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NONCONCENTRATION IN PARTIALLY RECTANGULAR BILLIARDS

LUC HILLAIRET AND JEREMY L. MARZUOLA

In specific types of partially rectangular billiards we estimate the mass of an eigenfunction of energy E in the region outside the rectangular set in the high-energy limit. We use the adiabatic ansatz to compare the Dirichlet energy form with a second quadratic form for which separation of variables applies. This allows us to use sharp one-dimensional control estimates and to derive the bound assuming that E is not resonating with the Dirichlet spectrum of the rectangular part.

1. Introduction

We study concentration and nonconcentration of eigenfunctions of the Laplace operator in stadium-like billiards. As predicted by the quantum/classical correspondence, such concentration is deeply linked with the classical underlying dynamics. In particular, the celebrated quantum ergodicity theorem roughly states that when the corresponding classical dynamics is ergodic then almost every sequence of eigenfunctions equidistributes in the high energy limit (see [Schnirelman 1974; Colin de Verdière 1985; Zelditch 1987] and [Gérard and Leichtnam 1993; Zelditch and Zworski 1996] in the billiard setting for a more precise statement). In strongly chaotic systems such as negatively curved manifolds, it is expected that every sequence of eigenfunctions equidistributes. This statement is the quantum unique ergodicity conjecture (Q.U.E.) and remains open in most cases despite several recent striking results (see for instance [Faure et al. 2003; Lindenstrauss 2006; Anantharaman 2008; Anantharaman and Nonnenmacher 2007]). On the other extreme, the Bunimovich stadium, although ergodic, is expected to violate Q.U.E. Indeed, it is expected that there exist bouncing ball modes, i.e., exceptional sequences of eigenfunctions concentrating on the cylinder of bouncing ball periodic orbits that sweep out the rectangular region (see [Bäcker et al. 1997] for instance). The existence of such bouncing ball modes is still open and only recently did Hassell prove that the generic Bunimovich stadium billiard indeed fails to be Q.U.E. (see [Hassell 2010]).

Our work is closely related to the search for bouncing ball modes but proceeds loosely speaking in the other direction. We actually aim at understanding how strong concentration of eigenfunctions in the rectangular part cannot be. We thus follow [Burq and Zworski 2005], where it is proved that even bouncing ball modes couldn't concentrate strictly inside the rectangular region. This was made precise by

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Burq, Hassell and Wunsch in [Burq et al. 2007], where the following estimate was proved:

$$\|u\|_{L^2(W)} \geq E^{-1}\|u\|_{L^2(\Omega)},$$

in which $\|u\|_{L^2(W)}$ and $\|u\|_{L^2(\Omega)}$ denote the L^2 norm of the eigenfunction u in the wings and in the billiard, respectively.

Our main result for the Bunimovich stadium is the following:

Theorem 1. *Let Ω be a Bunimovich stadium with rectangular part $R := [-B_0, 0] \times [0, L_0]$. We set $W = \Omega \setminus R$ and denote by Σ the Dirichlet spectrum of R, i.e.,*

$$\Sigma = \left\{ \frac{k^2\pi^2}{L_0^2} + \frac{l^2\pi^2}{B_0^2},\ k, l \in \mathbb{N} \right\}.$$

For any $\varepsilon \geq 0$ there exists E_0 and C such that if u is an eigenfunction of energy E such that $E > E_0$ and $\operatorname{dist}(E, \Sigma) > E^{-\varepsilon}$ *then the following estimates holds*:

$$\|u\|_{L^2(\Omega)} \leq CE^{\frac{5+8\varepsilon}{6}}\|u\|_{L^2(W)},$$

This bound improves on the Burq–Hassell–Wunsch bound provided that $\varepsilon < \frac{1}{8}$. It is natural that the smaller ε is the better the bound is. Indeed, the condition on the distance between E and Σ is comparable to a nonresonance condition and should imply heuristically that u must have some mass in the wing region. It is quite interesting to have a quantitative statement confirming this heuristics. We will actually give a more general statement concerning more general billiards (see Theorem 2). In particular we will consider billiards with smoother boundaries (see Section 2) disregarding the fact that these may not be ergodic. Here again we expect the bound to be better when the billiard becomes smoother and this statement is made quantitative in Theorem 2.

The method we propose relies on comparing the Dirichlet energy quadratic form with another quadratic form arising from the adiabatic ansatz presented in the numerical study of eigenfunctions by Bäcker, Schubert and Stifter [1997]. This adiabatic quadratic form has also appeared recently in [Hillairet and Judge 2009] in the study of the spectrum of the Laplacian on triangles. These two quadratic forms are close provided we do not enter too deeply into the wing region so that the nonconcentration estimate really takes place in a neighborhood of the rectangle that becomes smaller and smaller when the energy goes to infinity (see Sections 4.3.3 and 4.6.1). Since the new quadratic form may be addressed using separation of variables, we will show precise one-dimensional control estimates and then use them to prove our results. We have separated these one dimensional estimates in an appendix since they may be of independent interest. Finally, we remark that the method can be applied to quasimodes with some caution (see Remark 5.2) but there are no reasons to think that the bound we obtain is optimal.

2. The setting

Let L be a function defined on $[-B_0, B_1]$ with the following properties:

- For nonpositive x, $L(x) = L_0 > 0$.

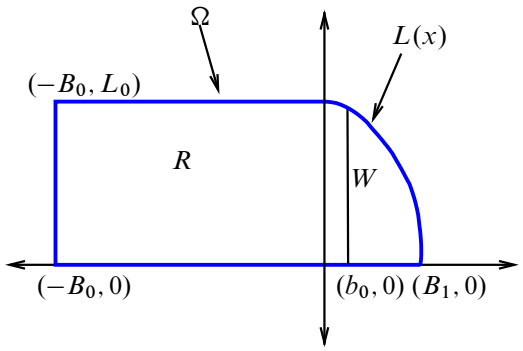

Figure 1. An example of a billiard Ω.

- On $(0, B_1)$, L is smooth, nonnegative and nonincreasing.
- When x goes to B_1, L' has a negative limit (either finite or $-\infty$).
- For small positive x, we have the asymptotic expansions

$$L(x) = L_0 - c_L x^\gamma + o(x^\gamma), \quad L'(x) = -c_L \gamma x^{\gamma-1} + o(x^{\gamma-1}) \tag{2-1}$$

for some positive c_L and $\gamma \geq \frac{3}{2}$.

The billiard Ω is then defined by

$$\Omega = \{(x, y) \mid -B_0 \leq x \leq B_1,\ 0 \leq y \leq L(x)\}.$$

See Figure 1 for an example of an applicable billiard. For any $b < B_1$, we will denote by $\Omega_b := \Omega \cap \{x \leq b\}$ and by $W_b := \Omega \cap \{0 \leq x \leq b\}$.

We study eigenfunctions of the positive Dirichlet Laplacian, Δ, on Ω. Namely, we study solutions u_E such that

$$\Delta u_E = -\left(\partial_x^2 + \partial_y^2\right) u_E = E u_E \quad \text{and} \quad u_E|_{\partial\Omega} = 0,$$

where $E > 0$.

We may formulate this equation using quadratic forms. We thus introduce q defined on $H^1(\Omega)$ by

$$q(u) = \int_\Omega |\nabla u|^2 dx\, dy.$$

The Euclidean Laplacian with Dirichlet boundary condition in Ω is the unique self-adjoint operator associated with q defined on $H_0^1(\Omega)$. We denote by q_b the restriction of q to $H^1(\Omega_b)$ and by Δ_b the Dirichlet Laplace operator on Ω_b. We will also denote by $\mathcal{D}_b$ the set of smooth functions with compact support in Ω_b.

3. Adiabatic approximation

Motivated by the well-known eigenvalue problem on a rectangular billiard and computational results in [Bäcker et al. 1997], we introduce a second family of quadratic forms a_b and compare it to q_b.

For any $b < B_1$ and any $u \in \mathcal{D}_b$, Fourier decomposition in y implies that

$$u(x, y) = \sum_k u_k(x) \sin\left(\frac{\pi k}{L(x)} y\right). \tag{3-1}$$

Since

$$\int_0^{L(x)} \left|\sin\left(k\pi \frac{y}{L(x)}\right)\right|^2 dy = \frac{L(x)}{2}$$

each Fourier coefficient u_k is given by

$$u_k(x) = \frac{2}{L(x)} \int_0^{L(x)} u(x, y) \sin\left(\frac{\pi k}{L(x)} y\right) dy.$$

For such u, we define

$$a_b(u) = \sum_{k\in\mathbb{N}} \int_{-B_0}^{b} \left(|u_k'(x)|^2 + \frac{k^2\pi^2}{L^2(x)} |u_k(x)|^2\right) \frac{L(x)}{2} dx,$$

$$N_b(u) = \sum_{k\in\mathbb{N}} \int_{-B_0}^{b} |u_k(x)|^2 \frac{L(x)}{2} dx.$$

Observe that for each fixed x, Plancherel's formula reads

$$\sum_{k\in\mathbb{N}} |u_k(x)|^2 \frac{L(x)}{2} = \int_0^{L(x)} |u(x, y)|^2 \, dy,$$

so that we get $N_b(u) = \|u\|^2_{L^2(\Omega_b)}$ by integration with respect to x.

Fixing some $0 < b_0 < B_1$, and using that L is uniformly bounded above and below on $[-B_0, b_0]$ we find a constant C such that for any $b \le b_0$ and $u \in L^2(\Omega_b)$:

$$C^{-1} \|u\|^2_{\Omega_b} \le \sum_{k=1}^{\infty} \|u_k\|^2_{L^2(-B_0,b)} \le C \|u\|^2_{\Omega_b}. \tag{3-2}$$

The quadratic form a_b appears as the direct sum of the following quadratic forms $a_{b,k}$ (that can be defined on the whole function space $H^1(-B_0, b)$):

$$a_{b,k}(u) := \int_{-B_0}^{b} \left(|u'|^2 + \frac{k^2\pi^2}{L^2(x)} |u|^2\right) \frac{L(x)}{2} dx. \tag{3-3}$$

Recall that, on an interval I, the standard H^1 norm is defined by

$$\|u\|_{H^1} := \left(\|u'\|^2_{L^2(I)} + \|u\|^2_{L^2(I)}\right)^{\frac{1}{2}}, \tag{3-4}$$

so that, for any k and $b < B_1$ and any $u \in \mathcal{C}_0^\infty(-B_0, b)$ we have

$$\min\left(L(b)\,,\, \frac{k^2\pi^2}{L_0}\right) \|u\|^2_{H^1} \le a_{b,k}(u) \le \max\left(L_0\,,\, \frac{k^2\pi^2}{L(b)}\right) \|u\|^2_{H^1}. \tag{3-5}$$

The norm $a_{b,k}^{\frac{1}{2}}$ thus defines on $H^1(-B_0, b)$ a norm that is equivalent to the standard H^1 norm.

3.1. ***Comparing a_b and q_b.*** To compare a_b and q_b, we introduce the following operators D and R defined on $\mathcal{D}_b$ by

$$\begin{aligned} Ru &= \frac{yL'(x)}{L(x)}\partial_y u, \\ Du &= \partial_x u + Ru. \end{aligned}$$

Using the Plancherel formula for each fixed x and then integrating, we obtain

$$a_b(u) = \int_{\Omega_b} |Du|^2 + |\partial_y u|^2 \, dx\, dy.$$

from which the following holds for any $u, v \in \mathcal{D}_b$:

$$\begin{aligned} a_b(u,v) - q_b(u,v) &= \langle Du, Dv\rangle - \langle \partial_x u, \partial_x v\rangle \\ &= \langle \partial_x u, Rv\rangle + \langle Ru, Dv\rangle, \qquad (3\text{-}6) \\ &= \langle \partial_x u, Rv\rangle + \langle Ru, \partial_x v\rangle + \langle Ru, Rv\rangle. \qquad (3\text{-}7) \end{aligned}$$

We thus obtain the following lemma.

Lemma 3.1. *Let δ be the function defined by*

$$\delta(b) = \sup_{(0,b]} |L'(x)| + \sup_{(0,b]} |L'(x)|^2.$$

Then for all $u, v \in \mathcal{D}_b$

$$|a_b(u,v) - q_b(u,v)| \le \delta(b) \cdot q_b^{\frac{1}{2}}(u) \cdot q_b^{\frac{1}{2}}(v).$$

Remark 3.1. The function δ is continuous on $(0, B_1)$ and $\delta(b) = O(b^{\gamma-1})$ when b goes to 0.

Proof. In (3-7), we use the Cauchy–Schwarz inequality, $\max(\|Du\|, \|\partial_y u\|) \le a_b^{\frac{1}{2}}(u)$, and the fact that $y/L(x)$ is uniformly bounded by 1 on Ω. □

The following corollary is then straightforward.

Corollary 3.2. *For any $0 < b < B_1$ and any $u \in H^1(\Omega)$, the linear functional Λ defined by $\Lambda(v) := a_b(u,v) - q_b(u,v)$ belongs to $H^{-1}(\Omega_b)$. Moreover*

$$\|\Lambda\|_{H^{-1}(\Omega_b)} \le \delta(b)\|u\|_{H^1(\Omega_b)}.$$

4. Nonconcentration

4.1. ***Preliminary reduction.*** Let u be an eigenfunction of q with eigenvalue E. And define the associated linear functional Λ using Corollary 3.2.

Integration by parts shows that for any $v \in H_0^1(\Omega_b)$ we have

$$q_b(u,v) = E \cdot \langle u, v\rangle_{L^2(\Omega)},$$

so that

$$a_b(u,v) - E \cdot N_b(u,v) \;=\; \Lambda(v). \tag{4-1}$$

We now deal with this equation using the adiabatic decomposition. We thus define Λ_k as the distribution over $\mathcal{D}_b$ such that, for any $v \in \mathcal{D}_b$,

$$\Lambda_k(v) := \Lambda\left(v(x)\sin\left(k\pi\frac{y}{L(x)}\right)\right). \tag{4-2}$$

Remark 4.1. From now on, u will always denote the eigenfunction that we are dealing with. We will denote by u_k the functions entering in the adiabatic decomposition of u, by Λ the linear functional associated with u and by Λ_k the one-dimensional linear functionals that are associated with Λ.

A straightforward computation yields, that for any $v \in \mathcal{D}_b$ we have

$$a_{b,k}(u_k,v) - E \cdot \int_{-B_0}^{b} u_k(x)v(x)\frac{L(x)}{2}\,dx \;=\; \Lambda_k(v),$$

where $a_{b,k}$ is the quadratic form defined in (3-3).

An integration by parts then shows that, in the distributional sense in $(-B_0, b)$, we have

$$-\frac{1}{L}\frac{d}{dx}(Lu_k') + \left(\frac{k^2\pi^2}{L^2} - E\right)u_k \;=\; \tilde{\Lambda}_k, \tag{4-3}$$

where the linear functional $\tilde{\Lambda}_k$ is defined by

$$\tilde{\Lambda}_k(v) := \Lambda_k\left(\frac{2}{L}\cdot v\right). \tag{4-4}$$

Remark 4.2. Since L is not smooth, this definition of $\tilde{\Lambda}_k$ doesn't make sense as a distribution. However, in the next section, we will prove that Λ_k actually is in H^{-1} and, since multiplication by $2/L$ is a bounded operator from $H^1(-B_0, b)$ into itself, we thus get that $\tilde{\Lambda}_k$ is a perfectly legitimate element of H^{-1}. Moreover, for any b_0 there exists $C(b_0)$ such that for any $b \le b_0$, and $v \in \mathcal{D}_b$, we have

$$\left\|\frac{2}{L}v\right\|_{H^1(-B_0,b)} \le C(b_0)\|v\|_{H^1(-B_0,b)}.$$

We denote by P_k the operator that is defined by

$$P_k(u) \;=\; -\frac{1}{L}\frac{d}{dx}(Lu') + \left(\frac{k^2\pi^2}{L^2} - E\right)u,$$

and we try to analyze the way a solution to equation (4-3) on $(-B_0, b)$ may be controlled by its behavior on $(0, b)$.

The strategy will depend upon whether k is large or not, but first we have to get a bound on Λ_k in some reasonable functional space of distributions.

4.2. ***Bounding*** $\boldsymbol{\Lambda_k}$. In this section, we prove that each Λ_k is actually in $H^{-1}(-B_0, b)$ and provide a bound for its H^{-1} norm.

We first note that, using (3-4), for any $F \in H^{-1}(-B_0, b)$:

$$\|F\|_{H^{-1}(-B_0,b)} := \sup_{\phi\in\mathcal{D}_b} \frac{|F(\phi)|}{\|\phi\|_{H^1}} \le \sup_{\phi\in\mathcal{D}_b} \frac{|F(\phi)|}{\|\phi'\|_{L^2}}. \tag{4-5}$$

Using (3-6) in the definition of Λ_k — see (4-2) — we obtain

$$\Lambda_k(v) = \left\langle \partial_x u, R\left(v(x)\sin\left(k\pi\frac{y}{L(x)}\right)\right)\right\rangle + \left\langle Ru, D\left(v(x)\sin\left(k\pi\frac{y}{L(x)}\right)\right)\right\rangle.$$

Denote by $A_k(v)$ the first term on the right and $B_k(v)$ the second term. By inspection, we have

$$A_k(v) := \frac{k\pi}{2}\int_0^b v(x)\frac{L'(x)}{L(x)}F_k(x)\,dx \quad \text{and} \quad B_k(v) := \frac{1}{2}\int_0^b v'(x)L'(x)G_k(x)\,dx,$$

where we have set

$$F_k(x) := \frac{2}{L(x)}\int_0^{L(x)} 1_W \cdot y\partial_x u(x,y)\cdot\cos\left(k\pi\frac{y}{L(x)}\right)dy, \tag{4-6}$$

$$G_k(x) := \frac{2}{L(x)}\int_0^{L(x)} 1_W \cdot y\partial_y u(x,y)\cdot\sin\left(k\pi\frac{y}{L(x)}\right)dy \tag{4-7}$$

Since $u \in H^1(\Omega)$, F_k and G_k are $L^2(0,b)$ and we can estimate the H^{-1} norm of Λ_k using them.

Lemma 4.1. *For any $b_0 < B_1$, and given Λ_k and F_k, G_k defined as above, there exists $C = C(\Omega_{b_0})$ such that*

$$\|\Lambda_k\|_{H^{-1}} \le C(kb^\gamma\|F_k\|_{L^2(0,b)} + b^{\gamma-1}\|G_k\|_{L^2(0,b)}). \tag{4-8}$$

Proof. We estimate $A_k(v)$, using first an integration by parts

$$A_k(v) := -\frac{k\pi}{2}\int_{-B_0}^b v'(x)\left(\int_0^x \frac{L'(\xi)}{L(\xi)}F_k(\xi)\,d\xi\right)dx.$$

Using the Cauchy–Schwarz inequality and the fact that $L'(\xi) = O(\xi^{\gamma-1})$ we have

$$\left|\int_0^x \frac{L'(\xi)}{L(\xi)}F_k(\xi)\,d\xi\right| \le Cx_+^{\gamma-\frac{1}{2}}\|F_k\|_{L^2(0,b)}.$$

Inserting into $A_k(v)$ and using the Cauchy–Schwarz inequality again we get

$$|A_k(v)| \le C\cdot(kb^\gamma)\|F_k\|_{L^2(0,b)}\cdot\|v'\|_{L^2(-B_0,b)},$$

which gives the claimed bound using (4-5).

Next, the second term is estimated using directly the Cauchy–Schwarz estimate and the fact that $\sup_{[0,b]}|L'(x)| \le Cb^{\gamma-1}$. We get

$$|B_k(v)| \le C\cdot b^{\gamma-1}\|G_k\|_{L^2(0,b)}\cdot\|v'\|_{L^2(-B_0,b)}.$$

That gives the claimed bound using again (4-5). □

Define $F := 1_W \partial_x u$ and $G := 1_W \partial_y u$. By definition, $F_k(x)$ is the Fourier coefficient of the function $F(x,\cdot)$ with respect to the Fourier basis

$$\left(y \mapsto \cos\left(k\pi \frac{y}{L(x)}\right)\right)_{k\in\mathbb{N}\cup\{0\}}.$$

Using the Plancherel formula we get

$$\sum_{k\geq 1} F_k(x)^2 \frac{L(x)}{2} \leq \int_0^{L(x)} |F(x,y)|^2 dy.$$

For the same reason, but using this time the sin basis, we have

$$\sum_{k\geq 1} G_k(x)^2 \frac{L(x)}{2} = \int_0^{L(x)} |G(x,y)|^2 dy.$$

Integrating with respect to x and bounding y from above and $L(x)$ from below uniformly we get:

Lemma 4.2. *For any b_0 there exists C depending only on the billiard and b_0 such that, for any $b < b_0$,*

$$\sum_{k\geq 1} \|F_k\|^2_{L^2(0,b)} \leq C\|\partial_x u\|^2_{L^2(W_b)}, \tag{4-9}$$

$$\sum_{k\geq 1} \|G_k\|^2_{L^2(0,b)} \leq C\|\partial_y u\|^2_{L^2(W_b)}. \tag{4-10}$$

We now switch to the control estimate. We begin by dealing with the modes for which $\dfrac{k^2\pi^2}{L_0^2} - E \geq E$.

4.3. *Large modes.*

4.3.1. *A control estimate.* Equation (4-3) may be rewritten as

$$-u_k'' + \left(\frac{k^2\pi^2}{L^2(x)} - E\right)u_k = h_k, \tag{4-11}$$

where h_k is the element of H^{-1} defined by

$$h_k := \tilde{\Lambda}_k + \frac{L'}{L}u_k' \tag{4-12}$$

The H^{-1} norm of h_k is now estimated as follows:

Lemma 4.3. *There exists a constant $C := C(b_0)$ such that for any $b \leq b_0$ and any k with $\dfrac{k^2\pi^2}{L_0^2} - E \geq E$ the following estimate holds*:

$$\|h_k\|_{H^{-1}(-B_0,b)} \leq C(b_0)\big(kb^\gamma \|F_k\|_{L^2(0,b)} + b^{\gamma-1}\|G_k\|_{L^2(0,b)} + b^{\gamma-1}\|u_k\|_{L^2(0,b)}\big). \tag{4-13}$$

Proof. Using Remark 4.2, the norm of $\tilde{\Lambda}_k$ is uniformly controlled by the norm of Λ_k and the latter is estimated using Lemma 4.1. To estimate the H^{-1} norm of $(L'/L)u_k'$, we first set $v = (L'/L)u_k'$ and remark that

$$v = \left(\frac{L'}{L}u_k\right)' - \left(\frac{L''}{L} - \frac{(L')^2}{L^2}\right)u_k.$$

We choose a test function ϕ and estimate

$$I_1 = \int_{-B_0}^{b} \left(\frac{L'}{L} u_k\right)' \phi \, dx.$$

We perform an integration by parts, use that $L'(x)/L(x) \le Cb^{\gamma-1} 1_{x>0}$, then apply the Cauchy–Schwarz inequality to get

$$\left| \int_{-B_0}^{b} \left(\frac{L'}{L} u_k\right)' \phi \, dx \right| = \left| \int_{-B_0}^{b} \frac{L'(x)}{L(x)} u_k(x) \phi'(x) \, dx \right| \le Cb^{\gamma-1} \|u_k\|_{L^2(0,b)} \|\phi'\|_{L^2(-B_0,b)}.$$

We then estimate

$$I_2 = \left| \int_{-B_0}^{b} \left(\frac{L''(x)}{L(x)} - \frac{(L'(x))^2}{L^2(x)} \right) u(x) \phi(x) \, dx \right|.$$

We perform an integration by parts, use that

$$\left| \frac{L''(x)}{L(x)} - \frac{(L'(x))^2}{L^2(x)} \right| \le C x_+^{\gamma-2},$$

then twice apply the Cauchy–Schwarz inequality to get

$$I_2 \le C \int_{-B_0}^{b} \left(\int_0^x \xi_+^{\gamma-2} |u(\xi)| \, d\xi \right) |\phi'(x)| \, dx \le Cb^{\gamma-1} \|u\|_{L^2(0,b)} \|\phi'\|_{L^2(0,b)}.$$

The claim follows using (4-5). □

The variational formulation of equation (4-11) is given by

$$\int_{-B_0}^{b} u_k' v' \, dx + \int_{-B_0}^{b} \left(\frac{k^2\pi^2}{L^2(x)} - E \right) u_k v \, dx = h_k(v). \tag{4-14}$$

Since $k^2\pi^2/L_0^2 - E \ge E$, the left-hand side is a continuous quadratic form on $H_0^1(-B_0, b)$, so that, by Lax–Milgram theory, there is a unique v_k in $H_0^1(-B_0, b)$ satisfying (4-11) in the distributional sense.

The following lemma allows us to estimate the L^2 norm of this v_k.

Lemma 4.4. *There exists a constant C depending only on b_0 but not on $b < b_0$, k, or E such that, if $E \ge 1$ and $k^2\pi^2/L_0^2 - E \ge E$, the variational solution v_k in $H_0^1(-B_0, b)$ to equation (4-11) satisfies*

$$\|v_k\|_{L^2(-B_0,b)} \le C(b_0)\big(b^\gamma \|F_k\|_{L^2(0,b)} + E^{-\frac{1}{2}} b^{\gamma-1} \|G_k\|_{L^2(0,b)} + E^{-\frac{1}{2}} b^{\gamma-1} \|u_k\|_{L^2(0,b)}\big). \tag{4-15}$$

Proof. Since v_k is a variational solution, putting $v = v_k$ in (4-14) we get

$$\int_{-B_0}^{b} |v_k'(x)|^2 dx + \int_{-B_0}^{b} \left(\frac{k^2\pi^2}{L^2(x)} - E \right) |v_k(x)|^2 dx = h_k(v_k). \tag{4-16}$$

In the regime we are considering the second integral on the left is positive, so that we obtain

$$\int_{-B_0}^{b} |v_k'(x)|^2 dx \le |h_k(v_k)| \le \|h_k\|_{H^{-1}} \|v_k\|_{H^1}.$$

Since v_k is in $H_0^1(-B_0, b)$, Poincaré's inequality gives $c(b)$, a positive continuous function of b defined for $b > -B_0$ and satisfying

$$\int_{-B_0}^{b} |v_k'(x)|^2 dx \geq c(b)\|v_k\|_{H^1}^2 .$$

This gives a constant C depending only on b_0 such that, for any $0 < b < b_0$, we have

$$\|v_k\|_{H^1} \leq C\|h_k\|_{H^{-1}} .$$

We now use (4-16) again to obtain

$$\left(\frac{k^2\pi^2}{L_0^2} - E\right)\int_{-B_0}^{b} |v_k(x)|^2 dx \leq \|h_k\|_{H^{-1}}\|v_k\|_{H^1} \leq C\|h_k\|_{H^{-1}}^2$$

with the preceding bound. Using the estimate (4-13) we obtain

$$\left(\frac{k^2\pi^2}{L_0^2} - E\right)^{\frac{1}{2}}\|v_k\|_{L^2(-B_0,b)} \leq C\big(kb^\gamma\|F_k\|_{L^2(0,b)} + b^{\gamma-1}\|G_k\|_{L^2(0,b)} + b^{\gamma-1}\|u_k\|_{L^2(0,b)}\big).$$

We divide both sides by $(k^2\pi^2/L_0^2 - E)^{\frac{1}{2}}$. The coefficient in front of $b^\gamma\|F_k\|_{L^2(0,b)}$ is bounded by a constant that is uniform in k, using the fact that

$$\sup_{k^2\pi^2/L_0^2 - E \geq E} \frac{k^2}{k^2\pi^2/L_0^2 - E} = \sup_{Z \geq E} \frac{L_0^2}{\pi^2}\left(1 + \frac{E}{Z}\right) = \frac{L_0^2}{\pi^2}\left(1 + \frac{E}{E}\right).$$

For the two other terms, we use simply that $k^2\pi^2/L_0^2 - E \geq E$. This gives the lemma. □

We can now let $w_k = u_k - v_k$. By construction, w_k is a solution to the homogeneous equation

$$-w'' + \left(\frac{k^2\pi^2}{L^2(x)} - E\right)w = 0. \tag{4-17}$$

Moreover, since both u_k and v_k satisfy Dirichlet boundary condition at $-B_0$ we have that $w_k(-B_0) = 0$.

Since the "potential" part in equation (4-17) is bounded below by E, concentration properties of solutions may be obtained using convexity estimates.

Lemma 4.5. *For any $b \leq b_0$, any solution w to (4-17) such that $w(-B_0) = 0$ satisfies*

$$b\int_{-B_0}^{b} |w|^2(x)\, dx \leq (B_0 + b_0)\int_0^b |w|^2(x)\, dx.$$

Proof. Multiplying the equation by w we find

$$-w''w + \left(\frac{k^2\pi^2}{L^2(x)} - E\right)w^2 = 0.$$

It follows that $(w^2)'' \geq \beta^2 w^2$, for some positive β (here $\beta^2 = 2E$).

Since $w(-B_0) = 0$, using the maximum principle on $[-B_0, \xi]$, we obtain for all $-B_0 \le x \le \xi \le b_0$

$$w^2(x) \le w^2(\xi) \frac{\sinh(\beta(x + B_0))}{\sinh(\beta(\xi + B_0))}.$$

For any $t \in [0, 1]$, define $x(t) = -B_0 + t(B_0 + b)$ and $\xi(t) = tb$. Since for any t we have $-B_0 \le x(t) \le \xi(t) \le b_0$, we may integrate the preceding relation:

$$\int_0^1 w^2(x(t))\, dt \le \int_0^1 w^2(\xi(t)) \frac{\sinh(\beta(x(t) + B_0))}{\sinh(\beta(\xi(t) + B_0))}\, dt.$$

Since sinh is increasing the quotient of sinh is bounded above by 1 and we obtain

$$b \int_{-B_0}^{b} w^2(x)\, dx \le (B_0 + b) \int_0^b w^2(x)\, dx. \qquad \square$$

Putting these two lemmas together we obtain:

Proposition 4.6. *There exists a constant C depending only on b_0 such that for any $b \le b_0$, for any k and E such that $k^2\pi^2/L_0^2 - E \ge E$ and $E \ge 1$,*

$$\|u_k\|_{L^2(-B_0,b)} \le C\big(b^{\gamma-\frac{1}{2}} \|F_k\|_{L^2(0,b)} + E^{-\frac{1}{2}} b^{\gamma-\frac{3}{2}} \|G_k\|_{L^2(0,b)} + b^{-\frac{1}{2}} \|u_k\|_{L^2(0,b)}\big) \tag{4-18}$$

for $C = C(b_0)$.

Proof. According to Lemma 4.5 we have

$$\|w_k\|_{L^2(-B_0,b)} \le C b^{-\frac{1}{2}} \|w_k\|_{L^2(0,b)},$$

where $w_k = u_k - v_k$ and v_k is the variational solution constructed above. Using the reverse triangle inequality, we obtain

$$\|u_k\|_{L^2(-B_0,b)} \le C b^{-\frac{1}{2}} \|u_k\|_{L^2(0,b)} + (C + b_0^{\frac{1}{2}}) b^{-\frac{1}{2}} \|v_k\|_{L^2(-B_0,b)}.$$

The claim will follow using estimate (4-15) of Lemma 4.4. Observe that the prefactor of $\|u_k\|_{L^2(0,b)}$ is at first (up to a constant prefactor)

$$b^{-\frac{1}{2}} + b^{-\frac{1}{2}} E^{-\frac{1}{2}} b^{\gamma-1}.$$

Since $E^{-\frac{1}{2}} b^{\gamma-1}$ is uniformly bounded we obtain the given estimate. $\square$

4.3.2. *Summing over k.* We will now sum the preceding estimates over k. We thus introduce

$$u_+(x, y) = \sum_{k^2\pi^2/L_0 - E \ge E} u_k(x) \sin \frac{k\pi y}{L(x)}$$

and prove the following proposition.

Proposition 4.7. *There exist b_0 and E_0 and a constant C depending only on E_0 and b_0 such that, if u is an eigenfunction with energy $E > E_0$ and $b < b_0$, then*

$$\|u_+\|^2_{L^2(R)} \le C\big(b^{2\gamma-1} \|\partial_x u\|^2_{W_b} + E^{-1} b^{2\gamma-3} \|\partial_y u\|^2_{L^2(W_b)} + b^{-1} \|u\|^2_{L^2(W_b)}\big).$$

Proof. We square estimate (4-18), sum over k, and use (3-2) and Lemma 4.2. □

Observe that the controlling term in the preceding estimate is supported in the wing region. However, compared to the usual bounds (as in [Burq et al. 2007]) there is a loss of derivatives since we need $\partial_x u$ and $\partial_y u$ in the wings.

Corollary 4.8. *Let b_0 and E_0 be fixed. There exists C depending on the billiard b_0 and E_0 but not on the eigenfunction nor on $b < b_0$ such that*

$$\|u_+\|^2_{L^2(R)} \le C\big((b^{2\gamma-1}E + b^{2\gamma-3})\|u\|^2_{L^2(\Omega)} + b^{-1}\|u\|^2_{L^2(W)}\big).$$

Proof. We bound $\|\partial_x u\|^2_{L^2(W_b)}$ and $\|\partial_y u\|^2_{L^2(W_b)}$ by $E\|u\|^2_{L^2(\Omega)}$ and use the fact that the norm over W_b is less than the norm over W. □

It remains to choose b in a clever way to obtain the desired bound.

4.3.3. *Optimizing b.* We will choose b to be of the form $M^{-1}E^{-\alpha}$ for some constants M and α to be chosen. As long as α is positive, there is some large E_0 such that for any $E \ge E_0$ then $b = ME^{-\alpha} < b_0$ so that we can use the preceding proposition.

We obtain

$$\|u_+\|^2_{L^2(R)} \le C\big((M^{1-2\gamma}E^{1-\alpha(2\gamma-1)} + E^{-\alpha(2\gamma-3)})\|u\|^2_{L^2(\Omega)} + ME^{\alpha}\|u\|^2_{L^2(W)}\big). \tag{4-19}$$

It remains to make good choices to obtain the following proposition.

Proposition 4.9. *There exists E_0 and C depending only on the billiard such that for any u eigenfunction with energy $E > E_0$ the following holds*:

$$\|u_+\|^2_{L^2(R)} \le \tfrac{1}{4}\|u\|^2_{L^2(\Omega)} + CE^{\frac{1}{2\gamma-1}}\|u\|^2_{L^2(W)} \tag{4-20}$$

Proof. We choose $\alpha := 1/(2\gamma-1)$ and M such that $CM^{1-2\gamma} = \frac{1}{8}$. For E large enough, $E^{-\alpha(2\gamma-3)}$ goes to zero. It is thus bounded by $1/(8C)$ for E large enough. Substituting in (4-19) we get (4-20). □

4.4. *Small modes.* We now consider modes for which $k^2\pi^2/L_0^2 - E \le E$, and this time we rewrite the equation $P_k(u_k) = \Lambda_k$ in the form

$$-u_k'' - z_k u_k = h_k, \tag{4-21}$$

in which we have set $z_k := E - k^2\pi^2/L_0^2$ and

$$h_k := \tilde{\Lambda}_k + \frac{L'}{L}u_k' + \frac{k^2}{\pi^2}\left(\frac{1}{L_0^2} - \frac{1}{L^2}\right)u_k.$$

4.4.1. *The control estimate.* Since $z_k \ge -E$ we can use the results of the Appendix to control the term $\|u_k\|_{L^2(-B_0,b)}$. To do so, we need to estimate the norm of h_k in $H^{-1}(-B_0, b)$.

Lemma 4.10. *There exists some constant C depending only on b_0 such that, for any $b \le b_0$ and any k such that $k^2\pi^2/L_0^2 - E \le E$, the following holds*:

$$\|h_k\|_{H^{-1}(-B_0,b)} \le C\big(kb^{\gamma}\|F_k\|_{L^2(0,b)} + b^{\gamma-1}\|G_k\|_{L^2(0,b)} + (b^{\gamma-1} + k^2b^{\gamma+1})\|u_k\|_{L^2(0,b)}\big). \tag{4-22}$$

Proof. From the definition,

$$h_k = \tilde{\Lambda}_k + \frac{L'}{L}u_k' + \frac{k^2}{\pi^2}\left(\frac{1}{L_0^2} - \frac{1}{L^2}\right)u_k$$

and estimate each term separately. The first term is estimated using Lemma 4.1 and Remark 4.2. The second is estimated as in the proof of Lemma 4.3. The same method applies to estimate the third term. We introduce

$$I_3 = \left|\int_{-B_0}^{b}\left(\frac{1}{L_0^2} - \frac{1}{L^2(x)}\right)u_k(x)\phi(x)\,dx\right|,$$

and observe that the quantity in parentheses is $O(x_+^\gamma)$. Integrating by parts and using the Cauchy–Schwarz inequality twice gives

$$I_3 \le Cb^{\gamma+1}\|\phi'\|_{L^2(0,b)}\|u_k\|_{L^2(0,b)}.$$

Using the definition of the H^{-1} norm (see (4-5)) and putting these estimates together yields the lemma. □

For any $E \in \mathbb{R}$, define

$$\nu(E) := \min\left\{\left|E - \frac{k^2\pi^2}{L_0^2} - \frac{l^2\pi^2}{B_0^2}\right|, (k,l) \in \mathbb{N}\times\mathbb{N}\right\}.$$

Remark 4.3. Taking $l = 1$ in the definition shows that, for E large, we have

$$\nu(E) < c\sqrt{E} \tag{4-23}$$

for some constant c.

Lemma 4.11. *For any $\beta > 0$, there exists some c such that the following holds. For any k such that $z_k = E - k^2\pi^2/L_0^2 \ge \beta^2$,*

$$|\sin(B_0\sqrt{z_k})| \ge c\cdot\frac{\nu(E)}{\sqrt{z_k}}.$$

Proof. First we use that there exists some c such that

$$|\sin x| \ge c\,\mathrm{dist}(x, \pi\mathbb{Z}) \quad \text{for all } x \in \mathbb{R}.$$

We denote by l_k the integer such that

$$\mathrm{dist}\left(\sqrt{z_k}, \frac{\pi}{B_0}\mathbb{Z}\right) = \left|\sqrt{z_k} - \frac{l_k\pi}{B_0}\right|,$$

so that we have

$$\left|\sin(B_0\sqrt{z_k})\right| \ge c\left|\sqrt{z_k} - \frac{l_k\pi}{B_0}\right| \ge c\frac{z_k - l_k^2\pi^2/B_0^2}{\sqrt{z_k} + l_k\pi/B_0} \ge c\frac{E - \dfrac{k^2\pi^2}{L_0^2} - \dfrac{l_k^2\pi^2}{B_0^2}}{\sqrt{z_k}},$$

where, for the last bound, we have used the Lemma 4.12 below.

The claim follows by the definition of $\nu(E)$. □

Lemma 4.12. *Fix $\alpha > 0$ and denote by l the (step-like) function on $[0,\infty)$ defined by*

$$|\lambda - l(\lambda)\alpha| \;=\; \operatorname{dist}(\lambda, \alpha\mathbb{Z})$$

Then there exists some C such that

$$\lambda + l(\lambda)\alpha \le C\lambda. \quad \textit{for all } \lambda \in [0,\infty).$$

Proof. Define f by

$$f(\lambda) \;=\; \frac{\lambda + l(\lambda)\alpha}{\lambda}.$$

Since l vanishes on $[0,\alpha/2]$, we have $f(\lambda) = 1$ on this interval. Next, f tends to the limit 2 when λ goes to infinity. Finally, on $[\alpha/2, M]$ we have

$$f(\lambda) \;=\; 1 + \frac{l(\lambda)}{\lambda}\alpha \le 1 + \frac{2M+1}{\alpha}. \qquad \square$$

Putting these estimates together, we get:

Proposition 4.13. *There exists b_0 and E_0 and a constant $C := C(b_0, E_0)$ such that the following holds. For any $E > E_0$, for any k such that $k^2\pi^2/L_0^2 - E \le E$ and for any $b < b_0$, we have the estimate*

$$\|u_k\|_{L^2(-B_0,b)} \le C\frac{E^{\frac12}}{\nu(E)}\big(E^{\frac12}b^{\gamma+\frac12}\|F_k\|_{L^2(0,b)} + b^{\gamma-\frac12}\|G_k\|_{L^2(0,b)} + (1 + Eb^{\gamma+2})b^{-\frac12}\|u_k\|_{L^2(0,b)}\big). \tag{4-24}$$

Proof. For any k we let $z_k = E - k^2\pi^2/L_0^2$ and use the estimates of the appendix combined with the bound on h_k given by Lemma 4.10. For k such that z_k corresponds to estimates (A-10) and (A-12) of Theorem 3 we obtain

$$\begin{aligned}\|u_k\|_{L^2(-B_0,b)} &\le C\big(b^{\frac12}\|h_k\|_{H^{-1}(-B_0,b)} + b^{-\frac12}\|u_k\|_{L^2(0,b)}\big)\\ &\le C\big(kb^{\gamma+\frac12}\|F_k\|_{L^2(0,b)} + b^{\gamma-\frac12}\|G_k\|_{L^2(0,b)} + (b^\gamma + k^2b^{\gamma+2} + 1)b^{-\frac12}\|u_k\|_{L^2(0,b)}\big).\end{aligned}$$

We now use that $k = O(E^{\frac12})$ in the regime we are considering. We also remark that $b^\gamma + k^2b^{\gamma+2} + 1 = O(1 + Eb^{\gamma+2})$.

In the opposite case (for k such that z_k corresponds to estimate (A-11)), we have to add a global $|\sin(B_0\sqrt{z_k})|^{-1}$ prefactor. Using Lemma 4.11, we have

$$|\sin(B_0\sqrt{z_k})|^{-1} \;\le\; C\frac{\sqrt{z_k}}{\nu(E)} \;\le\; C\frac{E^{\frac12}}{\nu(E)}.$$

We thus obtain that, for any k,

$$\begin{aligned}&\|u_k\|_{L^2(-B_0,b)}\\ &\quad\le C\cdot\max\left(1, \frac{E^{\frac12}}{\nu(E)}\right)\big(E^{\frac12}b^{\gamma+\frac12}\|F_k\|_{L^2(0,b)} + b^{\gamma-\frac12}\|G_k\|_{L^2(0,b)} + (1 + Eb^{\gamma+2})b^{-\frac12}\|u\|_{L^2(0,b)}\big).\end{aligned}$$

Using (4-23), for large E we have $E^{1/2}/\nu(E)$ bounded from below, so that the claim follows. $\square$

4.5. ***Summing over k.*** We use the estimates of the preceding sections to obtain a control on $\|u_-\|^2_{L^2(R)}$ in which we have set

$$u_-(x,y) = \sum_{k^2\pi^2/L_0 - E \le E} u_k(x)\sin\left(\frac{k\pi y}{L(x)}\right).$$

Proposition 4.14. *There exists b_0 and E_0 and a constant C depending only on E_0 and b_0 such that if u is an eigenfunction with energy $E > E_0$ and $b < b_0$, then*

$$\|u_-\|^2_{L^2(R)} \le C\frac{E}{\nu(E)^2}\left(Eb^{2\gamma+1}\|\partial_x u\|^2_{L^2(W)} + b^{2\gamma-1}\|\partial_y u\|^2_{L^2(W)} + (1+Eb^{\gamma+2})^2 b^{-1}\|u\|^2_{L^2(W)}\right).$$

Proof. We square (4-24) and sum with respect to k. The Lemma 4.2 controls $\sum \|F_k\|^2$ and $\sum \|G_k\|^2$. Plancherel formula takes care of $\sum \|u_k\|^2$. We also use as before that the norm over W_b is smaller than the norm over W. □

As for the large mode case, we get a corollary using the fact that $\|\partial_x u\|^2$ and $\|\partial_y u\|^2$ are bounded above by $E\|u\|^2_{L^2(\Omega)}$.

Corollary 4.15. *There exists b_0 and E_0 and a constant C depending only on E_0 and b_0 such that if u is an eigenfunction with energy $E > E_0$ and $b < b_0$, then*

$$\|u_-\|^2_{L^2(R)} \le C\left[\left(\frac{E^3}{\nu(E)^2}b^{2\gamma+1} + \frac{E^2}{\nu(E)^2}b^{2\gamma-1}\right)\|u\|^2_{L^2(\Omega)} + (1+Eb^{\gamma+2})^2\frac{Eb^{-1}}{\nu(E)^2}\|u\|^2_{L^2(W)}\right].$$

4.6. ***A nonresonance condition.*** We now want to make the previous estimates explicit with respect to E and b so that we can use a similar optimization procedure as for the large modes case. We thus impose some condition on $\nu(E)$. Namely, for any $\varepsilon \ge 0$, we introduce the set

$$\mathcal{X}_\varepsilon := \{E \in \mathbb{R} \mid \nu(E) \ge c_0 E^{-\varepsilon}\} = \left\{E \in \mathbb{R} \;\middle|\; \left|E - \frac{k^2\pi^2}{L_0^2} - \frac{l^2\pi^2}{B_0^2}\right| \ge c_0 E^{-\varepsilon} \text{ for } k,l \in \mathbb{N}\right\}.$$

In other words, the set $\mathcal{X}_\varepsilon$ consists in energies that are far from the Dirichlet spectrum of the rectangle $[-B_0, 0] \times [0, L_0]$. It is natural to say that such energies are *not resonating* with the rectangle. The coefficient c_0 which is irrelevant when $\varepsilon > 0$ has been chosen in such a way that Weyl's law for the rectangle implies that $\mathcal{X}_0$ is not empty. Note however that, although expected, it is not clear that there actually are eigenvalues in $\mathcal{X}_0$, nor for that matter in $\mathcal{X}_\varepsilon$.

Once ε is fixed, the estimate of the Corollary 4.15 becomes

$$\|u_-\|^2_{L^2(R)} \le C\left[(b^{2\gamma+1}E^{3+2\varepsilon} + b^{2\gamma-1}E^{2+2\varepsilon})\|u\|^2_{L^2(\Omega)} + (1+Eb^{\gamma+2})^2 b^{-1}E^{1+2\varepsilon}\|u\|^2_{L^2(W)}\right]. \tag{4-25}$$

4.6.1. *Optimizing b.* As before we let $b = ME^{-\alpha}$ for some positive α and try to optimize the bound.

Proposition 4.16. *Define α by*

$$\alpha = \max\left(\frac{3+2\varepsilon}{2\gamma+1}, \frac{2+2\varepsilon}{2\gamma-1}\right).$$

There exists E_0 and C such that for any u eigenfunction with energy E in $\mathcal{X}_\varepsilon$ such that $E > E_0$, the following holds:

$$\|u_-\|^2_{L^2(R)} \leq \tfrac{1}{4}\|u\|^2_{L^2(\Omega)} + C \cdot E^{1+2\varepsilon+\alpha} \cdot \|u\|^2_{L^2(W)}. \tag{4-26}$$

Proof. With the given choice of α it is possible to choose M so that the prefactor of $\|u\|^2_{L^2(\Omega)}$ is $\frac{1}{4}$ for E large enough. The claim follows remarking that the definition of α implies

$$\alpha \geq \frac{3}{2\gamma+1} > \frac{1}{\gamma+2},$$

so that the prefactor $(1 + Eb^{\gamma+2})^2$ is uniformly bounded above. □

5. Nonconcentration estimate

We now put all the estimates together to obtain the following theorem.

Theorem 2. *Fix ε, and define ρ by*

$$\rho := \max\left(\frac{2+\gamma+2(\gamma+1)\varepsilon}{2\gamma+1}, \frac{1+2\gamma+4\gamma\varepsilon}{4\gamma-2}\right).$$

There exists E_0 and C such that any eigenfunction u of Ω with energy E in $\mathcal{X}_\varepsilon$ such that $E > E_0$ satisfies:

$$\|u\|_{L^2(\Omega)} \leq C \cdot E^{\rho}\|u\|_{L^2(W)}.$$

Proof. We first remark that whatever the exponent α is we always have $1 + 2\varepsilon + \alpha \geq 1 > \frac{1}{2\gamma-1}$ so that the exponent for the small modes is always larger than the exponent for the large modes. Thus, adding the estimates from propositions 4.9 and 4.16, we obtain

$$\|u\|^2_{L^2(R)} \leq \tfrac{1}{2}\|u\|^2_{L^2(\Omega)} + CE^{1+2\varepsilon+\alpha}\|u\|^2_{L^2(W)}.$$

Since $\|u\|^2_{L^2(R)} = \|u\|^2_{L^2(\Omega)} - \|u\|^2_{L^2(W)}$ we get

$$\tfrac{1}{2}\|u\|^2_{L^2(\Omega)} \leq (1 + CE^{1+2\varepsilon+\alpha})\|u\|^2_{L^2(W)}$$

When E is large the constant 1 can be absorbed in the term with a power of E. The claim follows by computing $1 + 2\varepsilon + \alpha$ for both possible choices of α and taking square roots. □

We state as a corollary the corresponding statement for the Bunimovich billiard (see Theorem 1).

Corollary 5.1. *In the Bunimovich stadium, for any $\varepsilon \geq 0$ there exists E_0 and C such that if u is an eigenfunction of energy E in $\mathcal{X}_\varepsilon$ such that $E > E_0$ then the following estimate holds*:

$$\|u\|_{L^2(\Omega)} \leq CE^{\frac{5+8\varepsilon}{6}}\|u\|_{L^2(W)}.$$

Proof. We let $\gamma = 2$, so $\alpha = \max\left(\frac{4+6\varepsilon}{5}, \frac{5+8\varepsilon}{6}\right)$. Since $\frac{4+6\varepsilon}{5} \leq \frac{5+8\varepsilon}{6}$ for any nonnegative ε, the proof is complete. □

Remark 5.1. The bounds in [Burq et al. 2007] gives a similar control with 1 as the exponent of E. Our bound thus gives a better estimate as long as $\varepsilon < \frac{1}{8}$. As it has been recalled in the introduction, it is quite natural that the nonresonance condition allows to get better bounds.

Remark 5.2. We could deal with quasimodes by adding an error term to Λ that is controlled by some negative power of E. There will be mainly two differences in the analysis. First the second term Λ will not have support away from the rectangle anymore and second, in the optimization process, we will have to take care of the new error term (which will possibly change the range of applicable exponents).

Remark 5.3. By adding the estimates in propositions 4.7 and 4.14, we get a different control estimate, where the control still is in the wings but now with a loss in derivatives. We haven't tried to optimize this bound.

Appendix: One-dimensional control estimates

The aim of this appendix is to provide a control estimate for the equation

$$-u'' - z \cdot u = h$$

on $[-B_0, b]$ of the form

$$\|u\|_{L^2(-B_0,0)} \leq C_1 \|h\|_{H^{-1}(-B_0,b)} + C_2 \|u\|_{L^2(0,b)},$$

in which we want an explicit dependence of the constants C_1 and C_2 on z and b. It is now standard (see [Burq and Zworski 2005]) that if b is fixed then we can choose C_1 and C_2 to be independent of z but what we need is an estimate when b goes to 0.

We first need a few preparatory lemmas.

Lemma A.2. *For any $\varepsilon > 0$, there exists a constant $C := C(\varepsilon)$ such that for any b, for any $h \in H^{-1}(-B_0, b)$ and any z such that $z \leq (1-\varepsilon)\pi^2/b^2$, there exists a solution $v_p \in H_0^1(0,b)$ to*

$$-v_p'' - z v_p = h,$$

in $\mathcal{D}'(0,b)$ and

$$\|v_p\|_{L^2(0,b)} \leq C b \|h\|_{H^{-1}(-B_0,b)}. \tag{A-1}$$

Proof. First we note that h, when restricted to $(0,b)$ also belongs to $H^{-1}(0,b)$ and that $\|h\|_{H^{-1}(0,b)} \leq \|h\|_{H^{-1}(-B_0,b)}$. The proof follows from a standard resolvent estimate since, on $(0,b)$, the bottom of the spectrum of the self-adjoint operator $v \mapsto -v''$ with Dirichlet boundary condition is π^2/b^2. We include it for the convenience of the reader. We decompose v_p in Fourier series:

$$v_p(x) = \sum_{k \geq 1} a_k \sin\Big(\frac{k\pi}{b} x\Big).$$

We have

$$h(x) = \sum_{k \geq 1} \Big(\frac{k^2\pi^2}{b^2} - z\Big) a_k \sin\Big(\frac{k\pi}{b} x\Big);$$

hence

$$\|h\|^2_{H^{-1}(0,b)} = \sum_{k\geq 1} \frac{\left(k^2\pi^2/b^2 - z\right)^2}{k^2\pi^2/b^2}|a_k^2|$$

or

$$\|h\|^2_{H^{-1}(0,b)} \geq \pi^2 b^{-2}\left(\inf_{k\geq 1} \frac{\left(1-\frac{zb^2}{k^2\pi^2}\right)^2}{\pi^2 k^2}\right)\|v_p\|^2_{H^1(0,b)} \geq c\pi^2 b^{-2}\|v_p\|^2_{L^2(0,b)}.$$

The claim follows since the inf is bounded away from zero in the regime we are considering. □

Lemma A.3. *Given $z \leq (1-\varepsilon)\pi^2/b^2$, let $w \in H^1_0(-B_0, b)$ be a solution to*

$$-w'' - zw = 0$$

in $\mathcal{D}'((-B_0, b)\setminus\{0\})$. Then there exists a constant A such that $w = AG$, in which the function G is defined by

$$G(x) = \begin{cases} \dfrac{\sin(\sqrt{z}(x+B_0))}{\sqrt{z}}\dfrac{\sin(\sqrt{z}b)}{\sqrt{z}} & \text{if } x<0, \\ \dfrac{\sin(\sqrt{z}(b-x))}{\sqrt{z}}\dfrac{\sin(\sqrt{z}B_0)}{\sqrt{z}} & \text{if } x>0. \end{cases} \tag{A-2}$$

Proof. Let w be such a function then necessarily there exist two constants $A_\pm$ such that

$$w(x) = \begin{cases} A_-\dfrac{\sin(\sqrt{z}(x+B_0))}{\sqrt{z}} & \text{if } x<0, \\ A_+\dfrac{\sin(\sqrt{z}(b-x))}{\sqrt{z}} & \text{if } x>0. \end{cases}$$

By assumption $w \in H^1$ and hence is continuous at 0, so

$$A_-\frac{\sin(\sqrt{z}B_0)}{\sqrt{z}} = A_+\frac{\sin(\sqrt{z}b)}{\sqrt{z}}.$$

In the regime we are considering $\sin(\sqrt{z}b)/\sqrt{z} \neq 0$, hence we can divide by this expression and express A_- in terms of A_+. The claim follows. □

We finish these preparatory lemmas by establishing the control estimate for multiples of G.

Lemma A.4. (1) *For β such that $0<\beta\leq\pi/B_0$, there exists $B_1 = B_1(\beta)$ and $C := C(\beta)$ such that, for any $z\leq\beta^2$ and any $b<B_1$, the following estimate holds*:

$$\|G\|_{L^2(-B_0,0)} \leq Cb^{-\frac{1}{2}}\|G\|_{L^2(0,b)}. \tag{A-3}$$

(2) *For any $\beta,\varepsilon>0$ there exists $B_1 := B_1(\beta)$ and $C := C(\beta,\varepsilon)$ such that, for any $b\leq B_1$ and $\beta^2\leq z\leq(1-\varepsilon)\pi^2/b^2$, the following estimate holds*:

$$\|G\|_{L^2(-B_0,0)} \leq C\frac{b^{-\frac{1}{2}}}{\sin(\sqrt{z}B_0)}\|G\|_{L^2(0,b)}. \tag{A-4}$$

Proof. (1) We first assume that $z < -Z_0^2$ for some positive Z_0. We set $z = -\omega^2$ and compute

$$\int_{-B_0}^{0} |G(x)|^2\,dx = \frac{\sinh^2(\omega b)}{\omega^2}\int_{-B_0}^{0} \sinh^2(\omega(B_0+x))\,dx,$$
$$\int_{0}^{b} |G(x)|^2\,dx = \frac{\sinh^2(\omega B_0)}{\omega^2}\int_{0}^{b} \sinh^2(\omega(b-x))\,dx.$$

By a straightforward change of variables we get

$$\int_{-B_0}^{0} |G(x)|^2\,dx = \frac{\sinh^2(\omega b)}{\omega^3}\int_{0}^{\omega B_0} \sinh^2(\xi)\,d\xi,$$
$$\int_{0}^{b} |G(x)|^2\,dx = \frac{\sinh^2(\omega B_0)}{\omega^3}\int_{0}^{b} \sinh^2(\xi)\,d\xi.$$

We set $F(X) := \dfrac{\int_0^X \sinh^2(\xi)\,d\xi}{\sinh^2(X)}$, so that we finally obtain

$$\int_{-B_0}^{0} |G(x)|^2\,dx \;=\; \frac{F(\omega B_0)}{F(\omega b)}\cdot\int_{0}^{b} |G(x)|^2\,dx.$$

It is straightforward that $F(X)$ is positive, tends to 1 at infinity and that $F(X)/X$ tends to $\frac{1}{3}$ at 0. As a consequence, there exists some $C(Z_0)$ such that, for any $z < -Z_0^2$,

$$\int_{-B_0}^{0} |G(x)|^2\,dx \le C\max(1,(\omega b)^{-1})\int_{0}^{b} |G(x)|^2\,dx,$$

For $b < B_1$ and $\omega > Z_0$, we have $\max(1,(\omega b)^{-1}) \le \max(1,\omega^{-1})b^{-1} \le Cb^{-1}$ which gives the claim for this range of parameters.

We now assume that we have $-Z_0^2 < z < \beta^2$. We have

$$\begin{aligned}\int_{-B_0}^{0} |G(x)|^2\,dx &= \left|\frac{\sin(\sqrt{z}b)}{\sqrt{z}}\right|^2\int_{-B_0}^{0}\left|\frac{\sin(\sqrt{z}(x+B_0))}{\sqrt{z}}\right|^2 dx\\ &= b^2\left|\frac{\sin(\sqrt{z}b)}{\sqrt{z}b}\right|^2\int_{-B_0}^{0}\left|\frac{\sin(\sqrt{z}(x+B_0))}{\sqrt{z}(x+B_0)}\right|^2 (x+B_0)^2\,dx \le Cb^2,\end{aligned}$$

where the constant C comes from the fact that the function $\sin(w)/w$ is continuous and its argument belongs to a fixed compact set. On the other hand, by a simple change or variables we have

$$\int_{0}^{b} |G(x)|^2\,dx = \left|\frac{\sin(\sqrt{z}B_0)}{\sqrt{z}}\right|^2\int_{0}^{b}\left|\frac{\sin(\sqrt{z}x)}{\sqrt{z}}\right|^2 dx \ge cB_0^2\frac{b^3}{3},$$

in which c is given by

$$c \;=\; \left|\frac{\sin(\sqrt{z}B_0)}{\sqrt{z}B_0}\right|^2 \inf_{0\le x\le B_1}\left|\frac{\sin(\sqrt{z}x)}{\sqrt{z}x}\right|^2.$$

Using that $\sin(w)/w$ is continuous and does not vanish on $(-\infty, \pi)$ and choosing B_1 accordingly we obtain the first bound.

(2) We first use homogeneity and prove the bound for $\tilde{G} := zG$. We have

$$\int_{-B_0}^{0} \left|\tilde{G}(x)\right|^2 dx = \left|\sin(\sqrt{z}b)\right|^2 \int_{-B_0}^{0} \left|\sin(\sqrt{z}(x+B_0))\right|^2 dx \leq B_0 \left|\sin(X)\right|^2,$$

in which we have set $X := \sqrt{z}b$. On the other hand we have

$$\int_0^b \left|\tilde{G}(x)\right|^2 dx = \left|\sin(\sqrt{z}B_0)\right|^2 \int_0^b \left|\sin(\sqrt{z}x)\right|^2 dx = b \cdot \left|\sin(\sqrt{z}B_0)\right|^2 \cdot \frac{1}{X} \int_0^X |\sin(\xi)|^2 d\xi,$$

with the same X. Under the assumptions, X belongs to a compact subinterval of $[0, \pi)$. Since on this interval the function

$$X \mapsto \frac{1}{X|\sin(X)|^2} \int_0^X |\sin(\xi)|^2 d\xi$$

is continuous, the claim follows. □

Proposition A.5. *There exist β and $B_1 := B_1(\beta)$, such that if $b \leq B_1$ and $v \in H_0^1(-B_0, b)$ satisfies*

$$-v'' - zv = h,$$

with h that vanishes on $(-B_0, 0)$, then the following estimates hold:

(1) *If $z \leq \beta^2$, then*

$$\|v\|_{L^2(-B_0,0)} \leq C_1\big(b^{\frac{1}{2}}\|h\|_{H^{-1}(-B_0,b)} + b^{-\frac{1}{2}}\|v\|_{L^2(0,b)}\big), \tag{A-5}$$

(2) *If $\beta^2 \leq z \leq \dfrac{1}{b^2}$,*

$$\|v\|_{L^2(-B_0,0)} \leq C_1\left(\frac{b^{\frac{1}{2}}}{|\sin(B_0\sqrt{z})|}\|h\|_{H^{-1}(-B_0,b)} + \frac{b^{-\frac{1}{2}}}{|\sin(B_0\sqrt{z})|}\|v\|_{L^2(0,b)}\right), \tag{A-6}$$

(3) *If $\dfrac{1}{b^2} \leq z$ then*

$$\|v\|_{L^2(-B_0,0)} \leq C_3\big(b^{\frac{1}{2}}\|h\|_{H^{-1}(-B_0,b)} + b^{-\frac{1}{2}}\|v\|_{L^2(0,b)}\big). \tag{A-7}$$

Proof. In the first two cases, we have $z \leq \frac{1}{b^2} < \frac{\pi^2}{b^2}$. We may thus consider v_p as given by Lemma A.2 and define $\tilde{v}_p$ by extending v_p by 0 for negative x. Observe that $w := v - \tilde{v}_p$ is in $H_0^1(-B_0, b)$ and satisfies

$$-w'' - zw = 0$$

in $\mathcal{D}((-B_0, b) \setminus \{0\})$ so that $v - \tilde{v}_p = AG$ for some A according to Lemma A.3. Using Lemma A.4 we obtain in the first case

$$\|v - \tilde{v}_p\|_{L^2(-B_0,0)} \leq Cb^{-\frac{1}{2}}\|v - \tilde{v}_p\|_{L^2(0,b)}.$$

We use the triangle inequality on the right-hand side and the fact that $\tilde{v}_p$ is 0 for negative x and coincide with v_p for positive v. We obtain

$$\|v\|_{L^2(-b_0,0)} \leq Cb^{-\frac{1}{2}}\left(\|v\|_{L^2(0,b)} + \|v_p\|_{L^2(0,b)}\right).$$

The claim then follows from the estimate on $\|v_p\|_{L^2(0,b)}$ in Lemma A.2. We prove the second case by following the same argument, inserting the corresponding bound for G.

The third case will follow the same lines but we will introduce a different particular solution v_p, following then even more closely the proof of [Burq and Zworski 2005]. We set $\lambda = \sqrt{z}$.

Denote by H the unique L^2 function on $(-B_0, b)$ that vanishes on $(-B_0, 0)$ and such that $H' = h$ in the distributional sense. The L^2 norm of H is related to the H^{-1} norm of h by the relation

$$\|H - \Big(\int_0^b H(y)\,dy\Big)\|_{L^2(-B_0,b)} = \|h\|_{H^{-1}(-B_0,b)}.$$

The Cauchy–Schwarz inequality then implies that

$$\|H\|_{L^2(-B_0,b)} \geq (1 + b^{\frac{1}{2}})^{-1}\|h\|_{H^{-1}(-B_0,b)}. \tag{A-8}$$

Set

$$v_p(x) = \int_{-B_0}^{x} \frac{\sin(\lambda(x-y))}{\lambda} H'(y)\,dy.$$

Then v_p satisfies

$$-v_p'' - \lambda^2 v_p = H'$$

in $\mathcal{D}'(-B_0, b)$ and $v_p(-B_0) = 0$ but the boundary condition need not be satisfied at b. We thus have

$$v(x) = v_p(x) - v_p(b)\frac{\sin(\lambda(x+B_0))}{\sin(\lambda(B_0+b))}.$$

The function $v - v_p$ is thus a multiple of $\sin(\lambda(x + B_0))$.

We have

$$\int_{-B_0}^{0} \big|\sin(\lambda(x+B_0))\big|^2\,dx \leq B_0$$

and

$$\int_0^b |\sin(\lambda(x+B_0))|^2\,dx \geq \frac{1}{2}\Big(b - \frac{1}{2\lambda}\Big).$$

Hence, in the regime under consideration we have

$$\|v - v_p\|_{L^2(-B_0,b)} \leq Cb^{-\frac{1}{2}}\|v - v_p\|_{L^2(0,b)}. \tag{A-9}$$

We perform an integration by parts in v_p and observe that the boundary contributions vanish because H vanishes near $-B_0$ and $\sin(\lambda(y-x))$ vanishes at $y = x$.

Finally, we obtain

$$v_p(x) = \int_{-B_0}^{x} \cos(\lambda(x-y))H(y)\,dy.$$

It follows that v_p is identically 0 on $(-B_0, 0)$ and that, on $(0, b)$, it satisfies

$$|v_p(x)| \leq \|H\|_{L^2(-B_0,b)}\sqrt{x}.$$

Squaring and integrating, we get

$$\|v_p\|_{L^2(0,b)} \le b\|H\|_{L^2(-B_0,b)}.$$

Using the triangle inequality in (A-9) and inserting this bound, the result follows for $b \le \frac{1}{2}$ using (A-8). □

In the paper, we will need to relax the condition that $v(b) = 0$. This can be done using a standard construction related to a commutator method. We will get the following

Theorem 3. *There exist β and four constants B_1, C_1, C_2, C_3 depending only on β such that the following holds. For any $b \le B_1$, for any function u in $H^1(-B_0, b)$ that satisfies*

$$-u'' - zu = h,$$

with $h \in H^{-1}(-B_0, b)$ and such that $u(-B_0) = 0$ and h vanishes on $(-B_0, 0)$. Then, the following estimates hold:

(1) *If $z \le \beta^2$, then*

$$\|u\|_{L^2(-B_0,0)} \le C_1\big(b^{\frac{1}{2}}\|h\|_{H^{-1}(-B_0,b)} + b^{-\frac{1}{2}}\|u\|_{L^2(0,b)}\big). \tag{A-10}$$

(2) *If $\beta^2 \le z \le \dfrac{1}{b^2}$, then*

$$\|u\|_{L^2(-B_0,0)} \le C_1\left(\frac{b^{\frac{1}{2}}}{|\sin(B_0\sqrt{z})|}\|h\|_{H^{-1}(-B_0,b)} + \frac{b^{-\frac{1}{2}}}{|\sin(B_0\sqrt{z})|}\|u\|_{L^2(0,b)}\right). \tag{A-11}$$

(3) *If $\dfrac{1}{b^2} \le z$, then*

$$\|u\|_{L^2(-B_0,0)} \le C_3\big(b^{\frac{1}{2}}\|h\|_{H^{-1}(-B_0,b)} + b^{-\frac{1}{2}}\|u\|_{L^2(0,b)}\big). \tag{A-12}$$

Proof. Define a smooth cutoff function ρ_1 such that $\rho_1(x)$ is identically 1 if $x \le \frac{1}{2}$ and identically 0 if $x \ge 1$ and let ρ_b be the function $x \mapsto \rho_1(x/b)$. Define $v := \rho_b u$ then $v \in H_0^1(-B_0, b)$ and satisfies

$$-v'' - zv = h + 2(\rho_b' u)' - \rho_b'' u.$$

The right-hand side vanishes on $(-B_0, 0)$ so that, in order to use Proposition A.5, we have to estimate its H^{-1} norm. The strategy is the same as in the proofs of Lemmas 4.3 and 4.10.

An integration by parts followed by the use of the Cauchy–Schwarz inequality gives

$$\left|\int_{-B_0}^{B_1} (\rho_b' u)'\phi\right| \le \|\rho' u\|_{L^2(0,b)}\|\phi'\|_{L^2} \le \frac{C}{b}\|u\|_{L^2(0,b)}\|\phi'\|_{L^2}.$$

Thus,

$$\|(\rho_b' u)'\|_{H^{-1}} \le \frac{C}{b}\|u\|_{L^2(0,b)}.$$

The third term can be estimated using the same method. Indeed,

$$\left|\int \rho_b'' u\phi\right| = \left|\int_0^b \left(\int_0^x \rho_b''(y)u(y)\,dy\right)\phi'(x)\,dx\right| \le \|\int_0^x \rho_b''(y)u(y)\,dy\|_{L^2(0,b)}\|\phi'\|_{L^2}.$$

Using again Cauchy–Schwarz inequality and the fact that $|\rho_b''(y)| \le Cb^{-2}$ we get

$$\left| \int_0^x \rho_b''(y) u(y)\, dy \right| \le Cb^{-2} \|u\|_{L^2(0,b)} \sqrt{x}.$$

We obtain

$$\left\| \int_0^x \rho_b''(y) u(y)\, dy \right\|_{L^2(0,b)} \le Cb^{-2} \|u\|_{L^2(0,b)} \|\sqrt{x}\|_{L^2(0,b)} \le Cb^{-1} \|u\|_{L^2(0,b)}.$$

It follows that

$$\|h + 2(\rho_b' u)' - \rho_b'' u\|_{H^{-1}(-B_0,b)} \le \|h\|_{H^{-1}(-B_0,b)} + Cb^{-1} \|u\|_{L^2(0,b)}.$$

We obtain the theorem by plugging this bound into the estimates of the Proposition A.5. □

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