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Dirac spinors for Doubly Special Relativity and κ-Minkowski noncommutative spacetime

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ABSTRACT

We construct a Dirac equation that is consistent with one of the recently-proposed schemes for a "doubly-special relativity", a relativity with both an observer-independent velocity scale (still naturally identified with the speed-of-light constant) and an observer-independent length/momentum scale (possibly given by the Planck length/momentum). We find that the introduction of the second observer-independent scale only induces a mild deformation of the structure of Dirac spinors. We also show that our modified Dirac equation naturally arises in constructing a Dirac equation in the κ-Minkowski noncommutative spacetime. Previous, more heuristic, studies had already argued for a possible role of doubly-special relativity in κ-Minkowski, but remained vague on the nature of the consistency requirements that should be implemented in order to assure the observer-independence of the two scales. We find that a key role is played by the choice of a differential calculus in κ-Minkowski. A much-studied choice of the differential calculus does lead to our doubly-special relativity Dirac equation, but a different scenario is encountered for another popular choice of differential calculus.

1 Introduction

After more than 70 years of study [1, 2] the "quantum-gravity problem", the problem of reconciling/unifying gravity and quantum mechanics, is still unsolved. Even the best developed quantum-gravity theories [3, 4] still lack any observational support [5, 6, 7, 8] and are still affected by serious deficiencies in addressing some of the "conceptual issues" that arise at the interplay between gravity and quantum mechanics[1]. One can conjecture that the lack of observational support might be due to the difficulties of the relevant phenomenology [5, 6, 7, 8] and that the conceptual issues might be eventually settled, but on the other hand it is legitimate, as long as the quantum-gravity problem remains open, to explore possible alternative paths toward the solution of the problem. At present it is conceivable that the empasse in the study of the quantum-gravity problem might be due to the inadequacy of some of the key (and apparently most natural) common assumptions of quantum-gravity approaches. Over these past few years there has been growing interest in alternative quantum-gravity theories, as perhaps best illustrated by studies which take a condensed-matter perspective on the quantum-gravity problem [10, 11, 12]. Another possibility, recently proposed by one of us [13], is the one of a new starting point for the search of a quantum gravity: instead of assuming that the status of Lorentz symmetry remains unaffected by the interplay between gravity and quantum mechanics, one can explore the possibility that the Planck length L_p ($L_p \sim 10^{-33} cm$) should be taken into account in describing the rotation/boost transformations between inertial observers. This would amount to a deformation of special relativity, a so-called "doubly special relativity" [13, 14, 15, 16] (DSR), in which, in addition to the familiar[2] velocity scale c, also a second scale, a length scale λ (momentum scale $1/\lambda$), is introduced as observer-independent feature of the laws of transformation between inertial observers. λ can be naturally (though not necessarily) identified with the Planck length.

The fact that in some doubly-special-relativity scenarios the scale $1/\lambda$ turns out to set the maximum value of momentum [13, 14] and/or energy [14, 15] attainable by fundamental particles might be a useful tool for quantum-gravity research. In particular, it appears likely that [13, 14] the idea of a doubly special relativity may find applications in the study of certain noncommutative spacetimes (and we will provide here more evidence in favor of this possibility). Moreover, while the deformation is soft enough to be consistent with all presently-available data, some of the predictions of doubly-special-relativity scenarios are testable [13, 17] with forthcoming experiments [18], and therefore these theories may prove useful also in the wider picture of quantum-gravity research, as a training camp for the general challenge of setting up experiments capable of reaching sensitivity to very small (Planck-length suppressed) quantum-spacetime effects.

Some of these testable predictions, which concern spin-half particles, have been obtained at a rather heuristic level of analysis, since, so far, no DSR formulation of spinors had been presented[3]. We provide here this missing element of DSR theories.

[1]Examples of these conceptual issues are the "problem of time" and the "background-independence problem" [9].

[2]In presence of an observer-independent length scale the fact that our observations, on photons which inevitably have wavelengths that are much larger than the Planck length, are all consistent with a wavelength-independent speed of photons must be analyzed more cautiously [13]: it is only possible to identify the speed-of-light constant c as the speed of long-wavelength photons.

[3]Note however that in parallel with the present study, the issue of a description of spinors in DSR has also been considered, from a different perspective, in gr-qc/0207004 [19]. Moreover, after the appearance of the first version (gr-qc/0207003v1) of this manuscript, the problem of a DSR formulation of spinors has been considered, from yet another alternative perspective in hep-ph/0307205 [20]. Both in gr-qc/0207004 and in hep-ph/0307205 the possible connection with κ-Minkowski spacetime is not considered.

We focus on the specific DSR scheme used as illustrative example in the studies [13] that proposed the DSR idea, but our approach appears to be applicable to a wider class of DSR schemes, including the one recently proposed by Maguejio and Smolin in Ref. [15] and the wider class of DSR schemes considered in Refs. [14, 21]. In fact, in all of these DSR schemes the introduction of the second observer-independent scale relies on a nonlinear realization of the Lorentz group in energy-momentum space: the generators that govern the rules of transformation of energy-momentum between inertial observers still satisfy the Lorentz algebra, but their action is nonlinearly modified. This does not appear to be a necessary feature of DSR theories [13], but it does characterize all DSR schemes so far considered, and it plays a central role in the structure of our proposal.

A key aspect of our analysis is the fact that we first focus on how to formulate the relevant deformed Dirac equation in a way that involves exclusively energy-momentum space (which is the best understood sector of these DSR theories), and then we explore the possibility of a corresponding deformed Dirac equation in the spacetime sector. Confirming the indications of previous heuristic arguments [13, 14], we find that κ-Minkowski spacetime[22, 23, 24] provides a natural host for our DSR-deformed Dirac equation. But we also show that the presence of DSR symmetries in κ-Minkowski is not automatic: it requires a specific choice of the (noncommutative) differential calculus in κ-Minkowski. Our analysis shows that a generic claim that κ-Minkowski has DSR symmetries is incorrect. It is only once one makes a certain choice among the possible differential calculi in κ-Minkowski that the DSR symmetries emerge. It had been previously shown that a DSR-deformed Klein-Gordon equation could be obtained with two different choices of differential calculus, but we find that the richer structure of the DSR-deformed Dirac equation allows to select a specific choice of differential calculus in κ-Minkowski.

In the next section we start by deriving a DSR-deformed Dirac equation in energy-momentum space. Then in Section 3 we show that our DSR-deformed Dirac equation can be naturally introduced in κ-Minkowski if a certain appropriate choice of differential calculus is adopted. In Section 4 we show that with that choice of differential calculus ours is the only consistent Dirac equation that can be introduced in κ-Minkowski. In Section 5 we show that by adopting another differential calculus one is then not able to obtain a Dirac equation that is consistent with the DSR requirements. In Section 6 we compare our results with the ones of other deformed Dirac equations that had been previously considered in the literature. Section 7 contains some closing remarks.

2 Dirac spinors for Doubly Special Relativity

Before discussing our DSR formulation of the Dirac equation we start with a brief review of the structure of the ordinary Dirac equation, which will provide a useful starting point for our DSR deformation. The approach we adopt is based on the one of Ref. [25]. We start by introducing operators $\overrightarrow{A}$ and $\overrightarrow{B}$ that are related to the generators of rotations, $\overrightarrow{J}$, and boosts, $\overrightarrow{K}$, through

$$\overrightarrow{A} = \frac{1}{2}(\overrightarrow{J} + i\overrightarrow{K}) \tag{2.1}$$

$$\overrightarrow{B} = \frac{1}{2}(\overrightarrow{J} - i\overrightarrow{K}) \tag{2.2}$$

The usefulness of these generators $\overrightarrow{A}$ and $\overrightarrow{B}$ is due to the familiar relation between the Lorentz algebra and the algebra $SU(2) \otimes SU(2)$. In fact, from the Lorentz-algebra relations for $\overrightarrow{J}$ and $\overrightarrow{K}$ it follows that

$$[A_l, A_m] = i\epsilon_{lmn}A_n \ , \tag{2.3}$$

$$[B_l, B_m] = i\epsilon_{lmn} B_n \ , \tag{2.4}$$

$$[A_l, B_m] = 0 \ . \tag{2.5}$$

Spinors can be labelled with a pair of numbers (j, j') characteristic of the eigenvalues of $\overrightarrow{A}^2$ and $\overrightarrow{B}^2$. In particular, "left-handed" and "right handed" spinors correspond to the cases $\overrightarrow{A}^2 = 0$ and $\overrightarrow{B}^2 = 0$ respectively. Left-handed spinors are labelled by $(\frac{1}{2}, 0)$ and their transformation rules for generic Lorentz-boost "angle" (rapidity) $\overrightarrow{\xi}$ and rotation angle $\overrightarrow{\theta}$ are[4]

$$u_L \to \exp\left(i\frac{\overrightarrow{\sigma}}{2}\cdot\overrightarrow{\theta} - \frac{\overrightarrow{\sigma}}{2}\cdot\overrightarrow{\xi}\right) u_L \ , \tag{2.6}$$

where $\overrightarrow{\sigma}$ denotes the familiar 2×2 Pauli matrices. Analogously, right-handed spinors are labelled by $(0, \frac{1}{2})$ and transform according to

$$u_R \to \exp\left(i\frac{\overrightarrow{\sigma}}{2}\cdot\overrightarrow{\theta} + \frac{\overrightarrow{\sigma}}{2}\cdot\overrightarrow{\xi}\right) u_R \ . \tag{2.7}$$

It is sometimes convenient to describe a spinor with space momentum $\overrightarrow{p}$ in terms of a pure Lorentz boost from the rest frame:

$$u_R(\overrightarrow{p}) = e^{\frac{1}{2}\overrightarrow{\sigma}\cdot\overrightarrow{\xi}} u_R(0) = \left(\cosh\left(\frac{\xi}{2}\right) + \overrightarrow{\sigma}\cdot\overrightarrow{n} \sinh\left(\frac{\xi}{2}\right)\right) u_R(0) \ , \tag{2.8}$$

and

$$u_L(\overrightarrow{p}) = e^{-\frac{1}{2}\overrightarrow{\sigma}\cdot\overrightarrow{\xi}} u_L(0) = \left(\cosh\left(\frac{\xi}{2}\right) - \overrightarrow{\sigma}\cdot\overrightarrow{n} \sinh\left(\frac{\xi}{2}\right)\right) u_L(0) \ , \tag{2.9}$$

where $\overrightarrow{n}$ is the unit vector in the direction of the boost (and therefore characterizes the direction of the space momentum of the particle) and on the right-hand sides of Eqs. (2.8) and (2.9) the dependence on momentum is also present implicitly through the special-relativistic relations[5] between the boost parameter ξ and energy E,

$$\cosh \xi = \frac{E}{m} \ , \tag{2.10}$$

and (the "dispersion relation") between energy and spatial momentum

$$E^2 = \overrightarrow{p}^2 + m^2 \tag{2.11}$$

for given mass m of the particle.

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[4]The notation $u(\vec{p})$ is here introduced to denote the spinor wave function in energy-momentum space. It is natural to expect that $u(\vec{p})$ could be connected by a Fourier transform to a spinor defined on a suitable quantum spacetime. This expectation proves to be correct, as we show in the followings sections.

[5]In order to render some of our equations more compact we adopt conventions with $c \to 1$. This should not create any confusion since in DSR the speed-of-light constant preserves its role as observer-independent scale (but in DSR it is accompanied by a second observer-independent scale λ) and the careful reader can easily reinstate $c \neq 1$ by elementary dimensional-analysis considerations.

One must then codify the fact that left-handed and right-handed spinors cannot be distinguished at rest. One way to do this[6] relies on the condition $u_R(0) = u_L(0)$, from which it follows that

$$\begin{pmatrix} -I & F^{+}(\xi) \\ F^{-}(\xi) & -I \end{pmatrix} \begin{pmatrix} u_R(\overrightarrow{p}) \\ u_L(\overrightarrow{p}) \end{pmatrix} = 0 \ , \tag{2.12}$$

where

$$F^{\pm}(\xi) = 2\left(\cosh^2\left(\frac{\xi}{2}\right) - \frac{1}{2} \pm \overrightarrow{\sigma}\cdot\overrightarrow{n}\sinh\left(\frac{\xi}{2}\right)\cosh\left(\frac{\xi}{2}\right)\right) \ . \tag{2.13}$$

Using Eqs. (2.10) and (2.11) it is easy to establish the dependence on the particle's energy-momentum which is coded in the ξ-dependence of Eq. (2.12). This leads to the ordinary Dirac equation formulated in energy-momentum space

$$(\gamma^\mu p_\mu - m)\, u(\overrightarrow{p}) = 0 \ , \tag{2.14}$$

where γ^μ are the familiar "γ matrices" and

$$u(\overrightarrow{p}) \equiv \begin{pmatrix} u_R(\overrightarrow{p}) \\ u_L(\overrightarrow{p}) \end{pmatrix} \ . \tag{2.15}$$

The path we followed in reviewing the derivation of the ordinary special-relativistic Dirac equation provides a natural starting point for our announced deformation within the DSR framework. In fact, we relied exclusively on the algebraic properties of the generators of boosts and rotations (the properties of the Lorentz algebra, without making use of the specific representation of the generators of boosts and rotations as differential operators on energy-momentum space that is adopted in special relativity) and on Eqs. (2.10) and (2.11), the ordinary special-relativistic relations between energy and rapidity (boost parameter connecting to the rest frame) and between energy and momentum. The algebraic properties of the generators of boosts and rotations remain unmodified in the DSR scheme considered in Refs. [13] (and in the other DSR schemes considered in Refs. [14, 15, 21]). In fact, the nonlinearity needed in order to introduce the second observer-independent scale is implemented by adopting a deformed representation as differential operators on energy-momentum space of the generators of boosts and rotations, but these deformed generators still satisfy the Lorentz algebra. Therefore in the derivation of the Dirac equation the only changes are introduced by the DSR deformations of the relations between energy and rapidity and between energy and momentum. In the DSR scheme considered in Refs. [13], on which we focus here, the relation between energy and momentum (the dispersion relation) is

$$2\lambda^{-2}\cosh(\lambda E) - \overrightarrow{p}^2 e^{\lambda E} = 2\lambda^{-2}\cosh(\lambda m) \ . \tag{2.16}$$

The relation between rapidity and energy that holds in the DSR scheme considered in Refs. [13], can be deduced from the structure of the corresponding DSR-deformed boost transformations, which have been studied in Refs. [13, 14]. Focusing again on a pure Lorentz boost from the rest frame to an inertial frame in which the particle has spatial momentum $\overrightarrow{p} \equiv |\overrightarrow{p}|\overrightarrow{n}$ one easily finds [14]

$$E(\xi) = m + \lambda^{-1}\ln\left(1 - \sinh(\lambda m)\, e^{-\lambda m}(1 - \cosh\xi)\right) \ . \tag{2.17}$$

[6] Since we are here only concerned with the basics of the DSR deformation of Dirac spinors, we take the liberty to set aside the possible phase difference between $u_R(0)$ and $u_L(0)$.

Therefore the boost parameter ξ can be expressed as a function of the energy using

$$\cosh\xi = \frac{e^{\lambda E} - \cosh(\lambda m)}{\sinh(\lambda m)} \ . \tag{2.18}$$

In the DSR derivation of the Dirac equation the Eqs. (2.16) and (2.18) must replace the Eqs. (2.10) and (2.11) of the ordinary special-relativistic case. All the steps of the derivation that used the algebra properties of the boost generators apply also to the DSR context (since, as emphasized above, the Lorentz-algebra relations remain undeformed in the DSR scheme considered in Refs. [13], and in the other DSR schemes considered in Refs. [14, 15, 21]).

We are therefore ready[7] to write down the DSR-deformed Dirac equation:

$$\begin{pmatrix} -I & F_\lambda^+(E,m) \\ F_\lambda^-(E,m) & -I \end{pmatrix} \begin{pmatrix} u_R(\overrightarrow{p}) \\ u_L(\overrightarrow{p}) \end{pmatrix} = 0 \tag{2.19}$$

where

$$F_\lambda^\pm(E,m) = \frac{e^{\lambda E} - \cosh(\lambda m) \pm \overrightarrow{\sigma}\cdot\overrightarrow{n}\left(2e^{\lambda E}\left(\cosh(\lambda E) - \cosh(\lambda m)\right)\right)^{\frac{1}{2}}}{\sinh(\lambda m)} \ . \tag{2.20}$$

Introducing

$$D_0^\lambda(E,m) \equiv \frac{e^{\lambda E} - \cosh(\lambda m)}{\sinh(\lambda m)} \tag{2.21}$$

and

$$D_i^\lambda(E,m) \equiv \frac{n_i\left(2e^{\lambda E}\left(\cosh(\lambda E) - \cosh(\lambda m)\right)\right)^{\frac{1}{2}}}{\sinh(\lambda m)} \tag{2.22}$$

the DSR-deformed Dirac equation can be rewritten as

$$\left(\gamma^\mu D_\mu^\lambda(E,m) - I\right) u(\overrightarrow{p}) = 0 \tag{2.23}$$

where again the γ^μ are the familiar "γ matrices".

The nature of this DSR deformation of the Dirac equation becomes more transparent by rewriting (2.22) taking into account the DSR dispersion relation (2.16):

$$D_i^\lambda(\overrightarrow{p},m) = \frac{e^{\lambda E}}{\lambda^{-1}\sinh(\lambda m)} p_i \ . \tag{2.24}$$

In particular, as one should expect, in the limit $\lambda \to 0$ one finds

$$D_i^\lambda(E,m) \to \frac{E}{m} \ , \tag{2.25}$$

[7] We obtain here the DSR-deformed Dirac equation for the four-component spinor $u(\overrightarrow{p})$. We take some liberty in denoting with $u_R(\overrightarrow{p})$ two of the components of $u(\overrightarrow{p})$ and with $u_L(\overrightarrow{p})$ the remaining two components. In fact, especially if, as suggested in Refs. [13, 14], the DSR deformation should rely on a noncommutative spacetime sector the action of "space-Parity" transformations on energy-momentum space and on our spinors might involve some subtle issues [24]. The labels "R" and "L" on our DSR spinors are therefore at present only used for bookkeeping (they are reminders of the role that these components of the DSR Dirac spinor play in the $\lambda \to 0$ limit).

$$D_i^\lambda(\overrightarrow{p}, m) \to \frac{p_i}{m} \ , \tag{2.26}$$

and the familiar special-relativistic Dirac equation is indeed obtained in the $\lambda \to 0$ limit.

It is also easy to verify that the determinant of the matrix $(\gamma^\mu D_\mu^\lambda(E, m) - I)$ vanishes, as necessary. In fact,

$$\det\left(\gamma^\mu D_\mu^\lambda(E, m) - I\right) = \left(\sinh^2(\lambda m) - \left(e^{\lambda E} - \cosh(\lambda m)\right)^2 + \frac{e^{2\lambda E}}{\lambda^{-2}}\overrightarrow{p}^2\right)^2 =$$

$$= \left(\frac{e^{\lambda E}}{\lambda^{-2}}\left(-2\lambda^{-2}\cosh(\lambda E) + \overrightarrow{p}^2 e^{\lambda E} + 2\lambda^{-2}\cosh(\lambda m)\right)\right)^2 = 0 \ , \tag{2.27}$$

where the last equality on the right-hand side follows from the DSR dispersion relation.

Our DSR-deformed Dirac equation of course leads to the DSR-deformed Weyl equation in the case of massless particles. In terms of the "DSR helicity" of our massless spinors one finds:

$$(\overrightarrow{\sigma} \cdot \hat{p})\, u_{R,L}(\overrightarrow{p}) = \pm u_{R,L}(\overrightarrow{p}) \ , \tag{2.28}$$

where $\hat{p} \equiv \overrightarrow{p}/|\overrightarrow{p}|$. The operator $\overrightarrow{\sigma} \cdot \hat{p}$ still has eigenvalues ± 1 as in the ordinary special-relativistic case.

In summary the DSR description of spinors appears to require only a relatively mild deformation of the familiar special-relativistic formulas. Our DSR-deformed Dirac equation differs from the ordinary Dirac equation only through the dependence on energy-momentum of the coefficients of the γ^μ matrices. The difference between the DSR coefficients, $[D_0^\lambda(E, m), D_i^\lambda(\overrightarrow{p}, m)]$, and the ordinary ones, $[E/m, \overrightarrow{p}/m]$, is very small ($\lambda$-suppressed, Planck-length suppressed) for low-energy particles, and in particular the difference vanishes in the zero-momentum limit. Still it is plausible that the new effects might be investigated experimentally in spite of their smallness, following the strategy outlined in the recent literature [5, 6, 7, 26, 27] on the search of Planck-length suppressed effects.

3 DSR Dirac Equation in κ-Minkowski spacetime

3.1 DSR, κ-Minkowski and κ-Poincaré Hopf algebras

In the previous section our analysis has been performed entirely in the energy-momentum space. It turn out to be possible to specify completely the DSR-deformed Dirac equation in energy-momentum space using only the chosen DSR laws of transformation of energy-momentum, without any assumption about the nature and structure of spacetime. We now want to look for a spacetime realization of our DSR-deformed Dirac equation. We will verify, using a standard procedure [28] for the construction of a Dirac equation, that our DSR-deformed Dirac equation is appropriate for the description of spin-$\frac{1}{2}$ particles in the κ-Minkowski noncommutative spacetime (reviewed in the next subsection).

We consider κ-Minkowski spacetime because of various indications [13, 14] that the symmetries of this spacetime may be compatible with the DSR requirements. These indications are so far incomplete, especially for the analysis of multiparticle systems in κ-Minkowski, but at least in the one-particle sector the presence of DSR-deformed Lorentz symmetry in κ-Minkowski is rather well established. In work that preceded the proposal of DSR theories of Refs. [13], it had already been argued that the so-called κ-Poincaré Hopf algebra [23] could describe deformed infinitesimal symmetry transformations for κ-Minkowski, but it was believed that these algebra structures would not be compatible with a genuine symmetry group of finite transformations (on the basis of a few attempts [29] it was inferred that the emerging structure would be the one of a "quasigroup" [30], with dubious applicability in physics). However, already in Refs. [13], it was observed that at least one formulation of the κ-Poincaré Hopf algebra did allow for the emergence of a group of finite transformations of the energy-momentum of a particle (while the same is not true for other formulations, the so-called "bases", of the Hopf algebra). That result amounts to proving that the mathematics of the κ-Poincaré Hopf algebra (and therefore possibly κ-Minkowski) can meaningfully describe the one-particle sector of a physical theory in a way that involves DSR-deformed Lorentz symmetry. But it is still unclear whether there is a formulation of the κ-Poincaré Hopf algebra that can be used to construct a theory which genuinely enjoys deformed Lorentz symmetry throughout, including multiparticle systems.

The recipe adopted in the κ-Poincaré literature for the description of two-particle systems relies on the law of composition of momenta obtained through a "coproduct sum" $(p\dot{+}p')^\mu$ (where $(p\dot{+}p')^\mu = \delta^{\mu,0}(p^0+p'^0)+\delta^{\mu,j}(p^j+e^{p^0/\kappa}p'^j)$), and the action of boosts on the composed momenta which is induced by the action on each of the momenta entering the composition. This has been adopted in the κ-Poincaré literature even very recently [21], not withstanding the new DSR-deformed Lorentz symmetry perspective proposed in Ref. [13]. From a DSR perspective this κ-Poincaré description of two-particle systems is not acceptable: for a particle-producing collision process $a+b \rightarrow c+d$ laws [21] of the type $(p_a\dot{+}p_b)^\mu = (p_c\dot{+}p_d)^\mu$, are inconsistent [12] with the laws of transformation for the momenta of the four particles. In fact, one finds that the condition $(p_a\dot{+}p_b)^\mu = (p_c\dot{+}p_d)^\mu$ can be imposed in a given inertial frame but it will then be violated in other inertial frames (*i.e.* $(p_a\dot{+}p_b)^\mu - (p_c\dot{+}p_d)^\mu = 0 \rightarrow (p'_a\dot{+}p'_b)^\mu - (p'_c\dot{+}p'_d)^\mu \neq 0$).

Therefore the possibility that there would be a formulation of κ-Poincaré that is fully compatible with the DSR requirements remains an open problem. And, correspondingly, it remains to be established whether one can formulate a physical theory in κ-Minkowski spacetime that is acceptable from a DSR perspective. However, as mentioned, all the difficulties appear to emerge only outside the one-particle sector (for multiparticle processes), and it is therefore not surprising that, as we will show, the (single-particle) Dirac equation in κ-Minkowski spacetime turns out to be consistent with the DSR-deformed Dirac equation we obtained in the previous section.

Interestingly our analysis suggests that the presence of DSR symmetries in the one-particle sector of κ-Minkowski is not automatic: it requires a specific choice of the (noncommutative) differential calculus in κ-Minkowski. It is sometimes stated in very general terms that the one-particle sector of physical theories in κ-Minkowski should enjoy DSR symmetries. This probably comes from the analysis of the Klein-Gordon equation in κ-Minkowski which indeed inevitably takes a DSR-compatible form. But in our analysis of the Dirac equation in κ-Minkowski we find that there is a crucial choice between different formulations of the differential calculus in κ-Minkowski, and that the DSR-deformed Dirac equation is only obtained upon making an appropriate choice of differential calculus.

After a brief review of some important features of κ-Minkowski spacetime, given in the next subsection, in Subsection 3.3 we show that with a given choice of differential calculus and a given *ansatz* for the form of the Dirac equation one indeed obtains the DSR-deformed Dirac equation. Then in Section 4 we show that the chosen differential calculus inevitably leads to the DSR-deformed Dirac equation (independently of any *ansatz*). And in Section 5 we show that an alternative choice of differential calculus does not give us a DSR-deformed Dirac equation.

3.2 κ-Minkowski spacetime

κ-Minkowski [22, 23] is a Lie-algebra noncommutative spacetime [31] with coordinates satisfying the commutation relations

$$[\mathbf{x}_0, \mathbf{x}_j] = i\lambda \mathbf{x}_j \ , \quad [\mathbf{x}_j, \mathbf{x}_k] = 0 \tag{3.1}$$

where $j,k=1,2,3$. The noncommutativity parameter λ has dimensions of a length, in natural units $\hbar=c=1$. (In most of the κ-Minkowski literature one finds the equivalent parameter κ, which is $\kappa=1/\lambda$, but our formulas turn out to be more compact when expressed in terms of λ.) Of course, conventional commuting coordinates are recovered in the limit $\lambda\to 0$.

The elements of κ-Minkowski (the "functions of κ-Minkowski coordinates") are sums and products of the noncommuting coordinates $\mathbf{x}_\mu$. It is possible to establish a correspondence between elements of κ-Minkowski and analytic functions of four commuting variables x_μ. Such a correspondence is called "Weyl map", and is not unique [32, 33] since it depends on an ordering choice. For example the very simple commutative function $x_2 t$ can be mapped into different functions in κ-Minkowski: two possibilities are $\mathbf{x}_2\mathbf{x}_0$ and $\mathbf{x}_0\mathbf{x}_2=\mathbf{x}_2\mathbf{x}_0+i\lambda\mathbf{x}_2$ (of course the ordering issue disappears in the $\lambda\to 0$ limit, where $\mathbf{x}_2\mathbf{x}_0=\mathbf{x}_2\mathbf{x}_0$).

Many properties of a noncommutative spacetime are very naturally described in terms of a Weyl map [31]. While, as mentioned, different Weyl maps can be considered, in this paper for definiteness we will work with the "time-to-the-right-ordered map". It is sufficient to specify this Weyl map Ω on the complex exponential functions and extend it to the generic function $\phi(x)$, whose Fourier transform is $\tilde{\phi}(k)=\frac{1}{(2\pi)^4}\int d^4x\,\phi(x)e^{-ikx}$, by linearity

$$\Phi(\mathbf{x})\equiv\Omega(\phi(x))=\int d^4k\,\tilde{\phi}(k)\,\Omega(e^{ikx})=\int d^4k\,\tilde{\phi}(k)\,e^{-i\vec{k}\cdot\vec{\mathbf{x}}}e^{ik_0\mathbf{x}_0} \tag{3.2}$$

(Here and in the following we adopt conventions such that $kx\equiv k_\mu x^\mu\equiv k_0x^0-\vec{k}\cdot\vec{\mathbf{x}}$.)

On the basis of the analysis reported in Ref. [33] we expect that, consistently with this choice of Weyl map, translations, P_μ, rotation, M_j, and boosts $\mathcal{N}_j$ should be described as follows

$$\begin{aligned}
&P_\mu\Phi(\mathbf{x})=\Omega[-i\partial_\mu\phi(x)]\\
&M_j\Phi(\mathbf{x})=\Omega[i\epsilon_{jkl}x_k\partial_l\phi(x)]\\
&\mathcal{N}_j\Phi(\mathbf{x})=\Omega(-[ix_0\partial_j+x_j(\frac{1-e^{2i\lambda\partial_0}}{2\lambda}-\frac{\lambda}{2}\nabla^2)+\lambda x_l\partial_l\partial_j]\phi(x))
\end{aligned} \tag{3.3}$$

These generators satisfy the requirements for the Majid-Ruegg bicrossproduct basis of the κ-Poincaré Hopf algebra [22, 23], with the following commutation relations

$$\begin{aligned}
[P_\mu,P_\nu] &= 0\\
[M_j,M_k] &= i\varepsilon_{jkl}M_l,\quad [\mathcal{N}_j,M_k]=i\varepsilon_{jkl}\mathcal{N}_l,\quad [\mathcal{N}_j,\mathcal{N}_k]=-i\varepsilon_{jkl}M_l\\
[M_j,P_0] &= 0,\quad [M_j,P_k]=i\epsilon_{jkl}P_l\\
[\mathcal{N}_j,P_0] &= iP_j\\
[\mathcal{N}_j,P_k] &= i\left[\left(\frac{1-e^{-2\lambda P_0}}{2\lambda}+\frac{\lambda}{2}\vec{P}^2\right)\delta_{jk}-\lambda P_jP_k\right]
\end{aligned}$$

and the following co-algebra relations

$$\begin{aligned}
\Delta(P_0) &= P_0\otimes 1+1\otimes P_0\quad \Delta(P_j)=P_j\otimes 1+e^{-\lambda P_0}\otimes P_j\\
\Delta(M_j) &= M_j\otimes 1+1\otimes M_j\\
\Delta(\mathcal{N}_j) &= \mathcal{N}_j\otimes 1+e^{-\lambda P_0}\otimes\mathcal{N}_j-\lambda\epsilon_{jkl}P_k\otimes M_l\ .
\end{aligned} \tag{3.4}$$

The "mass-squared" Casimir operator in this Majid-Ruegg bicrossproduct basis takes the form

$$C_\lambda(P) = \cosh(\lambda P_0) - \frac{\lambda}{2} e^{\lambda P_0} \vec{P}^2 \tag{3.5}$$

and from this one easily obtains (see, *e.g.*, Ref. [33]) that the Klein-Gordon equation should be written as

$$\left(\Box_\lambda + M_{KG}^2\right) \Phi(\mathbf{x}) = 0 \tag{3.6}$$

in terms of the differential operator $\Box_\lambda$, which is a deformation of the familiar D'Alembert operator

$$\Box_\lambda = -\frac{2}{\lambda^2}\left[\cosh(\lambda P_0) - 1\right] + e^{\lambda P_0} \vec{P}^2 \tag{3.7}$$

and of the "Klein-Gordon mass parameter" M_{KG}.

Making use of the (3.2) we can write the Klein-Gordon equation in energy-momentum space

$$\left(2\lambda^{-2}\left[\cosh(\lambda k_0) - 1\right] - e^{\lambda k_0} \vec{k}^2 - M_{KG}^2\right) \tilde{\phi}(k) = 0 \tag{3.8}$$

from which it is easy to derive the dispersion relation for a free scalar particle in κ-Minkowski

$$\cosh(\lambda E) - \frac{\lambda^2}{2} e^{\lambda E} \vec{p}^2 = 1 + \frac{\lambda^2}{2} M_{KG}^2 \ , \tag{3.9}$$

where E and $\vec{p}$ denote the energy and the space-momentum of the particle ($E = E(\vec{p})$ from (3.9) for particles "on shell"). Clearly the mass parameter M_{KG} is not the rest energy, but M_{KG} and the rest energy (rest mass) m are connected by the relation $M_{KG} = \sqrt{2(\cosh(\lambda m) - 1)}/\lambda$.

The relation (3.9) is the same dispersion relation (4.4) which we considered in Section 2 on the basis of the DSR requirements.

3.3 A DSR Dirac equation in κ-Minkowski

We are basically ready to investigate whether the energy-momentum-space deformed Dirac equation obtained in Section 2 can be seen as the energy-momentum-space counter-part of a natural Dirac equation in κ-Minkowski noncommutative spacetime. We intend to follow closely the line of analysis originally adopted by Dirac in the conventional case of commutative Minkowski spacetime. This Dirac procedure consists in writing a partial differential equation linear in the derivatives with arbitrary coefficients:

$$(i\gamma^\mu \partial_\mu + mI)\psi(x) = 0 \tag{3.10}$$

where m is the particle mass, ψ is an n-plet of fields, and the γ_μ are $n \times n$ hermitian matrices to be determined by imposing the dispersion relation (that works as physical condition). It turns out that $n = 4$ or greater is required for consistency.

Also in κ-Minkowski, as in the commutative case, we introduce Lorentz spinor wave functions $\Psi(\mathbf{x})$, whose components are of the form:

$$\Psi_r(\mathbf{x}) = \int d^4k\, \tilde{\psi}_r(k) e^{-ik\mathbf{x}} e^{ik_0 \mathbf{x}_0} \tag{3.11}$$

where r is a spin index. $\Psi(\mathbf{x})$ will represent a physical state in κ-Minkowski when it satisfies a wave equation with a (deformed) $\not{D}_\lambda$ operator $[\not{D}_\lambda + M_D I]\Psi(\mathbf{x}) = 0$.

In seeking a suitable form for $\not{D}_\lambda$ it is important to notice that, while in the commutative case there is only one natural differential calculus (which involves the ordinary derivatives and gives rise to the Dirac equation (3.10), in the case of κ-Minkowski the introduction of a differential calculus is a more complex problem [34, 35]. As announced, in this section we focus on one possible choice of differential calculus, the "five-dimensional differential calculus" of Ref. [34]. In this 5D differential calculus the exterior derivative operator d of a generic κ-Minkowski element $F(\mathbf{x}) = \Omega(f(x))$ can be written in terms of vector fields $\mathcal{D}_a(P)$ as follows:

$$\begin{aligned} &dF(\mathbf{x}) = dx^a \mathcal{D}_a(P) F(\mathbf{x}), \quad a = 0, \dots, 4 \\ &\mathcal{D}_0(P) = \frac{i}{\lambda}[\sinh(\lambda P_0) + \frac{\lambda^2}{2} e^{\lambda P_0} P^2] \ , \\ &\mathcal{D}_j(P) = i P_j e^{\lambda P_0} \ , j = 1, 2, 3 \ , \\ &\mathcal{D}_4(P) = -\frac{1}{\lambda}(1 - \cosh(\lambda P_0) + \frac{\lambda^2}{2} P^2 e^{\lambda P_0}) \ . \end{aligned} \tag{3.12}$$

Of course, these κ-Minkowski derivative vector fields $\mathcal{D}_a$ reproduce their commutative-Minkowski counterparts in the $\lambda \to 0$ limit:

$$\lim_{\lambda \to 0} \mathcal{D}_\mu(P) = i P_\mu = \partial_\mu, \quad \lim_{\lambda \to 0} \mathcal{D}_4(P) = 0.$$

The introduction of this 5D calculus in our 4D spacetime may at first appear to be surprising, but it can be naturally introduced on the basis of the fact that the κ-Poincaré/κ-Minkowski framework can be obtained (and was originally obtained [36] by contraction of a quantum-deformed anti-de Sitter algebra). The fifth one-form generator is here denoted by "dx_4", but this is of course only a formal notation since there is no fifth κ-Minkowski coordinate $\mathbf{x}_4$. And the peculiar role of dx_4 in this differential calculus is also codified in the fact that, if one examines the deformed derivatives (3.12) on-shell, *i.e.* with $E = E(p)$ satisfying the dispersion relation (3.9)

$$\cosh(\lambda E) - \frac{\lambda^2}{2} e^{\lambda E} \vec{p}^2 = \cosh(\lambda m) \tag{3.13}$$

the last component $\mathcal{D}_4(P)$ can be written as a pure function of the mass:

$$\mathcal{D}_a(E, p) = \left(\frac{i}{\lambda}[e^{\lambda E} - \cosh \lambda m], i p_j e^{\lambda E}, \lambda^{-1}(\cosh(\lambda m) - 1) \right) \tag{3.14}$$

The deformed Klein-Gordon equation (3.6) takes a very simple form in terms of the five-dimensional differential calculus:

$$[\mathcal{D}^a \mathcal{D}_a + M_{KG}^2] \Phi(\mathbf{x}) = 0 \tag{3.15}$$

where $\mathcal{D}^a \mathcal{D}_a \equiv \mathcal{D}_0^2 - \sum_{j=1}^3 \mathcal{D}_j^2 + \mathcal{D}_4^2$. In fact, $\mathcal{D}^a \mathcal{D}_a = \Box_\lambda$, and therefore Eq. (3.15) is equivalent to the Klein-Gordon equation (3.6).

Let us now consider a Dirac equation in κ-Minkowski spacetime of the general form

$$(\not{D}_\lambda + M_D I) \Psi(\mathbf{x}) = 0 \tag{3.16}$$

where I is the identity matrix, M_D is a mass parameter analogous to M_{KG} (like M_{KG} it will be related to the rest energy m) and $\not{D}_\lambda$ is the deformed Dirac operator

The Dirac operator must satisfy three key requirements:

i) *Commutative limit*: in the limit $\lambda \to 0$ one must find that $\not{D}_\lambda$ reduces to the classical operator $\not{D} = i\partial_\mu\gamma^\mu$ in terms of usual Dirac γ^μ ($\mu = 0, ..., 3$) matrices.

ii)*Physical condition*: $\not{D}_\lambda$ must be such that the components of Ψ must satisfy the κ-deformed KG equation (4.15), *i.e.* "plane waves on shell", with momenta $(E, \vec{p})$ satisfying the dispersion relation (3.9), must be solutions of (3.16). The most general form of the "plane wave on shell" is:

$$u(\vec{p})e^{-ip_j\mathbf{x}_j}e^{iE\mathbf{x}_0} + v(\vec{p})e^{-iS(p_j)}e^{iS(E)\mathbf{x}_0} \tag{3.17}$$

where $S(E, \vec{p}) = (E, -e^{\lambda E}\vec{p})$ is the "antipode map", which generalizes the inversion operation in κ-Minkowski. In fact both $e^{-ip_j\mathbf{x}_j}e^{iE\mathbf{x}_0}$ and $e^{-iS(p_j)}e^{iS(E)\mathbf{x}_0}$ are solutions of the κ-deformed KG equation (if $E = E(p)$ satisfies the dispersion relation (3.9)). Thus, the following equations must be satisfied:

$$\begin{aligned}(\not{D}_\lambda - M_D I)u_r(\vec{p})e^{-ip_j\mathbf{x}_j}e^{iE(p)\mathbf{x}_0} = 0\\ (\not{D}_\lambda - M_D I)v_r(\vec{p})e^{-iS(p_j)\mathbf{x}_j}e^{-iE(p)\mathbf{x}_0} = 0.\end{aligned}$$

We focus our attention on the equation for $u(\vec{p})$. Then the equation for $v(\vec{p})$ will be straightforwardly found substituting $(E, \vec{p})$ with $(S(E), S(\vec{p}))$.

iii) *Covariance property*: $\not{D}_\lambda$ must be invariant, $[T, \not{D}_\lambda] = 0$, under the action of all generators T in the symmetry algebra.

We start by noticing that from

$$\left[\not{D}_\lambda(E(p), p) + M_D I\right] u(\vec{p}) = 0 \ , \tag{3.18}$$

one obtains, acting with $\not{D}_\lambda - M_D I$,

$$[\not{D}_\lambda - M_D I][\not{D}_\lambda + M_D I]u(\vec{p}) = 0 \ \Rightarrow \ (\not{D}_\lambda^2 - M_D^2)_{on-shell} = 0 \tag{3.19}$$

Observing that the components $\mathcal{D}_a$ introduced above transform under the κ-Poincaré action exactly as the standard derivatives transform under the standard Poincaré action, and in particular

$$[N_j, \mathcal{D}_0] = i\mathcal{D}_j \quad [N_j, \mathcal{D}_k] = i\mathcal{D}_0 \quad [N_j, \mathcal{D}_4] = 0 \ , \tag{3.20}$$

it appears natural to make the following *ansatz* for the Dirac operator $\not{D}_\lambda$:

$$\not{D}_\lambda(P) = i\gamma^\mu\mathcal{D}_\mu(P) \tag{3.21}$$

where γ^μ ($\mu = 0, \ldots, 3$) denotes again the usual (undeformed) Dirac matrices. Essentially in (3.21) on obtains the deformed Dirac operator using only the first four (more familiar) components of the vectorial field $\mathcal{D}_a$.

Our *ansatz* for the Dirac operator turns out to lead to a satisfactory Dirac equation; in fact, the requirement i) is self-evidently satisfied, the requirement ii) is satisfied with the condition that the parameter M_D and the rest energy m are related by $M_D = \sinh(\lambda m)/\lambda$; in fact:

$$\begin{aligned}\left[\not{\mathcal{D}}^2\right]_{onshell} &= [i\gamma^\mu \mathcal{D}_\mu(E,p)]^2 = -\gamma^\mu\gamma^\nu \mathcal{D}_\mu \mathcal{D}_\nu = -\frac{1}{2}\gamma^\mu\gamma^\nu(\mathcal{D}_\mu\mathcal{D}_\nu + \mathcal{D}_\mu\mathcal{D}_\nu) \\ &= -\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\mathcal{D}_\mu\mathcal{D}_\nu = -\mathcal{D}_\mu\mathcal{D}^\mu = \sum_j \mathcal{D}_j^2(E,p) - \mathcal{D}_0^2(E,p)\end{aligned}$$

and using (3.13),(3.14) one finds that

$$\begin{aligned}\left[\not{\mathcal{D}}^2\right]_{onshell} &= -e^{2\lambda E}\vec{p}^2 + \lambda^{-2}[e^{2\lambda E} - 2e^{\lambda E}\cosh(\lambda m) + \cosh(\lambda m)^2] \\ &= 2\lambda^{-2}e^{\lambda E}[\cosh(\lambda m) - \cosh(\lambda E)] + \lambda^{-2}[e^{2\lambda E} - 2e^{\lambda E}\cosh(\lambda m) + \cosh(\lambda m)^2] \\ &= \lambda^{-2}\sinh(\lambda m)^2 = M_D^2.\end{aligned}$$

Finally, the invariance requirement iii) is clearly satisfied in light of the covariant transformation properties (3.20) of the $\mathcal{D}_a$.

We are therefore led to the following spacetime (κ-Minkowski spacetime) formulation of the DSR-deformed Dirac equation

$$\left[i\mathcal{D}_\mu(P)\gamma^\mu + \frac{\sinh(\lambda m)}{\lambda}I\right]\Psi(\mathbf{x}) = 0 \tag{3.22}$$

(we remind the reader that the P_μ are defined as the differential operators of (3.3)).

The corresponding energy-momentum-space formulation of this DSR-deformed Dirac equation can be obtained through the Fourier transform (3.11), and takes the form

$$\left[i\mathcal{D}_\mu(k)\gamma^\mu + \frac{\sinh(\lambda m)}{\lambda}I\right]\tilde{\psi}(k) = 0 \tag{3.23}$$

Using the decomposition (3.17) of the Dirac spinor on shell, the equation (3.23) reproduces exactly the DSR-deformed equation (2.23):

$$\left[\frac{1}{\lambda}[e^{\lambda E} - \cosh\lambda m]\gamma^0 + e^{\lambda E}p_j\gamma^j - \frac{\sinh(\lambda m)}{\lambda}I\right]u(\vec{p}) = 0 \tag{3.24}$$

It is already noteworthy that our energy-momentum-space DSR-deformed Dirac equation (2.23), first derived in Section 2 from general DSR symmetry principles (without advocating in any way properties of the κ-Minkowski spacetime) emerges in κ-Minkowski spacetime, upon a suitable choice of differential calculus and within a natural *ansatz* for the formulation of the Dirac equation in terms of the elements of the chosen differential calculus. We are however hoping to establish an even more robust connection with κ-Minkowski spacetime, and therefore in the following sections we examine whether our result depends crucially on the *ansatz* (3.21) and/or on the choice of differential calculus.

4 Uniqueness of the Dirac equation for given choice of differential calculus

4.1 Constructing the deformed Dirac equation

The next step of our analysis relies once again on the differential calculus (3.12), but explores the structure of the Dirac equation in κ-Minkowski in otherwise completely general terms, without resorting to the *ansatz* (3.21). If the differential calculus is given by (3.12) the most general parametrization of the Dirac equation in κ-Minkowski is:

$$\left(i\mathcal{D}_0 + i\alpha^j \mathcal{D}_j + \alpha^4 \mathcal{D}_4 + \beta M_D\right) \Psi(\mathbf{x}) = 0 \tag{4.1}$$

where $\Psi(\mathbf{x})$ is again an n-plet (vector) of wave functions of noncommutative coordinates $\mathbf{x}$, and α^i $(i = 1, 2, 3)$, α^4, β are five hermitian matrices. M_D is once again a mass parameter which we expect to be some simple function of the physical mass m.

The *ansatz* considered in the previous section corresponds to $\alpha^i = \gamma^0\gamma^i$, $\alpha^4 = 0$, $\beta = \gamma^0$. In this section we look for all possible choices of α^i, α^4, β such that the components of Ψ satisfy the κ-deformed KG equation, *i.e.* such that a plane wave which obeys the dispersion relation (2.16)

$$\cosh(\lambda E) - \frac{\lambda^2}{2} e^{\lambda E} \vec{p}^2 = \cosh(\lambda m) \tag{4.2}$$

is a solution.

One can derive the Dirac equation in the familiar commutative Minkowski spacetime by following the same strategy (indeed Dirac obtained his equation in this way, by introducing some matrix coefficients of the elements of the differential calculus and imposing that these matrices be consistent with the KG equation). While in commutative Minkowski spacetime the consistency with the KG equation is sufficient to fully determine the Dirac equation, in κ-Minkowski this procedure only allows to determine the (deformed) Dirac equation "on shell" (one of the matrices that gives the general parametrization of the deformed Dirac equation remains undetermined, but it does not affect the form of the equation once the on-shell dependence of energy on momentum is imposed). In order to fully determine the equation it turns out to be necessary to impose a suitable condition of covariance under the action of the symmetry group (the covariance under symmetry-group transformations is instead automatically satisfied in the commutative-Minkowski case, once the consistency with the KG equation is imposed).

We start by making use of the requirement of consistency with the commutative ($\lambda \to 0$) limit (3.10):

$$\left(i\mathcal{D}_0 + i\alpha^j \mathcal{D}_j + \alpha^4 \mathcal{D}_4 + \beta M_D\right) \Psi(\mathbf{x}) = 0 \to \left(i\partial_0 + i\gamma^0\gamma^j \partial_j + \gamma^0 m\right) \psi(x) = 0 \tag{4.3}$$

where γ^μ are the familiar Dirac matrices satisfying the Clifford-algebra relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. From (4.3) one easily finds that

$$\begin{aligned} \lim_{\lambda=0} \alpha^i &= \gamma^0\gamma^i \\ \lim_{\lambda=0} \beta &= \gamma^0 \\ \lim_{\lambda=0} M_D(m) &= m \end{aligned} \tag{4.4}$$

and that the $\lambda \to 0$ limit of α^4 must be finite (if the $\lambda \to 0$ limit of α^4 is not singular the term in the Dirac equation that comes from the fifth "spurious" element of the differential calculus disappears, as needed, in the $\lambda \to 0$ limit).

Let us also observe that, whereas in commutative Minkowski one can safely assume that the entries of the matrices are just constant numbers (independent of any of the variables that characterize the system), the presence of the scale λ in κ-Minkowski forces us to allow for a possible dependence of the α^μ, β on the mass m ($\alpha^i = \alpha^i(\lambda m)$, $\beta = \beta(\lambda m)$, $\alpha^4 = \alpha^4(\lambda m)$). (We therefore allow for such a dependence, but eventually we find that it is not present.)

Next we observe that, by multiplying Eq. (4.1) by the operator $i\mathcal{D}_0 - (i\alpha^j\mathcal{D}_j + \alpha^4\mathcal{D}_4 + \beta M_D)$

$$\Big(i\mathcal{D}_0 - (i\alpha^j\mathcal{D}_j + \alpha^4\mathcal{D}_4 + \beta M_D)\Big)\Big(i\mathcal{D}_0 + i\alpha^j\mathcal{D}_j + \alpha^4\mathcal{D}_4 + \beta M_D\Big)\Psi(\mathbf{x}) = 0 \tag{4.5}$$

one obtains

$$[\mathcal{D}_0^2 + \Big(i\alpha^j\mathcal{D}_j + \alpha^4\mathcal{D}_4 + \beta M_D\Big)^2]\Psi(\mathbf{x}) = 0\ . \tag{4.6}$$

The requirement ii) of consistency with the deformed KG equation translates into the condition that a choice of Ψ given by on-shell plane waves (3.17)

$$u(\vec{p})e^{-ip_j\mathbf{x}_j}e^{iE\mathbf{x}_0} + v(\vec{p})e^{-iS(p_j)\mathbf{x}_j}e^{iS(E)\mathbf{x}_0}, \tag{4.7}$$

with $E = E(p)$ given by $\cosh(\lambda E) - \frac{\lambda^2}{2}e^{\lambda E}\vec{p}^2 = \cosh(\lambda m))$, should be a solution of our sought deformed Dirac equation. This allows us to obtain from (4.6) the following $n-plet$ of equations for $u(\vec{p})$:

$$\begin{aligned}
[\mathcal{D}_0^2(E,p) &- (\alpha^j)^2(\mathcal{D}_j(E,p))^2 + (\alpha^4)^2\mathcal{D}_4^2 + \beta^2 M_D^2 + \\
&- \sum_{j<l}\{\alpha^j,\alpha^l\}\mathcal{D}_j(E,p)\mathcal{D}_l(E,p) + [i\{\alpha^j,\alpha^4\}\mathcal{D}_4 + iM_D\{\alpha^j,\beta\}]\mathcal{D}_j(E,p) + \\
&+ M_D\{\alpha^4,\beta\}\mathcal{D}_4]u(\vec{p}) = 0
\end{aligned} \tag{4.8}$$

Using the fact that ordinary space-rotation symmetry should still be preserved the equations (4.8) straightforwardly lead to the consistency requirements:

$$\sum_{j<l}\{\alpha^j,\alpha^l\}\mathcal{D}_j(E,p)\mathcal{D}_l(E,p) = 0 \tag{4.9}$$

$$[\{\alpha^j,\alpha^4\}\mathcal{D}_4 + M_D\{\alpha^j,\beta\}]\mathcal{D}_j(E,p) = 0 \tag{4.10}$$

$$\sum_j(\alpha^j)^2(D_j(E,p))^2 \propto \sum_j(D_j(E,p))^2 \tag{4.11}$$

From this we conclude that

$$\{\alpha^i,\alpha^j\} = 0 \quad i \neq j \tag{4.12}$$

$$\{\alpha^i,\alpha^4\}\mathcal{D}_4(k) = -M_D\{\alpha^i,\beta\} \tag{4.13}$$

$$(\alpha^1)^2 = (\alpha^2)^2 = (\alpha^3)^2 \tag{4.14}$$

We can now use these results to write equation (4.8) as follows:

$$[\mathcal{D}_0^2(E,p) - |\vec{\mathcal{D}}(E,p)|^2\mathcal{G} + \mathcal{D}_4^2(\alpha^4)^2 + M_D^2\beta^2 + M_D\{\alpha^4,\beta\}\mathcal{D}_4]u(\vec{p}) = 0 \tag{4.15}$$

where we introduced the notation $\mathcal{G}$ for the common value (see (4.14)) of the $(\alpha^i)^2$ matrices, $\mathcal{G} \equiv (\alpha^1)^2 = (\alpha^2)^2 = (\alpha^3)^2$.

Next we can use the fact that, on the basis of (3.12), we know that the $D_a(k)$ have the following on-shell expressions:

$$\begin{aligned}
\mathcal{D}_0(E,p) &= \frac{i}{\lambda}[e^{\lambda E} - \cosh(\lambda m)] \\
\mathcal{D}_j(E,p) &= ie^{\lambda E}p_j \\
\mathcal{D}_4(E,p) &= \frac{1}{\lambda}[\cosh(\lambda m) - 1]\ .
\end{aligned} \tag{4.16}$$

This allows us to rewrite (4.15) as

$$\begin{aligned}
- &\ \lambda^{-2}[e^{2\lambda E} + \cosh^2(\lambda m) - 2\cosh(\lambda m) e^{\lambda E}] I + \\
+ &\ \mathcal{G} e^{2\lambda E} |\vec{p}|^2 + \\
+ &\ (\alpha^4)^2 \lambda^{-2} (1 - 2\cosh(\lambda m) + \cosh^2(\lambda m)) + M_D^2 \beta^2 + \\
+ &\ \lambda^{-1} M_D \{\alpha^4, \beta\}(\cosh(\lambda m) - 1) = 0 \ ,
\end{aligned} \tag{4.17}$$

which can also be cast in the form

$$\begin{aligned}
&\ -\sin^2(\lambda m) I + \lambda^2 [\mathcal{G} - I] e^{2\lambda E} |\vec{p}|^2 + \\
+ &\ (\alpha^4)^2 (1 - 2\cosh(\lambda m) + \cosh^2(\lambda m)) + \lambda^2 M_D^2 \beta^2 + \\
+ &\ \lambda M_D \{\alpha^4, \beta\}(\cosh(\lambda m) - 1) = 0 \ ,
\end{aligned} \tag{4.18}$$

using again the dispersion relation.

Since Eq.(4.18) must hold for every arbitrary value of $\vec{p}$ we can deduce that

$$\mathcal{G} = I \tag{4.19}$$

and

$$\sinh^2(\lambda m) I - (\alpha^4)^2 (1 - \cosh(\lambda m))^2 - \lambda^2 M_D^2 \beta^2 + M_D \lambda \{\alpha^4, \beta\}(1 - \cosh(\lambda m)) = 0 \tag{4.20}$$

which can be conveniently rewritten as

$$\sinh^2(\lambda m) I = (\alpha^4 (\cosh(\lambda m) - 1) + \lambda M_D \beta)^2 \ . \tag{4.21}$$

At this point we have reduced our search of consistent deformed Dirac equations to the search of matrices $\{\alpha^j, \alpha^4, \beta\}$ such that the following requirements (4.12-4.13-4.21) are satisfied:

$$\begin{aligned}
\{\alpha^j, \alpha^k\} &= 2\delta^{jk} I \\
\{\alpha^j, \alpha^4\}[\cosh(\lambda m) - 1] &= -\lambda M_D \{\alpha^j, \beta\} && (4.22) \\
\sinh^2(\lambda m) I &= (\alpha^4 (\cosh(\lambda m) - 1) + \lambda M_D \beta)^2 && (4.23)
\end{aligned}$$

In deriving from these requirements an explicit result for the matrices $\{\alpha^j, \alpha^4, \beta\}$ it is convenient to first consider the case in which $M_D m \neq 0$, the case of massive particles. It is then convenient to introduce the matrix A

$$A \equiv \frac{(\alpha^4 (\cosh(\lambda m) - 1) + \lambda M_D \beta)}{\sinh(\lambda m)} \tag{4.24}$$

which allows us to cast the deformed Dirac equation in the following form:

$$[i\mathcal{D}_0(k) + i\alpha^j \mathcal{D}_j(k) + \mathcal{D}_4(k) \frac{\sinh(\lambda m)}{\cosh(\lambda m) - 1} A + M_D[-\frac{\lambda \mathcal{D}_4(k)}{\cosh(\lambda m) - 1} + 1]\beta]\tilde{\psi}(k) = 0 \tag{4.25}$$

From (4.22) and (4.23) it follows that $\{A, \alpha^i\} = 0$ and $A^2 = I$.

We are at this point ready to obtain the most general DSR-deformed Dirac equation in κ-Minkowski. In fact, Eq. (4.25) for the (on-shell) spinor $u(\vec{p})$ (3.17) simplifies to

$$[i\mathcal{D}_0(E,p) + i\mathcal{D}_j(E,p)\alpha^j + \frac{\sinh(\lambda m)}{\lambda}A]u(\vec{p}) \tag{4.26}$$

where the matrices α^j, A satisfy the conditions[8]

$$\{\alpha^j, \alpha^k\} = 2\delta^{jk}I, \quad , \{\alpha^j, A\} = 0, \quad A^2 = I \tag{4.27}$$

Introducing $\gamma^0 \equiv A$ and $\gamma^j \equiv A\alpha^j$ one finds from (4.27) that the γ's must satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \mu = 0, 1, 2, 3. \tag{4.28}$$

i.e. they must be the usual (undeformed!) Dirac matrices. And we find (by multiplying (4.26) by A and making use of $\gamma^0 \equiv A$ and $\gamma^j \equiv A\alpha^j$) that in terms of the usual Dirac matrices there is a unique solution to our problem of finding the most general DSR-deformed Dirac equation in κ-Minkowski:

$$[\frac{e^{\lambda E} - \cosh(\lambda m)}{\lambda}\gamma^0 + p_j e^{\lambda E}\gamma^j - \frac{\sinh(\lambda m)}{\lambda}I]u(\vec{p}) = 0 \tag{4.29}$$

It is rather satisfactory that there is a unique deformed Dirac equation in κ-Minkowski, and that it reproduces the deformed Dirac equation which had been already derived, without any *a priori* assumption about the spacetime structure, in Section 2 .

4.2 Invariance under κ-Poincaré action

We established that the Dirac equation in κ-Minkowski must take the form (4.29) "on shell": our analysis so far has allowed us to fully determine the form of the Dirac equation with the exception of the matrix β which is still undetermined, but β does not affect the form of the equation once the on-shell dependence of energy on momentum is imposed. In order to determine the matrix β it is necessary to impose the covariance of our deformed Dirac equation in the κ-Poincaré sense. This will require us to introduce a representation of the κ-Poincaré (Hopf) algebra for spin-1/2 particles.

In preparation for this analysis we first briefly review for the reader the analogous analysis for the classical Poincaré (Lie) algebra. A key point is that for the Lorentz sector the representation can be described as the sum of two parts:

$$M_j^T = M_j + m_j \quad N_j^T = N_j + n_j \tag{4.30}$$

where (M_j, N_j) is a spinless unitary representation of $O(3,1)$ and acts in the "outer" space of the particle (the one the codifies the momenta and orbital momenta of the particle), whereas (m_j, n_j) is a finite-dimensional representation of $O(3,1)$ and acts in the "inner" space (spin indices). A representation of the whole Poincaré group is then obtained by

[8]From these conditions one can also infer that the $n\times n$ matrices we are seeking must have n even and $n \geq 4$ (not smaller than 4×4 matrices). In fact, from the anticommutation relations it follows that $TrA = 0$, $A^2 = 1$, and $detA = \pm 1$ which requires n to be even. The case $n = 2$ is also excluded since there are only 3 independent anticommuting 2×2 matrices (Pauli matrices). We take $n = 4$ just as in the $\lambda \to 0$ (commutative-Minkowski) limit.

introducing four translation generators $P_\mu = -i\partial_\mu$ that act only in the "outer" space of the particle. The spinorial representation of the classical Poincaré algebra is therefore given by

$$P^T_\mu = P_\mu \quad M^T_j = M_j + m_j \quad N^T_j = N_j + n_j \tag{4.31}$$

which of course satisfy the following familiar commutation relations:

$$\begin{aligned}
&[P^T_\mu, P^T_\nu] = 0 \\
&\left[M^T_j, M^T_k\right] = i\epsilon_{jkl}M^T_l \ , \quad [N^T_j, N^T_k] = -i\epsilon_{jkl}M^T_l \ , \quad [M^T_j, N^T_k] = i\epsilon_{jkl}N^T_l \\
&\left[M^T_j, P^T_0\right] = 0 \quad [M^T_j, P^T_k] = i\epsilon_{jkl}P^T_l \\
&\left[N^T_j, P^T_0\right] = iP^T_j \quad [N^T_j, P^T_k] = i\delta_{jk}P^T_0
\end{aligned}$$

The differential form of the spinless realization is given by:

$$P_\mu = -i\partial_\mu \ \ M_j = \epsilon_{jkl}x_kP_l \ \ N_j = x_jP_0 - x_0P_j \tag{4.32}$$

and the finite dimensional realization can be expressed in terms of the familiar γ matrices

$$m^j = \frac{i}{4}\epsilon^{jkl}\gamma^k\gamma^l \ , \quad n^j = \frac{i}{2}\gamma^j\gamma^0 \tag{4.33}$$

The action of the global generators of Poincaré over a Dirac spinor is:

$$\begin{aligned}
M^T_j \qquad \psi_r(x) &= \int d^4k \ [(M_je^{ikx})\tilde{\psi}_r(k) + e^{ikx}(m_j\tilde{\psi}_r(k))] \\
N^T_j \qquad \psi_r(x) &= \int d^4k \ [(N_je^{ikx})\tilde{\psi}_r(k) + e^{ikx}(n_j\tilde{\psi}_r(k))] \\
P^T_\mu \qquad \psi_r(x) &= \int d^4k \ [(P_\mu e^{ikx})\tilde{\psi}_r(k)]
\end{aligned}$$

and the Dirac operator is of course an invariant:

$$[P^T_\mu, \not{D}] = [M^T_j, \not{D}] = [N^T_j, \not{D}] = 0 \tag{4.34}$$

We intend to obtain analogous results for spinors and the Dirac operator in κ-Minkowski. Our deformed Dirac operator must be invariant,

$$[\mathcal{P}^T_\mu, \not{D}_\lambda] = [\mathcal{M}^T_j, \not{D}_\lambda] = [\mathcal{N}^T_j, \not{D}_\lambda] = 0 \tag{4.35}$$

under the action of κ-Poincaré generators $\mathcal{P}^T, \mathcal{M}^T, \mathcal{N}^T$, which satisfy the following commutation relations:

$$\begin{aligned}
\left[\mathcal{P}^T_\mu, \mathcal{P}^T_\nu\right] &= 0 \\
\left[\mathcal{N}^T_j, \mathcal{N}^T_k\right] &= -i\epsilon_{jkl}\mathcal{M}^T_l \quad \left[\mathcal{N}^T_j, \mathcal{M}^T_k\right] = i\epsilon_{jkl}\mathcal{N}^T_l \\
\left[\mathcal{M}^T_j, \mathcal{M}^T_k\right] &= i\epsilon_{jkl}\mathcal{M}^T_l \\
\left[\mathcal{M}^T_j, \mathcal{P}^T_0\right] &= 0 \quad \left[\mathcal{M}^T_j, \mathcal{P}^T_k\right] = i\epsilon_{jkl}\mathcal{P}^T_l \\
\left[\mathcal{N}^T_j, \mathcal{P}^T_0\right] &= iP^T_j \quad \left[\mathcal{N}^T_j, \mathcal{P}^T_k\right] = i\delta_{jk}[\frac{1-e^{-2\lambda\mathcal{P}^T_0}}{2\lambda} + \frac{\lambda}{2}(\mathcal{P}^T)^2] - i\lambda\mathcal{P}^T_j\mathcal{P}^T_k
\end{aligned}$$

Consistently with the results obtained so far we expect that it will not be necessary to deform the rotations:

$$\mathcal{M}_j^T = M_j^T = M_j + m_j \tag{4.36}$$

and in fact this satisfies all consistency requirements, as one can easily verify.

Boosts in general require a deformation, and we already know from the earlier points of our analysis that the differential form of the spinless realization must be given by the operators $\mathcal{N}_j$ (see (3.3)). We therefore need a suitable finite-dimensional realization $\tilde{n}_j$ of boosts, so that $\mathcal{N}_j^T$ will be given by $\mathcal{N}_j^T = \mathcal{N}_j + \tilde{n}_j$. Using the fact that

$$[\mathcal{N}_j, \mathcal{D}_0] = i\mathcal{D}_j, \quad [\mathcal{N}_j, \mathcal{D}_k] = i\mathcal{D}_0\delta_{jk}, \quad \left[\mathcal{N}_j, \mathcal{D}^4\right] = 0 \tag{4.37}$$

it is easy to verify[9] that with $\tilde{n}_j = n_j$ and $\beta = 0, \gamma^0, \gamma^0\gamma^5$ one has a form of $\mathcal{N}_j^T$

$$\mathcal{N}_j^T = \mathcal{N}_j + n_j \tag{4.38}$$

which satisfies (with $\mathcal{P}^T$ and $\mathcal{M}^T$) the Hopf-algebra requirement, and a Dirac equation which is invariant under these Hopf-algebra transformations. We will therefore describe $\mathcal{N}_j^T$ with (4.38) and we could consider three possibilities for the matrix β ($\beta = 0$, $\beta = \gamma^0$ and $\beta = \gamma^0\gamma^5$).

Actually only $\beta = \gamma^0$ is acceptable; in fact, both for $\beta = 0$ and for $\beta = \gamma^0\gamma^5$ it is easy to check that our deformed (off-shell) Dirac equation would not reproduce the correct $\lambda \to 0$ (classical-spacetime) limit.

We are left with a one-parameter family (M_D is the parameter) of deformed Dirac equations

$$\left[i\gamma^0\mathcal{D}_0(P) + i\gamma^j\mathcal{D}_j(P) + (\mathcal{D}_4(P)\frac{\sinh(\lambda m) - \lambda M_D}{\cosh(\lambda m) - 1} + M_D)I\right]\Psi(\mathbf{x}) = 0 \tag{4.39}$$

As far as we can see the free parameter M_D does not have physical consequences (it clearly does not affect the on-shell equation), and it appears legitimate to view it as a peculiarity associated with the nature of the (five-dimensional) differential calculus in κ-Minkowski. The choice $M_D = m$ is allowed (but not imposed upon us) by the formalism, which actually allows $M_D = mf(\lambda m)$ with any f such that $M_D \to m$ for $\lambda \to 0$ and $M_D \to 0$ for $m \to 0$. In addition to $M_D = m$, other noteworthy possibilities are $M_D = \frac{\sinh(\lambda m)}{\lambda}$, which corresponds to the *ansatz* $\alpha^4 = 0$ of Section 3.3, and $M_D = M_{KG} = \sqrt{2(\cosh(\lambda m) - 1)}/\lambda$. Since M_D does not affect the on-shell equation, our conclusion that the on-shell Dirac equation in κ-Minkowski reproduces the on-shell Dirac equation obtained, using only DSR criteria, in Section 2 is independent of this freedom for the parameter M_D.

4.3 Massless particles

Since the on-shell Dirac equation in κ-Minkowski is just the one already obtained in Section 2, clearly the case of on-shell massless particles ($m \to 0$) in κ-Minkowski is also consistent with the corresponding result already discussed in Section 2.

Concerning a space-time formulation of the deformed Dirac equation for massless particles we simply observe that (4.39) has a well-defined $m \to 0$ limit:

$$\left[i\gamma^0\mathcal{D}_0(P) + i\gamma^j\mathcal{D}_j(P)\right]\Psi(\mathbf{x}) = 0 \tag{4.40}$$

which is therefore well suited for the description of massless spin-1/2 particles.

[9]Details of this lengthy, but rather straightforward, analysis will be soon available on the arXiv [37].

4.4 Aside on a possible ambiguity in the derivation of the Dirac equation in κ-Minkowski

When we introduced

$$dF(\mathbf{x}) = dx^a \mathcal{D}_a(P)F(\mathbf{x}) \quad a = 0, \ldots, 4 \tag{4.41}$$

with

$$\mathcal{D}_a(P) = \left(\frac{i}{\lambda}[\sinh(\lambda P^0) + \frac{\lambda^2}{2}e^{\lambda P^0}P^2], iP_j e^{\lambda P_0}, -\frac{1}{\lambda}(1 - \cosh(\lambda P_0) + \frac{\lambda^2}{2}P^2 e^{\lambda P_0})\right) \tag{4.42}$$

we overlooked an equally valid way of introducing the exterior derivative operator d of a generic κ-Minkowski element $F(\mathbf{x}) = \Omega(f(x))$ in terms of the 5D differential calculus:

$$dF(\mathbf{x}) = \bar{\mathcal{D}}_a F(\mathbf{x}) dx^a \quad a = 0, \ldots, 4 \tag{4.43}$$

with

$$\bar{\mathcal{D}}_a(P) = \left(\frac{i}{\lambda}[\sinh(\lambda P^0) - \frac{\lambda^2}{2}e^{\lambda P^0}P^2], iP_j, \frac{1}{\lambda}(1 - \cosh(\lambda P_0) + \frac{\lambda^2}{2}P^2 e^{\lambda P_0})\right) \tag{4.44}$$

There is however a simple relation between $\mathcal{D}(k)$ and $\bar{D}(k)$ deformed derivatives:

$$\bar{\mathcal{D}}(k) = -\mathcal{D}(S(k)) \quad S(k) = (-k_0, -e^{\lambda k_0}k_j) \tag{4.45}$$

(where S is the antipode map, which generalizes the inversion operation in the way that is appropriate for κ-Minkowski studies [23, 24]), and the careful reader can easily verify[10] that there is no real ambiguity due to the choice of formulation of the exterior derivative operator d. The same physical Dirac theory is obtained in both cases.

5 An obstruction for a Dirac equation in κ-Minkowski based on a four-dimensional differential calculus

In alternative to the five-dimensional calculus which we have so far considered some studies (see, *e.g.*, Ref. [24]) of κ-Minkowski spacetime have used a four-dimensional differential calculus[11]:

$$[\mathbf{x}_\mu, dx_j] = 0, \quad [x_\mu, dx_0] = i\lambda dx_\mu \tag{5.1}$$

One can then express the derivative operator of the element $\Psi(\mathbf{x})$ of κ-Minkowski in the following way:

$$d\Psi = \tilde{\partial}_\mu \Psi(\mathbf{x}) dx^\mu \tag{5.2}$$

[10] Details of this straightforward, but tedious, analysis will be soon available on the arXiv [37].

[11] This four-dimensional differential calculus was originally obtained [38] as a generalization of a two-dimensional differential calculus over two-dimensional κ-Minkowski.

where $\tilde{\partial}_\mu$ are deformed derivatives that act on the time-to-the-right-ordered exponential as follows:

$$\tilde{\partial}_j e^{ik_0\mathbf{x}_0}e^{-ik\mathbf{x}} = \partial_j e^{ik_0\mathbf{x}_0}e^{-ik\mathbf{x}} = ik_j e^{ik_0\mathbf{x}_0}e^{-ik\mathbf{x}} \equiv d_j(k)e^{ik_0\mathbf{x}_0}e^{-ik\mathbf{x}}$$
$$\tilde{\partial}_0 e^{ik_0\mathbf{x}_0}e^{-ik\mathbf{x}} = \frac{i}{\lambda}(1-e^{-\lambda k_0})e^{ik_0\mathbf{x}_0}e^{-ik\mathbf{x}} \equiv d_0(k)e^{ik_0\mathbf{x}_0}e^{-ik\mathbf{x}}$$

It has been previously established [24] that it is possible to formulate the deformed Klein-Gordon equation (3.6) in terms of this four-dimensional calculus:

$$[\tilde{\partial}^\mu\tilde{\partial}_\mu L + M_{KG}^2]\Phi(\mathbf{x}) \tag{5.3}$$

where L is the shift operator $L\Phi(\vec{\mathbf{x}},\mathbf{x}_0) = \Phi(\vec{\mathbf{x}},\mathbf{x}_0 - i\lambda) = e^{i\lambda\partial_0}\Phi(\mathbf{x})$. In fact, one easily finds that $\tilde{\partial}^\mu\tilde{\partial}_\mu L = \mathcal{D}^a\mathcal{D}_a = \Box_\lambda$.

On the basis of the fact that one can write the deformed Klein-Gordon equation (3.6) equivalently in terms of the four-dimensional calculus and the five-dimensional calculus, one could guess that these two examples of differential calculus are equally well suited for implementing the DSR principles in κ-Minkowski. However, we find that this is not the case. The richer structure of the Dirac equation is more sensitive to the details of the differential calculus, and the choice of the five-dimensional differential calculus turns out to be most natural.

In support of this observation let us attempt to proceed with the four-dimensional calculus just as done for the five-dimensional calculus: we write a general parametrization of a deformed Dirac equation,

$$\left(id_0(k) + id_j(k)\rho^j + M_D'\sigma\right)\tilde{\psi}_\kappa(k) = 0 \tag{5.4}$$

where ρ^i, σ are four matrices (constant or at most dependent on λm) to be determined by imposing that an on-shell "plane wave" (3.17) (with $\cosh(\lambda E) - \lambda^2 e^{-\lambda E}k^2 = \cosh(\lambda m)$) is solution of the deformed Dirac equation and by imposing covariance in the κ-Poincaré sense.

The requirement that an on-shell plane wave is a solution leads to

$$\Big[d_0^2(E,p) - (\rho^j)^2(d_j(E,p))^2 + {M_D'}^2\sigma^2 + \\ - \sum_{j<k}\{\rho^j,\rho^k\}d_j(E,p)d_k(E,p) + M_D'\{\rho^j,\sigma\}d_j(E,p)\Big]u(\vec{p}) = 0 \tag{5.5}$$

from which one derives as necessary conditions:

$$\{\rho^j,\rho^k\} = 0,\ \ \{\rho^j,\sigma\} = 0,\ \ (\rho^1)^2 = (\rho^2)^2 = (\rho^3)^2 \tag{5.6}$$

These conditions are necessary but not sufficient, and actually there is no choice of the matrices ρ^j, σ of the type that we are seeking that allows to satisfy (5.5) for all values of the momentum p. To see this let us use (5.6) to rewrite (5.5) as

$$(d_0^2(E,p)I - d_j(E,p)^2\mathcal{Q} + {M_D'}^2\sigma^2)u(\vec{p}) = 0 \tag{5.7}$$

where $\mathcal{Q} \equiv (\rho^1)^2 = (\rho^2)^2 = (\rho^3)^2$. In this Eq. (5.7) we are left with two unknown matrices, $\mathcal{Q}, \sigma$, to be determined, and it is easy to see that there is no choice of $\mathcal{Q}, \sigma$ that allows to satisfy (5.7) for all values of the momentum p. For example, by looking at the form of the equation for $p = 0$ (and $E = m$) one is forced to conclude that

$$\sigma^2 = -d_0^2(m) I / {M'_D}^2(m) \tag{5.8}$$

but then, with this choice of σ^2, Eq. (5.7) turns into an equation for $\mathcal{Q}$ which does not admit any solution of the type we are seeking:

$$(d_0^2(E) - d_0^2(m) - d_j^2 \mathcal{Q}) = -(1 - e^{-\lambda E})^2 + (1 - e^{-\lambda m})^2 + \lambda^2 p^2 \mathcal{Q} = 0 \tag{5.9}$$

i.e. (using again the dispersion relation)

$$\mathcal{Q} = [(1 - e^{-\lambda E})^2 - (1 - e^{-\lambda m})^2][(e^{2\lambda E} + 1) - 2e^{\lambda E} \cosh(\lambda m)]^{-1} \tag{5.10}$$

What we have found is that there is no choice of energy-momentum-independent matrices ρ^j, σ that can be used in order to obtain a consistent Dirac equation for κ-Minkowski. The analogous problem for the 5D calculus did have a perfectly acceptable solution. Here, with the four-dimensional differential calculus, we would be led to consider energy-dependent matrices ρ^i, σ but this is unappealing on physical grounds and in any case the fact that this awkward assumption can be avoided in the five-dimensional calculus appears to be a good basis for preferring the five-dimensional calculus over the four-dimensional calculus.

6 Comparison with previous results on deformed Dirac equations

Our main objective was to formulate a DSR-deformed Dirac equation, and we showed that given the formulation of a DSR proposal in energy-momentum space one can very straightforwardly obtain a corresponding modification of the Dirac equation in energy-momentum space. We then used our result on DSR-deformed Dirac equation as an opportunity to explore the hypothesis that κ-Minkowski spacetime, at least the one-particle sector of theories in κ-Minkowski spacetime (see Subsection 3.1), might provide a spacetime arena for the DSR framework. From this perspective it was a key point for us to investigate under which assumptions about the formulation of spin-1/2 particles in κ-Minkowski one would reobtain the DSR-deformed Dirac equation. We considered alternative formulations of the differential calculus, and for each given choice of differential calculus we were interested in finding the most general compatible formulation of the Dirac equation. This allowed us to establish which choice of differential calculus was required for a DSR-compatible result, and to investigate whether in addition to the choice of differential calculus there were other choices to be made in order to obtain the DSR-deformed Dirac equation. We found that the "5D" differential calculus should be preferred to its "4D" alternative, and we found that, once the choice of the 5D differential calculus is made, one is then led automatically to the DSR-deformed Dirac equation.

While there was no previous study of deformed Dirac equation from the DSR perspective, there has been some previous work of a deformed Dirac equation for κ-Minkowski and on a deformed Dirac equation governed by the structure of the κ-Poincaré Hopf algebra. We find appropriate to comment here on these related studies.

One early investigation of a deformed Dirac equation governed by the structure of κ-Poincaré was reported in Ref. [36], where however the momentum generators were described as differential generators on a commutative spacetime.

The study reported in Ref. [39] adopts a general perspective which is closer to ours, but it focused on the so-called standard basis of κ-Poincaré for which (unlike the case we considered of the Majid-Ruegg bicrossproduct basis) it remains unclear [14, 40] whether a connection with the DSR criteria can be established. For the standard basis there is not even a clear connection with a choice of ordering convention for functions in κ-Minkowski, whereas in our analysis the connection between the Majid-Ruegg bicrossproduct basis and the time-to-the-right ordering convention played a key role. Moreover it was important for our line of analysis to consider that the construction of the deformed Dirac equation in full generality (including, for example, the search of acceptable forms for the matrix α^4 and for the M_D parameter) whereas in Ref. [39] a more limited class of possibilities was considered.

Ref. [41] took as starting point the κ-Minkowski spacetime and considered the construction of a Dirac equation for massless spin-1/2 particles. The perspective is considerably different from ours, since takes inspiration from the Connes criteria for a connection between differential operators and the Dirac operator. Besides the difference in perspective, the fact that Ref. [41] considers only massless spin-1/2 particles reduces its relevance to the problem we considered, where the particle mass (and its relation to various mass parameters) played a key role. The careful reader can easily verify that in the derivation of the Dirac equation following our strategy some of the conditions are multiplied by the mass parameter, and therefore those conditions are formally irrelevant in the massless case, leading to a less constrained framework. We have chosen to introduce massless particles at the end of the analysis, imposing continuity of the $m \rightarrow 0$ limit (so that, by continuity, the relevant conditions encountered for nonvanishing mass are taken into account also in the massless limit).

In Ref. [42] the emphasis is placed on the structure of the κ-Poincaré Hopf algebra and the analysis does not properly consider κ-Minkowski spacetime. In fact, Ref. [42] introduces a five-dimensional metric and a fifth spacetime coordinate (which commutes with the other four κ-Minkowski-type coordinates). A corresponding formulation of the Dirac equation is found by requiring that, in an appropriate sense, the Dirac operator should be a square root of the Klein-Gordon operator.

In Ref. [43] the analysis does concern κ-Minkowski spacetime. but the proposed deformation of the Dirac equation is obtained by enforcing certain criteria based on the search of the unitary representations of the so-called "κ-Poincaré group" with noncommuting group parameters. The physical meaning of noncommuting group parameters remains rather obscure[12]. And it appears difficult to establish whether the criteria proposed in Ref. [43] are as general as ours (we tentatively see one less free matrix introduced in the initial parametrization of the analysis).

7 Summary and outlook

The recent, rather strong, interest in the DSR framework has focused in part on some experimental contexts in which the kinematic properties of fundamental particles are analyzed. Some of these analyses involve spin-1/2 particles, but there was no direct derivation of a DSR-deformed Dirac equation. We have filled this gap in Section 2, where indeed we showed that a DSR-deformed Dirac equation can be derived straightforwardly on the basis of the laws of energy-momentum transformation in the DSR framework.

[12]In conventional theories, with conventional Lie-group symmetries, on obtains a group elements by exponentiation of the generators of the algebra, $e^{a_j T j}$, with commuting parameters a_j. For this situation the physical interpretation is well digested. It remains to be established whether a consistent physical interpretation can be given for the case in which the parameters a_j satisfy nontrivial algebraic relations (noncommutativity), as in the case of the "κ-Poincaré group" construction.

There has also been interest in the possibility that the κ-Minkowski noncommutative spacetime might provide an example of quantum spacetime in which DSR symmetries are present. Some difficulties have been encountered in enforcing DSR symmetries in the two-particle (and multi-particle) sector of theories in κ-Minkowski, but at least in the one-particle sector there is growing evidence of the connection between DSR and κ-Minkowski. We provided in Sections 3, 4 and 5 additional evidence of this connection. Our analysis showed however that it might be improper to state in full generality that κ-Minkowski spacetime is DSR invariant; in fact, in order to satisfy the DSR requirements some structures must be introduced consistently in κ-Minkowski. A noteworthy example of this fact is provided by the choice of differential calculus which we stressed. Previous analyses, focusing on the (deformed) Klein-Gordon equation appeared to suggest that both the 4D differential calculus and the 5D differential calculus should be equally well suited for the formulation of DSR-compatible theories in κ-Minkowski. Our analysis of the (deformed) Dirac equation, with its richer structure, shows that this is not the case: only the 5D differential calculus leads to a Dirac equation which is acceptable from a DSR perspective.

While most of our analysis focused on the DSR perspective and on the possible role of κ-Minkowski in DSR theories, as we stressed in Section 6 some of the results we obtained appear to contribute to the literature on various formulations of the Dirac equation motivated by κ-Poincaré and/or κ-Minkowski.

An interesting issue which could be considered in future studies is the one of the fate of Poincaré symmetries in a κ-Minkowski spacetime equipped with the "4D" differential calculus. This choice of differential calculus does not appear to lead to any pathologies from the perspective of κ-Minkowski mathematics, but it is clearly incompatible with classical Poincaré symmetries and it also fails to produce a DSR-deformed Dirac equation.

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