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Journal of Multivariate Analysis 90 (2004) 359–383
Journal of Multivariate Analysis
http://www.elsevier.com/locate/jmva

Correlation and spectral theory for periodically correlated random fields indexed on $\mathbf{Z}^2$

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Received 7 September 2001

Abstract

We show that a field $X(m,n)$ is strongly periodically correlated with period (M,N) if and only if there exist commuting unitary operators, U_1 and U_2 that shift the field unitarily by M and N along the respective coordinates. This is equivalent to a field whose shifts on a subgroup are unitary. We also define weakly PC fields in terms of other subgroups of the index set over which the field shifts unitarily. We show that every strongly PC field can be represented as $X(m,n) = \tilde{U}_1^m \tilde{U}_2^n P(m,n)$ where $\tilde{U}_1$ and $\tilde{U}_2$ are unitary and $P(m,n)$ is a doubly periodic vector-valued sequence. This leads to the Gladyshev representations of the field and to strong harmonizability. The 2- and 4-fold Wold decompositions are expressed for weakly commuting strongly PC fields. When the field is strongly commuting, a one-point innovation can be defined. For this case, we give necessary and sufficient conditions for a strongly commuting field to be PC and strongly regular, although possibly of deficient rank, in terms of periodicity and summability of the southwest moving average coefficients.

AMS 2000 subject classifications: 60G60; 60G25

Keywords: Random fields; Periodically correlated processes; Harmonizable processes; Spectral theory; Wold decomposition; Prediction

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0047-259X/$ - see front matter
doi:10.1016/j.jmva.2004.04.010

1. Introduction

Periodically correlated (PC) random processes are nonstationary processes for which the nonstationarity occurs in a manner that makes possible a spectral theory that is understandable and manageable. These processes occur, for example, when physical systems that generate random processes are perturbed or influenced periodically with respect to time. Physical examples are provided by meteorological processes, noise processes produced by rotating machinery, communications signals where randomness appears as the message or as additive noise, and periodicity comes through the communication format. In the communications context, periodically correlated processes are also called cyclostationary [7]. For a survey of univariate PC and almost PC processes, see Dehay and Hurd [4].

In this paper, we say that a collection of $L_2(\Omega)$ random variables $X(m,n)$, indexed on $\mathbf{Z}^2$ is a strongly[1] PC field with period (M,N) if its mean and covariance functions satisfy

$$\mu(m,n) = \mu(m+kM, n+lN) \tag{1}$$

$$R(m,n,m',n') = R(m+kM, n+lN, m'+kM, n'+lN) \tag{2}$$

for all integers m,n,m',n' and k,l in $\mathbf{Z}$, and M and N are each the smallest positive integers for which (1) and (2) are both true. It is clear that these preceding conditions also imply that the correlation function will satisfy (2) but we will throughout express the PC property as a condition on the covariance, and without loss in generality will take $\mu(m,n) \equiv 0$. For a field to be strongly PC we also require $M>0$ and $N>0$. The condition $M=1$ and $N>1$ means that for every m the field is PC with period N in the variable n and for every n is stationary with respect to m. As we will later see, it is equivalent to say that X shifts unitarily along its coordinates. We shall return to this topic below, but first we give some examples of some more general PC random fields that help motivate the investigation of this most elementary case where the index set is $\mathbf{Z}^2$.

- the acoustic pressure field in $\mathbf{R}^3$ produced by the propagation of a radiated PC acoustic source,
- the electromagnetic field in $\mathbf{R}^3$ produced by the propagation of communication signals (most of which possess a PC structure),
- the solutions to the three-dimensional Shrödinger equation in the presence of periodic potentials; this problem arose in the study of crystal structures and solutions in this context exhibiting a PC structure are often attributed to Bloch although the general ideas seem to have originated with Floquet. See Eastham [5, Chapters 1 and 6] and Kuchment [16, Chapter 3],
- texture fields such as fabric patterns (see [6]), crop photographs or object placement on a periodic grid with placement jitter [10],

[1] We will subsequently introduce fields with a weak PC property. The strong property given here is that studied by Alekseev [2].

- the product $f(m,n)Y(m,n)$ of scalar periodic function $f(m,n)$ with a stationary random field $Y(m,n)$.

One of our principal goals is to show how the basic one-dimensional results of Gladyshev [8] extend to fields. In doing this we find there is more than one way to define a PC structure on fields, and it is the *strong PC* fields to which the one-dimensional results nicely extend. In doing this we make use of the natural unitary operators whose existence are simple consequences of the periodicity of the covariance (2). These operators provide representations of the fields and help to show they are strongly harmonizable. The use of the spectral theorem for unitary operators clarifies the spectral representations and the characteristic nature of the random spectral measure produced by the harmonizability of strongly PC fields.

Our other principal goal is to establish some basic facts concerning the prediction of strongly PC fields. In particular, we present a 2-fold and a 4-fold Wold decomposition for such fields and determine the periodicity conditions imposed on the dimension of certain innovation subspaces. The aforementioned unitary operators play a key role in these results. In the case of strongly commuting fields we give necessary and sufficient conditions for a field to be PC and strongly regular in terms of the coefficients of a southwest moving average representation (among other things, the coefficients must satisfy a periodicity condition).

We begin with a review of a few facts about stationary random fields (indexed on $\mathbf{Z}^2$).

2. Stationary random fields indexed on $\mathbf{Z}^2$

A second-order random field X indexed on $\mathbf{Z}^2$ is a family of random variables $X(m,n)$ that are of second order on some probability space $(\Omega, \mathbf{F}, P)$ for each $(m,n)\in\mathbf{Z}^2$. Since $L_2(\Omega,\mathbf{F},P)$ is a Hilbert space, we have the usual inner product $\langle\cdot,\cdot\rangle$ produced by the expectation (or integral)

$$\langle x,y\rangle := E\{x\bar{y}\} = \int_\Omega x(\omega)\bar{y}(\omega)\,\mathrm{d}P.$$

In our considerations we can focus attention on the Hilbert subspace $\mathcal{H}(X) = \overline{sp}\mathcal{M}(X)$ where $M(X)$ is the (linear) space of finite linear combinations of elements from X; we write $\mathcal{M}(X) = sp\{X(m,n),(m,n)\in\mathbf{Z}^2\}$ and the closure is with respect to the L_2 norm. But since the L_2 inner product induces an inner product on $\mathcal{H}(X)$, we can take the view that $\mathcal{H}(X)$ has its own inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}(X)}$ and likewise its own norm. In our subsequent references to inner products and norms, this view is to be taken.

For $Q\in\mathbf{Z}^+$, a Q-dimensional second-order random field indexed on $\mathbf{Z}^2$ is just a finite collection $\{X_1((m,n)),X_2((m,n)),\dots,X_Q((m,n))\}$ of second-order random fields indexed on $\mathbf{Z}^2$. For any Q-dimensional second-order random field, denote

$$\mathcal{M}(X) = sp\{X_j((m,n)),(m,n)\in\mathbf{Z}^2,\ j\in[1,q]\},$$

and $\mathcal{H}(X) = \overline{\mathcal{M}(X)}$. Following the procedure outlined above we define the correlation

$$R_{jk}(m,n,m',n') = \langle X_j(m,n), X_k(m',n') \rangle_{H(X)} \tag{3}$$

Although we are primarily interested in finite dimensional second order *stationary* random fields indexed on $\mathbf{Z}^2$, we address a more general issue in the following:

Proposition 1. *A necessary and sufficient condition for some matrix of functions*

$$\{R_{jk}(m,n,m',n'), j \in [1,Q], k \in [1,Q]\}$$

to be the covariance matrix of a Q-dimensional vector random field indexed on $\mathbf{Z}^2$ *is that for any integer* v, *complex sequence* $\{\alpha_1, \alpha_2, \ldots, \alpha_v\}$, *any sequence* $\{k_1, k_2, \ldots, k_v\}$ *in* $[1,Q]$, *and any sequence* $\{(m_1,n_1),(m_2,n_2),\ldots,(m_v,n_v)\}$ *in* $\mathbf{Z}^2$, *the following inequality holds*:

$$\sum_{p=1}^{v}\sum_{p'=1}^{v} \alpha_p \overline{\alpha_{p'}} R_{k_p k_{p'}}(m_p,n_p,m_{p'},n_{p'}) \geqslant 0. \tag{4}$$

A Q-dimensional second-order random field indexed on $\mathbf{Z}^2$ is called *stationary* if (a) for every index j in $[1,Q]$, the mean $E\{X_j(m,n)\}$ is constant with respect to (m,n) and (b) for any j and k in $[1,Q]$, and any two vectors (m,n) and (m',n'), the correlation $R_{jk}(m,n,m',n') = \langle X_j(m,n), X_k(m',n') \rangle$ is a function only of $(m,n) - (m',n') \overset{def}{=} (m-m', n-n')$. In Proposition 1, by setting

$$R_{k_p k_{p'}}(m,n,m',n') = R_{k_p k_{p'}}((m,n)-(m',n')),$$

we obtain necessary and sufficient conditions for a matrix of functions to be the cross covariance matrix of some Q-dimensional *stationary* random field indexed on $\mathbf{Z}^2$.

Our approach to the spectral theory is through the family of unitary operators that occur naturally with stationary processes. We note the following results combine the theory for multivariate sequences indexed on $\mathbf{Z}$, which is nicely presented by Rozanov [19], and the case of univariate fields indexed on $\mathbf{Z}^2$ presented by Kallianpur and Mandrekar [13].

Proposition 2. *Suppose* $\{X_1((m,n)),\ldots,X_Q((m,n))\}$ *is a Q-dimensional stationary random field indexed on* $\mathbf{Z}^2$ *with covariance matrix* $R_{jk}((m,n)-(m',n'))$. *Then there exist a pair of commuting unitary operators,* U_1 *and* U_2, *operating in* $\mathcal{H}(X)$ *for which*

$$X_j(m,n) = U_1^m U_2^n [X_j(0,0)] \tag{5}$$

for every $(m,n) \in \mathbf{Z}$ *and* $j \in [1,Q]$.

The spectral representation of the unitary operators U_1 and U_2 leads to

$$U_1^m U_2^n = \int_0^{2\pi} \exp(\mathrm{i}\lambda_1 m)\ dE_1(\lambda_1) \int_0^{2\pi} \exp(\mathrm{i}\lambda_2 n)\ dE_2(\lambda_2) = \int_0^{2\pi}\int_0^{2\pi} \exp[(\mathrm{i}(\lambda_1 m + \lambda_2 n)]\ dE(\lambda_1, \lambda_2), \quad (6)$$

where the commuting of U_1 and U_2 implies that for A_1, A_2 intervals in $[0, 2\pi)$, the operator valued set function

$$E(A_1 \times A_2) = E_1(A_1)E_2(A_2) = E_2(A_2)E_1(A_1)$$

is a projection; this leads to the extension of E to the Borel sets $\mathcal{B}[0, 2\pi)^2$ and the notation $dE(\lambda_1, \lambda_2) = dE_1(\lambda_1)dE_2(\lambda_2)$.

Now using (5) and (6) gives, for all $(m, n) \in \mathbf{Z}^2$ and $j \in [1, Q]$,

$$X_j(m,n) = U(m,n)[X_j(0,0)] = \int_0^{2\pi}\int_0^{2\pi} \exp[\mathrm{i}(\lambda_1 m + \lambda_2 n)]\, dZ_j(\lambda_1, \lambda_2), \quad (7)$$

where $dZ_j(\lambda_1, \lambda_2) = dE(\lambda_1, \lambda_2)[X_j(0,0)]$ and $U(m,n) = U_1^m U_2^n$.

This leads immediately to an expression for the cross covariances:

$$\begin{aligned} R_{jk}((m,n) - (m',n')) &= \langle X(m,n), X(m',n') \rangle \\ &= \langle U(m,n)[X_j(0,0)], U(m',n')[X_k(0,0)] \rangle \\ &= \langle U((m,n) - (m',n'))[X_j(0,0)], X_k(0,0) \rangle \\ &= \int_0^{2\pi}\int_0^{2\pi} \exp[(\mathrm{i}\lambda_1(m - m') + \mathrm{i}\lambda_2(n - n')] dF_{jk}(\lambda_1, \lambda_2), \end{aligned} \quad (8)$$

where $dF_{jk}(\lambda_1, \lambda_2) = \langle dE(\lambda_1, \lambda_2)[X_j(0,0)], X_k(0,0) \rangle$.

Now Proposition 1 may be transformed into an equivalent statement about the matrix-valued distribution function F.

Proposition 3. *A necessary and sufficient condition for some matrix-valued distribution function* $\{F_{jk}(\lambda_1, \lambda_2), j \in [1, Q], k \in [1, Q]\}$ *to be the matrix-valued distribution function of an Q-dimensional random field is that for any integer* v, *any complex sequence* $\{\alpha_1, \alpha_2, \ldots, \alpha_v\}$, *any sequence* $\{k_1, k_2, \ldots, k_v\}$ *with* $k_j \in [1, Q]$ *and any Borel set* $\Delta \in [0, 2\pi]^2$, *the following inequality holds*:

$$\sum_{p=1}^{v} \sum_{p'=1}^{v} \alpha_p \overline{\alpha_{p'}} F_{k_p k_{p'}}(\Delta) \geqslant 0. \quad (9)$$

3. Periodically correlated random fields

A *periodically correlated* random field is a *second-order* random field whose covariance has a periodic structure. From this point forward we will omit the reference to second-order unless the emphasis is believed desirable. We shall see that there are two main types of periodic structure, *weak* and *strong*, for random fields indexed on $\mathbf{Z}^2$. In the last subsection of this section we discuss the connection between these types and subgroups of $\mathbf{Z}^2$. This provides a way to extend the notions of weak and strong periodic correlation to fields indexed on $\mathbf{Z}^d$ for arbitrary $d \geqslant 2$.

3.1. Strongly periodic fields

We say that $X(m,n)$ is *strongly periodic* with period (M,N) if M,N are the smallest positive integers for which $X(m,n)$ is *periodic* in the two indices independently:

$$X(m,n) = X(m+kM, n+lN) \tag{10}$$

for all integers m,n and k,l in $\mathbf{Z}$ where the equality is in the sense of $\|\cdot\|_{\mathcal{H}(X)}$.

If $M = N = 1$, the field is constant. The condition $M = 1$ and $N > 1$ means the field is periodic with period N in the second index and constant with respect to the first. To see the necessity of having $M > 0, N > 0$, suppose, for example, that $M = 0$ in (10). We observe that for every m, $X(m,n)$ is periodic in the index n with period N while there is no constraint whatsoever on the dependence upon n. We will see subsequently that this is a special case of a *weakly* periodic field.

We observe that the field $X(m,n)$ is *strongly periodic* if and only if it has a discrete Fourier series representation

$$X(m,n) = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} X^{jk} \exp(\mathrm{i}2\pi jm/M + \mathrm{i}2\pi kn/N), \tag{11}$$

where

$$X^{jk} = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} X(m,n) \exp(-\mathrm{i}2\pi jm/M - \mathrm{i}2\pi kn/N) \tag{12}$$

for $j \in [0, M-1], k \in [0, N-1]$. The proof, which holds for any periodic sequence (function) indexed on $\mathbf{Z}^2$ and taking values in a linear vector space over $\mathbf{C}$, is a straightforward extension of the familiar case and is omitted.

Since $X \in L_2(\Omega, \mathbf{F}, P)$ it's mean $\mu(m,n) = E\{X(m,n)\}$ will exist for all m,n and $\mu(m,n) = \mu(m+M,n) = \mu(m,n+N)$ for all $m,n \in \mathbf{Z}$. Without loss of generality we take $\mu(m,n) \equiv 0$.

The following proposition shows that strong periodicity of a field is equivalent to a strong periodicity of the covariance.

Proposition 4. *The random field X is strongly periodic if and only if*

$$R(m,n,m',n') = E\{X(m,n)\overline{X(m',n')}\} = R(m+kM, n+lN, m'+k'M, n'+l'N). \tag{13}$$

for every m, n, m', n' and k, l, k', l' in $\mathbf{Z}$.

Proof. If (10) holds for every m, n and k, l, then (13) holds also. Conversely if (13) is true for every m, n, m', n' and k, l, k', l', then

$$\begin{aligned}||X(m,n) - X(m+kM, n+lN)||^2_{H(X)} \\ = R(m,n,m,n) - R(m,n,m+kM,n+lN) - R(m+kM,n+lN,m,n) \\ + R(m+kM,n+lN,m+kM,n+lN) = 0\end{aligned} \tag{14}$$

which proves the result. □

We note that if $X(m,n)$ is strongly periodic with period (M,N), then it is PC with period (M,N) but the converse is not true. The relationship between strongly periodic fields and strongly PC fields is given in Proposition 9.

3.2. Weakly periodic random fields

We say that $X(m,n)$ is *weakly periodic* with period (M,N) if

$$X(m,n) = X(m+M, n+N) \tag{15}$$

for every m, n in $\mathbf{Z}$. Here we only require $M \geqslant 0$ and $N \geqslant 0$ but do not permit $M = N = 0$ because, following our previous discussion, this puts no constraint whatsoever on X. If (15) occurs for $M = 0$ and $N > 0$ with N minimal, we obtain periodicity of the field in the second index and no constraint on its behavior in the first index. That is, for every m, we have $X(m,n) = X(m, n+N)$ for all n and this is all that can be said; so this particular weakly periodic field may be viewed as a countable family of periodic sequences, each with period N. For arbitrary $M > 0$, $N > 0$, we require (M,N) be relatively prime, and then the result is essentially the same except the periodic sequences lie along the straight lines of slope N/M because all of $\mathbf{Z}^2$ can be expressed as a countable union

$$\mathbf{Z}^2 = \bigcup_{(m,n)\in B} D(m,n), \tag{16}$$

where the sets

$$D(m,n) = \{(m',n') \in \mathbf{Z}^2 : (m',n') = (m+kM, n+kN), k \in \mathbf{Z}\}$$

are disjoint provided the base set B is properly chosen; for example, if $B = \{(m,n) \in \mathbf{Z}^2 : 0 \leqslant m < M\}$.

Since $X \in L_2(\Omega, \mathbf{F}, P)$ its mean $\mu(m,n) = E\{X(m,n)\}$ will exist for all m, n and the weak periodicity implies $\mu(m,n) = \mu(m+M, n+N)$ for all $m, n \in \mathbf{Z}$. That is, $\mu(m,n)$ is also weakly periodic and so without loss of generality we may take $\mu(m,n) \equiv 0$.

There is a corresponding periodicity of the covariance, whose proof is omitted due to the similarity with the strongly periodic case given in Proposition 4.

Proposition 5. *The field X is weakly periodic if and only if*

$$R(m,n,m',n') = R(m+M,n+N,m'+M,n'+N) \tag{17}$$

for every m,n,m',n' in $\mathbf{Z}$.

These notions of strong and weak periodicity form the basis for the notions of strongly and weakly PC random fields. Our approach is to first give the definitions of strongly and weakly PC random fields in terms of the covariance functions, then to address the representation of the covariance functions and Gladyshev's theorem. Then we derive the relationship to unitary operators and address the harmonizability of PC fields.

3.3. Strongly periodically correlated fields

The random field X is called *strongly* PC with period $m,n=(M,N)$, if and only if there exists no smaller $M>0$ and $N>0$ for which the mean and covariance functions satisfy

$$\mu(m,n) = \mu(m+kM,n+lN), \tag{18}$$

$$\begin{aligned} R(m,n,m',n') &= E\{[X(m,n)-\mu(m,n)]\overline{[X(m',n')-\mu(m',n')]} \\ &= R(m+kM,n+lN,m'+kM,n'+lN) \end{aligned} \tag{19}$$

for all integers m,n,m',n' and k,l in $\mathbf{Z}$.

This condition is equivalent to $X(m,n)$ being PC in the two indices independently. If $M=N=1$, the field is stationary. The condition $M=1$ and $N>1$ means the field is PC with period N in the second variable and stationary with respect to the first.

Since the PC random fields we are considering are of second order, the existence of the mean and correlation are assured and the correlation of X is given by

$$E\{X(m,n)\overline{X(m',n')}\} = R(m,n,m',n') + \mu(m,n)\overline{\mu(m',n')}. \tag{20}$$

Thus it may be seen that the correlation also satisfies the periodicity condition (19) that defines the essential structure we wish to study. Hence, without any loss of generality we again can take $\mu(m,n)\equiv 0$.

The first result is that strongly PC fields are just finite collections of jointly stationary fields. This was first noticed by Gladyshev [8] for the univariate case, and we omit its straightforward proof.

Proposition 6. *A necessary and sufficient condition for a random field $X=\{X(m,n):m,n\in\mathbf{Z}^2\}$ to be strongly PC with period (M,N) is that the collection of fields*

$$Y_{jj'}(m,n) = X(j+mM,j'+nN), \quad (m,n)\in\mathbf{Z}^2 \tag{21}$$

for $j = 0, 1, \ldots, M-1$, $j' = 0, 1, \ldots, N-1$ *form a* $M \cdot N$*-dimensional stationary random field* $Y = \{Y_{jj'}(m,n), (m,n) \in \mathbf{Z}^2\}$.

In view of this proposition, the existence of a collection of unitary operators associated with a PC field is clear. We will return to the unitary operators subsequently.

To obtain a Fourier series decomposition of the covariance as in the case for PC sequences [8], we now define

$$B((m,n),(\tau_1,\tau_2)) = R(m+\tau_1, n+\tau_2, m, n) \tag{22}$$

and so the property (19) becomes

$$B((m,n),(\tau_1,\tau_2)) = B((m+M,n),(\tau_1,\tau_2)) = B((m,n+N),(\tau_1,\tau_2)) \tag{23}$$

for every (m,n) and $(\tau_1,\tau_2) \in \mathbf{Z}^2$. Thus, for every (τ_1,τ_2), the function $B((m,n),(\tau_1,\tau_2))$ is a scalar-valued strongly periodic function of (m,n) in the sense that it is periodic in the two indices independently with respective periods M and N. Hence for every (τ_1,τ_2), we have the discrete Fourier series representation

$$B((m,n),(\tau_1,\tau_2)) = \sum_{\vec{k}=(k_1,k_2)} B_{\vec{k}}(\tau_1,\tau_2)\, exp(\mathrm{i}2\pi k_1 m/M + \mathrm{i}2\pi k_2 n/N), \tag{24}$$

or in more explicit form,

$$B((m,n),(\tau_1,\tau_2)) = \sum_{k_1=0}^{M-1} \sum_{k_2=0}^{N-1} B_{k_1 k_2}(\tau_1,\tau_2)\, exp(\mathrm{i}2\pi k_1 m/M + \mathrm{i}2\pi k_2 n/N), \tag{25}$$

and where

$$B_{k_1 k_2}(\tau_1,\tau_2) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} B((m,n),(\tau_1,\tau_2))\, exp(-\mathrm{i}2\pi k_1 m/M - \mathrm{i}2\pi k_2 n/N). \tag{26}$$

Hence the covariance of strongly PC random fields is completely determined by a finite collection of coefficient functions $\{B_{\vec{k}}(\tau_1,\tau_2), \vec{k} \in [0, M-1] \times [0, N-1]\}$. The following gives conditions on the coefficient functions that ensure that $B((m,n),(\tau_1,\tau_2))$ arose from the covariance function of some strongly PC field. The proof is a straightforward extension of a result due to Gladyshev [8] for the case of univariate PC sequences, and so the proof is omitted.

Proposition 7. *A sequence of coefficient functions* $\{B_{\vec{k}}(\tau_1,\tau_2),\ \vec{k} \in [0, M-1] \times [0, N-1]\}$ *arises from some strongly PC random field having period* (M,N) *if and only if for every* v, *every sequence of complex numbers* $\{c_1, c_2, \ldots, c_v\}$, *integer pairs* $\{\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_v\}$, *and* $\{\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_v\}$ *each in* $[0, M-1] \times [0, N-1]$, *it follows that*

$$\sum_{p=1}^{v} \sum_{p'=1}^{v} c_p \bar{c}_{p'} \beta_{\vec{k}_p \vec{k}_{p'}}(\vec{t}_p - \vec{t}_{p'}) \geqslant 0, \tag{27}$$

where, denoting $\vec{j} = (j_1, j_2)$ *and* $\vec{k} = (k_1, k_2)$,

$$\beta_{\vec{j}\vec{k}}(\tau_1, \tau_2) = B_{\vec{k}-\vec{j}}(\tau_1, \tau_2)\, exp(\mathrm{i}2\pi j\tau_1/M + \mathrm{i}2\pi j'\tau_2/N). \tag{28}$$

We next turn to the connection between strongly PC fields and unitary operators. The following proposition extends the result of Kallianpur and Mandrekar [13] for the case of stationary fields indexed on $\mathbf{Z}^2$ and extends the result of Hurd and Kallianpur [11] for PC processes.

Proposition 8. *The zero mean random field* $X = \{X(m,n) : (m,n) \in \mathbf{Z}^2\}$ *is strongly PC with period* (M, N) *if and only if there exists a pair of commuting unitary operators,* U_1 *and* U_2 *on* $\mathcal{H}(X)$ *for which*

$$X(m + kM, n + lN) = U_1^k U_2^l X(m,n) \tag{29}$$

for every m, n *and* k, l *in* $\mathbf{Z}$, *and this occurs for no smaller* $M>0$ *and* $N>0$.

We will give a sketch of the proof. If the field satisfies (29) for unitary U_1 and U_2, then (19) holds. Conversely, suppose (19) holds for every m, n, m', n' and k, l. Then we define a collection of operators $\{V(k,l), k \in \mathbf{Z},\ l \in \mathbf{Z}\}$ on the linear span $\mathcal{M}(X) = \mathrm{sp}\{X(m,n), (m,n) \in \mathbf{Z}^2\}$ by the action on a typical element $\zeta = \sum_{p=1}^{v} \alpha_p X(m_p, n_p)$ by

$$V(k,l)[\zeta] = \sum_{p=1}^{v} \alpha_p X(m_p + kM, n_p + lN). \tag{30}$$

It is then easy to show that $V(k,l)$ preserves inner products on $\mathcal{M}(X)$ for any (k,l). Since $V(k,l)$ is clearly surjective, then it is unitary and hence continuous on $\mathcal{M}(X)$ and extends to $\mathcal{H}(X) = \overline{\mathcal{M}(X)}$.

Thus $V(1,0)$ and $V(0,1)$ are unitary and we set $U_1 = V(1,0),\ U_2 = V(0,1)$. By definition, we have $V(1,1) = V(1,0)V(0,1) = V(0,1)V(1,0)$ which shows that U_1 and U_2 commute. Thus $V(k,l) = U_1^k U_2^l$. □

The general idea here is that shift invariance of the covariance corresponds exactly to the existence of a shift mapping that is a unitary operator. This will also be the main idea in the case of *weakly PC* sequences described in the next section.

Now we give a characterization for the strong PC property in terms of unitary operators and strongly periodic fields. This result was observed for univariate continuous time PC processes in [9] and was more thoroughly investigated in [11]. The proof can also be found in the survey paper [4]. The proof here is omitted due to similarity with the univariate continuous time case.

Proposition 9. *The zero mean random field* $\{X(m,n), (m,n) \in \mathbf{Z}^2\}$ *is strongly PC with period* (M, N) *if and only if there exists a strongly periodic field* $P(m,n)$ *taking values in* $\mathcal{H}(X)$ *having the same period* (M, N), *and a pair of commuting unitary operators,*

U_1 and U_2 on $\mathcal{H}(X)$ for which

$$X(m,n) = U_1^{m/M} U_2^{n/N}[P(m,n)] \tag{31}$$

for every m,n and where $U_1^{m/M} \equiv (U_1^{1/M})^m$ and similarly for U_2.

The preceeding result immediately yields a representation of strongly PC fields in terms of a Fourier series having a finite collection of jointly stationary fields as coefficients. It gives another way in which a collection of jointly stationary fields give a representation of PC fields. For the case of PC sequences, the representation originated with Gladyshev [8].

Proposition 10. *The zero mean random field $\{X(m,n), m,n \in \mathbf{Z}^2\}$ is strongly PC with period (M,N) if and only if there exists a collection*

$$\{Z^{pp'}(m,n), p \in [0, M-1], p' \in [0, N-1]\}, (m,n) \in \mathbf{Z}^2\}$$

of jointly stationary (in the sense of Section 2) random fields whose spectral support is $[0, 2\pi/M) \times [0, 2\pi/N)$ and for which

$$X(m,n) = \sum_{p=0}^{M-1} \sum_{p'=0}^{N-1} Z^{pp'}(m,n) \exp(\mathrm{i}2\pi pm/M + \mathrm{i}2\pi p'n/N). \tag{32}$$

Harmonizability: The following facts are straightforward extensions of the notion of harmonizable strongly processes as presented by Loève [17] and in a more general context by Rao [3]. A two-dimensional random field $\{X(m,n), (m,n) \in \mathbf{Z}^2\}$ is called *strongly harmonizable* if it can be represented by the quadratic mean integral

$$X(m,n) = \int_{[0,2\pi)^2} \exp[i(m,n) \cdot \vec{\lambda})] Z(d\vec{\lambda}), \tag{33}$$

where $Z(\cdot) : \mathcal{B}[0, 2\pi)^2 \to H(X)$ is a random measure for which the set function $r_Z(\Delta_1 \times \Delta_2) = E\{Z(\Delta_1)\overline{Z(\Delta_2)}\}$ for $\Delta_1 \times \Delta_2 \in \mathcal{B}([0, 2\pi)^2 \times [0, 2\pi)^2)$ satisfies

$$\int_{[0,2\pi)^2 \times [0,2\pi)^2} |r_Z(d\vec{\alpha}, d\vec{\beta})| < \infty \tag{34}$$

and consequently r_Z is a measure that is sometimes called the spectral covariance measure, or just spectral measure, of $X(m,n)$. It follows that the covariance of $X(m,n)$ has the representation

$$R(m,n,m',n') = \int_{[0,2\pi)^2 \times [0,2\pi)^2} \exp[i(m,n) \cdot \vec{\alpha} - (m',n') \cdot \vec{\beta})] r_Z(d\vec{\alpha}, d\vec{\beta}). \tag{35}$$

Conversely, if the covariance of a process is expressed by (35) where $r_Z(\cdot,\cdot)$ satisfies (34), then there is a random measure $Z(\cdot)$ such that $X(m,n)$ is represented by (33).

Following the one-dimensional case [8], the next propositions (a) characterize the spectral measure for harmonizable strongly PC random fields, (b) show that all strongly PC random fields (on $\mathbf{Z}^2$) are harmonizable and (c) give the relationship between the multi-dimensional spectral distributions of Proposition 6 and of Proposition 10.

Proposition 11. *A zero mean strongly harmonizable random field* $X = \{X(m,n), (m,n) \in \mathbf{Z}^2\}$ *is strongly PC with period* (M,N) *if and only if the support of the spectral measure* r_Z *is contained in the set*

$$S_{M,N} = \{(\alpha_1, \alpha_2, \beta_1, \beta_2) \in [0, 2\pi)^2 \times [0, 2\pi)^2 \text{ for which } \\ \beta_1 = \alpha_1 - 2\pi k/M,\ k \in [-(M-1), (M-1)] \\ \beta_2 = \alpha_2 - 2\pi k'/N,\ k' \in [-(N-1), (N-1)]\}. \quad (36)$$

Proposition 12. *Every strongly PC random field* $X(m,n)$ *is strongly harmonizable.*

The next proposition extends to fields another result of Gladyshev [8]. It relates the matrix-valued spectral distribution

$$F(\vec{\lambda}) = \{F_{(j_1,j_2),(k_1,k_2)}(\vec{\lambda}), j_1, k_1 \in [0, M-1], j_2, k_2 \in [0, N-1]\},$$

of the multi-dimensional stationary field

$$Y = \{Y_{j_1 j_2}(m,n),\ (m,n) \in \mathbf{Z}^2, j_1 \in [0, M-1], j_2 \in [0, N-1]\}$$

given in Proposition 6 to the matrix-valued spectral distribution

$$\mathcal{F}(\vec{\lambda}) = \{\mathcal{F}_{(p_1,p_2),(q_1,q_2)}(\vec{\lambda}), p_1, q_1 \in [0, M-1], p_2, q_2 \in [0, N-1]\}.$$

of the multi-dimensional stationary field

$$Z = \{Z^{p_1 p_2}(m,n),\ (m,n) \in \mathbf{Z}^2, p_1 \in [0, M-1], p_2 \in [0, N-1]\}$$

resulting from Proposition 10.

We note that the methods used by Gladyshev [8] to prove Proposition 12, when applied here show that $\mathcal{F}(\vec{\lambda})$ may be interpreted directly in terms of r_Z.

Proposition 13. *If* $X = \{X(m,n), (m,n) \in \mathbf{Z}^2\}$ *is strongly PC with period* (M,N), *then*

$$dF(\vec{\lambda}) = MNV(\vec{\lambda})d\mathcal{F}\left(\frac{\lambda_1}{M}, \frac{\lambda_2}{N}\right)V^{-1}(\vec{\lambda}), \quad (37)$$

where $V(\vec{\lambda})$ *is a unitary transformation from* $\mathbf{C}^M \times \mathbf{C}^N$ *to itself defined by*

$$(V(\vec{\lambda}))_{pqjk} = (V_1(\lambda_1))_{pj}(V_2(\lambda_2))_{qk}, \quad (38)$$

where $V_1(\lambda_1)$ *and* $V_2(\lambda_2)$ *are linear transformations (matrices) on* $\mathbf{C}^M$ *and* $\mathbf{C}^N$. *The pjth element of* $V_1(\lambda_1)$ *is*

$$(V_1(\lambda_1))_{pj} = \frac{1}{\sqrt{M}} \exp(\mathrm{i}2\pi pj/M + \mathrm{i}\lambda_1 j/M) \quad (39)$$

and the qkth element of $V_2(\lambda_2)$ *is*

$$(V_2(\lambda_2))_{qk} = \frac{1}{\sqrt{N}} \exp(\mathrm{i}2\pi qk/N + \mathrm{i}\lambda_2 k/N). \tag{40}$$

3.4. Weakly periodically correlated random fields

The zero mean random field $X = \{X(m,n), (m,n) \in \mathbf{Z}^2\}$ is called *weakly* PC with *period* (M,N) if

$$R(m,n,m',n') = R(m+M, n+N, m'+M, n'+N) \tag{41}$$

for every m,n,m',n'. As in the case of *weakly periodic fields*, we require $M \geqslant 0$ and $N \geqslant 0$ but do not permit $M = N = 0$ because this would put no constraint whatsoever on the covariance structure of the field. If (41) occurs for $M = 0$ and $N > 0$ with N minimal, we obtain a field that is PC in the second index and has no constraint on its behavior in the first index. For arbitrary $M > 0$, $N > 0$, we require (M,N) be relatively prime, and then the result is essentially the same except the field is periodically correlated along the straight lines of slope N/M. Since all of $\mathbf{Z}^2$ can be expressed as a countable union as in (16), a weakly PC random field is essentially a countable collection of PC sequences arranged along parallel lines of slope N/M in $\mathbf{Z}^2$. If X is of zero mean and strongly PC, then it is also weakly PC. The following proposition connects (41) to unitary operators. The proof, which follows along the same lines as the proof of Proposition 8, is omitted.

Proposition 14. *The zero mean random field* $\{X(m,n), (m,n) \in \mathbf{Z}^2\}$ *is weakly PC with period* (M,N) *if and only if there exists a unitary operator*, U *operating in* $\mathcal{H}(X)$ *for which*

$$X(m+M, n+N) = U[X(m,n)] \tag{42}$$

for every $(m,n) \in \mathbf{Z}^2$.

Again, as in Proposition 8, the shift invariance of the covariance corresponds exactly to the existence of a shift mapping that is a unitary operator. Except for the following paragraphs, we defer any further analysis of *weakly* PC fields to subsequent efforts.

3.5. The role of subgroups

The set $\mathbf{Z}^2$, taken as a group under the usual addition $(a,b) + (c,d) = (a+c, b+d)$ has non-trivial subgroups of only two types. The first type we shall call a *strong* subgroup and for a given pair (M,N) with $M > 0$ and $N > 0$, it is the set

$$S_{M,N} = \{(m,n) : m = kM, n = lN,\ (k,l) \in \mathbf{Z}^2\}. \tag{43}$$

Given again a pair (M,N) with $M \geqslant 0$ and $N \geqslant 0$ but not $M = N = 0$, a *weak* subgroup is a set

$$W_{M,N} = \{(m,n) : m = kM, n = kN,\ k \in \mathbf{Z}\}. \tag{44}$$

Now we can see that a random field $X = \{X(m,n), (m,n) \in \mathbf{Z}^2\}$ is *strongly periodic* with period (M, N) if and only if $X(m,n)$ is *invariant* under translation by any element of $S_{M,N}$. Similarly, X is *weakly periodic* with period (M,N) if and only if $X(m,n)$ is *invariant* under translation by any element of $W_{M,N}$.

Further, a field X is *strongly PC* with period (M,N) if and only if $X(m,n)$ is *unitarily related* via (29) to its translation by any element of $S_{M,N}$. Finally, X is *weakly PC* with period (M,N) if and only if $X(m,n)$ is *unitarily related* via (42) to its translation by any element of $W_{M,N}$. We note that the preceeding may be seen as an $L_2(\Omega)$ version of the following idea, the roots of which may be found in Jain and Kallianpur [12, p. 24, definition 4.1]. Suppose $\{X(s,\omega),\ s \in S,\ \omega \in \Omega\}$ is a random process defined on the index set S and let Γ be a collection of bijections from S to S. Then $\{X(s,\omega), s \in S\}$ is called (X,Γ) stationary if the probability distributions of X are invariant under any $\gamma \in \Gamma$. In our case, $S = \mathbf{Z}^2$ and the bijections $\gamma_{k,l}$ are shifts by (kM, lN); that is $\gamma_{k,l}(m,n) = (m + kM, n + lN)$ for every (m,n) and each pair $(k,l) \in \mathbf{Z}^2$ produces one such bijection. And finally, in our case it is only the covariance structure that is invariant.

4. The Wold decomposition

Here we shall present a few elementary results concerning Wold decompositions for zero mean *strongly* PC random fields. The main purpose of this section is to illustrate that the commuting of the operators U_1 and U_2 (Proposition 8) and their relationship to the various subspaces that are of interest provide for the straightforward extension of many results that have been obtained for stationary fields. See Kallianpur and Mandrekar [13] for a general discussion of the role of commuting isometries in the prediction context. The results we have chosen to present are essentially extensions of the 2- and 4-fold decompositions of Kallianpur, Miamee and Niemi [14,15], where for the latter case, weak commutativity is assumed.

For a second-order random field $X = \{X(m,n), (m,n) \in \mathbf{Z}^2\}$ we define $\mathcal{H} = \overline{\text{sp}}\{X(j,k), (j,k) \in \mathbf{Z}^2\}$ to be the Hilbert space of X. If the context requires a symbol for the field we will use subscripts, such as $\mathcal{H}^1_y(m)$ or $\mathcal{H}^1_{y,-\infty}$, to refer to the field y. The absence of the subscript means we are referring to the field X. Further,

1. $\mathcal{H}(m,n) = \overline{\text{sp}}\{X(j,k), j \leqslant m, k \leqslant n\}$ is the subspace of the *lower left* or *south-west* (*SW*) *quarter plane* at (m,n), and $\mathcal{H}^{12}_{-\infty} = \bigcap_{m,n} \mathcal{H}(m,n)$ is the subspace of the SW (southwest) remote past. The field is called *southwest purely non-deterministic* (*or regular*) if $\mathcal{H}^{12}_{-\infty} = \{0\}$ and *southwest deterministic* (*or singular*) if $\mathcal{H}(X; m,n) = \mathcal{H}^{12}_{-\infty}$ for all m, n in $\mathbf{Z}$.
2. $\mathcal{H}^1(m) = \overline{\text{sp}}\{X(j,k), j \leqslant m, k \in \mathbf{Z}\}$ is the subspace of the *left half plane* at m, or the left-horizontal past at m, and $\mathcal{H}^1_{-\infty} = \bigcap_m \mathcal{H}^1(m)$ is the subspace of the horizontal remote past. The field is called *horizontally purely non-deterministic or horizontally*

regular if $\mathcal{H}^1_{-\infty} = \{0\}$; it is called *horizontally deterministic or horizontally singular* if $\mathcal{H}^1_{-\infty} = \mathcal{H}$, or equivalently, if $\mathcal{H}^1(m_1) = \mathcal{H}^1(m_2)$ for all m_1, m_2 in $\mathbf{Z}$;
3. $\mathcal{H}^2(n) = \overline{\text{sp}}\{X(j,k), j \in \mathbf{Z}, k \leqslant n\}$ is the subspace of the *lower half plane* at n, or the bottom-vertical past at n, and $\mathcal{H}^2_{-\infty} = \bigcap_n \mathcal{H}^2(n)$ is the subspace of the vertical remote past. The field is called *vertically purely non-deterministic* if $\mathcal{H}^2_{-\infty} = \{0\}$; it is called *vertically deterministic or vertically singular* if $\mathcal{H}^2_{-\infty} = \mathcal{H}$ or equivalently, if $\mathcal{H}^2(n_1) = \mathcal{H}^2(n_2)$ for all n_1, n_2 in $\mathbf{Z}$;
4. the field is called *strongly purely non-deterministic* (*or regular*) if $\mathcal{H}^1_{-\infty} = \mathcal{H}^2_{-\infty} = \{0\}$ and *weakly deterministic* (*or singular*) if $\mathcal{H}^1_{-\infty} = \mathcal{H}^2_{-\infty} = \mathcal{H}$, or equivalently, $\mathcal{H}^1(m) \bigcap \mathcal{H}^2(n) = \mathcal{H}$ for all $(m,n) \in \mathbf{Z}$.

Wold decompositions follow, in essence, from the fact that certain subspaces are invariant under the unitary operators that describe the evolution of the process. In our current case, we summarize the pertinent results in the following proposition.

Proposition 15. *If $X(m,n)$ is strongly PC with period (M,N) and with associated unitary operators U_1 and U_2, then*

1. $\mathcal{H}(m+kM, n+lN) = U_1^k U_2^l \mathcal{H}(m,n)$ *for arbitrary m,n,k,l in $\mathbf{Z}$,*
2. $\mathcal{H}^1(m+kM) = U_1^k U_2^l \mathcal{H}^1(m)$ *for arbitrary m,k,l in $\mathbf{Z}$,*
3. $\mathcal{H}^2(n+lM) = U_1^k U_2^l \mathcal{H}^2(n)$ *for arbitrary n,k,l in $\mathbf{Z}$,*
4. $\mathcal{H}^1_{-\infty} = U_1^k U_2^l \mathcal{H}^1_{-\infty}$ *for arbitrary k,l in $\mathbf{Z}$,*
5. $\mathcal{H}^2_{-\infty} = U_1^k U_2^l \mathcal{H}^2_{-\infty}$ *for arbitrary k,l in $\mathbf{Z}$,*
6. $\mathcal{H}^{12}_{-\infty} = U_1^k U_2^l \mathcal{H}^{12}_{-\infty}$ *for arbitrary k,l in $\mathbf{Z}$.*

Proof. To prove (1), define $\mathcal{M}(m,n) = \text{sp}\{X(j,k), j \leqslant m, k \leqslant n\}$ then it is easy to see that $\mathcal{M}(m+kM, n+lN) = U_1^k U_2^l \mathcal{M}(m,n)$ for arbitrary m,n,k,l and the relationship extends to the closure $\mathcal{H}(m+kM, n+lN)$ by the unitarity of the operators U_1 and U_2. Items (2) and (3) are similar. Statements (4)–(6) follow from the first three. Taking statement (6) for example, if $x \in \mathcal{H}^{12}_{-\infty}$ then $x \in \mathcal{H}(m,n)$ for every m,n; but then by statement (1) the element $z = U_1^{-k} U_2^{-l} x \in \mathcal{H}(m-kM, n-lN)$ for every m,n and k,l in $\mathbf{Z}$, so that also $z \in \mathcal{H}^{12}_{-\infty}$, and hence $x = U_1^k U_2^l z \in U_1^k U_2^l \mathcal{H}^{12}_{-\infty}$. Conversely, if $x \in U_1^k U_2^l \mathcal{H}^{12}_{-\infty}$ then $x = U_1^k U_2^l z$ for $z \in \mathcal{H}^{12}_{-\infty}$; hence $z \in \mathcal{H}(m,n)$ for every m,n and so by (1) the same is true for x and thus $x \in \mathcal{H}^{12}_{-\infty}$. □

Remark. From standard results in the theory of Hilbert space (see Akheizer and Glazman [1, Sections 40–42]), we can conclude that any of the spaces $\mathcal{H}^1_{-\infty}, \mathcal{H}^2_{-\infty}, \mathcal{H}^{12}_{-\infty}$ together with their orthogonal complements *reduce* the unitary operator $U_1^k U_2^l$ for every k,l. That is, taking $\mathcal{H}^{12}_{-\infty}$ to be specific, item 6. in Proposition 15 shows that the subspace $\mathcal{H}^{12}_{-\infty}$ is invariant under both $U_1^k U_2^l$ and its inverse $U_1^{-k}U_2^{-l}$. This implies (see [1, Section 42]) that $U_1^kU_2^l$ commutes with the orthogonal projection onto $\mathcal{H}_{-\infty}^{12}$ and that

$$U_1^kU_2^l = U_1^kU_2^l|_{\mathcal{H}_{-\infty}^{12}} + U_1^kU_2^l|_{(\mathcal{H}_{-\infty}^{12})^\perp},$$

meaning the operator $U_1^kU_2^l$ can be split into it's restriction to $\mathcal{H}_{-\infty}^{12}$ and $(\mathcal{H}_{-\infty}^{12})^\perp$.

An elementary result is the 2-fold horizontal Wold decomposition; the vertical decomposition follows similarly.

Proposition 16 (Horizontal 2-fold decomposition). *If $X(m,n)$ is a strongly PC random field with period (M,N), then*

$$X(m,n) = X_s(m,n) + X_r(m,n), \tag{45}$$

where

1. *X_s is horizontally deterministic* (*singular*);
2. *X_r is horizontally purely non-deterministic* (*regular*);

Further, these two components are mutually orthogonal, are strongly PC with the same period (M,N) and

$$\mathcal{H}^1(m) = \mathcal{H}_s^1(m) \oplus \mathcal{H}_r^1(m). \tag{46}$$

Furthermore, the two subspaces $\mathcal{H}_{-\infty}^1$ and $(\mathcal{H}_{-\infty}^1)^\perp$ reduce the operator $U_1^kU_2^l$ for every $k,l \in \mathbf{Z}$.

Proof. The result follows primarily from the fact that $\mathcal{H}_{-\infty}^1$ is invariant under $U_1^kU_2^l$ for every k,l in $\mathbf{Z}$; see Proposition 15, item 4. Then defining $X_s(m,n) = P_{\mathcal{H}_{-\infty}^1}X(m,n)$ and $X_r(m,n) = X(m,n) - X_s(m,n)$, it follows that the X_s and X_r are singular and regular, respectively, and orthogonal: $X_s(m,n) \perp X_r(m',n')$ for every m,n,m',n'; expression (46) follows from the fact that projections are continuous operators. The argument in the remark following Proposition 15 implies that $U_1^kU_2^l$ commutes with $P_{\mathcal{H}_{-\infty}^1}$; hence (1) X_s and X_r are strongly PC with period (M,N) and (2) the two subspaces $\mathcal{H}_{-\infty}^1$ and $\mathcal{H}_{-\infty}^{1\ \perp}$ reduce the operator $U_1^kU_2^l$ for every $k,l \in \mathbf{Z}$. □

We begin our discussion of 4-fold decompositions with the notion of weak commutativity.

Definition 1. A second-order random field $X(m,n)$ is said to have the weak commutation property if

$$P_{\mathcal{H}^1(m)}P_{\mathcal{H}^2(n)} = P_{\mathcal{H}^2(n)}P_{\mathcal{H}^1(m)} = P_{\mathcal{H}^1(m)\cap\mathcal{H}^2(n)} \tag{47}$$

for every $m,n \in \mathbf{Z}$.

Given any second-order random field $X(m,n)$, let us consider the two following decompositions of $\mathcal{H} = \overline{sp}\{X(j,k),(j,k)\in\mathbf{Z}^2\}$

$$\mathcal{H} = \mathcal{H}^1_{-\infty} \oplus (\mathcal{H}^1_{-\infty})^\perp$$
$$\mathcal{H} = \mathcal{H}^2_{-\infty} \oplus (\mathcal{H}^2_{-\infty})^\perp \tag{48}$$

and define

$$\mathcal{H}_{ss} = \mathcal{H}^1_{-\infty} \cap \mathcal{H}^2_{-\infty},$$
$$\mathcal{H}_{sr} = \mathcal{H}^1_{-\infty} \cap (\mathcal{H}^2_{-\infty})^\perp,$$
$$\mathcal{H}_{rs} = (\mathcal{H}^1_{-\infty})^\perp \cap \mathcal{H}^2_{-\infty},$$
$$\mathcal{H}_{rr} = (\mathcal{H}^1_{-\infty})^\perp \cap (\mathcal{H}^2_{-\infty})^\perp. \tag{49}$$

Lemma 1. *The subspaces $\mathcal{H}_{ss}, \mathcal{H}_{sr}, \mathcal{H}_{rs}$ and $\mathcal{H}_{rr}$ are all invariant under $U_1^k U_2^l$ for arbitrary integers k, l.*

Proof. The subspaces in question are all intersection of subspaces whose invariance under $U_1^k U_2^l$ has already been determined. Then apply the following. If subspace $\mathcal{A}$ is invariant under unitary operators U and U^{-1}, then $\mathcal{A}$ and $\mathcal{A}^\perp$ reduce U and thus $\mathcal{A}^\perp$ is also invariant under U and U^{-1} (see [1, Section 42]). It follows easily that if $\mathcal{A}$ and $\mathcal{B}$ are invariant under U, then also is $\mathcal{A}\cap\mathcal{B}$. For if $x\in\mathcal{A}\cap\mathcal{B}$, then $Ux\in\mathcal{A}$ and $Ux\in\mathcal{B}$ so $Ux\in U(\mathcal{A}\cap\mathcal{B})$. Conversely if $x\in U(\mathcal{A}\cap\mathcal{B})$ then there exists $z\in\mathcal{A}\cap\mathcal{B}$ with $x = Uz$; but then since $\mathcal{A}$ and $\mathcal{B}$ are invariant under U, we have $x\in\mathcal{A}\cap\mathcal{B}$. □

It is clear that the subspaces $\mathcal{H}_{ss}, \mathcal{H}_{sr}, \mathcal{H}_{rs}$ and $\mathcal{H}_{rr}$ are mutually orthogonal and

$$\mathcal{H} \supset \mathcal{H}_{ss} \oplus \mathcal{H}_{sr} \oplus \mathcal{H}_{rs} \oplus \mathcal{H}_{rr}.$$

The opposite inclusion requires something additional. The following result shows that weak commutativity of a strongly PC random field is sufficient (since we already have commutativity of U_1 and U_2 and Proposition 15).

Proposition 17 (Four-fold decomposition). *If the random field $X(m,n)$ is weakly commuting and strongly PC with period (M,N), then*

$$X(m,n) = X_{ss}(m,n) + X_{sr}(m,n) + X_{rs}(m,n) + X_{rr}(m,n) \tag{50}$$

where

1. *X_{ss} is horizontally and vertically singular* (*weakly deterministic*),
2. *X_{sr} is horizontally singular and vertically regular* (2-*purely non-deterministic*),
3. *X_{rs} is horizontally regular and vertically singular* (1-*purely non-deterministic*),
4. *X_{rr} is horizontally and vertically regular* (*strongly purely non-deterministic*).

Further, each of these four components has the weak commutation property; they are mutually orthogonal, strongly PC with period (M,N) *and for all* $(m,n)\in\mathbf{Z}^2$,

$$\mathcal{H}(m,n)=\mathcal{H}_{ss}(m,n)\oplus\mathcal{H}_{sr}(m,n)\oplus\mathcal{H}_{rs}(m,n)\oplus\mathcal{H}_{rr}(m,n), \tag{51}$$

where

$$\mathcal{H}_{ss}(m,n)=\overline{sp}\{X_{ss}(j,k), j\leqslant m, k\leqslant n\}$$

and similarly for $\mathcal{H}_{sr}(m,n), \mathcal{H}_{rs}(m,n)$ *and* $\mathcal{H}_{rr}(m,n)$.

Proof. Since the field is weakly commutative, (47) holds, and since, for example, $\mathcal{H}^1_{-\infty}$ can be considered a monotone limit of subspaces, it follows by a limiting argument (see [1, Section 33]) that as $m,n\to-\infty$, the projections $P_{\mathcal{H}^1(m)}P_{\mathcal{H}^2(n)}\to P_{\mathcal{H}^1_{-\infty}}P_{\mathcal{H}^2_{-\infty}}$. Applying the same technique to the other three cases we conclude

$$\mathcal{H}=\mathcal{H}_{ss}\oplus\mathcal{H}_{sr}\oplus\mathcal{H}_{rs}\oplus\mathcal{H}_{rr}$$

and the Wold decomposition is just the projection onto these four subspaces. Expression (51) naturally follows. The weak commutativity of the four components follows in exactly the same manner as part (c), Theorem I.7 of [14], a cornerstone of which is Lemma 2.1 of [13]. □

The commuting of the projections $P_{\mathcal{H}^1_{-\infty}}$ and $P_{\mathcal{H}^2_{-\infty}}$ also yields, via the Wold-Halmos decomposition (see [13,14]), the following.

Corollary 1. *If the random field* $X(m,n)$ *is weakly commuting and strongly PC with period* (M,N), *then for all* m,n

$$\mathcal{H}_{ss}(m,n)=\mathcal{H}^1_{-\infty}\cap\mathcal{H}^2_{-\infty}, \tag{52}$$

$$\mathcal{H}_{rs}(m,n)=\sum_{j\leqslant m}\oplus[\mathcal{H}^1(j)\ominus\mathcal{H}^1(j-1)]\cap\mathcal{H}^2_{-\infty}, \tag{53}$$

$$\mathcal{H}_{sr}(m,n)=\mathcal{H}^1_{-\infty}\cap\sum_{k\leqslant n}\oplus[\mathcal{H}^1(k)\ominus\mathcal{H}^1(k-1)], \tag{54}$$

$$\mathcal{H}_{rr}(m,n)=\sum_{j\leqslant m}\oplus[\mathcal{H}^1(j)\ominus\mathcal{H}^1(j-1)]\cap\sum_{k\leqslant n}\oplus[\mathcal{H}^1(k)\ominus\mathcal{H}^1(k-1)]. \tag{55}$$

5. Innovations

Let us now denote

$$I_m^1 = [\mathcal{H}^1(m) \ominus \mathcal{H}^1(m-1)] \cap \mathcal{H}^2_{-\infty},$$
$$I_n^2 = [\mathcal{H}^2(n) \ominus \mathcal{H}^2(n-1)] \cap \mathcal{H}^1_{-\infty},$$
$$I_{mn} = [\mathcal{H}^1(m) \ominus \mathcal{H}^1(m-1)] \cap [\mathcal{H}^2(n) \ominus \mathcal{H}^2(n-1)] \quad (56)$$

and

$$M_1(m) = dim(I_m^1),$$
$$M_2(n) = dim(I_n^2),$$
$$M_0(m,n) = dim(I_{mn}). \quad (57)$$

We can interpret I_m^1 as the innovation space of a vertical strip intersected with the vertical remote past and similarly I_n^2 is the innovation space of a horizontal strip intersected with the horizontal remote past. We interpret I_{mn} as the subspace of the intersection of a vertical strip at m with a horizontal strip at n.

Lemma 2. *If the random field $X(m,n)$ is strongly PC with period (M,N), then*

$$I^1_{m+kM} = U_1^k U_2^l I_m^1 \quad for \quad every \quad k,l,m \in \mathbf{Z},$$
$$I^2_{n+lN} = U_1^k U_2^l I_n^2 \quad for \quad every \quad k,l,n \in \mathbf{Z},$$
$$I_{m+kM,n+lN} = U_1^k U_2^l I_{mn} \quad for \quad every \quad k,l,m,n \in \mathbf{Z} \quad (58)$$

and

$$M_1(m) = M_1(m+M) \quad for \quad every \quad m \in \mathbf{Z},$$
$$M_2(n) = M_2(n+N) \quad for \quad every \quad n \in \mathbf{Z},$$
$$M_0(m,n) = M_0(m+kM, n+lN) \quad for \quad every \quad k,l,m,n \in \mathbf{Z}. \quad (59)$$

The results all follow from the invariance of the subspaces $\mathcal{H}^1(m)$, $\mathcal{H}^2(n)$, $\mathcal{H}^1_{-\infty}$, $\mathcal{H}^2_{-\infty}$ under $U_1^k U_2^l$ for arbitrary $(k,l) \in \mathbf{Z}^2$.

Generally we cannot say too much about these dimensions without adding some other conditions. The following is a direct extension of a result due to Kallianpur, Miamee and Niemi [14,15] for the stationary case.

Proposition 18. *If the random field $X(m,n)$ is strongly PC with period (M,N) and weakly commuting, then there exists m_0 such that $M_1(m_0) \neq 0$ if and only if there is an m', n' for which*

$$||P_{\mathcal{H}_{rs}} X(m,n)|| \neq 0 \quad \begin{array}{l} for \quad m = m' + jM, \quad j \in \mathbf{Z}, \\ and \quad n = n' + kN, \quad k \in \mathbf{Z}. \end{array} \quad (60)$$

Proof. Note that $M_1(m_0) \neq 0$ iff $M_1(m_0 + jM) \neq 0, j \in \mathbf{Z}$. From the decomposition

$$\mathcal{H}_{rs} = \sum_{m=-\infty}^{\infty} I_m^1 \tag{61}$$

we can conclude that $||P_{\mathcal{H}_{rs}} X(m,n)|| \neq 0$ for some m,n if and only if $\mathcal{H}_{rs} \neq \{0\}$ which occurs if and only if $I_{m_0}^1 \neq \{0\}$ for some m_0. $\square$

The following example shows why $||P_{\mathcal{H}_{rs}} X(m,n)|| \neq 0$ for some m,n does not imply $M_1(m) \neq 0$ but only that $M_1(m_0) \neq 0$ for some m_0. Suppose $Y(m,n)$ is a stationary random field (PC with period $(1,1)$) whose unitary shift operators are U_1 and U_2. Consider the following diagram of $Y(m,n)$ near $Y(0,0)$:

$$\begin{array}{c} \vdots \\ \dots Y(0,2)\ Y(1,2)\ Y(2,2) \dots \\ \dots Y(0,1)\ Y(1,1)\ Y(2,1) \dots \\ \dots Y(0,0)\ Y(1,0)\ Y(2,0) \dots \\ \vdots \end{array}$$

and we now construct a new field, call it $X(m,n)$, by replacing $Y(m,n)$ with $Y(m,n)\ Y(m,n)$ thus producing the diagram

$$\begin{array}{c} \vdots \\ \dots Y(0,2)\ Y(0,2)\ Y(1,2)\ Y(1,2)\ Y(2,2)\ Y(2,2) \dots \\ \dots Y(0,1)\ Y(0,1)\ Y(1,1)\ Y(1,1)\ Y(2,1)\ Y(2,1) \dots \\ \dots Y(0,0)\ Y(0,0)\ Y(1,0)\ Y(1,0)\ Y(2,0)\ Y(2,0) \dots \\ \vdots \end{array}$$

That is, let $X(m,n) = Y([m/2],n)$. It is evident that $X(m,n)$ is PC with period (2,1) and the corresponding shift operators are U_1, U_2 inherited from $Y(m,n)$. It follows from the construction that $\mathcal{H}_X^1(m) = \mathcal{H}_y^1([m/2])$, $\mathcal{H}_X^2(n) = \mathcal{H}_y^2(n), \mathcal{H}_{X,-\infty}^1 = \mathcal{H}_{Y,-\infty}^1$ and $\mathcal{H}_{X,-\infty}^2 = \mathcal{H}_{Y,-\infty}^2$. It is also readily seen that

$$\mathcal{H}_X^1(m) \ominus \mathcal{H}_X^1(m-1) = \begin{cases} \mathcal{H}_Y^1([m/2]) \ominus \mathcal{H}_Y^1([(m-1)/2]) & m \text{ even}, \\ \{0\} & \text{otherwise} \end{cases}$$

and that

$$\mathcal{H}_X^2(n) \ominus \mathcal{H}_X^2(n-1) = \mathcal{H}_Y^2(n) \ominus \mathcal{H}_Y^2(n-1)$$

for all m and n.

Now if $dim\{[\mathcal{H}_Y^1(m) - \mathcal{H}_Y^1(m-1)] \cap \mathcal{H}_{Y,-\infty}^2\} \neq 0$ for some m then it is true for every m from stationarity. Then

$$M_1(m) = dim\{[\mathcal{H}_X^1(m) - \mathcal{H}_X^1(m-1)] \cap \mathcal{H}_{-\infty}^2(X)\} = \begin{cases} \neq 0 & m \text{ even}, \\ 0 & m \text{ odd}. \end{cases}$$

Suppose now that $||P_{\mathcal{H}_{rs}}X(m,n)||\neq 0$ for some even integer m so that $M_1(m)\neq 0$. But also $||P_{\mathcal{H}_{rs}}X(m+1,n)||\neq 0$ and yet $M_1(m+1)=0$.

5.1. Innovations under strong commutativity

Definition 2. A random field is said to have the strong commutativity property if

$$P_{\mathcal{H}^1(m)}P_{\mathcal{H}^2(n)} = P_{\mathcal{H}(m,n)} \tag{62}$$

for all $m,n\in\mathbf{Z}$.

Proposition 19. *If the random field $X(m,n)$ is strongly commuting, then*

$$\mathcal{H}(m,n) = I_{mn}\oplus\mathcal{H}_-(m,n), \tag{63}$$

where

$$\mathcal{H}_-(m,n) = \overline{sp}\{X(j,k),\ j\leqslant m,\ k\leqslant n,\ (j,k)\neq(m,n)\}. \tag{64}$$

Proof. It is equivalent to show that

$$I_{mn} = \mathcal{H}(m,n)\ominus\mathcal{H}_-(m,n).$$

First by strong commutativity (and since strong implies weak)

$$\begin{aligned} P_{I_{mn}} &= [P_{\mathcal{H}^1(m)}\ominus P_{\mathcal{H}^1(m-1)}][P_{\mathcal{H}^2(n)}\ominus P_{\mathcal{H}^2(n-1)}] \\ &= P_{\mathcal{H}^1(m)}P_{\mathcal{H}^2(n)} - P_{\mathcal{H}^1(m-1)}P_{\mathcal{H}^2(n)} \\ &\quad - P_{\mathcal{H}^1(m)}P_{\mathcal{H}^2(n-1)} + P_{\mathcal{H}^1(m-1)}P_{\mathcal{H}^2(n-1)} \end{aligned} \tag{65}$$

so that

$$\begin{aligned} I_{mn} &= \mathcal{H}(m,n)\ominus\mathcal{H}(m-1,n)\ominus[\mathcal{H}(m,n-1)\ominus\mathcal{H}(m-1,n-1)], \\ &= \mathcal{H}(m,n)\ominus\mathcal{H}(m,n-1)\ominus[\mathcal{H}(m-1,n)\ominus\mathcal{H}(m-1,n-1)]. \end{aligned} \tag{66}$$

Now if $x\in\mathcal{H}(m,n)\ominus\mathcal{H}_-(m,n)$, then $x\in\mathcal{H}^1(m)$ and $x\in\mathcal{H}^2(n)$. Furthermore, $x\perp\mathcal{H}_-(m,n)$ implies $x\perp\mathcal{H}(m-1,n)$ and $x\perp\mathcal{H}(m,n-1)$. Thus by (66), $x\in I_{mn}$.

Conversely, if $x\in I_{mn}$ then by (66) $x\in\mathcal{H}(m,n)$, $x\perp\mathcal{H}(m-1,n)$ and $x\perp\mathcal{H}(m,n-1)$ so that x is orthogonal to all the random variables $\{X(j,k), j\leqslant m, k\leqslant n, (j,k)\neq(m,n)\}$ that generate $\mathcal{H}_-(m,n)$. Thus $x\in\mathcal{H}(m,n)\ominus\mathcal{H}_-(m,n)$. □

When $X(m,n)$ is strongly commuting we can say something useful about $M_0(m,n)$.

Corollary 2. *If the random field $X(m,n)$ is strongly commuting, then $M_0(m,n)=1$ if and only if $X(m,n)\notin\mathcal{H}_-(m,n)$ and otherwise $M_0(m,n)=0$.*

We finish this work with some remarks and a proposition about one-sided moving average representations of strongly PC fields. First, if $X(m,n)$ is PC with period M,N, is weakly commutative and strongly regular ($\mathcal{H} = \mathcal{H}_{rr}$), then from (55)

$$\mathcal{H}_{rr}(m,n) = \sum_{p\leqslant m} \sum_{q\leqslant n} I_{pq}, \tag{67}$$

and since it is readily seen that $I_{pq} \perp I_{p'q'}$ unless $(p,q) = (p',q')$, then every $X(m,n)$ has the decomposition

$$X(m,n) = \sum_{p\leqslant m} \sum_{q\leqslant n} \eta_{pq}(m,n), \tag{68}$$

where $\eta_{pq}(m,n) \in I_{pq}$. Hence we already have a one-sided representation in terms of the "past". Under the assumption of strong commutativity, the vectors $\eta_{pq}(m,n)$ are zero or in a subspace of dimension 1 because $dim(I_{pq})$ is either 0 or 1, but still not a moving average. Adding the PC structure then gives the moving average with respect to orthogonal vectors in one-dimensional subspaces. But we first need the following.

Definition 3. If the random field $X(m,n)$ is strongly commuting and strongly PC with period (M,N), then its rank is

$$rank(X) = card(\{(m,n) : M_0(m,n) \neq 0, m = 0,1,\dots,M-1,\ n = 0,1,\dots,N-1\}). \tag{69}$$

Thus the largest rank possible for such a process is $M \cdot N$, and a PC field with period (M,N) is said to be of full rank if $rank(X) = M \cdot N$. Following Miamee and Salehi [18], it is clear that the rank of a PC field is closely related to the rank of a stationary vector-valued field having $M \cdot N$ components and satisfying some appropriate strong commutativity property. We will not pursue this idea further in this paper but will use the rank as we have defined it.

In order to treat the case where $rank(X) < M \cdot N$ we define

$$D^+ = \{(m,n) : M_0(m,n) > 0\} \tag{70}$$

to be the set of indices where the field has positive innovation dimension according to $M_0(m,n)$. We note that D^+ is a periodic set in the sense that if $(m,n) \in D^+$ then also $(m+kM, n+lN) \in D^+$ for every $k,l \in \mathbf{Z}$. We define $\mathbf{M} \times \mathbf{N} = [0,1,\dots,M-1] \times [0,1,\dots,N-1]$ as the principal rectangle having sides M,N.

Proposition 20. *If the random field $X(m,n)$ is strongly commuting, then it is strongly PC with period (M,N), and strongly regular ($\mathcal{H} = \mathcal{H}_{rr}$) and of rank Q if and only if there exists a periodic set D^+ of period (M,N) having $Q = card(D^+ \cap \mathbf{M} \times \mathbf{N})$, and a sequence of orthonormal innovation vectors*

$$I = \{\xi_{p,q}, (p,q) \in D^+\} \tag{71}$$

such that for every m,n

$$X(m,n) = \sum_{r\geqslant 0, s\geqslant 0 : (m-r,n-s) \in D^+} a_{r,s}(m,n)\xi_{m-r,n-s}, \tag{72}$$

$$\sum_{r\geqslant 0, s\geqslant 0:(m-r,n-s)\in D^+} |a_{r,s}(m,n)|^2 < \infty, \tag{73}$$

and

$$a_{r,s}(m+kM, n+lN) = a_{r,s}(m,n) \tag{74}$$

for every r,s,k,l,m,n *such that* $(m-r,n-s)\in D^+$.

Remark. To clarify the notation, we first observe that if $(m-r,n-s)\notin D^+$, then $\xi_{m-r,n-s}$ does not exist nor does $\xi_{m+kM-r,n+lN-s}$ for $(k,l)\in \mathbf{Z}^2$ (there do not exist vectors with these indices).

Proof. The orthonormality of the $\xi_{p,q}$ and the square summability (73) together ensure that $X_{m,n}$ is a L_2 random variable for every m,n. It also follows from the orthogonality that

$$R(m,n,m',n') = \sum_{r\geqslant 0, s\geqslant 0:(m-r,n-s)\in D^+} a_{r,s}(m,n)\overline{a_{r+m'-m,s+n'-n}}(m',n') \tag{75}$$

and hence (19) is satisfied. The orthogonality of the $\xi_{p,q}$ imply that $[\mathcal{H}^1(m)\ominus\mathcal{H}^1(m-1)]\perp[\mathcal{H}^1(m')\ominus\mathcal{H}^1(m'-1)]$ for $m\neq m'$ and hence $\mathcal{H}^1_{-\infty}=\{0\}$; similarly $\mathcal{H}^2_{-\infty}=\{0\}$ and therefore $\mathcal{H}=\mathcal{H}_{rr}$, or in other words, $X(m,n)$ is strongly regular. To see that $X(m,n)$ is of rank Q we note from (72) that if we consider $X(m,n)$ for $(m,n)\in\mathbf{M}\times\mathbf{N}$, then by the definition of D^+ there are only Q values of (m,n) for which $X(m,n)$ depends on $\xi_{m,n}$; for the others, $X(m,n)$ depends only on the past innovations ($r\geqslant 0, s\geqslant 0$ but $r=s=0$ not permitted). Said another way, $X(m,n)$ has exactly Q non zero innovations for $(m,n)\in\mathbf{M}\times\mathbf{N}$ and therefore $M_0(m,n)=dim(I_{mn})=1$ for exactly these Q values of m,n and this implies $rank(X)=Q$.

Conversely if the strongly PC field $X(m,n)$ is strongly regular and strongly commuting then the innovation spaces I_{pq} appearing in (67) are of dimension at most one and thus in (68) we may write $\eta_{p,q}(m,n)=\alpha_{p,q}(m,n)\xi_{p,q}$, where $\xi_{p,q}$ is given below by (77), but, to emphasize the point, $\eta_{p,q}\in I_{pq}$ and hence $\xi_{p,q}$ are defined only when $(p,q)\in D^+$. To amplify this, the assumption $rank(X)=Q$ means there are only Q values of $(m,n)\in\mathbf{M}\times\mathbf{N}$ for which there is a nontrivial innovation, meaning that (68) may be replaced with

$$X(m,n) = \sum_{p\leqslant m, q\leqslant n:(p,q)\in D^+} \alpha_{p,q}(m,n)\xi_{p,q}. \tag{76}$$

Since $X(p,q)-P_{\mathcal{H}_-(p,q)}X(p,q)$ is a non-zero vector (in I_{pq}) only when $(p,q)\in D^+$ we then define

$$\xi_{p,q} = \frac{X(p,q)-P_{\mathcal{H}_-(p,q)}X(p,q)}{||X(p,q)-P_{\mathcal{H}_-(p,q)}X(p,q)||}\in I_{pq} \tag{77}$$

which satisfies $\xi_{p+kM,q+lN}=U_1^k U_2^l \xi_{p,q}$ for every $k,l\in\mathbf{Z}$ and $(p,q)\in D^+$ and the collection $\{\xi_{p,q},(p,q)\in D^+\}$ is clearly orthonormal. Now setting

$$a_{p,q}(m,n) = \alpha_{m-p,n-q}(m,n)$$

we may now rewrite (76) as (72) where (73) must apply for every p, q, k, l, m, n such that $(m-p, n-q) \in D^+$.

To obtain the periodicity of the coefficients we consider

$$X(m+M,n) = U_1 X(m,n) = \sum_{r \geqslant 0, s \geqslant 0 : (m-r, n-s) \in D^+} a_{r,s}(m,n) U_1 \xi_{m-r,n-s}$$
$$= \sum_{r \geqslant 0, s \geqslant 0 : (m-r, n-s) \in D^+} a_{r,s}(m,n) \xi_{m+M-r,n-s}$$

but also

$$X(m+M,n) = \sum_{r \geqslant 0, s \geqslant 0 : (m-r, n-s) \in D^+} a_{r+M,s}(m,n) \xi_{m+M-r,n-s}$$

which shows that

$$a_{r,s}(m+M,n) = a_{r,s}(m,n)$$

for every r, s, k, l, m, n such that $(m-r, n-s) \in D^+$. Repeating the exercise for the variable n leads to

$$a_{r,s}(m,n+N) = a_{r,s}(m,n)$$

for every r, s, k, l, m, n such that $(m-r, n-s) \in D^+$ and hence the claimed result. □

Acknowledgments

We acknowledge with appreciation the very helpful remarks of a referee. This work was supported by the Office of Naval Research under contract N00014-95-C-0093, and by the US Army Research Office under contracts DAAH04-96-C-0027 and DAAD19-02-C-0045.

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