# COHOMOLOGY OF A FLAG VARIETY AS A BETHE ALGEBRA

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*To the memory of V. I. Arnold*

Abstract. We interpret the equivariant cohomology $H^*_{GL_n}(\mathcal{F}_{\boldsymbol\lambda},\mathbb{C})$ of a partial flag variety $\mathcal{F}_{\boldsymbol\lambda}$ parametrizing chains of subspaces $0=F_0\subset F_1\subset\dots\subset F_N=\mathbb{C}^n$, $\dim F_i/F_{i-1}=\lambda_i$, as the Bethe algebra $\mathcal{B}^\infty(\mathcal{V}^\pm_{\boldsymbol\lambda})$ of the $\mathfrak{gl}_N$-weight subspace $\mathcal{V}^\pm_{\boldsymbol\lambda}$ of a $\mathfrak{gl}_N[t]$-module $\mathcal{V}^\pm$.

## 1. Introduction

A Bethe algebra of a quantum integrable model is a commutative algebra of linear operators (Hamiltonians) acting on the space of states of the model. An interesting problem is to describe the Bethe algebra as the algebra of functions on a suitable scheme. Such a description can be considered as an instance of the geometric Langlands correspondence, see [MTV2], [MTV3]. The $\mathfrak{gl}_N$ Gaudin model is an example of a quantum integrable model [G1], [G2]. The Bethe algebra $\mathcal{B}^K$ of the $\mathfrak{gl}_N$ Gaudin model is a commutative subalgebra of the current algebra $U(\mathfrak{gl}_N[t])$. The algebra $\mathcal{B}^K$ depends on the parameters $K=(K_1,\dots,K_N)\in\mathbb{C}^N$. Having a $\mathfrak{gl}_N[t]$-module $M$, one obtains the commutative subalgebra $\mathcal{B}^K(M)\subset\operatorname{End}(M)$ as the image of $\mathcal{B}^K$. The geometric interpretation of the algebra $\mathcal{B}^K(M)$ as the algebra of functions on a scheme leads to interesting objects. For example, the Bethe algebra $\mathcal{B}^{K=0}((\otimes_{s=1}^n L_{\boldsymbol\Lambda_s}(z_s))^{sing}_{\boldsymbol\lambda})$ of the subspace of singular vectors of the $\mathfrak{gl}_N$-weight $\boldsymbol\lambda$ of the tensor product of finite-dimensional evaluation modules $\otimes_{s=1}^n L_{\boldsymbol\Lambda_s}(z_s)$ is interpreted as the space of functions on the intersection of suitable Schubert cycles in a Grassmannian variety, see [MTV2]. This interpretation gives a relation between representation theory and Schubert calculus useful in both directions.

One of the most interesting $\mathfrak{gl}_N[t]$-modules is the vector space $\mathcal{V}=V^{\otimes n}\otimes\mathbb{C}[z_1,\dots,z_n]$ of $V^{\otimes n}$-valued polynomials in $z_1,\dots,z_n$, where $V=\mathbb{C}^N$ is the standard vector representation

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of $\mathfrak{gl}_N$. The Lie algebra $\mathfrak{gl}_N[t]$ naturally acts on $\mathcal{V}$ as well as the symmetric group $S_n$, which permutes the factors of $V^{\otimes n}$ and variables $z_1, \dots, z_n$ simultaneously. We denote by $\mathcal{V}^+$ and $\mathcal{V}^-$ the $S_n$-invariant and antiinvariant subspaces of $\mathcal{V}$, respectively. The actions of $\mathfrak{gl}_N[t]$ and $S_n$ on $\mathcal{V}$ commute, so $\mathcal{V}^+$ and $\mathcal{V}^-$ are $\mathfrak{gl}_N[t]$-submodules of $\mathcal{V}$. The Bethe algebra $\mathcal{B}^K$ preserves the $\mathfrak{gl}_N$-weight decompositions $\mathcal{V}^+ = \oplus_{\boldsymbol\lambda} \mathcal{V}^+_{\boldsymbol\lambda}$ and $\mathcal{V}^- = \oplus_{\boldsymbol\lambda} \mathcal{V}^-_{\boldsymbol\lambda}$, $\boldsymbol\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol\lambda| = n$. The Bethe algebra $\mathcal{B}^K(\mathcal{V}^+_{\boldsymbol\lambda})$ was described in [MTV3] as the algebra of functions on a suitable space of quasiexponentials $\{e^{K_i u}(u^{\lambda_i} + \Sigma_{i1} u^{\lambda_i - 1} + \dots + \Sigma_{i\lambda_i}),\ i = 1, \dots, N\}$. In this paper we give a similar description for $\mathcal{B}^K(\mathcal{V}^-_{\boldsymbol\lambda})$ and study the limit of the algebras $\mathcal{B}^K(\mathcal{V}^+_{\boldsymbol\lambda})$, $\mathcal{B}^K(\mathcal{V}^-_{\boldsymbol\lambda})$ as all coordinates of the vector $K$ tend to infinity so that $K_i/K_{i+1} \to \infty$ for all $i$. We show that in this limit both Bethe algebras $\mathcal{B}^\infty(\mathcal{V}^+_{\boldsymbol\lambda})$, $\mathcal{B}^\infty(\mathcal{V}^-_{\boldsymbol\lambda})$ can be identified with the algebra of the equivariant cohomology $H^*_{GL_n}(\mathcal{F}_{\boldsymbol\lambda}, \mathbb{C})$ of the partial flag variety $\mathcal{F}_{\boldsymbol\lambda}$ parametrizing chains of subspaces

$$0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathbb{C}^n,$$

$\dim F_i/F_{i-1} = \lambda_i$. This identification was motivated for us by the considerations in [RV], [RSV], where the equivariant cohomology of the partial flag varieties were used to construct certain conformal blocks in $V^{\otimes n}$.

Our identification of the Bethe algebra with the algebra of multiplication operators of the equivariant cohomology $H^*_{GL_n}(\mathcal{F}_{\boldsymbol\lambda}, \mathbb{C})$ can be considered as a degeneration of the recent description in [O] of the equivariant quantum cohomology of the partial flag varieties as the Bethe algebra of a suitable Yangian model associated with $V^{\otimes n}$, cf. [BMO].

In Section 2 we introduce the Bethe algebra. Section 3 contains the main results — Theorems 3.3, 3.4. Theorem 3.3 identifies the algebra of equivariant cohomology $H^*_{GL_n}(\mathcal{F}_{\boldsymbol\lambda}, \mathbb{C})$ and the Bethe algebras $\mathcal{B}^\infty(\mathcal{V}^+_{\boldsymbol\lambda})$, $\mathcal{B}^\infty(\mathcal{V}^-_{\boldsymbol\lambda})$. Theorem 3.4 says that the Shapovalov pairing of $\mathcal{V}^+_{\boldsymbol\lambda}$ and $\mathcal{V}^-_{\boldsymbol\lambda}$ is nondegenerate. In Section 4 we show that the isomorphisms of Theorem 3.3 are limiting cases of a geometric Langlands correspondence. In Section 5 we explain how the Bethe algebras $\mathcal{B}^\infty(\mathcal{V}^+_{\boldsymbol\lambda})$, $\mathcal{B}^\infty(\mathcal{V}^-_{\boldsymbol\lambda})$ are related to the quantum equivariant cohomology $QH_{GL_n \times \mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol\lambda})$ of the cotangent bundle $T^*\mathcal{F}_{\boldsymbol\lambda}$ of the flag variety $\mathcal{F}_{\boldsymbol\lambda}$. Appendix contains the topological description of $\mathfrak{gl}_N[t]$-actions on $\oplus_{\boldsymbol\lambda} H^*_{GL_n}(\mathcal{F}_{\boldsymbol\lambda}, \mathbb{C})$.

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## 2. Representations of current algebra $\mathfrak{gl}_N[t]$

2.1. **Lie algebra $\mathfrak{gl}_N$.** Let $e_{ij}$, $i, j = 1, \dots, N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$ satisfying the relations $[e_{ij}, e_{sk}] = \delta_{js} e_{ik} - \delta_{ik} e_{sj}$. We denote by $\mathfrak{h} \subset \mathfrak{gl}_N$ the subalgebra generated by $e_{ii}$, $i = 1, \dots, N$. For a Lie algebra $\mathfrak{g}$, we denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

A vector $v$ of a $\mathfrak{gl}_N$-module $M$ has weight $\boldsymbol\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ if $e_{ii} v = \lambda_i v$ for $i = 1, \dots, N$. We denote by $M_{\boldsymbol\lambda} \subset M$ the weight subspace of weight $\boldsymbol\lambda$.

Let $V = \mathbb{C}^N$ be the standard vector representation of $\mathfrak{gl}_N$ with basis $v_1, \dots, v_N$ such that $e_{ij} v_k = \delta_{jk} v_i$ for all $i, j, k$. A tensor power $V^{\otimes n}$ of the vector representation has a basis given by the vectors $v_{i_1} \otimes \dots \otimes v_{i_n}$, where $i_j \in \{1, \dots, N\}$. Every such sequence $(i_1, \dots, i_n)$ defines a decomposition $I = (I_1, \dots, I_N)$ of $\{1, \dots, n\}$ into disjoint subsets $I_1, \dots, I_N$: $I_j = \{k \mid i_k = j\}$. We denote the basis vector $v_{i_1} \otimes \dots \otimes v_{i_n}$ by $v_I$.

Let

$$V^{\otimes n} = \bigoplus_{\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0},\,|\boldsymbol{\lambda}|=n} (V^{\otimes n})_{\boldsymbol{\lambda}}$$

be the weight decomposition. Denote $\mathcal{I}_{\boldsymbol{\lambda}}$ the set of all indices $I$ with $|I_j| = \lambda_j,\ j = 1,\dots N$. The vectors $\{v_I, I \in \mathcal{I}_{\boldsymbol{\lambda}}\}$ form a basis of $(V^{\otimes n})_{\boldsymbol{\lambda}}$. The dimension of $(V^{\otimes n})_{\boldsymbol{\lambda}}$ equals the multinomial coefficient $d_{\boldsymbol{\lambda}} := \frac{n!}{\lambda_1!\dots\lambda_N!}$.

Let $\mathcal{S}$ be the bilinear form on $V^{\otimes n}$ such that the basis $\{v_I\}$ is orthonormal. We call $\mathcal{S}$ the Shapovalov form.

2.2. **Current algebra $\mathfrak{gl}_N[t]$.** Let $\mathfrak{gl}_N[t] = \mathfrak{gl}_N \otimes \mathbb{C}[t]$ be the Lie algebra of $\mathfrak{gl}_N$-valued polynomials with pointwise commutator. We identify $\mathfrak{gl}_N$ with the subalgebra $\mathfrak{gl}_N \otimes 1$ of constant polynomials in $\mathfrak{gl}_N[t]$. Hence any $\mathfrak{gl}_N[t]$-module has the canonical structure of a $\mathfrak{gl}_N$-module.

The Lie algebra $\mathfrak{gl}_N[t]$ has a basis $e_{ij} \otimes t^r$, $i, j = 1,\dots,N$, $r \in \mathbb{Z}_{\geqslant 0}$, such that

$$[e_{ij} \otimes t^r, e_{sk} \otimes t^p] \,=\, \delta_{js} e_{ik} \otimes t^{r+p} - \delta_{ik} e_{sj} \otimes t^{r+p}\,.$$

It is convenient to collect elements of $\mathfrak{gl}_N[t]$ in generating series of a variable $u$. For $g \in \mathfrak{gl}_N$, set $g(u) = \sum_{s=0}^{\infty} (g \otimes t^s) u^{-s-1}$.

The subalgebra $\mathfrak{z}_N[t] \subset \mathfrak{gl}_N[t]$ with basis $\sum_{i=1}^N e_{ii} \otimes t^r$, $r \in \mathbb{Z}_{\geqslant 0}$, is central.

2.3. **The $\mathfrak{gl}_N[t]$-modules $\mathcal{V}^{\pm}$.** Let $S_n$ be the permutation group on $n$ elements. For an $S_n$-module $M$ we denote by $M^+$ (resp. $M^-$) the subspace of $S_n$-invariants (resp. antiinvariants).

The group $S_n$ acts on $\mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1,\dots,z_n]$ by permuting the variables. Denote by $\sigma_s(\boldsymbol{z})$, $s = 1,\dots,n$, the $s$th elementary symmetric polynomial in $z_1,\dots,z_n$.

Let $\mathcal{V}$ be the vector space of polynomials in variables $z_1,\dots,z_n$ with coefficients in $V^{\otimes n}$:

$$\mathcal{V} \,=\, V^{\otimes n} \otimes_{\mathbb{C}} \mathbb{C}[\boldsymbol{z}]\,.$$

The symmetric group $S_n$ acts on $\mathcal{V}$ by permuting the factors of $V^{\otimes n}$ and the variables $z_1,\dots,z_n$ simultaneously,

$$\sigma\big(v_1 \otimes \dots \otimes v_n \otimes p(z_1,\dots,z_n)\big) \,=\, v_{(\sigma^{-1})_1} \otimes \dots \otimes v_{(\sigma^{-1})_n} \otimes p(z_{\sigma_1},\dots,z_{\sigma_n})\,, \quad \sigma \in S_n\,.$$

We are interested in the subspaces $\mathcal{V}^+, \mathcal{V}^- \subset \mathcal{V}$ of $S_n$-invariants and antiinvariants.

The space $\mathcal{V}$ is a $\mathfrak{gl}_N[t]$-module,

$$g \otimes t^r \left(v_1 \otimes \dots \otimes v_n \otimes p(\boldsymbol{z})\right) \,=\, \sum_{s=1}^n v_1 \otimes \dots \otimes g v_s \otimes \dots \otimes v_n \otimes z_s^r p(\boldsymbol{z})\,.$$

The image of the subalgebra $U(\mathfrak{z}_N[t]) \subset U(\mathfrak{gl}_N[t])$ in $\mathrm{End}(\mathcal{V})$ is the algebra of operators of multiplication by elements of $\mathbb{C}[\boldsymbol{z}]^+$. The $\mathfrak{gl}_N[t]$-action on $\mathcal{V}$ commutes with the $S_n$-action. Hence $\mathcal{V}^+$ and $\mathcal{V}^-$ are $\mathfrak{gl}_N[t]$-submodules of $\mathcal{V}$. The subspaces $\mathcal{V}^+$ and $\mathcal{V}^-$ are free $\mathbb{C}[\boldsymbol{z}]^+$-modules of rank $N^n$.

Consider the $\mathfrak{gl}_N$-weight decompositions

$$\mathcal{V}^+ = \oplus_{\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0},|\boldsymbol{\lambda}|=n} \mathcal{V}^+_{\boldsymbol{\lambda}}, \qquad \mathcal{V}^- = \oplus_{\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0},|\boldsymbol{\lambda}|=n} \mathcal{V}^-_{\boldsymbol{\lambda}}.$$

For any $\boldsymbol{\lambda}$, the subspaces $\mathcal{V}^+_{\boldsymbol{\lambda}}$ and $\mathcal{V}^-_{\boldsymbol{\lambda}}$ are free $\mathbb{C}[\boldsymbol{z}]^+$-modules of rank $d_{\boldsymbol{\lambda}}$.

Denote by $\frac{1}{D}\mathcal{V}^-$ the vector space of all $V^{\otimes n}$-valued rational functions of the form $\frac{1}{D}x$, $x \in \mathcal{V}^-$, $D = \prod_{1\leqslant i<j\leqslant n}(z_j - z_i)$. The Shapovalov form induces a $\mathbb{C}[\boldsymbol{z}]^+$-bilinear map

$$\mathcal{S}_{+-} : \mathcal{V}^+ \otimes \frac{1}{D}\mathcal{V}^- \to \mathbb{C}[\boldsymbol{z}]^+.$$

The $\mathfrak{gl}_N[t]$-module structures on $\mathcal{V}^+$ and $\frac{1}{D}\mathcal{V}^-$ are contravariantly related through the Shapovalov form,

$$\mathcal{S}_{+-}\big((e_{ij}\otimes t^r)x, \frac{1}{D}\,y\big) \,=\, \mathcal{S}_{+-}\big(x, (e_{ji}\otimes t^r)\frac{1}{D}\,y\big) \qquad \text{for all } \ i,j,x,y.$$

2.4. **Bethe algebra.** Given an $N\times N$ matrix $A$ with possibly noncommuting entries $a_{ij}$, we define its row determinant to be

$$\operatorname{rdet} A \,=\, \sum_{\sigma\in S_N} (-1)^\sigma\, a_{1\sigma(1)}a_{2\sigma(2)}\dots a_{N\sigma(N)}\,.$$

Let $K = (K_1,\dots,K_N)$ be a sequence of complex numbers. Let $\partial$ be the operator of differentiation in a variable $u$. Define the universal differential operator $\mathcal{D}^K$ by

$$\mathcal{D}^K \,=\, \operatorname{rdet}\begin{pmatrix} \partial - K_1 - e_{11}(u) & -e_{21}(u) & \dots & -e_{N1}(u) \\ -e_{12}(u) & \partial - K_2 - e_{22}(u) & \dots & -e_{N2}(u) \\ \dots & \dots & \dots & \dots \\ -e_{1N}(u) & -e_{2N}(u) & \dots & \partial - K_N - e_{NN}(u) \end{pmatrix}.$$

It is a differential operator in the variable $u$, whose coefficients are formal power series in $u^{-1}$ with coefficients in $U(\mathfrak{gl}_N[t])$,

$$\mathcal{D}^K \,=\, \partial^N + \sum_{i=1}^N B^K_i(u)\,\partial^{N-i}, \qquad B^K_i(u) \,=\, \sum_{j=0}^\infty B^K_{ij}\,u^{-j}$$

and $B^K_{ij} \in U(\mathfrak{gl}_N[t])$ for $i = 1,\dots,N,\ j \geqslant 0$.

Denote by $\mathcal{B}^K$ the unital subalgebra of $U(\mathfrak{gl}_N[t])$ generated by $B^K_{ij}$ with $i = 1,\dots,N$, $j \geqslant 0$. The subalgebra $\mathcal{B}^K$ is called the Bethe algebra with parameters $K$.

**Theorem 2.1.** *The algebra $\mathcal{B}^K$ is commutative. The algebra $\mathcal{B}^K$ commutes with the subalgebra $U(\mathfrak{h}) \subset U(\mathfrak{gl}_N[t])$. If $K = 0$, then the algebra $\mathcal{B}^{K=0}$ commutes with the subalgebra $U(\mathfrak{gl}_N) \subset U(\mathfrak{gl}_N[t])$.* □

The theorem is proved for $K = 0$ in [T] and for nonzero $K$ in [CT], [MTV1].

Each element $B^K_{ij}$ is a polynomial in $K_1,\dots,K_N$. We define $\mathcal{B}^\infty$ to be the unital subalgebra of $U(\mathfrak{gl}_N[t])$ generated by the leading terms of the elements $B^K_{ij}$, $i = 1,\dots,N,\ j \geqslant 0$, as $K$ tends to infinity so that $K_i/K_{i+1} \to \infty$ for all $i$.

**Lemma 2.2.** *The algebra $\mathcal{B}^\infty$ is the unital subalgebra generated by the elements $e_{ii}\otimes t^j$ with $i = 1,\dots,N,\ j \geqslant 0$.*

*Proof.* We have $B^K_{i0} = (-1)^i K_1 \dots K_i \bigl(1 + o(1)\bigr)$, and

$$B^K_{ij} = (-1)^i K_1 \dots K_{i-1} \Bigl(\sum_{m=i}^{N} e_{mm} \otimes t^{j-1} + o(1)\Bigr)$$

for $j > 0$, where $o(1)$ stands for the terms vanishing as $K$ tends to infinity. □

**Remark.** There are $N!$ asymptotic zones labeled by elements of $S_N$ in which $K$ may tend to infinity. For $\sigma \in S_N$ we may assume that all coordinates of $K$ tend to infinity and $K_{\sigma_i}/K_{\sigma_{i+1}} \to \infty$ for all $i$. It is easy to see that the limiting Bethe algebra $\mathcal{B}^\infty$ does not depend on $\sigma$.

The algebra $\mathcal{B}^\infty$ is commutative and contains $U(\mathfrak{z}_N[t])$. The algebra $\mathcal{B}^\infty$ commutes with the subalgebra $U(\mathfrak{h}) \subset U(\mathfrak{gl}_N[t])$.

As a subalgebra of $U(\mathfrak{gl}_N[t])$, the Bethe algebra $\mathcal{B}^K$ acts on any $\mathfrak{gl}_N[t]$-module $M$. Since $\mathcal{B}^K$ commutes with $U(\mathfrak{h})$, it preserves the weight subspaces $M_{\boldsymbol{\lambda}}$. If $K = 0$, then $\mathcal{B}^{K=0}$ preserves the singular weight subspaces $M^{sing}_{\boldsymbol{\lambda}}$. We will study the action of $\mathcal{B}^\infty$ on the weight subspaces $\mathcal{V}^+_{\boldsymbol{\lambda}}$, $\mathcal{V}^-_{\boldsymbol{\lambda}}$.

**Lemma 2.3.** *The element $\sum_{i=1}^N e_{ii} \otimes t^r \in U(\mathfrak{z}_N[t])$ acts on $\mathcal{V}$ as the operator of multiplication by $\sum_{s=1}^n z_s^r$.* □

If $L \subset M$ is a $\mathcal{B}^K$-invariant subspace, then the image of $\mathcal{B}^K$ in $\mathrm{End}(L)$ will be called the Bethe algebra of $L$ and denoted by $\mathcal{B}^K(L)$.

## 3. Equivariant cohomology of partial flag varieties

3.1. **Partial flag varieties.** For $\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}| = n$, consider the partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$ parametrizing chains of subspaces

$$0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathbb{C}^n$$

with $\dim F_i/F_{i-1} = \lambda_i$, $i = 1, \dots, N$.

Let $T^n \subset GL_n$ be the torus of diagonal matrices. The group $T^n \subset GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$ and hence on $\mathcal{F}_{\boldsymbol{\lambda}}$. The set of fixed points $\mathcal{F}^{T^n}_{\boldsymbol{\lambda}}$ of the torus action consists of coordinate flags $F_I = (F_0 \subset \dots \subset F_N)$, $I = (I_1, \dots, I_N) \in \mathcal{I}_{\boldsymbol{\lambda}}$, where $F_i$ is the span of the basis vectors $v_j \in \mathbb{C}^n$ with $j \in I_1 \cup \dots \cup I_i$. The fixed points are in a one-to-one correspondence with the elements of $\mathcal{I}_{\boldsymbol{\lambda}}$ and hence with the basis vectors of $V_{\boldsymbol{\lambda}}$.

We consider the $GL_n(\mathbb{C})$-equivariant cohomology

$$H_{\boldsymbol{\lambda}} = H^*_{GL_n}(\mathcal{F}_{\boldsymbol{\lambda}}, \mathbb{C}).$$

Denote by $\Gamma_i = \{\gamma_{i1}, \dots, \gamma_{i\lambda_i}\}$ the set of the Chern roots of the bundle over $\mathcal{F}_{\boldsymbol{\lambda}}$ with fiber $F_i/F_{i-1}$. Denote by $\boldsymbol{z} = \{z_1, \dots, z_n\}$ the Chern roots corresponding to the factors of the torus $T^n$. Then

$$H_{\boldsymbol{\lambda}} = \mathbb{C}[\boldsymbol{z}; \Gamma_1; \dots; \Gamma_N]^{S_n \times S_{\lambda_1} \times \dots \times S_{\lambda_N}} \Big/ \Bigl\langle \prod_{i=1}^{N} \prod_{j=1}^{\lambda_i} (1 + u\gamma_{ij}) = \prod_{i=1}^{n} (1 + uz_i) \Bigr\rangle. \tag{3.1}$$

The cohomology $H_{\boldsymbol{\lambda}}$ is a module over $H^*_{GL_n}(pt, \mathbb{C}) = \mathbb{C}[\boldsymbol{z}]^+$.

Let $J_H \subset H_{\boldsymbol\lambda}$ be the ideal generated by the polynomials $\sigma_i(\boldsymbol z)$, $i = 1, \dots, n$. Then $H_{\boldsymbol\lambda}/J_H = H^*(\mathcal F_{\boldsymbol\lambda}, \mathbb C)$ .

3.2. **Integration over $\mathcal F_{\boldsymbol\lambda}$.** We will need the integration map $\int : H_{\boldsymbol\lambda} \to H^*_{GL_n}(pt, \mathbb C)$. The following formula (3.2) gives the integration map in terms of the fixed point set $\mathcal F_{\boldsymbol\lambda}^{T^n}$.

For a subset $A \subset \{1, \dots, N\}$ denote $\boldsymbol z_A = \{z_a,\, a \in A\}$. For $I = (I_1, \dots, I_N) \in \mathcal I_{\boldsymbol\lambda}$ denote

$$R(\boldsymbol z_{I_1}|\boldsymbol z_{I_2}| \dots |\boldsymbol z_{I_m}) \,=\, \prod_{i<j}\ \prod_{a\in I_i,\, b\in I_j} (z_b - z_a).$$

The Atiyah-Bott equivariant localization theorem [AB] says that for any $[h(\boldsymbol z, \Gamma_1, \dots, \Gamma_N)] \in H_{\boldsymbol\lambda}$,

$$\int [h] \,=\, \sum_{I\in\mathcal I_{\boldsymbol\lambda}} \frac{h(\boldsymbol z, \boldsymbol z_{I_1}, \dots, \boldsymbol z_{I_N})}{R(\boldsymbol z_{I_1}|\boldsymbol z_{I_2}| \dots |\boldsymbol z_{I_N})}\ . \tag{3.2}$$

More precisely, it is enough to verify this identity after the base change $H^*_{GL_n}(\mathcal F_{\boldsymbol\lambda}) \to H^*_{T^n}(\mathcal F_{\boldsymbol\lambda})$, where $T^n \subset GL_n$ is the maximal torus, and in the $T^n$-equivariant cohomology this is the Atiyah-Bott localization formula. Note that the expression for $R$ is nothing else but the Euler class of the tangent space at a $T^n$-fixed point, see formula (2.5) in [RSV].

Clearly, the right hand side in (3.2) lies in $\mathbb C[\boldsymbol z]^+$. The integration map induces the pairing

$$(\,,\,)\,:\, H_{\boldsymbol\lambda} \otimes H_{\boldsymbol\lambda} \to \mathbb C[\boldsymbol z]^+, \qquad [h] \otimes [g] \mapsto \int [hg]\ .$$

After factorization by the ideal $J_H$ we obtain the nondegenerate Poincare pairing

$$(\,,\,)\,:\, H^*(\mathcal F_{\boldsymbol\lambda}, \mathbb C) \otimes H^*(\mathcal F_{\boldsymbol\lambda}, \mathbb C) \ \to\ \mathbb C.$$

3.3. $H_{\boldsymbol\lambda}$ **and** $\mathcal V^\pm$**.**

**Lemma 3.1.** *The maps*

$$i^+_{\boldsymbol\lambda}\,:\, H_{\boldsymbol\lambda} \to \mathcal V^+_{\boldsymbol\lambda}, \qquad [h(\boldsymbol z, \Gamma_1, \dots, \Gamma_N)] \,\mapsto\, \sum_{I\in\mathcal I_{\boldsymbol\lambda}} v_I \otimes h(\boldsymbol z, \boldsymbol z_{I_1}, \dots, \boldsymbol z_{I_N}),$$

$$i^-_{\boldsymbol\lambda}\,:\, H_{\boldsymbol\lambda} \to \frac{1}{D}\mathcal V^-_{\boldsymbol\lambda}, \qquad [h(\boldsymbol z, \Gamma_1, \dots, \Gamma_N)] \,\mapsto\, \sum_{I\in\mathcal I_{\boldsymbol\lambda}} v_I \otimes \frac{h(\boldsymbol z, \boldsymbol z_{I_1}, \dots, \boldsymbol z_{I_N})}{R(\boldsymbol z_{I_1}|\boldsymbol z_{I_2}| \dots |\boldsymbol z_{I_N})}$$

*are well-defined isomorphisms of* $\mathbb C[\boldsymbol z]^+$*-modules.* □

*Proof.* If $h$ belongs to the ideal of relations in (3.1) then $h(z, z_{I_1}, \dots, z_{I_N}) = 0$ for any $I$, because the $\Gamma_i = z_{I_i}$ substitution makes the generators of the ideal identities. This proves well-definedness.

Consider the $\mathbb C[\boldsymbol z]^+$-module $\mathbb C[\boldsymbol z]^{S_{\lambda_1}\times\dots\times S_{\lambda_N}}$ of polynomials symmetric in the first $\lambda_1$ variables, the next $\lambda_2$ variables, etc. In Schubert calculus it is known that this module is free of rank $d_{\boldsymbol\lambda}$, and that it is isomorphic to $H_{\boldsymbol\lambda}$ under the correspondence

$$p \in \mathbb C[\boldsymbol z]^{S_{\lambda_1}\times\dots\times S_{\lambda_N}} \ \longleftrightarrow\ \big[p(\Gamma_1, \dots, \Gamma_N)\big] \in H_{\boldsymbol\lambda}. \tag{3.3}$$

An element $\sum_{I\in\mathcal I_{\boldsymbol\lambda}} v_I \otimes p_I(\boldsymbol z)$ of $\mathcal V_{\boldsymbol\lambda}$ belongs to $\mathcal V^+_{\boldsymbol\lambda}$, if and only if $p_I(\boldsymbol z) = p(z_{I_1}, \dots, z_{I_N})$ for a polynomial $p \in \mathbb C[\boldsymbol z]^{S_{\lambda_1}\times\dots\times S_{\lambda_N}}$. This shows that $\mathcal V^+_{\boldsymbol\lambda}$ is isomorphic to $\mathbb C[\boldsymbol z]^{S_{\lambda_1}\times\dots\times S_{\lambda_N}}$, and that $i^+_{\boldsymbol\lambda}$ is the composition of this isomorphism with (3.3).

A similar argument shows that $i^-_{\boldsymbol\lambda}$ is also an isomorphism. $\square$

**Corollary 3.2.** *The Shapovalov form and the Poincare pairing are related by the formula*

$$\mathcal{S}_{+-}(i_+[h], i_-[g]) = \int [h][g]\ .$$

$\square$

Let $A$ be a commutative algebra. The algebra $A$ considered as an $A$-module is called the regular representation of $A$. Here is our main result.

**Theorem 3.3.**

(i) *The maps* $\xi^{\pm}_{\boldsymbol\lambda} : e_{ii} \otimes t^r|_{\mathcal{V}^{\pm}_{\boldsymbol\lambda}} \mapsto \sum_{j=1}^{\lambda_i} \gamma^r_{ij}$ *define isomorphisms of the algebras* $\mathcal{B}^\infty(\mathcal{V}^{\pm}_{\boldsymbol\lambda})$ *and* $H_{\boldsymbol\lambda}$ *.*
(ii) *The maps* $\xi^+_{\boldsymbol\lambda}$, $i^+_{\boldsymbol\lambda}$ *identify the* $\mathcal{B}^\infty(\mathcal{V}^+_{\boldsymbol\lambda})$*-module* $\mathcal{V}^+_{\boldsymbol\lambda}$ *with the regular representation of* $H_{\boldsymbol\lambda}$*.*
(iii) *The maps* $\xi^-_{\boldsymbol\lambda}$, $i^-_{\boldsymbol\lambda}$ *identify the* $\mathcal{B}^\infty(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$*-module* $\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}$ *with the regular representation of* $H_{\boldsymbol\lambda}$*.*

The theorem follows from Lemmas 2.2, 2.3 and 3.1.

3.4. **Cohomology as $\mathfrak{gl}_N[t]$-modules.** Let $J$ be the ideal of $\mathbb{C}[\boldsymbol z]^+$ generated by the elementary symmetric functions $\sigma_i(\boldsymbol z)$, $i = 1, \dots, n$. Define $J^+ = J\mathcal{V}^+$ and $J^- = \frac{1}{D}J\mathcal{V}^-$. Clearly, $J^+$ is a $\mathfrak{gl}_N[t]$-submodule of $\mathcal{V}^+$ and $J^-$ is a $\mathfrak{gl}_N[t]$-submodule of $\frac{1}{D}\mathcal{V}^-$. The $\mathfrak{gl}_N[t]$-module $\mathcal{V}^+/J^+$ is graded and has dimension $N^n$ over $\mathbb{C}$, see [MTV2]. Similarly, $\frac{1}{D}\mathcal{V}^-/J^-$ is a graded $\mathfrak{gl}_N[t]$-module of the same dimension.

**Theorem 3.4.** *The Shapovalov form establishes a nondegenerate pairing*

$$\mathcal{S}_{+-}\ :\ \mathcal{V}^+/J^+ \otimes \frac{1}{D}\mathcal{V}^-/J^-\ \to\ \mathbb{C}.$$

The theorem follows from Lemmas 3.1, 3.2 and the nondegeneracy of the Poincare pairing.

**Corollary 3.5.** *The* $\mathfrak{gl}_N[t]$*-modules* $\mathcal{V}^+/J^+$ *and* $\frac{1}{D}\mathcal{V}^-/J^-$ *are contravariantly related through the Shapovalov form,* $\mathcal{S}_{+-}\big((e_{ij}\otimes t^r)x, \frac{1}{D}y\big) = \mathcal{S}_{+-}\big(x, (e_{ji}\otimes t^r)\frac{1}{D}y\big)$ *for all* $i, j, x, y$*.*

Let $W_n$ be the $\mathfrak{gl}_N[t]$-module generated by a vector $w_n$ with the defining relations:

$$\begin{aligned} e_{ii}(u)w_n &= \delta_{1i}\,\frac{n}{u}\,w_n\,, && i = 1, \dots, N\,, \\ e_{ij}(u)w_n &= 0\,, && 1 \leqslant i < j \leqslant N\,, \\ (e_{ji}\otimes 1)^{n\delta_{1i}+1}w_n &= 0\,, && 1 \leqslant i < j \leqslant N\,. \end{aligned}$$

As an $\mathfrak{sl}_N[t]$-module, the module $W_n$ is isomorphic to the Weyl module from [CL], [CP], corresponding to the weight $n\omega_1$, where $\omega_1$ is the first fundamental weight of $\mathfrak{sl}_N$.

In [MTV2] an isomorphism of $\mathcal{V}^+/J^+$ and the Weyl module $W_n$ is constructed.

**Corollary 3.6.** *The Shapovalov form* $\mathcal{S}_{+-}$ *establishes an isomorphism of* $\frac{1}{D}\mathcal{V}^-/J^-$ *and the contravariantly dual of the Weyl module* $W_n$*.*

Here is an application of this fact. For $\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}| = n$, $\lambda_1 \geqslant \cdots \geqslant \lambda_N$, denote

$$(\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol{\lambda}} \;=\; \{v \in \frac{1}{D}\mathcal{V}^-/J^- \mid e_{ij}v = 0 \text{ for } i<j,\ e_{ii}v = \lambda_i v \ \text{ for } \ i = 1,\dots,N\}\,.$$

This is a graded space. Denote by $\big((\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol{\lambda}}\big)_k$ the subspace of all elements of $\boldsymbol{z}$-degree $k$. Define the graded character by the formula

$$\mathrm{ch}\big((\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol{\lambda}}\big) \;=\; \sum_k q^k \dim((\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol{\lambda}})_k.$$

**Corollary 3.7.** *We have*

$$\mathrm{ch}\big((\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol{\lambda}}\big) \;=\; \frac{(q)_n \prod_{1\leqslant i<j\leqslant N}(1-q^{\lambda_i-\lambda_j+j-i})}{\prod_{i=1}^N (q)_{\lambda_i+N-i}}\; q^{-\sum_{1\leqslant i<j\leqslant N}\lambda_i\lambda_j}, \tag{3.4}$$

*where* $(q)_a = \prod_{j=1}^a (1-q^j)$ .

The corollary follows from Lemma 2.2 in [MTV2] and Corollary 3.6.

The isomorphisms

$$i^+ = \bigoplus_{\boldsymbol{\lambda}} i^+_{\boldsymbol{\lambda}} \,:\, \bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}} \,\to\, \mathcal{V}^+, \qquad i^- = \bigoplus_{\boldsymbol{\lambda}} i^-_{\boldsymbol{\lambda}} \,:\, \bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}} \,\to\, \frac{1}{D}\mathcal{V}^- \tag{3.5}$$

induce two graded $\mathfrak{gl}_N[t]$-module structures on $\oplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$ denoted by $\rho^+$ and $\rho^-$, respectively. These module structures descend to two graded $\mathfrak{gl}_N[t]$-module structures on the cohomology with constant coefficients

$$H(\mathbb{C}) \,:=\, \bigoplus_{\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0},\, |\boldsymbol{\lambda}|=n} H^*(\mathcal{F}_{\boldsymbol{\lambda}},\mathbb{C})\,,$$

denoted by the same letters $\rho^+$ and $\rho^-$.

**Corollary 3.8.** *The $\mathfrak{gl}_N[t]$-module $H(\mathbb{C})$ with the $\rho^+$-structure is isomorphic to the Weyl module $W_n$. The $\mathfrak{gl}_N[t]$-module $H(\mathbb{C})$ with the $\rho^-$-structure is isomorphic to the contravariant dual of the Weyl module $W_n$.*

The $\rho^\pm$ structures can be defined topologically, see [RSV] and Appendix. The $\rho^-$-structure appears to be more preferable. It was used in [RV], [RSV] to construct conformal blocks in the tensor power $V^{\otimes n}$.

## 4. Isomorphisms $i^\pm_{\boldsymbol{\lambda}}$ as a geometric Langlands correspondence

4.1. **The $\mathcal{V}^+_{\boldsymbol{\lambda}}$ case.** The following geometric description of the $\mathcal{B}^K$-action on $\mathcal{V}^+_{\boldsymbol{\lambda}}$ was given in [MTV3] as an example of the geometric Langlands correspondence.

Let $K = (K_1,\dots,K_N)$ be a sequence of distinct complex numbers. Let $\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0}$ , $|\boldsymbol{\lambda}| = n$. Introduce the polynomial algebras

$$\mathbb{C}[\boldsymbol{\Sigma}] \,:=\, \mathbb{C}[\Sigma_{ij},\, i=1,\dots,N,\, j=1,\dots,\lambda_i]\,, \qquad \mathbb{C}[\boldsymbol{\sigma}] \,:=\, \mathbb{C}[\sigma_1,\dots,\sigma_n]\,.$$

Define

$$\Sigma_i(u) \;=\; e^{K_i u}\,(u^{\lambda_i} + \Sigma_{i1}u^{\lambda_i-1} + \cdots + \Sigma_{i\lambda_i})\,, \qquad i = 1,\dots,N\,.$$

For arbitrary functions $g_1(u),\dots,g_N(u)$, introduce the Wronskian determinant by the formula

$$\mathrm{Wr}(g_1(u),\dots,g_N(u)) \,=\, \det\begin{pmatrix} g_1(u) & g_1'(u) & \dots & g_1^{(N-1)}(u)\\ g_2(u) & g_2'(u) & \dots & g_2^{(N-1)}(u)\\ \dots & \dots & \dots & \dots\\ g_N(u) & g_N'(u) & \dots & g_N^{(N-1)}(u)\end{pmatrix}.$$

We have

$$\mathrm{Wr}(\varSigma_1(u),\dots,\varSigma_N(u)) \,=\, e^{\sum_{i=1}^N K_i u}\prod_{1\leqslant i<j\leqslant N}(K_j-K_i)\cdot\Bigl(u^n+\sum_{s=1}^n(-1)^s\,A_s^K(\boldsymbol{\varSigma})\,u^{n-s}\Bigr),$$

where $A_1^K(\boldsymbol{\varSigma}),\dots,A_n^K(\boldsymbol{\varSigma})\in\mathbb{C}[\boldsymbol{\varSigma}]$. Define an algebra homomorphism

$$\mathcal{W}^K\,:\,\mathbb{C}[\boldsymbol{\sigma}]\to\mathbb{C}[\boldsymbol{\varSigma}],\qquad \sigma_s\mapsto A_s^K(\boldsymbol{\varSigma})\,.$$

The homomorphism defines a $\mathbb{C}[\boldsymbol{\sigma}]$-module structure on $\mathbb{C}[\boldsymbol{\varSigma}]$.

Define a differential operator $\mathcal{D}^K_{\boldsymbol{\Sigma}}$ by

$$\mathcal{D}^K_{\boldsymbol{\Sigma}} = \frac{1}{\mathrm{Wr}(\varSigma_1(u),\dots,\varSigma_N(u))}\,\mathrm{rdet}\begin{pmatrix} \varSigma_1(u) & \varSigma_1'(u) & \dots & \varSigma_1^{(N)}(u)\\ \varSigma_2(u) & \varSigma_2'(u) & \dots & \varSigma_2^{(N)}(u)\\ \dots & \dots & \dots & \dots\\ 1 & \partial & \dots & \partial^N\end{pmatrix}.$$

It is a differential operator in the variable $u$, whose coefficients are formal power series in $u^{-1}$ with coefficients in $\mathbb{C}[\boldsymbol{\varSigma}]$,

$$\mathcal{D}^K_{\boldsymbol{\Sigma}} = \partial^N+\sum_{i=1}^N F_i^K(u)\,\partial^{N-i},\qquad F_i^K(u)\,=\,\sum_{j=0}^\infty F_{ij}^K\,u^{-j}\,, \tag{4.1}$$

and $F_{ij}^K\in\mathbb{C}[\boldsymbol{\varSigma}],\ i=1,\dots,N,\ j\geqslant 0$.

**Theorem 4.1** ([MTV3])**.** *The map*

$$\tau_{\boldsymbol{\lambda}}^{K+}\,:\,B_{ij}^K|_{\mathcal{V}^+_{\boldsymbol{\lambda}}}\,\mapsto\,F_{ij}^K$$

*defines an isomorphism of the Bethe algebra $\mathcal{B}^K(\mathcal{V}^+_{\boldsymbol{\lambda}})$ and the algebra $\mathbb{C}[\boldsymbol{\varSigma}]$. The isomorphism $\tau_{\boldsymbol{\lambda}}^{K+}$ becomes an isomorphism of the $U(\mathfrak{z}_N[t])|_{\mathcal{V}^+_{\boldsymbol{\lambda}}}$-module $\mathcal{B}^K(\mathcal{V}^+_{\boldsymbol{\lambda}})$ and the $\mathbb{C}[\boldsymbol{\sigma}]$-module $\mathbb{C}[\boldsymbol{\varSigma}]$ if we identify the algebras $U(\mathfrak{z}_N[t])|_{\mathcal{V}^+_{\boldsymbol{\lambda}}}$ and $\mathbb{C}[\boldsymbol{\sigma}]$ by the map $\sigma_s[\boldsymbol{z}]\mapsto\sigma_s,\ s=1,\dots,n$.*

Denote

$$v^+\,=\,\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}v_I\in\mathcal{V}^+_{\boldsymbol{\lambda}}.$$

**Theorem 4.2** ([MTV3])**.** *The map*

$$\mu_{\boldsymbol{\lambda}}^{K+}\,:\,B_{ij}^K v^+\,\mapsto\,F_{ij}^K\,,$$

*defines a linear isomorphism $\mathcal{V}^+_{\boldsymbol{\lambda}}\to\mathbb{C}[\boldsymbol{\varSigma}]$. The maps $\tau_{\boldsymbol{\lambda}}^{K+},\mu_{\boldsymbol{\lambda}}^{K+}$ give an isomorphism of the $\mathcal{B}^K(\mathcal{V}^+_{\boldsymbol{\lambda}})$-module $\mathcal{V}^+_{\boldsymbol{\lambda}}$ and the regular representation of the algebra $\mathbb{C}[\boldsymbol{\varSigma}]$.*

4.2. **The limit of $\tau_{\boldsymbol\lambda}^{K+}$ and $\mu_{\boldsymbol\lambda}^{K+}$ as $K\to\infty$.** Let all the coordinates of the vector $K$ tend to infinity so that $K_i/K_{i+1}\to\infty$ for $i=1,\dots,N-1$. Then the homomorphism $\mathcal W^K$ has a limit $\mathcal W^\infty$. Namely, define $A_s^\infty(\boldsymbol\Sigma)$ by the formula

$$\prod_{i=1}^N\,(u^{\lambda_i}+\Sigma_{i1}u^{\lambda_i-1}+\dots+\Sigma_{i\lambda_i})\,=u^n+\sum_{s=1}^n\,(-1)^s A_s^\infty(\boldsymbol\Sigma)\,u^{n-s}\,.$$

Then

$$\mathcal W^\infty:\,\mathbb C[\boldsymbol\sigma]\,\to\,\mathbb C[\boldsymbol\Sigma],\qquad \sigma_s\,\mapsto\,A_s^\infty(\boldsymbol\Sigma)\,. \tag{4.2}$$

Define algebra homomorphisms

$$\mathbb C[\boldsymbol\sigma]\,\to\,\mathbb C[\boldsymbol z]^+,\qquad \eta:\mathbb C[\boldsymbol\Sigma]\,\to\,H_{\boldsymbol\lambda}\,, \tag{4.3}$$

by the agreement that the first one sends $\sigma_s$ to $\sigma_s(\boldsymbol z)$ for all $s$, and the second one sends $(-1)^s\Sigma_{is}$ to the $s$-th elementary symmetric function of $\gamma_{i1},\dots,\gamma_{i\lambda_i}$ for all $i,s$. Clearly, the defined maps are isomorphisms.

**Lemma 4.3.** *The isomorphisms* (4.3) *identify the* $\mathbb C[\boldsymbol\sigma]$*-module* $\mathbb C[\boldsymbol\Sigma]$ *defined by formula* (4.2) *and the* $\mathbb C[\boldsymbol z]^+$*-module* $H_{\boldsymbol\lambda}$*.*

Let $p_i(u)\,=\,u^{\lambda_i}+\Sigma_{i1}u^{\lambda_i-1}+\dots+\Sigma_{i\lambda_i}$ for all $i=1,\dots N$. Notice that

$$\eta\bigl(p_i(u)\bigr)\,=\,\prod_{j=1}^{\lambda_i}\,(u-\gamma_{ij})\,,\qquad \eta\Bigl(\frac{p_i'(u)}{p_i(u)}\Bigr)\,=\,\sum_{r=0}^\infty\,\sum_{j=1}^{\lambda_i}\,\gamma_{ij}^r\,u^{-r-1}\,. \tag{4.4}$$

**Lemma 4.4.** *We have* $F_{i0}^K\,=\,(-1)^i\,K_1\dots K_i\,\bigl(1+o(1)\bigr)$, *and*

$$\sum_{j=1}^\infty\,F_{ij}^Ku^{-j}\,=\,(-1)^i\,K_1\dots K_{i-1}\Bigl(\,\sum_{m=i}^N\,\frac{p_m'(u)}{p_m(u)}+o(1)\Bigr)\,,$$

*where* $o(1)$ *stands for the terms vanishing as* $K$ *tends to infinity.*

*Proof.* Let $y_i(u)\,=\,\mathrm{Wr}\bigl(\Sigma_i(u),\dots\Sigma_N(u)\bigr)$, $i\,=\,1,\dots N$. Then the operator $\mathcal D_{\boldsymbol\Sigma}^K$ can be factorized:

$$\mathcal D_{\boldsymbol\Sigma}^K\,=\,\Bigl(\partial-\frac{y_1'(u)}{y_1(u)}+\frac{y_2'(u)}{y_2(u)}\Bigr)\,\dots\,\Bigl(\partial-\frac{y_{N-1}'(u)}{y_{N-1}(u)}+\frac{y_N'(u)}{y_N(u)}\Bigr)\Bigl(\partial-\frac{y_N'(u)}{y_N(u)}\Bigr)\,, \tag{4.5}$$

see [MV]. Since

$$y_i(u)\,=\,(-1)^{(N-i)(N-i-1)/2}\,K_i^{N-i}\dots K_{N-1}\,\bigl(p_i(u)\dots p_N(u)+o(1)\bigr)\,e^{\sum_{m=i}^N K_m u}$$

as $K$ tends to infinity, the claim follows from formulae (4.1) and (4.5). $\square$

**Theorem 4.5.**

(i) *The map* $\eta\circ\tau_{\boldsymbol\lambda}^{K+}:\mathcal B^K(\mathcal V_{\boldsymbol\lambda}^+)\to H_{\boldsymbol\lambda}$ *tends to the isomorphism* $\xi_{\boldsymbol\lambda}^+:\mathcal B^\infty(\mathcal V_{\boldsymbol\lambda}^+)\to H_{\boldsymbol\lambda}$, *see Theorem 3.3, as* $K$ *tends to infinity.*

(ii) *The map* $\eta\circ\mu_{\boldsymbol\lambda}^{K+}:\mathcal V_{\boldsymbol\lambda}^+\,\to\,H_{\boldsymbol\lambda}$ *tends to the isomorphism* $(i_{\boldsymbol\lambda}^+)^{-1}:\mathcal V_{\boldsymbol\lambda}^+\,\to\,H_{\boldsymbol\lambda}$, *see Lemma 3.1, as* $K$ *tends to infinity.*

*Proof.* The statement follows from the definitions of the maps, Lemma 4.4, formulae (4.4), and the proof of Lemma 2.2. □

4.3. **The $\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}$ case.** Theorem 3.4 allows us to establish a geometric description of the $\mathcal{B}^K$-action on $\frac{1}{D}\mathcal{V}^-$ which is analogous to the description of the $\mathcal{B}^K$-action on $\mathcal{V}^+$.

**Theorem 4.6.** *The map*

$$\tau^{K-}_{\boldsymbol\lambda} \,:\, B^K_{ij}|_{\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}} \,\mapsto\, F^K_{ij}$$

*defines an isomorphism of the Bethe algebra $\mathcal{B}^K(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$ and the algebra $\mathbb{C}[\boldsymbol\Sigma]$. The isomorphism $\tau^{K-}_{\boldsymbol\lambda}$ becomes an isomorphism of the $U(\mathfrak{z}_N[t])|_{\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}}$-module $\mathcal{B}^K(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$ and the $\mathbb{C}[\boldsymbol\sigma]$-module $\mathbb{C}[\boldsymbol\Sigma]$ if we identify the algebras $U(\mathfrak{z}_N[t])|_{\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}}$ and $\mathbb{C}[\boldsymbol\sigma]$ by the map $\sigma_s[\boldsymbol z] \mapsto \sigma_s$, $s = 1, \dots, n$.*

Denote

$$v^- \,=\, \sum_{I\in\mathcal{I}_{\boldsymbol\lambda}} v_I \otimes \frac{1}{R(\boldsymbol z_{I_1}|\boldsymbol z_{I_2}|\dots|\boldsymbol z_{I_N})} \in \frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}\,.$$

**Theorem 4.7.** *The map*

$$\mu^{K-}_{\boldsymbol\lambda} \,:\, B^K_{ij} v^- \,\mapsto\, F^K_{ij}\,,$$

*defines a linear isomorphism $\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda} \to \mathbb{C}[\boldsymbol\Sigma]$. The maps $\tau^{K-}_{\boldsymbol\lambda}, \mu^{K-}_{\boldsymbol\lambda}$ give an isomorphism of the $\mathcal{B}^K(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$-module $\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}$ and the regular representation of the algebra $\mathbb{C}[\boldsymbol\Sigma]$.*

The proofs of Theorems 4.6 and 4.7 are basically word by word the same as the proofs of Theorems 4.1 and 4.2 in [MTV3].

It is interesting to note that the element $v^-$ becomes a conformal block under certain conditions and satisfies a *KZ* equation with respect to $\boldsymbol z$, see [V], [RV], [RSV].

4.4. **The limit of $\tau^{K-}_{\boldsymbol\lambda}$ and $\mu^{K-}_{\boldsymbol\lambda}$ as $K\to\infty$.** Let all the coordinates of the vector $K$ tend to infinity so that $K_i/K_{i+1}\to\infty$ for $i = 1, \dots, N-1$.

**Theorem 4.8.**

(i) *The map $\eta\circ\tau^{K-}_{\boldsymbol\lambda} : \mathcal{B}^K(\mathcal{V}^-_{\boldsymbol\lambda}) \to H_{\boldsymbol\lambda}$ tends to the isomorphism $\xi^-_{\boldsymbol\lambda} : \mathcal{B}^\infty(\mathcal{V}^-_{\boldsymbol\lambda}) \to H_{\boldsymbol\lambda}$, see Theorem 3.3, as $K$ tends to infinity.*

(ii) *The map $\eta\circ\mu^{K-}_{\boldsymbol\lambda} : \mathcal{V}^-_{\boldsymbol\lambda} \to H_{\boldsymbol\lambda}$ tends to the isomorphism $(i^-_{\boldsymbol\lambda})^{-1} : \mathcal{V}^-_{\boldsymbol\lambda} \to H_{\boldsymbol\lambda}$, see Lemma 3.1, as $K$ tends to infinity.*

The proof is similar to the proof of Theorem 4.5.

4.5. **The $(\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol\lambda}$ case.** Formula (3.4) for the graded character of $(\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol\lambda}$ is the analog of the formula for the graded character of $(\mathcal{V}^+/J^+)^{sing}_{\boldsymbol\lambda}$ in [MTV2]. The latter formula was used in [MTV2] to obtain a geometric description of the $\mathcal{B}^{K=0}$-action on $(\mathcal{V}^+)^{sing}_{\boldsymbol\lambda}$. Using formula (3.4) we can obtain a similar geometric description of the $\mathcal{B}^{K=0}$-action on $(\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol\lambda}$.

Let $\boldsymbol\lambda\in\mathbb{Z}^N_{\geqslant 0}$, $\lambda_1\geqslant\dots\geqslant\lambda_N, |\boldsymbol\lambda| = n$. Introduce $P = \{d_1, \dots, d_N\}$, $d_i = \lambda_i + N - i, i = 1, \dots, N$. Let

$$\varSigma_i(u) \,=\, u^{d_i} + \sum_{j=1,\ d_i-j\notin P}^{d_i} \varSigma_{ij} u^{d_i-j}.$$

Consider the polynomial algebras

$$\mathbb{C}[\boldsymbol{\Sigma}]^{sing} := \mathbb{C}[\Sigma_{ij}, i=1,\dots,N, j\in\{1,\dots,d_i\}, d_i-j\notin P], \qquad \mathbb{C}[\boldsymbol{\sigma}] := \mathbb{C}[\sigma_1,\dots,\sigma_n].$$

We have

$$\mathrm{Wr}(\Sigma_1(u),\dots,\Sigma_N(u)) = \prod_{1\leqslant i<j\leqslant N}(d_j-d_i)\cdot\Big(u^n+\sum_{s=1}^n(-1)^s A_s(\boldsymbol{\Sigma})\,u^{n-s}\Big),$$

where $A_1(\boldsymbol{\Sigma}),\dots,A_n(\boldsymbol{\Sigma})\in\mathbb{C}[\boldsymbol{\Sigma}]^{sing}$. Define an algebra homomorphism

$$\mathcal{W}\,:\,\mathbb{C}[\boldsymbol{\sigma}]\to\mathbb{C}[\boldsymbol{\Sigma}]^{sing}, \qquad \sigma_s\mapsto A_s(\boldsymbol{\Sigma}).$$

The homomorphism defines a $\mathbb{C}[\boldsymbol{\sigma}]$-module structure on $\mathbb{C}[\boldsymbol{\Sigma}]^{sing}$. Define a differential operator $\mathcal{D}_{\boldsymbol{\Sigma}}$ by

$$\mathcal{D}_{\boldsymbol{\Sigma}} = \frac{1}{\mathrm{Wr}(\Sigma_1(u),\dots,\Sigma_N(u))}\,\mathrm{rdet}\begin{pmatrix}\Sigma_1(u) & \Sigma_1'(u) & \dots & \Sigma_1^{(N)}(u)\\ \Sigma_2(u) & \Sigma_2'(u) & \dots & \Sigma_2^{(N)}(u)\\ \dots & \dots & \dots & \dots\\ 1 & \partial & \dots & \partial^N\end{pmatrix}.$$

It is a differential operator in the variable $u$, whose coefficients are formal power series in $u^{-1}$ with coefficients in $\mathbb{C}[\boldsymbol{\Sigma}]^{sing}$,

$$\mathcal{D}_{\boldsymbol{\Sigma}} = \partial^N+\sum_{i=1}^N F_i(u)\,\partial^{N-i}, \qquad F_i(u) = \sum_{j=i}^\infty F_{ij}\,u^{-j},$$

and $F_{ij}\in\mathbb{C}[\boldsymbol{\Sigma}]^{sing},\ i=1,\dots,N,\ j\geqslant i$.

**Theorem 4.9.** *The map*

$$\tau^-_{\boldsymbol{\lambda}}\,:\,B^{K=0}_{ij}\big|_{(\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}}\mapsto F_{ij}$$

*defines an isomorphism of the Bethe algebra* $\mathcal{B}^{K=0}\big((\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}\big)$ *and the algebra* $\mathbb{C}[\boldsymbol{\Sigma}]^{sing}$. *The isomorphism* $\tau^-_{\boldsymbol{\lambda}}$ *becomes an isomorphism of the* $U(\mathfrak{z}_N[t])|_{(\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}}$*-module* $\mathcal{B}^{K=0}\big((\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}\big)$ *and the* $\mathbb{C}[\boldsymbol{\sigma}]$*-module* $\mathbb{C}[\boldsymbol{\Sigma}]^{sing}$ *if we identify the algebras* $U(\mathfrak{z}_N[t])|_{(\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}}$ *and* $\mathbb{C}[\boldsymbol{\sigma}]$ *by the map* $\sigma_s[\boldsymbol{z}]\mapsto\sigma_s,\ s=1,\dots,n$.

Fix a vector $v^-\in(\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}$ of degree $\sum_{i=1}^N(1-i)\lambda_i$. By formula (3.4) such a vector is unique up to proportionality.

**Theorem 4.10.** *The map*

$$\mu^-_{\boldsymbol{\lambda}}\,:\,B^{K=0}_{ij}v^-\mapsto F_{ij},$$

*defines a linear isomorphism* $(\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}\to\mathbb{C}[\boldsymbol{\Sigma}]^{sing}$. *The maps* $\tau^-_{\boldsymbol{\lambda}},\mu^-_{\boldsymbol{\lambda}}$ *give an isomorphism of the* $\mathcal{B}^{K=0}\big((\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}\big)$*-module* $(\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}$ *and the regular representation of the algebra* $\mathbb{C}[\boldsymbol{\Sigma}]^{sing}$.

The proofs of Theorems 4.9 and 4.10 are basically word by word the same as the proofs of Theorems 5.3 and 5.6 in [MTV2].

## 5. Relations with quantum cohomology

In lectures [O] Okounkov, in particular, considers the equivariant quantum cohomology $QH_{GL_n\times\mathbb{C}^*}(T^*F_{\boldsymbol{\lambda}})$ of the cotangent bundle $T^*F_{\boldsymbol{\lambda}}$ of a flag variety $F_{\boldsymbol{\lambda}}$. More precisely, he considers the standard equivariant cohomology $H^*_{GL_n\times\mathbb{C}^*}(T^*F_{\boldsymbol{\lambda}})$ as a module over the algebra of quantum multiplication and describes this module as the Yangian Bethe algebra of the $XXX$ model associated with $V^{\otimes n}$.

The algebra $H^*_{GL_n\times\mathbb{C}^*}(T^*F_{\boldsymbol{\lambda}})$ has $n+1$ equivariant parameters $z_1,\dots,z_n,u$. The parameters $z_1,\dots,z_n$ correspond to the $GL_n$-action on $T^*F_{\boldsymbol{\lambda}}$ and $u$ corresponds of the $\mathbb{C}^*$-action on $T^*F_{\boldsymbol{\lambda}}$ stretching the cotangent vectors. The operators of quantum multiplication depend on additional parameters $q_1,\dots,q_N$ corresponding to quantum deformation.

It is well-known how the Yangian Bethe algebra degenerates into the Gaudin Bethe algebra, see for example [T], [MTV1]. This degeneration construction gives us the following fact. Introduce new parameters $K_1,\dots,K_N$ by the formula $q_i = 1+K_iu, i=1,\dots,N$, and consider the limit of the algebra of quantum multiplication on $H^*_{GL_n\times\mathbb{C}^*}(T^*F_{\boldsymbol{\lambda}})$ as $u\to 0$. Then this limit is isomorphic to the $\mathcal{B}^K(\mathcal{V}^+_{\boldsymbol{\lambda}})$-module $\mathcal{V}^+_{\boldsymbol{\lambda}}$. This limit is also isomorphic to the $\mathcal{B}^K(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$-module $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$.

## Appendix. Topological description of the $\mathfrak{gl}_N[t]$-module structures on the cohomology of flag varieties

Given $\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0}$ define

$$\begin{aligned} e_{a,a+1}\boldsymbol{\lambda} &= (\lambda_1,\dots,\lambda_{a-1},\lambda_a+1,\lambda_{a+1}-1,\lambda_{a+2},\dots,\lambda_N)\,,\\ e_{a+1,a}\boldsymbol{\lambda} &= (\lambda_1,\dots,\lambda_{a-1},\lambda_a-1,\lambda_{a+1}+1,\lambda_{a+2},\dots,\lambda_N)\,,\\ \boldsymbol{\lambda}' &= (\lambda_1,\dots,\lambda_{a-1},\lambda_a,1,\lambda_{a+1}-1,\lambda_{a+2},\dots,\lambda_N)\,,\\ \boldsymbol{\lambda}'' &= (\lambda_1,\dots,\lambda_{a-1},\lambda_a-1,1,\lambda_{a+1},\lambda_{a+2},\dots,\lambda_N)\,. \end{aligned}$$

Let $A'$ (resp. $B',C'$) be the rank $\lambda_a$ (resp. rank 1, $\lambda_{a+1}-1$) bundle over $\mathcal{F}_{\boldsymbol{\lambda}'}$ whose fiber over the flag $L_1\subset\ldots\subset L_{N+1}$ is $L_a/L_{a-1}$ (resp. $L_{a+1}/L_a$, $L_{a+2}/L_{a+1}$). Let $A''$ (resp. $B'',C''$) be the rank $\lambda_a-1$ (resp. rank 1, $\lambda_{a+1}$) bundle over $\mathcal{F}_{\boldsymbol{\lambda}''}$ whose fiber over the flag $L_1\subset\ldots\subset L_{N+1}$ is $L_a/L_{a-1}$ (resp. $L_{a+1}/L_a$, $L_{a+2}/L_{a+1}$).

Consider the obvious projections

$$\mathcal{F}_{\boldsymbol{\lambda}}\xleftarrow{\pi_1'}\mathcal{F}_{\boldsymbol{\lambda}'}\xrightarrow{\pi_2'}\mathcal{F}_{e_{a,a+1}\boldsymbol{\lambda}}\qquad\text{and}\qquad\mathcal{F}_{\boldsymbol{\lambda}}\xleftarrow{\pi_1''}\mathcal{F}_{\boldsymbol{\lambda}''}\xrightarrow{\pi_2''}\mathcal{F}_{e_{a+1,a}\boldsymbol{\lambda}}\,.$$

For an equivariant map $f$ (eg. $f=\pi_1'$ or $\pi_1''$) the induced pull-back map on equivariant cohomology will be denoted by $f^*$. For an equivariant fibration $f$ (eg. $f=\pi_2'$ or $\pi_2''$) its Gysin map (a.k.a. *push-forward* map, or *integration along the fibers* map) will be denoted by $f_*$. The equivariant Euler class of a vector bundle $X$ will be denoted by $e(X)$.

The following theorem was announced in [RSV].

**Theorem A.1.**

(i) *The map* $\rho^-(e_{a,a+1}\otimes t^j): H_{\boldsymbol{\lambda}}\to H_{e_{a,a+1}\boldsymbol{\lambda}}$

$$x\mapsto \pi'_{2*}\Big({\pi_1'}^*(x)\cdot e\big(\mathrm{Hom}(B',C')\big)\cdot e(B')^j\Big)$$

*makes the diagram*

$$\begin{array}{ccc} H_{\boldsymbol{\lambda}} & \xrightarrow{\rho^-(e_{a,a+1}\otimes t^j)} & H_{e_{a,a+1}\boldsymbol{\lambda}} \\ \Big\downarrow{\scriptstyle i^-} & & \Big\downarrow{\scriptstyle i^-} \\ \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}} & \xrightarrow{e_{a,a+1}\otimes t^j} & \frac{1}{D}\mathcal{V}^-_{e_{a,a+1}\boldsymbol{\lambda}} \end{array}$$

*commutative.*

(ii) *The map* $\rho^-(e_{a+1,a}\otimes t^j): H_{\boldsymbol{\lambda}} \to H_{e_{a+1,a}\boldsymbol{\lambda}}$

$$x \mapsto \pi''_{2*}\Big(\pi_1''^*(x)\cdot e\big(\mathrm{Hom}(A'',B'')\big)\cdot e(B'')^j\Big)$$

*makes the diagram*

$$\begin{array}{ccc} H_{\boldsymbol{\lambda}} & \xrightarrow{\rho^-(e_{a+1,a}\otimes t^j)} & H_{e_{a+1,a}\boldsymbol{\lambda}} \\ \Big\downarrow{\scriptstyle i^-} & & \Big\downarrow{\scriptstyle i^-} \\ \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}} & \xrightarrow{e_{a+1,a}\otimes t^j} & \frac{1}{D}\mathcal{V}^-_{e_{a+1,a}\boldsymbol{\lambda}} \end{array}$$

*commutative.*

*Proof.* We will prove part (i), the proof of part (ii) is similar. Let $K$ be the index in $\mathcal{I}_{e_{a,a+1}\boldsymbol{\lambda}}$ with $K_1 = \{1,\dots,(e_{a,a+1}\boldsymbol{\lambda})_1\}$, $K_2 = \{(e_{a,a+1}\boldsymbol{\lambda})_1+1,\dots,(e_{a,a+1}\boldsymbol{\lambda})_1+(e_{a,a+1}\boldsymbol{\lambda})_2\}$, etc.

Consider $x = [h(\boldsymbol{z},\Gamma_1,\dots,\Gamma_N)] \in H_{\boldsymbol{\lambda}}$. Its $i^-$-image is

$$\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} v_I \otimes \frac{h(\boldsymbol{z},\boldsymbol{z}_{I_1},\dots,\boldsymbol{z}_{I_N})}{R(\boldsymbol{z}_{I_1}|\dots|\boldsymbol{z}_{I_N})}\,.$$

The coefficient of $v_K$ of the $e_{a,a+1}\otimes t^j$-image of this is

$$\sum_{i\in K_a} \frac{h(\boldsymbol{z},\boldsymbol{z}_{K_1},\dots,\boldsymbol{z}_{K_{a-1}},\boldsymbol{z}_{K_a-i},\boldsymbol{z}_{K_{a+1}\cup i},\boldsymbol{z}_{K_{a+2}},\dots,\boldsymbol{z}_{K_N})\,z_i^j}{R(\boldsymbol{z}_{K_1},\dots,\boldsymbol{z}_{K_{a-1}},\boldsymbol{z}_{K_a-i},\boldsymbol{z}_{K_{a+1}\cup i},\boldsymbol{z}_{K_{a+2}},\dots,\boldsymbol{z}_{K_N})} = \tag{A.1}$$

$$= \frac{1}{R(\boldsymbol{z}_{K_1}|\dots|\boldsymbol{z}_{K_N})}\sum_{i\in K_a}\frac{h(\boldsymbol{z},\boldsymbol{z}_{K_1},\dots,\boldsymbol{z}_{K_a-i},\boldsymbol{z}_{K_{a+1}\cup i},\dots,\boldsymbol{z}_{K_N})\,z_i^j\,R(z_i|\boldsymbol{z}_{K_{a+1}})}{R(\boldsymbol{z}_{K_a-i},z_i)}\,.$$

On the other hand, the $\rho^-(e_{a,a+1}\otimes t^j)$-image of $x$ (using a version of the Atiyah-Bott localization formula for $\pi'_{2*}$) is

$$\sum_{\delta\in\Delta_a}\frac{h(\boldsymbol{z},\Delta_1,\dots,\Delta_{a-1},\Delta_a-\delta,\delta,\Delta_{a+1},\dots,\Delta_N)R(\delta|\Delta_{a+1})\delta^j}{R(\Delta_a-\delta|\delta)}, \tag{A.2}$$

where we denoted the Chern roots of the natural bundles over $\mathcal{F}_{e_{a,a+1}\boldsymbol{\lambda}}$ by $\Delta_1,\dots,\Delta_N$. The coefficient of $v_K$ of $i^-$-image of (A.2) is the right hand side of (A.1). The theorem is proved. $\square$

The topological interpretation of generators of the $\rho^+$-representation is similar, its proof is left to the reader.

**Theorem A.2.**

(i) *For the map* $\rho^+(e_{a,a+1}\otimes t^j) : H_{\boldsymbol\lambda} \to H_{e_{a,a+1}\boldsymbol\lambda}$,

$$x \mapsto \pi'_{2*}\Big(\pi_1'^{\,*}(x)\cdot e\big(\mathrm{Hom}(A',B')\big)\cdot e(B')^j\Big)$$

*we have* $i^+\circ\rho^+(e_{a,a+1}\otimes t^j) = (e_{a,a+1}\otimes t^j)\circ i^+$.

(ii) *For the map* $\rho^+(e_{a+1,a}\otimes t^j) : H_{\boldsymbol\lambda} \to H_{e_{a+1,a}\boldsymbol\lambda}$,

$$x \mapsto \pi''_{2*}\left(\pi_1''^{*}(x)\cdot e(\mathrm{Hom}(B'',C''))\cdot e(B'')^j\right)$$

*we have* $i^+\circ\rho^+(e_{a+1,a}\otimes t^j) = (e_{a+1,a}\otimes t^j)\circ i^+$.

The $\mathfrak{gl}_N[t]$-module structures $\rho^\pm$ on $\bigoplus_{\boldsymbol\lambda} H_{\boldsymbol\lambda}$ descend to $\mathfrak{gl}_N[t]$-module structures on $H(\mathbb{C})$, also denoted by $\rho^\pm$ in Section 3.4. The topological interpretation of the actions of $e_{a,a+1}\otimes t^j$ and $e_{a+1,a}\otimes t^j$ for these representations is the same as that for $\bigoplus_{\boldsymbol\lambda} H_{\boldsymbol\lambda}$ given in Theorems A.1 and A.2.

Topological description of the $\mathfrak{gl}_N[t]$-module structures on the cohomology of flag varieties, presented in this Appendix, is a version of a construction due to Ginzburg and Vasserot, cf. [Vas1], [Vas2]. We thank Eric Vasserot for pointing out to us these references. However, unlike these authors, we work with flag varieties themselves and not with their cotangent bundles. Also our proof is different.

## References


[AB] M. F. Atiyah, R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28

[BMO] A. Braverman, D. Maulik, A. Okounkov, *Quantum cohomology of the Springer resolution*, Preprint (2010), 1–35, `arXiv:1001.0056`

[CL] V. Chari, S. Loktev, *Weyl, Fusion and Demazure modules for the current algebra of* $\mathfrak{sl}_{r+1}$, Adv. Math. 207 (2006), no. 2, 928–960

[CP] V. Chari, A. Pressley *Weyl Modules for Classical and Quantum Affine algebras*, Represent. Theory 5 (2001), 191–223 (electronic)

[CT] A. Chervov, D. Talalaev, *Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence*, Preprint (2006), 1–54; `hep-th/0604128`

[G1] M. Gaudin, *Diagonalisation d'une classe d'Hamiltoniens de spin*, J. Physique **37** (1976), no. 10, 1089–1098

[G2] M. Gaudin, *La fonction d'onde de Bethe*, Collection du Commissariat à l'Énergie Atomique: Série Scientifique, Masson, Paris, 1983

[MTV1] E. Mukhin, V. Tarasov, A. Varchenko, *Bethe Eigenvectors of Higher Transfer Matrices*, J. Stat. Mech. (2006), no. 8, P08002, 1–44

[MTV2] E. Mukhin, V. Tarasov, A. Varchenko, *Schubert calculus and representations of general linear group*, J. Amer. Math. Soc. **22** (2009), no. 4, 909–940

[MTV3] E. Mukhin, V. Tarasov, A. Varchenko, *Spaces of quasi-exponentials and representations of* $\mathfrak{gl}_N$, J. Phys. A **41** (2008), no. 19, 194017, 28 pp.

[MV] E. Mukhin, A. Varchenko, *Critical points of master functions and flag varieties*, Commun. Contemp. Math. **6** (2004), no. 1, 111–163

[O] A. Okounkov, *Quantum Groups and Quantum Cohomology*, Lectures at the 15th Midrasha Mathematicae on: "Derived categories of algebro-geometric origin and integrable systems", December 19–24, 2010, Jerusalem

[RV] R. Rimányi, A. Varchenko, *Conformal blocks in the tensor product of vector representations and localization formulas*, Preprint (2009), 1–21, `arXiv:0911.3253`

[RSV] R. Rimányi, V. Schechtman, A. Varchenko, *Conformal blocks and equivariant cohomology*, Preprint (2010), 1–23, `arXiv:1007.3155`

[T] D. Talalaev, *Quantization of the Gaudin System*, Preprint (2004), 1–19; `hep-th/0404153`

[V] A. Varchenko, *A Selberg integral type formula for an $\mathfrak{sl}_2$ one-dimensional space of conformal blocks*, Mosc. Math. J. **10** (2010), no. 2, 469–475, 480

[Vas1] E. Vasserot, *Représentations de groupes quantiques et permutations*, Ann. Sci. ENS **26** (1993), 747–773

[Vas2] E. Vasserot, *Affine quantum groups and equivariant $K$-theory*, Transformation groups, **3** (1998), 269–299