Inverse Problems for Obstacles in a Waveguide *

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Abstract

In this paper we prove that a particular entry in the scattering matrix, if known for all energies, determines certain rotationally symmetric obstacles in a generalized waveguide. The generalized waveguide X can be of any dimension and we allow either Dirichlet or Neumann boundary conditions on the boundary of the obstacle and on ∂X. In the case of a two-dimensional waveguide, two particular entries of the scattering matrix suffice to determine the obstacle, without the requirement of symmetry.

1 Introduction

The purpose of this paper is to show that knowledge of a single specific entry in the scattering matrix suffices to determine certain analytic obstacles $\mathcal{O} \subset X$, where X is an $n+1$ dimensional generalized waveguide. We remark that an entry in the scattering matrix is a scalar function; see (1.4) and subsequent discussion for the definition. The results of this paper extend the inverse results of [2], both by allowing higher-dimensional waveguides and by considering either Dirichlet or Neumann boundary conditions on $\partial\mathcal{O}$.

A cylindrical waveguide in $\mathbb{R}^{n+1}$ has the form

$$X = \mathbb{R} \times \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$; we assume $\partial\Omega$ is smooth. If $\mathcal{O} \subset\subset \mathbb{R} \times \Omega$ is a smoothly bounded connected compact set, we desire to identify properties of $\mathcal{O}$ from the scattering matrix of $X \setminus \mathcal{O}$. In fact, both X and $X \setminus \mathcal{O}$ are examples of manifolds with infinite cylindrical ends, and it is natural to set up inverse scattering problems for more general such manifolds.

We recall that a smooth Riemannian manifold X with infinite cylindrical ends has the form

$$X = X_c \cup \Big(\bigcup_{i=1}^{i_o} X_i \Big), \tag{1.2}$$

where X_c is compact, each $X_i \approx [0, \infty) \times Y_i$, and each Y_i is a compact, connected manifold, possibly with boundary. In case (1.1), there are two ends, $Y_1 = Y_2 = \Omega$. The Riemannian

* *2000 Math Subject Classification* 35L20, 35P25, 35S30.
Key words: waveguide, cylindrical ends, scattering matrix, inverse problems

metric on X is a product metric on each X_i; $G\big|_{X_i} = dx^2 + g_i$, where g_i is a Riemannian metric tensor on Y_i.

Extending the class of domains (1.1), we say that a domain $X \subset \mathbb{R}^{n+1}$ with smooth boundary is a *generalized waveguide* if X is a manifold with infinite cylindrical ends. Given such a generalized waveguide, suppose one has a smoothly bounded compact subset $\mathcal{O}$ of the interior of X, with the property that $\mathcal{O}$ is known to have nonempty intersection with one of the ends, say with $X_1 = [0,\infty) \times Y_1$. Without loss of generality, we can assume $\mathcal{O}$ has empty intersection with the ends X_i for $i \geq 2$. We want to draw conclusions about $\mathcal{O}$ from information on the scattering matrix of $X \setminus \mathcal{O}$, which is also a manifold with cylindrical ends. In this case the ends are $[a_1,\infty) \times Y_1$ and $[0,\infty) \times Y_i,\ i \geq 2$, given that

$$\overline{\mathcal{O}} \cap [a_1,\infty) \times Y_1 = \emptyset. \tag{1.3}$$

Before describing results we obtain about $\mathcal{O}$, we recall the definition of the scattering matrix associated to $\widetilde{X} = X \setminus \mathcal{O}$. Let $\Delta_{\widetilde{X}}$ denote the (non-negative) Laplace operator on $\widetilde{X}$. We impose either the Dirichlet boundary condition or the Neumann boundary condition on ∂X, and we impose either the Dirichlet boundary condition or the Neumann boundary condition on $\partial\mathcal{O}$. Let Δ_Y denote the Laplace operator on $Y = Y_1 \cup \cdots \cup Y_{i_0}$, with the same type of boundary condition on ∂Y as given on ∂X (Dirichlet or Neumann). Let $\{\sigma_j^2\}$ be the eigenvalues of Δ_Y, repeated according to multiplicity, with $\sigma_1^2 \leq \sigma_2^2 \leq \cdots$. Let $\{\phi_j\}$ be a corresponding real orthonormal set of eigenfunctions of Δ_Y, so $\Delta_Y \phi_j = \sigma_j^2 \phi_j$. Arrange that each ϕ_j is supported on one connected component of Y.

For each $j \in \mathbb{N}$, let $r_j(\lambda) = (\lambda^2 - \sigma_j^2)^{1/2}$, with the square root chosen so $\operatorname{Im} r_j(\lambda) > 0$ when $\operatorname{Im}\lambda > 0$, and extended by continuity to real λ. Hence if $\lambda \in \mathbb{R}$ and $|\lambda| > |\sigma_j|$, then $\operatorname{sgn} r_j(\lambda) = \operatorname{sgn}\lambda$. For each $\ell \in \mathbb{N}$, there is a generalized eigenfunction $\Psi_\ell(\lambda)$ of $\Delta_{\widetilde{X}}$, satisfying

$$\begin{aligned} &(\Delta_{\widetilde{X}} - \lambda^2)\Psi_\ell = 0 \quad \text{on} \quad X \setminus \mathcal{O}, \\ &\qquad \Psi_\ell\big|_{X_e} = e^{-ir_\ell(\lambda)\overline{x}}\phi_\ell + \sum_m S_{m\ell}(\lambda)\Big(\frac{r_\ell(\lambda)}{r_m(\lambda)}\Big)^{1/2} e^{ir_m(\lambda)\overline{x}}\phi_m, \end{aligned} \tag{1.4}$$

where $X_e = [a_1,\infty) \times Y_1 \cup [0,\infty) \times Y_2 \cup \cdots \cup [0,\infty) \times Y_{i_0}$, with coordinates $(\overline{x}, y)$ on each end. Moreover, Ψ_ℓ satisfies the specified boundary conditions on $\partial\mathcal{O}$ and ∂X. The scattering matrix is $(S_{m\ell}(\lambda))_{\sigma_m^2,\sigma_\ell^2 \leq \lambda^2}$. The scattering matrix depends in a mild way on the choice of decomposition of X into X_c and $X_e = \cup X_i$. This can be realized in another way as the choice of coordinate $\overline{x}$ on each end X_i. Thus, it is important to think of the scattering matrix coming with a fixed choice of such coordinate or decomposition. For more details on the scattering matrix and (1.4), cf. [1], [8].

We now give further conditions on an obstacle $\mathcal{O}$ in a generalized waveguide, which will permit us to deduce the nature of $\mathcal{O}$ from knowledge of one or two elements of the scattering matrix, for all $\lambda \in [|\sigma_1|,\infty)$. With coordinates $(\overline{x}, y)$ on $[0,\infty) \times Y_1$, suppose $\overline{x}|_{\partial O \cap X_1}$ is maximal at $p \in \partial\mathcal{O}$. We assume

$$\begin{aligned} &\text{such } p \text{ is unique;} \\ &\mathcal{O} \ \text{ is strongly convex in a neighborhood of } \ p \cap \mathcal{O}; \text{ and} \\ &\partial\mathcal{O} \ \text{ has positive Gauss curvature at } \ p. \end{aligned} \tag{1.5}$$

Let

$$p = (\overline{x}_0, q), \quad q \in Y_1, \quad \overline{x}_0 \in (0, a_1). \tag{1.6}$$

Consider the line in $\mathbb{R}^{n+1}$ of the form $L = \{(s\overline{x}_0, q) : s \in \mathbb{R}\}$. The following will be proved in §6.

Theorem 1.1 *In addition to hypothesis (1.5), assume*

$$\partial\mathcal{O} \ \text{ is real analytic and connected,} \tag{1.7}$$

and

$$\mathcal{O} \ \text{ is rotationally symmetric about the axis } \ L. \tag{1.8}$$

Assume q in (1.6) is known. Take $k \in \mathbb{N}$ such that supp $\phi_k \subset Y_1$ and $\phi_k(q) \neq 0$. Then knowledge of $S_{kk}(\lambda)$ for all $\lambda \in [|\sigma_1|, \infty)$ determines $\mathcal{O}$.

REMARK. We do not assume knowledge of $\overline{x}_0$ in (1.6). Identification of $\overline{x}_0$ will follow from the scattering data.

Theorem 1.1 extends Theorems 1.1 and 5.1 of [2], which deal with obstacles in planar cylindrical waveguides. In such a case, hypothesis (1.8) becomes the hypothesis of bilateral symmetry of $\mathcal{O}$ about its axis (taken in [2] to run down the middle of $X = \mathbb{R} \times [\gamma_1, \gamma_2]$). Another way in which the current result improves on the results of [2] is that we treat the Neumann boundary condition on $\partial\mathcal{O}$ as well as the Dirichlet boundary condition.

Theorem 1.2 of [2] is a result about unique determination of $\mathcal{O} \subset \mathbb{R} \times (\gamma_1, \gamma_2)$ without bilateral symmetry, given knowledge of two elements of the scattering matrix, namely $S_{11}(\lambda)$ and $S_{13}(\lambda)$ in the terminology of [2]. We give some extensions of this result to other planar waveguide settings in §7, to which we refer for specific statements.

Theorem 1.1 bears some resemblance to results of Zelditch [9, 10, 11] on recovering planar domains with one symmetry from the spectrum of the Laplacian. Zelditch's proof uses the study of singularities of the fundamental solution of the wave equation which propagate along a single isolated periodic broken geodesic. Here we do a detailed study of the singularity of a solution of the wave equation corresponding to a broken geodesic after a single reflection. Our techniques and results have some similarities to those of [7], though we do a much more detailed analysis of the singularity resulting from the reflection of the wave.

Some inverse scattering results for waveguides from an applied mathematics point of view can be found in [3], [5] and references. In [4] the authors show that for a strip in the plane, knowledge of the scattering matrix at all energies suffices to determine the compactly supported perturbation of a sound speed.

In outline, our approach to the proof of Theorem 1.1 is parallel to the strategy used in [2]. First it is shown that $\{S_{jk}(\lambda) : \lambda \in [|\sigma_1|, \infty)\}$ determines the singularities of a certain family of distributions derived from the fundamental solution to the wave equation on $\mathbb{R} \times (X \setminus \mathcal{O})$. In particular, it determines the singularity arising when a certain progressing wave in the end $[0, \infty) \times Y_1$ reflects off $\partial\mathcal{O}$ in a small neighborhood of the tip p. Calculations involving a wave equation parametrix for transversal reflection of singularities give an asymptotic expansion of this singularity. There is a (more or less) explicit formula, involving the derivatives at q of the function h, defining $\partial\mathcal{O}$ near p by $\overline{x} = h(y)$. Under the hypothesis of rotational symmetry about the axis L, these formulas simplify, and one sees that all the derivatives of h at q are determined by the data specified in Theorem 1.1. The real analyticity and connectedness hypotheses of (1.7) then guarantee uniqueness.

We implement this strategy in the following sections. In §2 we associate a certain family of distributions $u_{j,k,\alpha,\beta} \in \mathcal{D}'(\mathbb{R})$ to the fundamental solution to the wave equation on $\mathbb{R} \times (X \setminus \mathcal{O})$ and to ϕ_j, ϕ_k, and recall from [2] that knowledge of $S_{jk}(\lambda)$ for $\lambda \geq |\sigma_1|$ uniquely determines $u_{j,k,\alpha,\beta}$, mod $C^\infty(\mathbb{R})$. We describe which singularity of $u_{j,k,\alpha,\beta}$ we need to compute in detail in order to obtain information on $D^\gamma h(q)$.

In §3 we construct a wave equation parametrix when $\partial\mathcal{O}$ has the Dirichlet boundary condition, extending to higher dimensions a construction of [2]. As there, we switch from $(\overline{x}, y)$ coordinates on $[0, \infty) \times Y_1$ to (x, y) coordinates, with $x = \overline{x} - h(y)$, so $\partial\mathcal{O}$ near p is given by $x = 0$. The main task is to evaluate the power series in x at $x = 0$ of the solutions to eikonal and transport equations rather precisely. In §4 we make such a construction when $\partial\mathcal{O}$ has the Neumann boundary condition. Our method involves the Dirichlet-to-Neumann map, which allows us to use many calculations from §3. We apply the results of §§3–4 in §5 to the needed singularity analysis of $u_{j,k,\alpha,\beta}$, or more precisely of its symmetrization $u^s_{j,k,\alpha,\beta}(t) = (u_{j,k,\alpha,\beta}(t) + u_{j,k,\alpha,\beta}(-t))/2$. This sets us up to prove Theorem 1.1 in §6. Finally, in §7, we obtain some two-dimensional uniqueness results without symmetry hypotheses, as advertised above.

Acknowledgments. This project was initiated while both authors were visitors at MSRI and the authors are grateful to MSRI for partial support and hospitality. In addition, the authors gratefully acknowledge the partial support of T.J.C. by NSF grant DMS-0500267 and an MU Research Leave, and the partial support of M.T. by NSF grant DMS-0758320. T.J.C. thanks the UC Berkeley math department for its hospitality in spring 2009.

2 From $S_{jk}(\lambda)$ to $u^s_{j,k,\alpha,\beta}(t)$

Let $U(t,z,z')$ denote the Schwartz kernel of $e^{it\Delta_{\tilde{X}}}$, so, for $f \in C_0^\infty(\widetilde{X})$,

$$\left(e^{it\Delta_{\tilde{X}}} f\right)(z) = \int_{\widetilde{X}} U(t,z,z') f(z')\, dz'. \tag{2.1}$$

On each end $[0,\infty) \times Y_i$, use coordinates $z = (\overline{x}, y)$, and for $\alpha, \beta > 0$, set

$$u_{j,k,\alpha,\beta}(t) = \int_Y \int_Y U(t,\alpha,y,\beta,y') \phi_j(y) \phi_k(y')\, dy\, dy'. \tag{2.2}$$

Recall that each ϕ_j and ϕ_k is supported on one component of Y. If $\operatorname{supp} \phi_j \subset Y_1$ we require $\alpha > a_1$ and if $\operatorname{supp} \phi_k \subset Y_1$ we require $\beta > a_1$. This construction is treated in detail in §3 of [2]. Proposition 3.3 of [2] is the identity

$$\begin{aligned} u_{j,k,\alpha,\beta}(t) = \frac{1}{\pi} \int_{|\sigma_1|}^\infty & e^{it\lambda} \lambda \operatorname{Re}\Big(\big(r_j(\lambda) r_k(\lambda)\big)^{-1/2} e^{i\alpha r_j(\lambda) + i\beta r_k(\lambda)} S_{kj}(\lambda) \Big)\, d\lambda \\ & + \frac{1}{\pi} \delta_{jk} \int_{|\sigma_j|}^\infty e^{it\lambda} \frac{\lambda}{r_j(\lambda)} \operatorname{Re}\big(e^{i(\alpha-\beta) r_j(\lambda)} \big)\, d\lambda \\ & + R_{j,k,\alpha,\beta}(t), \end{aligned} \tag{2.3}$$

where $R_{j,k,\alpha,\beta} \in C^\infty(\mathbb{R})$. Recall from §1 that, with $\Delta_Y \phi_j = \sigma_j^2 \phi_j$,

$$r_j(\lambda) = (\lambda^2 - \sigma_j^2)^{1/2}, \quad \text{and} \quad |\lambda| > |\sigma_j| \Rightarrow \operatorname{sgn} r_j(\lambda) = \operatorname{sgn} \lambda. \tag{2.4}$$

Consequently the singularities of $u_{j,k,\alpha,\beta}(t)$ are determined by $\{S_{kj}(\lambda) : \lambda \geq |\sigma_1|\}$.

We look more closely at the symmetrization:

$$u^s_{j,k,\alpha,\beta}(t) = \frac{1}{2}\big[u_{j,k,\alpha,\beta}(t) + u_{j,k,\alpha,\beta}(-t)\big]. \tag{2.5}$$

We have

$$u^s_{j,k,\alpha,\beta}(t) = \int_Y v_{k\beta}(t,\alpha,y) \phi_j(y)\, dy, \tag{2.6}$$

where $v_{k\beta}(t,\overline{x},y)$ solves

$$(D_t^2 - \Delta) v_{k\beta} = 0 \text{ on } \tilde{X}. \tag{2.7}$$

Moreover, the initial conditions are

$$v_{k\beta}(0,\overline{x},y) = \delta(\overline{x} - \beta)\phi_k(y), \quad \partial_t v_{k\beta}(0,\overline{x},y) = 0, \tag{2.8}$$

and $v_{k\beta}$ must satisfy whatever boundary condition on $[0,\infty) \times \partial Y_i$ as was imposed on Δ_X on ∂X, and also whatever boundary condition on $\partial\mathcal{O}$ as was imposed for $\Delta_{\widetilde{X}}$ (either Dirichlet or Neumann, in either case).

From here on we assume j and k are chosen so that

$$\operatorname{supp}\phi_j,\ \operatorname{supp}\phi_k \subset Y_1, \tag{2.9}$$

so that

$$(D_t^2 - D_{\overline{x}}^2 - \Delta_{Y_1})v_{k\beta}(t,\overline{x},y) \tag{2.10}$$

holds on $(0,\infty)\times Y_1 \setminus \mathcal{O}$. Recall from §1 that

$$\mathcal{O}\cap[a_1,\infty)\times Y_1 = \emptyset \Longleftrightarrow a_1 \geq \overline{x}_0. \tag{2.11}$$

Pick α and β to satisfy

$$\overline{x}_0 < \alpha < \beta < \infty. \tag{2.12}$$

(Later we will take $\alpha \searrow \overline{x}_0$.) Recalling that $\Delta_{Y_1}\phi_k = \sigma_k^2\phi_k$, we have from (2.10) and (2.8) that

$$|t| < \beta - \overline{x}_0 \Longrightarrow v_{k\beta}(t,\overline{x},y) = u_{k\beta}(t,\overline{x})\phi_k(y), \tag{2.13}$$

with

$$\begin{aligned}&(\partial_t^2 - \partial_{\overline{x}}^2 + \sigma_k^2)u_{k\beta} = 0,\\ &u_{k\beta}(0,\overline{x}) = \delta(\overline{x}-\beta), \quad \partial_t u_{k\beta}(0,\overline{x}) = 0.\end{aligned} \tag{2.14}$$

Consequently,

$$|t| < \beta - \overline{x}_0 \Longrightarrow u^s_{j,k,\alpha,\beta}(t) = \delta_{jk}u_{k\beta}(t,\alpha), \tag{2.15}$$

so

$$\operatorname{sing\,supp} u^s_{k,k,\alpha,\beta} \cap (\overline{x}_0-\beta,\beta-\overline{x}_0) = \{\pm(\beta-\alpha)\}\cap(\overline{x}_0-\beta,\beta-\overline{x}_0). \tag{2.16}$$

Of course, the nature of the singularity at $t=\pm(\beta-\alpha)$ is independent of the obstacle $\mathcal{O}$. Standard results on propagation of singularities imply that $u^s_{j,k,\alpha,\beta}$ is smooth for $\beta-\alpha < t < (\beta-\overline{x}_0)+(\alpha-\overline{x}_0)$. One expects a singularity at $t=(\beta-\overline{x}_0)+(\alpha-\overline{x}_0)$ and that the singularity depends on the behavior of $\partial\mathcal{O}$ at the tip p. Note that the observation of such a singularity would serve to identify $\overline{x}_0$. Furthermore, in the limit $\alpha\searrow\overline{x}_0$ one expects to observe a singularity of $u^s_{j,k,\overline{x}_0,\beta}$ at $t=\beta-\overline{x}_0$, depending on the behavior of $\partial\mathcal{O}$ at p. Verification of these expectations, together with a detailed account of this dependence, will occupy §§3–5.

3 Dirichlet parametrix in special coordinates

We introduced in §2 functions $v_{k\beta}(t,\overline{x},y)$ on $(0,\infty)\times Y_1\setminus\mathcal{O}$ (given (2.9)), solving the wave equation (2.13), with initial condition (2.8), and with boundary conditions described there. We saw that $v_{k\beta}$ is given by

$$v^-_{k\beta}(t,\overline{x},y) = u_{k\beta}(t,\overline{x})\phi_k(y), \tag{3.1}$$

for $|t| < \beta-\overline{x}_0$, where $u_{k\beta}$ solves (2.14). Note that such $u_{k\beta}$ is naturally defined for $|t| < \beta-\overline{x}_0$. Our goal here is to obtain a useful parametrix for $v_{k\beta}(t,\overline{x},y)$, in a neighborhood of $(t, \overline{x}, y) = (\beta - \overline{x}_0, \overline{x}_0, q)$, where $p = (\overline{x}_0, q)$ is the tip of $\partial\mathcal{O}$, in case $v_{k\beta}$ satisfies the Dirichlet boundary condition on $\partial\mathcal{O}$:

$$v_{k\beta}\big|_{\mathbb{R}\times\partial\mathcal{O}} = 0. \tag{3.2}$$

We construct $v_{k\beta}$ as

$$v_{k\beta} = v_{k\beta}^- + v_{k\beta}^+, \tag{3.3}$$

where $v_{k\beta}^+$ solves

$$(D_t^2 - \Delta_{\widetilde{X}})v_{k\beta}^+ = 0 \tag{3.4}$$

on $(\overline{x}, y) \in (0, \infty) \times Y_1 \setminus \mathcal{O}$, with the same homogeneous boundary conditions on $(0, \infty) \times \partial Y_1$ as $v_{k\beta}$, while

$$v_{k\beta}^+ = -v_{k\beta}^- \quad \text{on} \quad \mathbb{R} \times \partial\mathcal{O}, \tag{3.5}$$

and

$$v_{k\beta}^+ = 0 \quad \text{for} \quad t < \beta - \overline{x}_0. \tag{3.6}$$

(Note that $v_{k\beta}^-|_{\mathbb{R}\times\partial\mathcal{O}}$ vanishes for $|t| < \beta - \overline{x}_0$.)

Moving towards this goal, we set up eikonal and transport equations for a parametrix

$$v = \int e^{i\varphi} a\, \widehat{\Phi}(\xi, \eta)\, d\xi\, d\eta, \tag{3.7}$$

for solutions to (3.4) on $[0, \infty) \times Y_1 \setminus \mathcal{O}$. We find it useful to change coordinates from $(\overline{x}, y)$ to curvilinear coordinates

$$(x, y) = (\overline{x} - h(y), y), \tag{3.8}$$

where, as stated in §1, $\partial\mathcal{O}$ is defined near p by $\overline{x} = h(y)$. We will obtain eikonal and transport equations for φ and a in these coordinates.

It is convenient first to present such equations in a coordinate independent fashion. We have (with $\Box = D_t^2 - \Delta$),

$$\begin{aligned}(D_t^2 - \Delta)(ae^{i\varphi}) = [\varphi_t^2 - \langle d\varphi, d\varphi\rangle]ae^{i\varphi} - 2i[\varphi_t a_t - \langle d\varphi, da\rangle]e^{i\varphi} \\ + (\Box a + ia\Box\varphi)e^{i\varphi}.\end{aligned} \tag{3.9}$$

Here $\langle d\varphi, da\rangle$ is the inner product of cotangent vectors, i.e., elements of $T_z^*\widetilde{X}$. If we set

$$a \sim \sum_{k\le 0} a_k, \tag{3.10}$$

with a_k homogeneous of degree k in (ξ, η), for $|\xi|^2 + \eta^2 \ge 1$, we get the eikonal equation

$$\langle d\varphi, d\varphi\rangle - \varphi_t^2 = 0, \tag{3.11}$$

and transport equations

$$\varphi_t \partial_t a_k - \langle d\varphi, da_k\rangle = \frac{1}{2i}(ia_k\Box\varphi + \Box a_{k+1}), \quad k \le 0, \tag{3.12}$$

with the convention that $a_1 \equiv 0$.

Now for the presentation in the curvilinear coordinates (x, y). Expanding

$$d\overline{x}^2 + \sum dy_j^2 = \Big(dx + \sum \partial_{y_j} h \, dy_j\Big)^2 + \sum dy_j^2, \tag{3.13}$$

we obtain the inner product on tangent vectors:

$$\langle \partial_x, \partial_x \rangle = 1, \quad \langle \partial_x, \partial_{y_j} \rangle = \frac{\partial h}{\partial y_j}, \quad \langle \partial_{y_j}, \partial_{y_k} \rangle = \delta_{jk} + \frac{\partial h}{\partial y_j}\frac{\partial h}{\partial y_k}. \tag{3.14}$$

Linear algebra yields the following formula for the inner product on cotangent vectors:

$$\langle dx, dx \rangle = 1 + |dh|^2, \quad \langle dx, dy_j \rangle = -\frac{\partial h}{\partial y_j}, \quad \langle dy_j, dy_k \rangle = \delta_{jk}. \tag{3.15}$$

Here

$$|dh|^2 = \sum \Big(\frac{\partial h}{\partial y_j}\Big)^2. \tag{3.16}$$

We have

$$\begin{aligned} \langle d\varphi, d\varphi \rangle &= \langle \varphi_x \, dx + d_y\varphi, \varphi_x \, dx + d_y\varphi \rangle \\ &= (1 + |dh|^2)\varphi_x^2 - 2\varphi_x \langle dh, d_y\varphi \rangle + \langle d_y\varphi, d_y\varphi \rangle, \end{aligned} \tag{3.17}$$

where

$$\langle dh, d_y\varphi \rangle = \sum_j \frac{\partial h}{\partial y_j}\frac{\partial \varphi}{\partial y_j}, \quad \langle d_y\varphi, d_y\varphi \rangle = \sum_j \Big(\frac{\partial \varphi}{\partial y_j}\Big)^2. \tag{3.18}$$

Hence the eikonal equation (3.11) becomes

$$(1 + |dh|^2)\varphi_x^2 - 2\langle dh, d_y\varphi \rangle \varphi_x + \langle d_y\varphi, d_y\varphi \rangle - \varphi_t^2 = 0. \tag{3.19}$$

We also have

$$\begin{aligned} \langle d\varphi, da_k \rangle &= \langle \varphi_x \, dx + d_y\varphi, a_{kx} \, dx + d_y a_k \rangle \\ &= [(1 + |dh|^2)\varphi_x - \langle dh, d_y\varphi \rangle]\partial_x a_k + \langle d_y\varphi - \varphi_x \, dh, d_y a_k \rangle, \end{aligned} \tag{3.20}$$

where again the remaining inner products are given as in (3.18). Hence the transport equation (3.12) becomes

$$\begin{aligned} \varphi_t \partial_t a_k - [(1 + |dh|^2)\varphi_x - \langle dh, d_y\varphi \rangle]\partial_x a_k - \langle d_y\varphi - \varphi_x \, dh, d_y a_k \rangle \\ = \frac{1}{2i}(i a_k \Box\varphi + \Box a_{k+1}). \end{aligned} \tag{3.21}$$

Returning to the eikonal equation (3.11), we have

$$\begin{aligned} \varphi_x = {} & \frac{\langle dh, d_y\varphi \rangle}{1 + |dh|^2} \\ & \pm \frac{\sqrt{(1 + |dh|^2)(\varphi_t^2 - \langle d_y\varphi, d_y\varphi \rangle) + \langle dh, d_y\varphi \rangle^2}}{1 + |dh|^2}. \end{aligned} \tag{3.22}$$

It is natural to define the phase functions $\varphi^\pm(x,y,t,\xi,\eta)$ as solutions to

$$\begin{aligned}\varphi^\pm_x = {} & \frac{\langle dh, d_y\varphi^\pm\rangle}{1+|dh|^2} \\ & \mp (\operatorname{sgn}\eta)\frac{\sqrt{(1+|dh|^2)((\varphi^\pm_t)^2 - \langle d_y\varphi^\pm, d_y\varphi^\pm\rangle) + \langle dh, d_y\varphi^\pm\rangle^2}}{1+|dh|^2}.\end{aligned} \tag{3.23}$$

with data at $x = 0$:

$$\varphi^\pm(0,y,t,\xi,\eta) = y\cdot\xi + t\eta. \tag{3.24}$$

This is well defined for $|x|$ small provided $|\xi|^2 < \eta^2$. The following result parallels Lemma 4.2 of [2].

Lemma 3.1 *For $|\xi|^2 < \eta^2$ the phase functions $\varphi^\pm(x,y,t,\xi,\eta)$ have asymptotic expansions at $x = 0$ given by*

$$\varphi^\pm(x,y,t,\xi,\eta) \sim \sum_{j\geq 0} x^j \varphi^\pm_j(y,t,\xi,\eta), \tag{3.25}$$

with $\varphi^\pm_0$ given by the right side of (3.24) and

$$\begin{aligned}\varphi^\pm_1(y,t,\xi,\eta) = {} & \frac{\xi\cdot\nabla h}{1+|dh|^2} \\ & \mp (\operatorname{sgn}\eta)\frac{\sqrt{(1+|dh|^2)(\eta^2 - |\xi|^2) + (\xi\cdot\nabla h)^2}}{1+|dh|^2}.\end{aligned} \tag{3.26}$$

Moreover, for $j \geq 1$, $\varphi^\pm_j$ are independent of t, and $\varphi^\pm_j(y,t,\xi,\eta)$ are determined by ξ,η, and $D^\alpha h(y)$ for $|\alpha| \leq j$.

Proof. The identification of $\varphi^\pm_0$ is clear. Setting $x = 0$ in (3.23) gives (3.26). The stated result on $\varphi^\pm_j$ follows by a straightforward induction, applying x-derivatives to (3.23) and evaluating at $x = 0$. $\square$

Returning to the transport equations, we have for each choice of phase function $\varphi^\pm$ the following, from (3.21):

$$\begin{aligned}\varphi^\pm_t \partial_t a^\pm_k - [(1+|dh|^2)\varphi^\pm_x - \langle dh, d_y\varphi^\pm\rangle]\partial_x a^\pm_k - \langle d_y\varphi^\pm - \varphi^\pm_x\, dh, d_y a^\pm_k\rangle & \\ = \frac{1}{2i}(i a^\pm_k \Box\varphi^\pm + \Box a^\pm_{k+1}), & \end{aligned} \tag{3.27}$$

for $k \leq 0$, as before with the convention $a^\pm_1 \equiv 0$. Here $a^\pm_k = a^\pm_k(x,y,\xi,\eta)$ (independent of t). We take data at $x = 0$:

$$a^\pm_0(0,y,\xi,\eta) = 1, \quad a^\pm_k(0,y,\xi,\eta) = 0 \;\text{ for }\; k < 0. \tag{3.28}$$

Then $a^\pm_k$ are well defined and homogeneous of degree k in (ξ,η) for $|x|$ small, provided $|\xi|^2 < \eta^2$. Regarding the right side of (3.27), note that

$$\Box\varphi^\pm = -\Delta\varphi^\pm, \quad \Box a^\pm_{k+1} = -\Delta a^\pm_{k+1}, \tag{3.29}$$

and, in curvilinear (x, y) coordinates,

$$\Delta f = -(1 + |dh|^2)\partial_x^2 f + \Delta_Y f + 2\langle dh, d_y \partial_x f\rangle - (\Delta_Y h)\partial_x f. \tag{3.30}$$

In fact, the principal symbol of $\Delta = \Delta_{\widetilde{X}}$ is given by the inner product (3.15) on cotangent vectors and the first order term arises because $\Delta_{\widetilde{X}} x = \Delta_{\widetilde{X}}(\overline{x} - h(y)) = -\Delta_Y h(y)$.

The following result parallels Lemma 4.3 of [2].

Lemma 3.2 *For $|\xi|^2 < \eta^2$ each $a_k^\pm(x, y, \xi, \eta)$ has an asymptotic expansion of the form*

$$a_k^\pm(x, y, \xi, \eta) \sim \sum_{j \geq 0} x^j a_{kj}^\pm(y, \xi, \eta), \tag{3.31}$$

with $a_{kj}^\pm$ homogeneous of degree k in (ξ, η), and $a_{kj}^\pm(y, \xi, \eta)$ determined by ξ, η, and $D^\alpha h(y)$ for $|\alpha| \leq j + |k| + 1$.

Proof. Clearly $a_{00}^\pm = 1$ and $a_{k0}^\pm = 0$ for $k < 0$. The assertions about $a_{kj}^\pm$ for $j \geq 1$ follow inductively, by evaluating (3.27) and its x-derivatives at $x = 0$ and making use of Lemma 3.1 and of (3.29)–(3.30). $\square$

Remark. Note in particular that

$$\pm(\operatorname{sgn} \eta)\sqrt{(1 + |dh|^2)(\eta^2 - |\xi|^2) + (\xi \cdot \nabla h)^2}\, \partial_x a_0^\pm\big|_{x=0} = \frac{1}{2}\Delta\varphi^\pm\big|_{x=0}, \tag{3.32}$$

which provides the asserted information on $a_{01}^\pm$. Additionally, for $k < 0$, (3.27) yields

$$\pm(\operatorname{sgn} \eta)\sqrt{(1 + |dh|^2)(\eta^2 - |\xi|^2) + (\xi \cdot \nabla h)^2}\, \partial_x a_k^\pm\big|_{x=0} = \frac{1}{2i}\Delta a_{k+1}^\pm\big|_{x=0}. \tag{3.33}$$

Returning to the parametrix construction previewed in (3.7), we have a pair of parametrices:

$$\mathcal{S}_D^\pm \Phi = \int e^{i\varphi^\pm} a^\pm\, \widehat{\Phi}_\pm(\xi, \eta)\, d\xi\, d\eta, \tag{3.34}$$

satisfying

$$\mathcal{S}_D^\pm \Phi = v^\pm, \quad \text{mod} \ \ C^\infty(U), \tag{3.35}$$

where U is a neighborhood of $(t, \overline{x}, y) = (\beta - \overline{x}_0, \overline{x}_0, q)$, i.e., in (t, x, y) coordinates, of $(t, x, y) = (\beta - \overline{x}_0, 0, q)$, in $\mathbb{R} \times ([0, \infty) \times Y_1 \setminus \mathcal{O})$. Here $v^\pm$ solves

$$(D_t^2 - \Delta_{\widetilde{X}})v^\pm = 0 \tag{3.36}$$

on U,

$$v^\pm\big|_{U \cap \mathbb{R} \times \partial\mathcal{O}} = \Phi_\pm, \quad \text{mod} \ \ C^\infty(U), \tag{3.37}$$

(with $U \cap \mathbb{R} \times \partial\mathcal{O}$ identified via (x, y) coordinates with a subset of $\mathbb{R} \times \mathbb{R}^n$) and singularities of v^+ lie on rays leaving $\mathbb{R} \times \partial\mathcal{O}$ with $t \nearrow$, while singularities of v^- lie on rays leaving

$\mathbb{R} \times \partial\mathcal{O}$ with $t \searrow$. (We say v^- is an incoming solution and v^+ an outgoing solution.) In light of our construction of $\varphi^{\pm}$ and $a^{\pm}$, we require

$$\operatorname{supp} \widehat{\Phi}_{\pm} \subset \{(\xi, \eta) \in \mathbb{R}^{n+1} : |\xi| < (1-\varepsilon)|\eta|, \ |\eta| \geq 1\}, \tag{3.38}$$

for some $\varepsilon > 0$, which is a restriction on the wave front set of $\Phi_{\pm}$.

We can now achieve the original goal of this section, which was to construct a parametrix for $v_{k\beta}$ in a neighborhood of $(t, \overline{x}, y) = (\beta - \overline{x}_0, \overline{x}_0, q)$ in $[0, \infty) \times Y_1 \setminus \mathcal{O}$. By virtue of (3.2)–(3.6), such a parametrix has the form

$$v_{k\beta}^- + \mathcal{S}_D^+ \Phi_+, \tag{3.39}$$

where we take $\psi \in C_0^\infty(\partial\mathcal{O})$, supported on a small neighborhood of the tip p, equal to 1 on a smaller neighborhood, and set

$$\Phi_+ = -\Phi_- = -\psi \, v_{k\beta}^- \big|_{\mathbb{R}\times\partial\mathcal{O}}, \quad \operatorname{mod} \ C^\infty. \tag{3.40}$$

Note that the wave front set of the right side of (3.40) is contained in a small conic neighborhood of $\{(\xi, \eta) : \xi = 0\}$, so one can take such Φ_+ and arrange that (3.38) hold.

We summarize our result.

Proposition 3.3 *On a neighborhood of $(t, \overline{x}, y) = (\beta - \overline{x}_0, \overline{x}_0, q)$ in $[0, \infty) \times Y_1 \setminus \mathcal{O}$, the solution $v_{k\beta}$ to (2.7)–(2.8) with Dirichlet boundary condition (3.2) has the form, mod C^∞,*

$$v_{k\beta} = v_{k\beta}^- - \int e^{i\varphi^+} a^+ \, \widehat{\Phi}_-(\xi, \eta) \, d\xi \, d\eta, \tag{3.41}$$

with φ^+, a^+ as in Lemmas 3.1–3.2 and Φ_- as in (3.39).

We end this section with a formula for $\widehat{\Phi}_-(\xi, \eta)$, which will be useful in the analysis of (3.41) made in §5. First, note that $u_{k\beta}$ as defined via (2.13), is given by

$$\begin{aligned} u_{k\beta}(t, \overline{x}) &= \cos t\sqrt{-\partial_{\overline{x}}^2 + \sigma_k^2}\, \delta(\overline{x} - \beta) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mu(\overline{x}-\beta)} \cos t\sqrt{\mu^2 + \sigma_k^2}\, d\mu \\ &= \frac{1}{2\pi} \int\limits_{|\lambda| \geq |\sigma_k|} (\cos t\lambda)\, e^{ir_k(\lambda)(\overline{x}-\beta)} \frac{\lambda}{r_k(\lambda)}\, d\lambda, \end{aligned} \tag{3.42}$$

where $r_k(\lambda)$ is given as in (2.4). This is the sum of a wave moving to the left plus a wave moving to the right as t increases. At $\partial\mathcal{O}$ we need only the wave moving to the left, which is captured by replacing $\cos t\lambda$ by $e^{it\lambda}$ in the formula above. Compare (7) of [2]. Then, from (3.40) and (3.1), we get

$$\begin{aligned} \Phi_-(y, t) &= \psi(y)\phi_k(y) u_{k\beta}(t, h(y)) \\ &= \frac{1}{2\pi} \Big(\int\limits_{|\lambda| \geq |\sigma_k|} e^{it\lambda} e^{ir_k(\lambda)(h(y)-\beta)} \frac{\lambda}{r_k(\lambda)}\, d\lambda \Big) \psi(y)\phi_k(y), \end{aligned} \tag{3.43}$$

mod C^∞. This yields

$$\widehat{\Phi}_-(\xi,\eta)=\frac{\mathfrak{h}_k(\eta)}{2}\int e^{-iy\cdot\xi}e^{ir_k(\eta)(h(y)-\beta)}\frac{\eta}{r_k(\eta)}\psi(y)\phi_k(y)\,dy. \tag{3.44}$$

Here

$$\mathfrak{h}_k(\eta)=\begin{cases}1 & \text{if } |\eta|\ge|\sigma_k|\\ 0 & \text{if } |\eta|<|\sigma_k|.\end{cases} \tag{3.45}$$

Compare (11) of [2]. The right side of (3.44) is rapidly decreasing in (ξ,η) away from $\{(\xi,\eta):\xi=\eta\nabla h(y)\ \text{ for some } y\in\operatorname{supp}\psi\}$, hence rapidly decreasing for (ξ,η) away from a small conic neighborhood of $\{\xi=0\}$, if supp ψ is a sufficiently small neighborhood of q. It follows that we can throw a factor of the form

$$\chi\Bigl(\frac{|\xi|}{\eta}\Bigr),\quad \chi\in C_0^\infty[(-1,1)], \tag{3.46}$$

into the right side of (3.44), and still have (3.40). With this done, the condition (3.38) is satisfied.

4 Neumann parametrix

Our setting here is parallel to that of §3, but we want a parametrix for $v_{k\beta}(t,\overline{x},y)$, in a neighborhood of $(t,\overline{x},y)=(\beta-\overline{x}_0,\overline{x}_0,q)$, in case $v_{k\beta}$ satisfies the Neumann boundary condition on $\partial\mathcal{O}$:

$$\partial_\nu v_{k\beta}\big|_{\mathbb{R}\times\partial\mathcal{O}}=0. \tag{4.1}$$

Here ∂_ν denotes the normal derivative. We construct $v_{k\beta}$ as

$$v_{k\beta}=v_{k\beta}^-+v_{k\beta}^+, \tag{4.2}$$

where $v_{k\beta}^+$ solves (3.4) and (3.6), but with (3.5) replaced by

$$\partial_\nu v_{k\beta}^+=-\partial_\nu v_{k\beta}^-\ \text{ on }\ \mathbb{R}\times\partial\mathcal{O}. \tag{4.3}$$

The distribution $v_{k\beta}^-$ is exactly as in §3, i.e., given by (3.1), but (4.3) yields a different $v_{k\beta}^+$ from that constructed in §3.

We look for such a parametrix for $v_{k\beta}^+$ in the form

$$\mathcal{S}_D^+\Phi_+=\int e^{i\varphi^+}a^+\,\widehat{\Phi}_+(\xi,\eta)\,d\xi\,d\eta. \tag{4.4}$$

In this section $\varphi^\pm$ and $a^\pm$ are as constructed in §3. We play (4.4) off against

$$\mathcal{S}_D^-\Phi_-=\int e^{i\varphi^-}a^-\,\widehat{\Phi}_-(\xi,\eta)\,d\xi\,d\eta, \tag{4.5}$$

which is equal to $v^-_{k\beta}$ (mod C^∞) near $(t,\overline{x},y)=(\beta-\overline{x}_0,\overline{x}_0,q)$ provided

$$\Phi_- = \psi v^-_{k\beta}\big|_{\mathbb{R}\times\partial\mathcal{O}}, \quad \text{mod} \;\; C^\infty, \tag{4.6}$$

where, as in (3.40), $\psi\in C_0^\infty(\partial\mathcal{O})$ is supported on a small neighborhood of p, equal to 1 on a smaller neighborhood. On such a neighborhood, we have

$$\partial_\nu v^\pm_{k\beta} = \int e^{i(y\cdot\xi+t\eta)}(i\varphi^\pm_\nu + a^\pm_\nu)\widehat{\Phi}_\pm(\xi,\eta)\,d\xi\,d\eta, \tag{4.7}$$

where $\varphi^\pm_\nu=\partial_\nu\varphi^\pm$ and $a^\pm_\nu=\partial_\nu a^\pm$. The parametrix construction thus leads to the task of producing Φ_+ so that (4.1) holds, mod C^∞, near $(\beta-\overline{x}_0,\overline{x}_0,q)$. That is to say, we want Φ_+ to satisfy

$$\int e^{i(y\cdot\xi+t\eta)}(i\varphi^+_\nu + a^+_\nu)\widehat{\Phi}_+(\xi,\eta)\,d\xi\,d\eta = -\int e^{i(y\cdot\xi+t\eta)}(i\varphi^-_\nu + a^-_\nu)\widehat{\Phi}_-(\xi,\eta)\,d\xi\,d\eta, \tag{4.8}$$

mod C^∞. A key observation is that

$$\varphi^+_\nu = -\varphi^-_\nu, \tag{4.9}$$

which follows from the eikonal equation (3.11); see (4.13) below for explicit formulas. We will also see that $\varphi^\pm_\nu$ are elliptic, microlocally on the set (3.38), so (4.8) is an elliptic equation for Φ_+, given Φ_-.

Hence we want to obtain explicit formulas for $\varphi^\pm_\nu$ and $a^\pm_\nu$, in the curvilinear coordinates $(x,y)=(\overline{x}-h(y),y)$. We compute ∂_ν, which must satisfy

$$\langle\partial_\nu,\partial_{y_j}\rangle = 0, \quad \langle\partial_\nu,\partial_\nu\rangle = 1. \tag{4.10}$$

A straightforward computation using (3.14) gives

$$\partial_\nu f = \sqrt{1+|dh|^2}\,\partial_x f - \frac{1}{\sqrt{1+|dh|^2}}\langle dh, d_y f\rangle. \tag{4.11}$$

We note parenthetically that $\langle dx, df\rangle = (1+|dh|^2)\partial_x f - \langle dh, d_y f\rangle$, and hence an alternative formula is

$$\partial_\nu f = \frac{1}{\sqrt{1+|dh|^2}}\langle dx, df\rangle, \tag{4.12}$$

where the inner product on the right side of (4.12) is defined by (3.15). Recalling (3.23), we have

$$\begin{aligned}\partial_\nu\varphi^\pm &= \sqrt{1+|dh|^2}\,\partial_x\varphi^\pm\big|_{x=0} - \frac{\langle dh, d_y\varphi^\pm\rangle}{\sqrt{1+|dh|^2}}\Big|_{x=0} \\ &= \mp(\operatorname{sgn}\eta)\frac{\sqrt{(1+|dh|^2)(\eta^2-|\xi|^2)+(\xi\cdot\nabla h)^2}}{\sqrt{1+|dh|^2}}.\end{aligned} \tag{4.13}$$

Note how the signs on the last line verify (4.9). Ellipticity in the region (3.38) is also apparent. Next, we have

$$\partial_\nu a^\pm_k = \sqrt{1+|dh|^2}\,\partial_x a^\pm_k\big|_{x=0}, \tag{4.14}$$

since $d_y a_k^\pm|_{x=0} = 0$. In the terminology of Lemma 3.2, this reads

$$\partial_\nu a_k^\pm = \sqrt{1 + |dh|^2} a_{k1}^\pm (y, \xi, \eta), \tag{4.15}$$

which, according to Lemma 3.2, is determined by ξ, η, and $D^\alpha h(y)$ for $|\alpha| \le |k| + 2$.

To pursue the task of solving (4.8) for Φ_+, let us write

$$A_\pm \Phi = \pm \int e^{i(y\cdot\xi + t\eta)} (i\varphi_\nu^\pm + a_\nu^\pm) \widehat{\Phi} \, d\xi \, d\eta, \tag{4.16}$$

so (4.8) is

$$A_+ \Phi_+ = A_- \Phi_-. \tag{4.17}$$

In (4.16), we can extend $\varphi_\nu^\pm$ and $a_\nu^\pm$ from (ξ, η) as in (3.38) to all (ξ, η), in such a fashion that $A_\pm$ are elliptic operators in OPS^1. By (4.9),

$$A_- = A_+ + B, \quad B \in OPS^0, \tag{4.18}$$

where the symbol of B is $-a_\nu^- - a_\nu^+$. Let

$$E_+ \in OPS^{-1} \text{ be a parametrix for } A_+. \tag{4.19}$$

Then (4.17) becomes

$$\Phi_+ = (I + R)\Phi_-, \quad R = E_+ B \in OPS^{-1}. \tag{4.20}$$

Standard pseudodifferential operator calculus yields the following.

Proposition 4.1 *We have*

$$R\Phi = \int e^{i(y\cdot\xi + t\eta)} r(y, \xi, \eta) \widehat{\Phi}(\xi, \eta) \, d\xi \, d\eta, \tag{4.21}$$

with

$$r \sim \sum_{k \le -1} r_k(y, \xi, \eta), \tag{4.22}$$

where r_k is homogeneous of degree k in (ξ, η) and r_k depends on ξ, η, and $D^\alpha h(y)$ for $|\alpha| \le |k| + 1$.

Proof. This follows from (4.13) and (4.15) and accompanying remark, together with the standard elliptic parametrix construction, via repeated use of the formula

$$\sigma_{PQ}(y, \xi, \eta) \sim \sum_{\alpha \ge 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_P(y, \xi, \eta) \partial_y^\alpha \sigma_Q(y, \xi, \eta) \tag{4.23}$$

for the product of classical pseudodifferential operators whose symbols depend on (y, ξ, η). (Recall that ξ is the dual variable to y.) □

We deduce that $\mathcal{S}_D^+\Phi_+$ in (4.4) is given by

$$\mathcal{S}_D^+\Phi_+ = \mathcal{S}_D^+\Phi_- + \mathcal{S}_D^+ R\Phi_-, \tag{4.24}$$

with Φ_- as in (4.6) and R as in (4.21). Application of symbol calculus of Fourier integral operators in concert with Lemmas 3.1–3.2 gives the following.

Proposition 4.2 *We have*

$$\mathcal{S}_D^+ R\Phi_- = \int e^{i\varphi^+} b\, \widehat{\Phi}_-(\xi,\eta)\, d\xi\, d\eta, \tag{4.25}$$

with

$$b \sim \sum_{k\le -1} b_k(x,y,\xi,\eta), \tag{4.26}$$

where b_k is homogeneous of degree k in (ξ,η). Each b_k has the asymptotic expansion

$$b_k(x,y,\xi,\eta) \sim \sum_{j\ge 0} x^j b_{kj}(y,\xi,\eta), \tag{4.27}$$

with b_{kj} homogeneous of degree k in (ξ,η), and $b_{kj}(y,\xi,\eta)$ is determined by ξ,η, and $D^\alpha h(y)$ for $|\alpha| \le j+|k|+1$.

Proof. The fact that $\mathcal{S}_D^+ R$ has the form (4.25)–(4.26) is a standard application of the symbol calculus for Fourier integral operators. It remains to prove the asserted properties of $b_{kj}(y,\xi,\eta)$, arising in (4.27). First, evaluating (4.25) at $x=0$ gives

$$b(0,y,\xi,\eta) = r(y,\xi,\eta), \tag{4.28}$$

so the case $j=0$ of the results on (4.27) follows directly from Proposition 4.1. To tackle the case $j=1$, note that

$$\partial_x \mathcal{S}_D^+\Phi\big|_{x=0} = \int e^{i(y\cdot\xi+t\eta)}(i\varphi_1^+ + a_x^+\big|_{x=0})\widehat{\Phi}(\xi,\eta)\, d\xi\, d\eta \tag{4.29}$$

where here and below the φ_j^+ are as in Lemma 3.1. Denote the right side of (4.29) by $A_1\Phi$, with $A_1 \in OPS^1$. Applying ∂_x to the right side of (4.25) and evaluating at $x=0$ gives

$$A_1 R\Phi = B_1\Phi = \int e^{i(y\cdot\xi+t\eta)}(i\varphi_1^+ b + b_x)\big|_{x=0}\widehat{\Phi}\, d\xi\, d\eta. \tag{4.30}$$

Using the formula (4.23) to evaluate σ_{A_1R}, we get

$$\begin{aligned}\sigma_{A_1R} &\sim i\varphi_1^+ r + a_x^+\big|_{x=0} r + \sum_{|\alpha|\ge 1} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha(i\varphi_1^+ + a_x^+\big|_{x=0})\cdot\partial_y^\alpha r \\ &= (i\varphi_1^+ b + b_x)\big|_{x=0},\end{aligned} \tag{4.31}$$

which we can solve for $b_x|_{x=0}$, obtaining

$$b_x\big|_{x=0} \sim -i\varphi_1^+ b\big|_{x=0} + i\varphi_1^+ r + a_x^+\big|_{x=0} r + \sum_{|\alpha|\geq 1} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha (i\varphi_1^+ + a_x^+\big|_{x=0}) \cdot \partial_y^\alpha r. \tag{4.32}$$

(Note that $i\varphi_1^+ r$ and $i\varphi_1^+ b|_{x=0}$ cancel, but we will not need to make use of this. Our lack of need to track such cancellation will be useful below.) We examine the terms of order $k \leq -1$ in (ξ, η) in $b_x|_{x=0}$:

$$b_x\big|_{x=0} \sim \sum_{k\leq -1} b_{k1}(y, \xi, \eta). \tag{4.33}$$

Inspection reveals that all the terms of order k in (ξ, η) occurring in the right side of (4.32) depend on $D^\gamma h(y)$ only for $|\gamma| \leq |k| + 2 = 1 + |k| + 1$. This gives the asserted result on (4.27) for $j = 1$.

For general j, we have

$$\partial_x^j \mathcal{S}_D^+ \Phi\big|_{x=0} = A_j \Phi, \quad \partial_x^j \mathcal{S}_D^+ R\Phi\big|_{x=0} = B_j \Phi, \tag{4.34}$$

with

$$\begin{aligned} \sigma_{A_j}(y, \xi, \eta) &= e^{-i\varphi^+}\, \partial_x^j (a^+ e^{i\varphi^+})\big|_{x=0}, \\ \sigma_{B_j}(y, \xi, \eta) &= e^{-i\varphi^+}\, \partial_x^j (b e^{i\varphi^+})\big|_{x=0}. \end{aligned} \tag{4.35}$$

An expansion gives

$$\begin{aligned} \partial_x^j (a^+ e^{i\varphi^+}) &= \sum_{i=0}^{j} \binom{j}{i} (\partial_x^{j-i} a^+)\, \partial_x^i e^{i\varphi^+} \\ &= \sum_{i=0}^{j} \sum_{\ell} \tilde{c}(j, i, \ell)(\partial_x^{j-i} a^+)(\partial_x^{\ell_1}\varphi^+) \cdots (\partial_x^{\ell_\nu}\varphi^+) e^{i\varphi^+}, \end{aligned} \tag{4.36}$$

for certain constants $\tilde{c}(j, i, \ell)$. Here, if $i \geq 1$, we have $\ell = (\ell_1, \dots, \ell_\nu)$ with $\ell_1 + \cdots + \ell_\nu = i$ and each $\ell_\mu \geq 1$. If $i = 0$, by convention $\ell = \{\emptyset\}$, and the associated term appearing in (4.36) is simply $(\partial_x^j a^+) e^{i\varphi^+}$. Hence, with the same conventions,

$$\sigma_{A_j}(y, \xi, \eta) = \sum_{i=0}^{j} \sum_{\ell} c(j, i, \ell) \varphi_{\ell_1}^+ \cdots \varphi_{\ell_\nu}^+\, \partial_x^{j-i} a^+\big|_{x=0}, \tag{4.37}$$

with $c(j, i, \ell) = \ell!\tilde{c}(j, i, \ell)$. Similarly,

$$\sigma_{B_j}(y, \xi, \eta) = \sum_{i=0}^{j} \sum_{\ell} c(j, i, \ell) \varphi_{\ell_1}^+ \cdots \varphi_{\ell_\nu}^+\, \partial_x^{j-i} b\big|_{x=0}. \tag{4.38}$$

Meanwhile, $B_j = A_j R$, so

$$\sigma_{B_j} \sim \sum_{\alpha\geq 0} \frac{(-i)^{|\alpha|}}{\alpha!} (\partial_\xi^\alpha \sigma_{A_j})(\partial_y^\alpha r). \tag{4.39}$$

Note that $\partial_x^j b|_{x=0}$ occurs as one term in (4.38), so we get

$$\begin{aligned}\partial_x^j b\big|_{x=0} \sim\ &-\sum_{i=1}^{j}\sum_{\ell} c(j,i,\ell)\varphi_{\ell_1}^{+}\cdots\varphi_{\ell_\nu}^{+}\,\partial_x^{j-i}b\big|_{x=0}\\ &+\sum_{\alpha\ge 0}\frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha\sigma_{A_j})(\partial_y^\alpha r),\end{aligned}\tag{4.40}$$

with σ_{A_j} given by (4.37), so $(\partial_\xi^\alpha\sigma_{A_j})(\partial_y^\alpha r)$ is a sum of terms of the form

$$\partial_\xi^\alpha(\varphi_{\ell_1}^{+}\cdots\varphi_{\ell_\nu}^{+}\,(\partial_x^{j-i}a)\big|_{x=0})\cdot\partial_y^\alpha r,\tag{4.41}$$

with

$$\alpha\ge 0,\quad \ell_\mu\ge 1,\quad \ell_1+\cdots+\ell_\nu=i,\quad \nu\le i,\tag{4.42}$$

for $1\le i\le j$, and with the convention given above for $i=0$. Recall from Lemmas 3.1–3.2 that $\varphi_{\ell_\mu}^{+}$ depends on $D^\gamma h(y)$ for $|\gamma|\le\ell_\mu$ and

$$\begin{aligned}&a\sim\sum_{k\le 0}a_k, \qquad a_k\ \text{ order }\ k\ \text{ in }\ (\xi,\eta),\\ &\partial_x^{j-i}a_k\big|_{x=0}\ \text{ depends on }\ D^\gamma h(y),\ \ |\gamma|\le j-i+|k|+1.\end{aligned}\tag{4.43}$$

Inspection reveals that each term of order $k\le -1$ in (ξ,η) arising from (4.41) depends on $D^\gamma h(y)$ only for $|\gamma|\le j+|k|+1$. Furthermore, given $j\in\mathbb{N}$ and given that $b_{k,j-i}$ depends on $D^\gamma h(y)$ only for $|\gamma|\le j-i+|k|+1$ whenever $i\ge 1,\ k\le -1$, we see that the first set of terms on the right side of (4.40) also depend on $D^\gamma h(y)$ only for $|\gamma|\le j+|k|+1$. Inductively, this yields the desired result on b_{kj}, proving Proposition 4.2. □

We summarize our results.

Proposition 4.3 *On a neighborhood of $(t,\overline{x},y)=(\beta-\overline{x}_0,\overline{x}_0,q)$ in $\mathbb{R}\times([0,\infty)\times Y_1\setminus\mathcal{O})$, the solution $v_{k\beta}$ to (2.7)–(2.8) with Neumann boundary condition (4.1) has the form, mod C^∞,*

$$v_{k\beta}=v_{k\beta}^{-}+\int e^{i\varphi^{+}}(a^{+}+b)\widehat{\Phi}_{-}(\xi,\eta)\,d\xi\,d\eta,\tag{4.44}$$

with φ^{+}, a^{+} as in Lemmas 3.1–3.2, b as in Proposition 4.2, and Φ_{-} as in (4.6).

Note that Φ_{-} here is the same as in Proposition 3.3. In particular, (3.43)–(3.46) apply.

5 Singularity analysis of $u^s_{j,k,\alpha,\beta}(t)$

Here we apply the results of §§3–4 to study the nature of the singularity in t, for t near $\beta-\overline{x}_0$ and α near $\overline{x}_0$, of

$$\begin{aligned}&u^s_{j,k,\alpha,\beta}(t)\\ &=\iint e^{i\varphi^{+}(\alpha-h(y),y,t,\xi,\eta)}a^b(\alpha-h(y),y,\xi,\eta)\phi_j(y)\psi(y)\widehat{\Phi}_{-}(\xi,\eta)\,d\xi\,d\eta\,dy\\ &\quad+\delta_{jk}\,u_{k\beta}(t,\alpha)+R(t)\end{aligned}\tag{5.1}$$

where $R(t)$ is smooth on $(\beta - \alpha - \epsilon, \beta - \alpha + \epsilon)$. Here $\widehat{\Phi}_-$ is as in (3.44)–(3.46) and

$$a^b = -a^+, \quad a^b = a^+ + b, \tag{5.2}$$

in the case of Dirichlet and Neumann boundary conditions on $\partial\mathcal{O}$, respectively, as in (3.41) and (4.28). As in (2.12) we take $\overline{x}_0 < \alpha < \beta < \infty$, and we are particularly interested in the limit $\alpha \searrow \overline{x}_0$, i.e., in

$$\begin{aligned} &\iint e^{i\varphi^+(\overline{x}_0 - h(y), y, t, \xi, \eta)} a^b(\overline{x}_0 - h(y), y, \xi, \eta) \phi_j(y) \psi(y) \widehat{\Phi}_-(\xi, \eta)\, d\xi\, d\eta\, dy \\ &+ \delta_{jk}\, u_{k\beta}(t, \overline{x}_0). \end{aligned} \tag{5.3}$$

Since the last term in (5.3) is independent of $\mathcal{O}$, we need only concentrate on the first term. In other words, we are looking at the singularity at $t = \beta - \overline{x}_0$ of

$$I(t) = \iint e^{i\varphi^+(\overline{x}_0 - h(y), y, t, \xi, \eta)} a^b(\overline{x}_0 - h(y), y, \xi, \eta) \phi_j(y) \psi(y) \widehat{\Phi}_-(\xi, \eta)\, d\xi\, d\eta\, dy. \tag{5.4}$$

It is convenient first to analyze the corresponding integral with a^b replaced by 1. Recalling the condition (3.38) on the support of $\widehat{\Phi}_-$, let us set

$$\xi = \eta\zeta, \quad \zeta \in \mathbb{R}^n, \quad |\zeta| < 1, \tag{5.5}$$

and analyze

$$I(\eta, t) = \iint e^{i\varphi^+(\overline{x}_0 - h(y), y, t, \eta\zeta, \eta)} \phi_j(y) \psi(y) \widehat{\Phi}_-(\eta\zeta, \eta)\, d\zeta\, dy. \tag{5.6}$$

The following result parallels Proposition 4.6 of [2]. Here, $h''(q)$ denotes the Hessian of h at q, i.e., the $n \times n$ matrix of second order partial derivatives.

Proposition 5.1 *Given $M \in \mathbb{N}$ and $I(\eta, t)$ as in (5.6), as $|\eta| \to \infty$,*

$$I(\eta, t) = e^{i\eta(t + \overline{x}_0 - \beta)} \left(\frac{\pi^{3n}}{|\eta|^{3n} |\det h''(q)|} \right)^{1/2} e^{-i\pi n \cdot \mathrm{sgn}(\eta)/4} \Big(\sum_{m=0}^{M} \eta^{-m} b_{mjk} + O(|\eta|^{-M-1}) \Big). \tag{5.7}$$

Moreover,

$$b_{0jk} = \phi_j(q) \phi_k(q), \tag{5.8}$$

and, for $m \geq 1$,

$$b_{mjk} = C_{m1} \phi_j(q) \phi_k(q) \left(\nabla_y \cdot h''(q)^{-1} \nabla_y \right)^{m+1} h(y) \big|_{y=q} + b^R_{mjk}. \tag{5.9}$$

Here $C_{m1} \neq 0$ and depends only on m and the dimension n. Also, for $m \geq 1$, b^R_{mjk} is determined by ϕ_j, ϕ_k, and a finite number of their derivatives at q, and the derivatives of h of order $\leq 2m+1$ at q.

Proof. Bringing in (3.44)–(3.46), we have

$$I(\eta,t)=\frac{\mathfrak{h}^k(\eta)}{2}\frac{\eta}{r_k(\eta)}\iiint e^{i\varphi^+(\overline{x}_0-h(y),y,t,\eta\zeta,\zeta)-i\eta y'\cdot\zeta+ir_k(\eta)(h(y')-\beta)} \times\phi_j(y)\psi(y)\chi(\zeta)\psi(y')\phi_k(y')\,dy'\,d\zeta\,dy. \tag{5.10}$$

Note that $\varphi^+(\overline{x}_0-h(y),y,t,\eta\zeta,\eta)=\eta\varphi^+(\overline{x}_0-h(y),y,t,\zeta,1)$, both for $\eta>0$ and for $\eta<0$. Also, we can write

$$e^{ir_k(\eta)(h(y')-\beta)}\sim e^{i\eta(h(y')-\beta)}\Big(1+\sum_{\ell\geq 0}\gamma_\ell(h(y')-\beta)\eta^{-\ell-1}\Big), \tag{5.11}$$

where $\gamma_\ell(0)=0$ and $\gamma_\ell(s)$ is a polynomial of degree $\leq \ell+1$ in s. Then Theorem 7.7.5 of [6] applies to the phase function

$$f(y,y',\zeta)=\varphi^+(\overline{x}_0-h(y),y,t,\zeta,1)-y'\cdot\zeta+h(y')-\beta. \tag{5.12}$$

Note that

$$\begin{aligned}\partial_y f&=-\varphi^+_x(\overline{x}_0-h(y),y,t,\zeta,1)\partial_y h(y)+(\nabla_y\varphi^+)(\overline{x}_0-h(y),y,t,\zeta,1),\\ \partial_{y'} f&=-\zeta+\partial_{y'}h(y'),\\ \partial_\zeta f&=\partial_\zeta\varphi^+(\overline{x}_0-h(y),y,t,\zeta,1)-y'.\end{aligned} \tag{5.13}$$

Clearly there is a stationary point at $(y,y',\zeta)=(q,q,0)$. We have

$$f(q,q,0)=t+\overline{x}_0-\beta. \tag{5.14}$$

A calculation gives

$$f''(q,q,0)=\begin{pmatrix}h''(q) & 0 & I\\ 0 & h''(q) & -I\\ I & -I & 0\end{pmatrix}, \tag{5.15}$$

the right side being a $(3n)\times(3n)$ matrix of second order partial derivatives. Consequently,

$$\det f''(q,q,0)=-2\det h''(q). \tag{5.16}$$

Theorem 7.7.5 of [6] yields (5.7), with

$$\begin{aligned}&\sum_{m\geq 0} b_{mjk}\eta^{-m}\\ &\sim\sum_{m\geq 0}\eta^{-m}L_m\Big(\phi_j(y)\psi(y)\chi(\zeta)\psi(y')\phi_k(y')\Big(1+\sum_{\ell\geq 0}\gamma_\ell(h(y')-\beta)\eta^{-\ell-1}\Big)\Big).\end{aligned} \tag{5.17}$$

Here

$$L_m w=\sum_{\nu-\mu=m}\sum_{2\nu\geq 3\mu}\frac{i^{-m}2^{-\nu}}{\mu!\nu!}Q^\nu(g^\mu w)(q,q,0), \tag{5.18}$$

where g and Q are defined as follows. First,

$$\begin{aligned} & g(y, y', \zeta) \\ & = \varphi^+(\overline{x}_0 - h(y), y, t, \zeta, 1) + h(y') - \overline{x}_0 - t \\ & \quad - \frac{1}{2}\Big((y - q) \cdot h''(q)(y - q) + 2y \cdot \zeta + (y' - q) \cdot h''(q)(y' - q)\Big). \end{aligned} \tag{5.19}$$

Next

$$\begin{aligned} Q = \frac{1}{2}\Big[& \nabla_y \cdot h''(q)^{-1}\nabla_y + 2\nabla_y \cdot h''(q)^{-1}\nabla_{y'} + \nabla_{y'} \cdot h''(q)^{-1}\nabla_{y'} \\ & + 2\nabla_y \cdot \nabla_\zeta - 2\nabla_{y'} \cdot \nabla_\zeta - \nabla_\zeta \cdot h''(q)\nabla_\zeta\Big]. \end{aligned} \tag{5.20}$$

Given these formulas, (7.7.12)–(7.7.13) of [6] yield the expansion (5.7). The asserted results about b_{mjk} in (5.8)–(5.9) involve the same arguments as in Proposition 4.6 of [2], with Lemma 3.1 of this paper playing the same role as Lemma 4.2 of [2]. □

To proceed from Proposition 5.1 to an analysis of the first term in (5.3), we bring in the following, which parallels Proposition 4.7 of [2].

Proposition 5.2 *Consider*

$$I_\ell(\eta, t) = \iint e^{i\varphi^+(\overline{x}_0 - h(y), y, t, \eta\zeta, \eta)} a_\ell(\overline{x}_0 - h(y), y, \eta\zeta, \eta)\phi_j(y)\psi(y)\widehat{\Phi}_-(\eta\zeta, \eta)\, d\zeta\, dy, \tag{5.21}$$

with

$$a_1 = a^+ - 1, \quad a_2 = b, \tag{5.22}$$

and with a^+, b as in (3.41) and (4.44). Then, as $|\eta| \to \infty$,

$$I_\ell(\eta, t) = e^{i\eta(t + \overline{x}_0 - \beta)} \left(|\eta|^{3n}|\det h''(q)|\right)^{-1/2} e^{-i\pi n \cdot \mathrm{sgn}(\eta)/4}\Big(\sum_{m=1}^{M} \eta^{-m} b^\ell_{mjk} + O(|\eta|^{-M-1})\Big). \tag{5.23}$$

Each b^ℓ_{mjk} is determined by $D^\alpha h(q)$, $|\alpha| \le 2m$, and ϕ_j, ϕ_k and a finite number of their derivatives at q.

Proof. The proof is similar to that of Proposition 5.1, and parallels that [2, Proposition 4.7]. In addition to Lemma 3.1, it makes use of Lemma 3.2 and, for the analysis of b^2_{mjk}, also use of Proposition 4.2. □

We are now ready to analyze the singularity of (5.4), i.e., of

$$I_\ell(t) = \int I_\ell(\eta, t)|\eta|^n\, d\eta, \tag{5.24}$$

for $\ell = 3, 4$, where $I_\ell(\eta, t)$ is given by (5.21) with

$$a_3 = -a^+, \quad a_4 = a^+ + b, \tag{5.25}$$

the extra factor of $|\eta|^n$ arising from the relation $\xi = \eta\zeta$. The following, which parallels Theorem 4.8 of [2], is a direct consequence of (5.24) and Propositions 5.1–5.2. Recall that ϕ_k, ϕ_k are chosen to both be supported on Y_1.

Theorem 5.3 *For Dirichlet and Neumann boundary conditions on $\partial\mathcal{O}$, we have, with $\ell = 3$ or 4 respectively,*

$$I_\ell(t) = |\det h''(q)|^{-1/2} \sum_{m=0}^{M} B^\ell_{mjk}(t - \beta + \overline{x}_0)_+^{n/2-1+m} + R_M(t), \tag{5.26}$$

where $R_M(t)$ is $C^{M+n/2-1}$ in a neighborhood of $t = \beta - \overline{x}_0$. Moreover,

$$B^\ell_{0jk} = C^\ell_0 \phi_j(q)\phi_k(q), \tag{5.27}$$

and, for $m \geq 1$,

$$B^\ell_{mjk} = C^\ell_{m1}\phi_j(q)\phi_k(q)\big(\nabla_y \cdot h''(q)^{-1}\nabla_y\big)^{m+1} h(y)\big|_{y=q} + B^{\ell R}_{mjk}. \tag{5.28}$$

Here $C^\ell_{m1} \neq 0$ and depends only on ℓ and m. Also, for $m \geq 1$, $B^{\ell R}_{mjk}$ is determined by ϕ_j, ϕ_k and a finite number of their derivatives at q and the derivatives of h of order $\leq 2m+1$ at q. In particular, the discussion surrounding (5.1)-(5.4) and the identity (2.3) mean we can, from knowledge of $S_{jk}(\lambda)$ for all $\lambda > |\sigma_1|$, recover the singularities of I_3 at $\beta - \overline{x}_0$ if the boundary conditions on $\partial\mathcal{O}$ are Dirichlet, and those of I_4 if the boundary conditions on $\mathcal{O}$ are Neumann.

6 Inverse problems with symmetry

Here we prove Theorem 1.1, that knowledge of $S_{kk}(\lambda)$ for all $\lambda \in [|\sigma_1|, \infty)$ determines the obstacle $\mathcal{O}$ uniquely, given hypotheses (1.5)–(1.8), and given knowledge of the point $q \in Y_1$ and that $\phi_k(q) \neq 0$. As we have seen, such information uniquely determines the tip $p = (\overline{x}_0, q)$ of $\mathcal{O}$ via the first positive time for which a singularity of $u^s_{j,k,\alpha,\beta}$ occurs for α, $\beta \gg 0$. Let us translate coordinates in Y_1 to make q the origin: $q = 0$. Then hypothesis (1.8) says $h(y)$ is a radial function of y; hence

$$h(y) = \sum_{k\geq 0} \tau_{2k}|y|^{2k}. \tag{6.1}$$

Comparison with the standard power series formula

$$h(y) = \sum_{\alpha \geq 0} \frac{h^{(\alpha)}(0)}{\alpha!} y^\alpha, \tag{6.2}$$

via the multinomial formula

$$|y|^{2k} = (y_1^2 + \cdots + y_n^2)^k = \sum_{|\alpha|=2k} C^n_k(\alpha) y^\alpha, \tag{6.3}$$

yields the identities

$$|\alpha| = 2k \Longrightarrow h^{(\alpha)}(0) = \alpha! C_k^n(\alpha)\tau_{2k}, \tag{6.4}$$

while, of course,

$$|\alpha| = 2k+1 \Longrightarrow h^{(\alpha)}(0) = 0. \tag{6.5}$$

Also the formula

$$\Delta = -(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}) \tag{6.6}$$

for the Laplacian on radial functions gives

$$\Delta\, |y|^{2k} = -2k(2k+n-2)|y|^{2k-2}, \tag{6.7}$$

and then, inductively,

$$\Delta^k\, |y|^{2k} = D_k^n, \tag{6.8}$$

a combinatorial function of n and k. Hence, for a radial function $h(y)$,

$$|\alpha| = 2k \Longrightarrow h^{(\alpha)}(0) = \frac{\alpha! C_k^n(\alpha)}{D_k^n}\,\Delta^k h(0) = E_k^n(\alpha)\,\Delta^k h(0). \tag{6.9}$$

According to Theorem 5.3, under the hypotheses of Theorem 1.1, the scattering data uniquely determine $\det h''(0)$. Given radial symmetry, we have

$$h''(0) = -\frac{1}{n}\Delta h(0)\, I, \tag{6.10}$$

where I is the $n \times n$ identity matrix, and hence

$$\det h''(0) = n^{-n}\big(-\Delta h(0)\big)^n. \tag{6.11}$$

Strict convexity at p implies $-\Delta h(0) < 0$, and then $\Delta h(0)$ is determined uniquely. Hence, by (6.9), $h^{(\alpha)}(0)$ is determined uniquely for all $|\alpha| \le 2$. Looking at (5.28), we see that

$$\big(\nabla_y \cdot h''(0)^{-1}\nabla_y\big)^{m+1} h(y)\big|_{y=0} = n^{m+1}\big(-\Delta h(0)\big)^{-(m+1)}\,(-\Delta)^{m+1}h(0). \tag{6.12}$$

Hence B^ℓ_{mjk} is a known nonzero multiple of $\Delta^{m+1}h(0)$ plus a quantity that depends, in a known fashion, on $h^{(\alpha)}(0)$ for $|\alpha| \le 2m+1$, hence on $\Delta^i h(0)$ for $i \le m$. Inductively, we conclude that $\Delta^{m+1}h(0)$ is uniquely determined by the scattering data, for all $m \ge 0$. Hence, by (6.9), the power series coefficients of h are uniquely determined. Analyticity implies $h(y)$ is uniquely determined for small y, hence $\partial\mathcal{O}$ is uniquely determined, locally. Connectivity then implies $\partial\mathcal{O}$ is uniquely determined globally. This proves Theorem 1.1.

7 Two-dimensional inverse problems without symmetry

In this section we study the inverse problem for $\mathcal{O} \subset X$, where $X \subset \mathbb{R}^2$ is a two-dimensional manifold. By working in two dimensions and assuming that we know two particular entries in the scattering matrix, we are able to recover $\mathcal{O}$. Theorem 7.1 extends [2, Theorem 1.2], which considered the case of Dirichlet boundary conditions on both ∂X and $\partial\mathcal{O}$.

Recall our convention that $0 \leq \sigma_1^2 \leq \sigma_2^2 \leq \cdots$ are the eigenvalues of the (nonnegative) Laplacian on $Y = \sqcup_{i=1}^{i_0} Y_i$ with Dirichlet or Neumann boundary conditions as inherited from the boundary conditions on ∂X, and that $\Delta_Y \phi_j = \sigma_j^2 \phi_j$, with each ϕ_j supported on a single connected component of Y. Note that if $Y_1 \simeq (0, \gamma)$ with the usual metric, then the eigenvalues of the Laplacian on Y_1 are $k^2(\pi/\gamma)^2$, $k \in \mathbb{N}$ for Dirichlet boundary conditions, and $k^2(\pi/\gamma)^2$, $k \in \mathbb{N} \cup \{0\}$ for Neumann boundary conditions. Choose j_k so that $\phi_{j_k} \in C^\infty(Y)$ is supported on Y_1 and satisfies $\Delta_{Y_1}\phi_{j_k} = k^2(\pi/\gamma)^2\phi_{j_k}$ and the boundary conditions on ∂Y_1.

Theorem 7.1 *Let $X \subset \mathbb{R}^2$, and let $\mathcal{O} \subset X$ satisfy assumption (1.5) and, in addition, assume $\partial\mathcal{O}$ is real analytic and connected. Consider the Laplacian on $X \setminus \mathcal{O}$ with either Dirichlet or Neumann boundary conditions on ∂X and Dirichlet or Neumann boundary conditions on $\mathcal{O}$. Then knowledge of both $S_{j_0 j_0}(\lambda)$ and $S_{j_0 j_1}(\lambda)$ (for Neumann boundary conditions on ∂X) or both $S_{j_1 j_1}(\lambda)$ and $S_{j_1 j_2}(\lambda)$ (for Dirichlet boundary conditions on ∂X) for all $\lambda > |\sigma_1|$ determines $\mathcal{O}$.*

We comment explicitly that it is not necessary to have the same boundary conditions on $\partial\mathcal{O}$ and ∂X.

Proof. The case of Dirichlet boundary conditions on both ∂X and $\partial\mathcal{O}$ is exactly [1, Theorem 1.2]. We give the proof for Neumann boundary conditions on both ∂X and $\partial\mathcal{O}$, and indicate how to adapt the proof for the remaining two combinations of boundary conditions.

As we saw earlier, $\overline{x}_0$ is determined by the given data, since ϕ_{j_0} is nonvanishing on Y_1. Note that given that $y_0 \in (0, \gamma)$, y_0 is determined by knowledge of $\cos(\pi y_0/\gamma) = \sqrt{2}\phi_{j_1}(y_0)/\sqrt{\gamma}$ with the identification of Y_1 with $(0, \gamma)$. Using Theorem 5.3 we can determine $\phi_{j_1}(q)$ by taking the ratios of the leading order singularities of $u^s_{j_1,j_0,\overline{x}_0,\beta}(t)$ and $u^s_{j_0,j_0,\overline{x}_0,\beta}(t)$ at $t = \beta - \overline{x}_0$. Thus q is determined by the data. Now $h''(q)$, which is a scalar here, can be recovered from S_{j_0,j_0}, using the fact that we know $h''(q) < 0$.

Now suppose we know $h^{(l)}(q)$ for $l \leq 2m$, $m \geq 1$. Combining [2, Proposition 4.6] with Proposition 4.3, Proposition 5.2 and the proof of Theorem 5.3, we see that from knowledge of $S_{j_0,j_0}(\lambda)$ and $S_{j_0,j_1}(\lambda)$, we can recover

$$[c_{m1}h''(q)h^{(2m+2)}(q) + c_{m2}h^{(3)}(q)h^{(2m+1)}(q)]\phi_j(q)\phi_{j_0}(q) + c_{m3}h''(q)h^{(2m+1)}(q)[\phi'_{j_0}(q)\phi_j(q) + \phi_{j_0}(q)\phi'_j(q)] \quad (7.1)$$

where c_{ml} are known nonzero constants and j is j_0 or j_1. Note that ϕ_{j_0} is a constant so that $\phi'_{j_0} \equiv 0$ and that ϕ'_{j_1} is nonzero on the interior of Y. Thus, from (7.1) with $j = j_0$ and

$j = j_1$ we are able to recover first $h^{(2m+1)}(q)$ and then $h^{(2m+2)}(q)$. Inductively, then, we are able to recover all the derivatives of h at q, and, using the assumption of analyticity and connectivity of $\partial\mathcal{O}$, we recover $\mathcal{O}$.

If we have Neumann boundary conditions on ∂X and Dirichlet conditions on $\partial\mathcal{O}$, the proof is essentially unchanged, using the fact that Theorem 5.3 handles both boundary conditions.

In the case of Dirichlet boundary conditions on ∂X and Neumann conditions on $\partial\mathcal{O}$, the proof can be adapted by first using [2, Lemma 6.1], which ensures, together with the results on the singularities of $u^s_{j,k,x_0^+,\beta}$ for Neumann boundary conditions on $\partial\mathcal{O}$ of this paper, that q can be determined from the given data. For the inductive step, assume one knows $h^{(l)}$ for $l \leq 2m$, $m \geq 1$. Then from $S_{j_1 j_1}(\lambda)$ and $S_{j_1 j_2}(\lambda)$ one can recover

$$[c_{m1}h''(q)h^{(2m+2)}(q) + c_{m2}h^{(3)}(q)h^{(2m+1)}(q)]\phi_j(q)\phi_{j_1}(q) + c_{m3}h''(q)h^{(2m+1)}(q)[\phi'_{j_1}(q)\phi_j(q) + \phi_{j_1}(q)\phi'_j(q)] \quad (7.2)$$

with $j = j_1$ or $j = j_2$; cf (7.1). But then [2, Lemma 6.2] ensures that from this data we can recover $h^{(2m+1)}(q)$ and $h^{(2m+2)}(q)$. □

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