RG}^{(n)}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})(t)$. Then, for every $t > 0$ such that $\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) < 1$, there exists $A \in (0, \infty)$ such that*

$$\mathbb{P}(vol_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}}(t)) > A \log^4 n) \to 0, \text{ as } n \to \infty.$$

Proof: We will use Lemma 5.1(ii). Define

$$\Lambda_n := \{(s, w) \in \mathcal{X} : w(t) < B \log n\},$$

where B will be chosen appropriately later in the proof. Now consider the function $g(n)$ in Lemma 5.1(ii) with Λ_n defined as above and (μ, ϕ, κ) there replaced by $(\mu_t, \phi_t, \kappa_t)$, where (μ_t, κ_t) is the type measure and kernel associated with $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$. Note that

$$\kappa_t(\mathbf{x}, \mathbf{y}) = \int_0^t \beta(u)w(u)\tilde{w}(u)du \le t||\beta||_\infty w(t)\tilde{w}(t)$$

and therefore

$$g(n) \le 8B \log n(1 + 3\mu_t(\mathcal{X}) \cdot t||\beta||_\infty B^2 \log^2 n). \tag{5.18}$$

Writing $m_n = A \log^4 n$, the bound in Lemma 5.1(ii) then gives

$$\mathbb{P}(vol_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}}(t)) > m_n) \le n\mu_t(\Lambda_n^c) + 2n\mu_t(\mathcal{X}) \exp\left(-\Delta^2 A \log^4 n / g(n)\right), \tag{5.19}$$

where $\Delta = 1 - \rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) > 0$. Using Lemma 5.13 with $l = B \log n$ gives

$$n\mu_t(\Lambda_n^c) \le nt||\boldsymbol{\alpha}||_\infty \cdot n^{-Be^{-T||\boldsymbol{\gamma}||_\infty}/2K} = o(1) \tag{5.20}$$

for $B > 2Ke^{T||\boldsymbol{\gamma}||_\infty}$. Now fix $B > e^{T||\boldsymbol{\gamma}||_\infty/2K}$, and choose A large such that

$$n\mu_t(\mathcal{X}) \exp\left(-\Delta^2 A \log^4 n / g(n)\right) \to 0$$

as $n \to \infty$. The result follows. $\square$

Proof of Theorem 1.3: Let, for $t \ge 0$, (μ_t, κ_t) be the type measure and the kernel associated with rate functions $(\mathbf{a}, b, \mathbf{c})$. We will prove Theorem 1.3 with this choice of (μ_t, κ_t). From Proposition 5.11 we have that $\rho(t) = \rho_t(\mathbf{a}, b, \mathbf{c})$ is continuous and strictly increasing in t and there is a unique $t'_c \in (0, \infty)$ such that $\rho(t'_c) = 1$. It now suffices to show that: (a) For $t < t'_c$, $|\mathcal{C}_1(t)|$ (the size of the largest component in $\mathbf{BSR}^*(t)$) is $O(\log^4 n)$; and (b) for $t > t'_c$, $|\mathcal{C}_1(t)| = \Omega(n)$.

Consider first (a). Fix $t < t'_c$. For $\delta > 0$, define rate functions $(\mathbf{a}^+, b^+, \mathbf{c}^+) = (\mathbf{a} + \delta, b + \delta, \mathbf{c} + \delta)$. Since $\rho(t) < 1$, by Proposition 5.5, we can choose δ sufficiently small so that $\rho_t(\mathbf{a}^+, b^+, \mathbf{c}^+) < 1$. Denote $\mathcal{I}_1^{\mathbf{RG}^+}(t)$ for the component of the largest volume in $\mathbf{RG}^+(t) := \mathbf{RG}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)$. From Lemma 5.14 there exists $A \in (0, \infty)$ such that

$$\mathbb{P}(\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^+}(t)) > A \log^4 n) \to 0, \text{ as } n \to \infty.$$

Combining this result with Corollary 4.5 we see that

$$\mathbb{P}(\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\Gamma}(t)) > A \log^4 n) \to 0, \text{ as } n \to \infty,$$

where $\mathcal{I}_1^{\Gamma}(t)$ is the component with the largest volume in $\Gamma(t)$. Part (a) is now immediate from the one to one correspondence between the components in $\Gamma(t)$ and $\mathbf{BSR}^*(t)$ (see (4.20)).

We now consider (b). Fix $t > t'_c$. Then $\rho(t) > 1$. From Proposition 5.5 we can find $\delta > 0$ such that $\rho_t(\mathbf{a}^-, b^-, \mathbf{c}^-) > 1$, where $(\mathbf{a}^-, b^-, \mathbf{c}^-) = ((\mathbf{a} - \delta)^+, (b - \delta)^+, (\mathbf{c} - \delta)^+)$. Let $\mathcal{C}_1^{\mathbf{RG}^-}(t)$ be the component in $\mathbf{RG}^-(t) := \mathbf{RG}^{(n)}(\mathbf{a}^-, b^-, \mathbf{c}^-)(t)$ with the largest number of vertices. By Theorem 3.1 of [8], $|\mathcal{C}_1^{\mathbf{RG}^-}(t)| = \Theta(n)$. Since $\mathrm{vol}_{\phi_t}(\mathcal{C}_1^{\mathbf{RG}^-}(t)) \geq |\mathcal{C}_1^{\mathbf{RG}^-}(t)|$, we have $\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^-}(t)) = \Omega(n)$, where $\mathcal{I}_1^{\mathbf{RG}^-}(t)$ is the component with the largest volume in $\mathbf{RG}^-(t)$. Finally, in view of Corollary 4.5, we have the same result with $\mathcal{I}_1^{\mathbf{RG}^-}(t)$ replaced by $\mathcal{I}_1^{\Gamma}(t)$ and the result follows once more from the one to one correspondence between the components in $\Gamma(t)$ and $\mathbf{BSR}^*(t)$. $\qquad\square$

5.5. Barely subcritical regime for Bounded-size rules.
In this section we complete the proof of Theorem 1.2. Throughout this section we fix $\gamma \in (0, 1/4)$ and let $t_n = t_c - n^{-\gamma}$. The main ingredient in the proof is the following proposition.

Proposition 5.15. *There exist $\bar{B}, \bar{C}, \bar{N} \in (0, \infty)$ such that for all $n \geq \bar{N}$ and all $0 \leq t \leq t_n$*

$$\mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| \geq \bar{m}(n, t)\} \leq \frac{\bar{C}}{n^2}, \text{ where } \bar{m}(n, t) = \frac{\bar{B}(\log n)^4}{(t_c - t)^2}.$$

Let us first prove Theorem 1.2 assuming the above proposition.

Proof of Theorem 1.2: Write $\tau = \inf\{t \geq 0 : |\mathcal{C}_1^{(n)}(t)| \geq m(n, t)\}$, where $m(n, t) = \frac{2\bar{B}(\log n)^4}{(t_c - t)^2}$. Then

$$\mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| \geq m(n, t) \text{ for some } t \leq t_n\} = \mathbb{P}\{\tau \leq t_n\}. \tag{5.21}$$

Note that

$$\{\tau = t\} \subset \bigcup_{v, v' \in [n], v \neq v'} E^{v, v'}, \tag{5.22}$$

where, denoting the component in $\mathbf{BSR}(t)$ that contains the vertex $v \in [n]$ by $\mathcal{C}_v^{(n)}(t)$ and its size by $|\mathcal{C}_v^{(n)}(t)|$,

$$E^{v, v'} = \{\max\{|\mathcal{C}_v^{(n)}(t-)|, |\mathcal{C}_{v'}^{(n)}(t-)|\} < m(n, t); \mathcal{C}_v^{(n)}(t-) \neq \mathcal{C}_{v'}^{(n)}(t-)\}$$
$$\bigcap\{|\mathcal{C}_v^{(n)}(t-)| + |\mathcal{C}_{v'}^{(n)}(t-)| \geq m(n, t)\}\bigcap\{\mathcal{C}_v^{(n)}(t) = \mathcal{C}_{v'}^{(n)}(t)\}. \tag{5.23}$$

Note that

$$\mathbb{P}\{|\mathcal{C}_v^{(n)}(t)| + |\mathcal{C}_{v'}^{(n)}(t)| \geq m(n, t)\} \leq 2\mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| \geq m(n, t)/2\} \tag{5.24}$$

and, on the set, $\{\max\{|\mathcal{C}_v^{(n)}(t)|, |\mathcal{C}_{v'}^{(n)}(t)|\} < m(n, t)\}$, the rate at which $\mathcal{C}_v^{(n)}(t)$ and $\mathcal{C}_{v'}^{(n)}(t)$ merge can be bounded by

$$\frac{1}{2n^3} \cdot 4|\mathcal{C}_v^{(n)}(t)||\mathcal{C}_{v'}^{(n)}(t)|n^2 \leq \frac{2m^2(n, t)}{n}.$$

Combining this observation with (5.22) and (5.24), we have

$$\mathbb{P}\{\tau \leq t_n\} \leq \sum_{v,v' \in [n], v \neq v'} \int_0^{t_n} \mathbb{P}\{|\mathcal{C}_v^{(n)}(t)| + |\mathcal{C}_{v'}^{(n)}(t)| \geq m(n,t)\} \cdot \frac{2m^2(n,t)}{n} dt$$

$$\leq 2n^2 \int_0^{t_n} \mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| \geq m(n,t)/2\} \cdot \frac{2m^2(n,t)}{n} dt$$

$$\leq 4nt_c \sup_{t \leq t_n} \{m^2(n,t)\mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| \geq \bar{m}(n,t)\}\}$$

$$= O(n \cdot n^{4\gamma}(\log n)^8 \cdot n^{-2}) = O(n^{-1+4\gamma}(\log n)^8) = o(1),$$

where the last line follows from Proposition 5.15 and the fact that $\gamma < 1/4$. Using the above estimate in (5.21) we have the result. $\qquad\square$

We will need the following lemma in the proof of Proposition 5.15.

Lemma 5.16. *Let* $(\mathbf{a}^+, b^+, \mathbf{c}^+) = (\mathbf{a} + \delta_n, b + \delta_n, \mathbf{c} + \delta_n)$, *where* $\delta_n = n^{-2\gamma_0}$ *and* $\gamma_0 \in (\gamma, 1/4)$. *Let* $\rho_t^{(n),+} = \rho_t(\mathbf{a}^+, b^+, \mathbf{c}^+)$. *Then there exists* $C_9, N_0 \in (0, \infty)$ *such that for all* $n \geq N_0$,

$$\rho_t^{(n),+} < 1 - C_9(t_c - t) \text{ for all } 0 \leq t \leq t_n.$$

Proof of Lemma 5.16: From Proposition 5.5, there is a $d_1 \in (0, \infty)$ such that

$$\rho_t^{(n),+} \leq \rho_t(\mathbf{a}, b, \mathbf{c}) + d_1 n^{-\gamma_0} \log^2 n, \text{ for all } t \leq t_c.$$

By Lemma 5.12 and since $\rho_{t_c}(\mathbf{a}, b, \mathbf{c}) = 1$, there exists $d_2 \in (0, \infty)$ such that

$$\rho_t(\mathbf{a}, b, \mathbf{c}) \leq 1 - d_2(t_c - t), \text{ for all } t \leq t_n.$$

Thus, since $\gamma < \gamma_0$, we have for some $N_0 > 0$

$$\rho_t^{(n),+} \leq 1 - d_2(t_c - t) + d_1 n^{-\gamma_0}(\log n)^2 < 1 - \frac{d_2}{2}(t_c - t),$$

for all $n \geq N_0$ and $0 \leq t \leq t_c - n^{-\gamma}$. The result follows. $\qquad\square$

Proof of Proposition 5.15: Recall the rate functions $(\mathbf{a}, b, \mathbf{c})$ introduced in Section 4.2. Choose $\gamma_0 \in (\gamma, 1/4)$ and let $(\mathbf{a}^+, b^+, \mathbf{c}^+)$ be as in Lemma 5.16. Fix $t < t_n$ and consider the random graph $\mathbf{RG}^{(n)}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)$. From Lemma 4.4, we can couple $\Gamma(t)$ and $\mathbf{RG}^{(n)}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)$ such that

$$\mathbb{P}(\Gamma(t) \subseteq \mathbf{RG}^{(n)}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)) \geq 1 - C_3 \exp(-C_4 n^{1-4\gamma_0}), \text{ for all } t \in [0, T].$$

Recalling the one to one correspondence between components in $\mathbf{BSR}^*(t)$ and $\Gamma(t)$, and (4.20), we have for any $m \geq 1$,

$$\mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| > m\} \leq \mathbb{P}\{\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^+}(t)) \geq m\} + C_3 \exp\{-C_4 n^{1-4\gamma_0}\}, \qquad (5.25)$$

where $\mathcal{I}_1^{\mathbf{RG}^+}(t)$ is the component with the largest volume in $\mathbf{RG}^{(n)}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)$. From Lemma 5.16, there is a $N_0 > 0$ such that $\Delta_t^{(n),+} = 1 - \rho_t(\mathbf{a}^+, b^+, \mathbf{c}^+)$ satisfies

$$\Delta_t^{(n),+} \geq C_9(t_c - t), \text{ for all } t \leq t_n, \ n \geq N_0. \qquad (5.26)$$

Using Lemma 5.1 and arguing as in equation (5.19) we have for all $t \in [0, T]$ and all $m \geq 1$,

$$\mathbb{P}\{\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^+}(t)) \geq m\} \leq nd_1 \exp\{-(\Delta_t^{(n),+})^2 m/(d_2 \log^3 n)\} + d_3 n^{-2}, \qquad (5.27)$$

where d_1, d_2, d_3 are suitable constants. Using (5.26) in (5.27) we get

$$\mathbb{P}\{\text{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^+}(t)) \geq m\} \leq nd_1 \exp\{-d_4(t_c - t)^2 m/\log^3 n\} + d_3 n^{-2}.$$

The result now follows on substituting $m = m(n, t) = \frac{\bar{B}(\log n)^4}{(t_c-t)^2}$, with $\bar{B} > 3/d_4$, in the above inequality. $\qquad\square$

References

[1] D. Achlioptas, R. M. D'Souza, and J. Spencer, *Explosive percolation in random networks*, Science **323** (2009), no. 5920, 1453.

[2] D. Aldous, *Brownian excursions, critical random graphs and the multiplicative coalescent*, The Annals of Probability **25** (1997), no. 2, 812–854.

[3] S. Bhamidi, A. Budhiraja, and X. Wang, *Aggregation models with limited choice and the multiplicative coalescent*, To appear in Random Structures and Algorithms (2012).

[4] S Bhamidi, A Budhiraja, and X Wang, *The augmented multiplicative coalescent and emergence of the giant and complexity for dynamic random graph models at criticality*, Arxiv preprint (2012).

[5] T. Bohman and A. Frieze, *Avoiding a giant component*, Random Structures and Algorithms **19** (2001), no. 1, 75–85.

[6] T. Bohman, A. Frieze, and N. C. Wormald, *Avoidance of a giant component in half the edge set of a random graph*, Random Structures & Algorithms **25** (2004), no. 4, 432–449.

[7] B. Bollobás, *Random graphs*, Second, Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001. MR1864966 (2002j:05132)

[8] B. Bollobás, S. Janson, and O. Riordan, *The phase transition in inhomogeneous random graphs*, Random Structures Algorithms **31** (2007), no. 1, 3–122. MR2337396 (2008e:05124)

[9] P. Brémaud, *Point processes and queues*, Springer-Verlag, New York, 1981. Martingale dynamics, Springer Series in Statistics. MR636252 (82m:60058)

[10] J. Ding, E. Lubetzky, and Y. Peres, *Mixing time of near-critical random graphs*, Ann. Probab. **40** (2012), no. 3, 979–1008. MR2962084

[11] P. Erdős and A. Rényi, *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **5** (1960), 17–61. MR0125031 (23 #A2338)

[12] P. Erdős and A. Rényi, *On the evolution of random graphs*, Bull. Inst. Internat. Statist. **38** (1961), 343–347. MR0148055 (26 #5564)

[13] S. Janson, D. Knuth, T. Luczak, and B. Pittel, *The birth of the giant component, with an introduction by the editors*, Random Struct. Alg **4** (1994), 231–358.

[14] S. Janson and J. Spencer, *Phase Transitions for Modified Erdos–Rényi Processes*, Arxiv preprint arXiv:1005.4494 (2010).

[15] S. Janson, *The largest component in a subcritical random graph with a power law degree distribution*, Ann. Appl. Probab. **18** (2008), no. 4, 1651–1668. MR2434185 (2009d:05227)

[16] S. Janson and M. J. Luczak, *Susceptibility in subcritical random graphs*, J. Math. Phys. **49** (2008), no. 12, 125207, 23. MR2484338 (2010b:05152)

[17] M. Kang, W Perkins, and J. Spencer, *The Bohman-Frieze Process Near Criticality*, Arxiv preprint arXiv:1106.0484v1 (2010).

[18] J. H. Kim, *Poisson cloning model for random graphs*, International Congress of Mathematicians. Vol. III, 2006, pp. 873–897. MR2275710 (2007j:05196)

[19] R. Sh. Liptser and A. N. Shiryayev, *Theory of martingales*, Mathematics and its Applications (Soviet Series), vol. 49, Kluwer Academic Publishers Group, Dordrecht, 1989. Translated from the Russian by K. Dzjaparidze [Kacha Dzhaparidze]. MR1022664 (90j:60046)

[20] J. R. Norris, *Markov chains*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 2, Cambridge University Press, Cambridge, 1998. Reprint of 1997 original. MR1600720 (99c:60144)

[21] B. G. Pittel, *On the largest component of a random graph with a subpower-law degree sequence in a subcritical phase*, Ann. Appl. Probab. **18** (2008), no. 4, 1636–1650. MR2434184 (2009e:60020)

[22] O. Riordan and L. Warnke, *Achlioptas process phase transitions are continuous*, Arxiv preprint arXiv:1102.5306 (2011).

[23] O. Riordan and L. Warnke, *The evolution of subcritical Achlioptas processes*, arXiv preprint arXiv:1204.5068 (2012).

[24] J. Spencer and N. Wormald, *Birth control for giants*, Combinatorica **27** (2007), no. 5, 587–628.

[25] R. van der Hofstad and A. Nachmias, *Hypercube percolation*, arXiv preprint arXiv:1201.3953 (2012).