TWISTED AFFINE SCHUBERT VARIETIES AND TWISTED AFFINE DEMAZURE MODULES

Marc Besson

A dissertation submitted to the faculty at the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the College of Arts and Sciences.

Chapel Hill
2021

Approved by:

Jiuzu Hong

Prakash Belkale

Ivan Cherednik

Shrawan Kumar

Richárd Rimányi

ABSTRACT

Marc Besson: Twisted affine Schubert varieties and twisted affine Demazure modules
(Under the direction of Jiuzu Hong)

This work studies line bundles of level one on twisted affine Schubert varieties. Following work of Zhu, we describe the restriction map from global sections of a level one line bundle to the T^{σ}-fixed subscheme. Our technique uses global methods and studies this map using the associated map for untwisted affine Schubert varieties. As a corollary we are able to describe the smooth locus of many twisted affine Schubert varieties.

à mes parents

ACKNOWLEDGEMENTS

First and foremost, I must acknowledge Professor Jiuzu Hong. As my adviser and mentor for four years, he has spent a huge amount of his time and energy with me. He has fielded countless questions, and has taught me his way of approaching mathematics. From him I have gained my mathematical abilities, and my mathematical taste; I hope that I can also emulate his energy and vigor.

I must also thank the members on my committee, Professor Belkale, Professor Cherednik, Professor Kumar and Professor Rimányi. They have taught me, in classes and office hours and conversations, for the past six years and I owe them a great debt. Moreover they have kindly agreed to participate in this final stage of my Ph.D., so I am triply grateful.

I must just as soon acknowledge my peers Sam Jeralds and Josh Kiers. My success in graduate school is due almost entirely to our interminable "Ginzburg" seminars. In Josh and Sam I found collaborators with whom I can talk for hours and hours. More than collaborators, I have also found true friends, and they were a very important part of making grad school survivable.

I must also thank the other graduate students who have spent so much time with me, discussing math and other things; my longtime officemate Michael Strayer and housemates Paul Kruse and Yiyan Shou being friends of mine who endured countless basic questions (often the very same ones) asked repeatedly at intervals of roughly two months.

All of my acknowledgements thus far has been of a mathematical nature more or less, but one does not subsist on math alone for six years. I want to also deeply thank the people in life who made graduate school a bearable experience, and who were able to pull me away from my books from time to time. I want to thank Laryssa Abdala, Mack Krone, Audrey Morrow, and Aaron Taggart from the Princess Palace, as well as Ian Von Wald, Seth Baldwin, Logan Tatham, Gonzalo Cazes-Nasitiqui, and Blake Keeler.

In a similar vein I want to thank my friends Omar Hyjek, Andrew Tolbert, Peter Sisson, Pat Horan and Daivik Orth for their consistent friendship.

I must also of course thank Laurie and Sarah, without whom this department would surely cease to function.

Going further back, I would like to thank Prof. Schaefer and Prof. Tesman; they played instrumental roles in teaching and mentoring me during my undergraduate years.

Lastly, and yet really deserving true precedence, I must thank my family; my father, mother, and sister.

TABLE OF CONTENTS

LIST OF FIGURES . . . . **ix**

LIST OF TABLES . . . . **x**

CHAPTER 1: INTRODUCTION **1**

CHAPTER 2: BASICS OF AFFINE GRASSMANNIANS **5**

2.1 Basic Notation 5

2.2 Affine Grassmannians 5

2.2.1 Stratification 6

2.2.2 Components 7

2.3 Global Affine Grassmannians 7

2.3.1 Global Arc Groups 8

2.3.2 Global Schubert Varieties 9

2.4 Summary of some work of Zhu 10

2.4.1 Step 1: 12

2.4.2 Step 2: 12

2.4.3 Step 3: 12

CHAPTER 3: ABSOLUTELY SPECIAL AUTOMORPHISMS **13**

3.1 Twisted Affine Algebras 13

3.2 Absolutely Special Automorphisms 14

3.2.1 Root Systems 15

CHAPTER 4: BASICS OF TWISTED AFFINE GRASSMANNIANS **17**

4.1 Group Schemes and first construction 17

4.2 Twisted Affine Grassmannians 18

4.2.1 Stratification 18

4.2.2 Components 19

4.3 Global Twisted Affine Grassmannians 22

4.3.1 Global Twisted Schubert Varieties 25

CHAPTER 5: CONSTRUCTION OF LINE BUNDLES OF LEVEL 1 27

5.1 Conformal Blocks, Line Bundles 27

5.1.1 Construction of level one line bundles on $\mathrm{Bun}_{\mathcal{G}}$ 34

CHAPTER 6: FLAT DEGENERATIONS AND SMOOTHNESS 38

6.1 Flat Degeneration and Smoothness Results 38

6.1.1 A duality isomorphism for twisted Schubert varieties 40

6.1.2 Application: Smooth locus of twisted affine Schubert varieties 45

CHAPTER 7: PARTIAL RESULTS ON E_6 49

7.1 Partial Results on E6 49

7.2 $V_{\omega_4,-3}$ 51

7.2.1 $V_{\omega_4,-3}(\omega_4)$ 51

7.2.2 $V_{\omega_4,-3}(\omega_1+\omega_6)$ 51

7.2.3 $V_{\omega_4,-3}(\omega_2)$ 51

7.2.4 $V_{\omega_4,-3}(0)$ 53

7.3 $V_{\omega_2,-2}$ 54

7.3.1 Heisenberg Action 54

7.3.2 $V_{\omega_2,-2}(0)$ 56

7.3.3 $V_{\omega_2,-2}(\omega_2)$ 56

7.3.4 Conclusion 57

REFERENCES 59

LIST OF FIGURES

7.1 This is the poset of extremal weights μ such that $\omega_2 < \mu \leq \omega_4$ and such that the support of $\mu - \omega_2$ is not a proper subdiagram. 52

LIST OF TABLES

3.1 The semisimple types, the orders of their absolutely special automorphisms, and the fixed subalgebra. 15

4.1 Coinvariants of fundamental groups. This also provides the number of components of $\mathtt{Gr}_{\mathscr{G}}$ for G adjoint. 20

4.2 Twisted affine algebras and their Dynkin diagrams. 21

CHAPTER 1

Introduction

A trois heures et demie, le docteur découvre la valeur de x, l'inconnue cherchée; ce qui lui cause une joie sans mélange. – Nous prions les esprits superficiels de s'abstenir de toute réflexion sur la valeur de x, et de ne point prétendre que Zéphyrin a beaucoup travaillé pour peu de chose.

— Christophe, *L'Idée fixe du savant Cosinus*

This work follows in a long tradition of studying homogeneous spaces for reductive groups. Over an algebraically closed field, these are some of the most well-studied geometric objects available, and whose geometry can almost completely be described using the combinatorics of Weyl groups and root systems associated to the reductive group G.

This situation is largely paralleled when G is a reductive group over a local field; while slighly more recent, this topic still dates back at least over half a century. When the local field in question is the field of formal Laurent series, this topic is extremely closely related to the combinatorical and algebraic framework of Kac-Moody groups and algebras.

An important feature in this case is the existence of line bundles of "level" one on these homogeneous spaces. Let us introduce some notation. We will write $\mathcal{K}$ and $\mathcal{O}$ for the field of formal Laurent series and for formal power series. We write G for a semisimple reductive group, and $\mathtt{Gr}_G$ for the homogeneous space in question, whose $\mathbb{C}$-points are $G(\mathcal{K})/G(\mathcal{O})$. For simplicity in the introduction, we will discuss only the case where G is simply-connected; in this case the affine Grassmannian has only one component. In this case, $Pic(\mathtt{Gr}_G) \simeq \mathbb{Z}$. If L is the ample generator of $Pic(\mathtt{Gr}_G)$, then the space $H^0(\mathtt{Gr}_G, L)^*$ is usually called the basic representation.

A beautiful structure of the basic representation associated to affine Kac-Moody groups of type A,D and E was exposed first by Frenkel-Kac and Segal, and later generalized by Kac-Peterson. Roughly, the Frenkel-Kac formulation describes the branching of the basic representation over the homogeneous (infinite-dimensional) Heisenberg subalgebra. A weak form of Frenkel-Kac states that the basic representation $\mathscr{H}$ decomposes into a direct sum of Fock spaces

$$\mathscr{H} = \bigoplus_{\lambda \in X_*(T)} \pi_\lambda$$

where π_λ are Fock modules for the homogeneous Heisenberg subalgebra $\hat{\mathfrak{t}}$.

Xinwen Zhu revisited this result in his paper [Zh1], reinterpreting it as a geometric statement about the T-fixed subscheme of the affine Grassmannian $\mathtt{Gr}_G$. The closed T-fixed points of $\mathtt{Gr}_G$ are easily described, and they correspond precisely to coweights $\lambda \in X_*(T)$. However, the T-fixed subscheme $\mathtt{Gr}_G^T$ is a highly non-reduced scheme. Zhu's geometric interpretation of the Frenkel-Kac isomorphism involves interpreting sections of the ample line bundle L restricted to a T-fixed subscheme supported over the point $[t^\lambda]$ as dual to the Fock module π_λ. Moreover, this result holds for very many affine Schubert varieties. Let $\overline{\mathtt{Gr}}_G^\lambda$ be an affine Schubert variety, and let I denote the ideal sheaf of the T-fixed subscheme $(\overline{\mathtt{Gr}}_G^\lambda)^T$. We have a short exact sequence of sheaves

$$0 \to I^\lambda(1) \to L|_{\overline{\mathtt{Gr}}_G^\lambda} \to \mathscr{O}_{(\overline{\mathtt{Gr}}_G^\lambda)^T} \otimes L \to 0. \tag{1.1}$$

Zhu shows that in very many situations for G of type ADE, upon taking global sections we have an isomorphism

$$H^0(\overline{\mathtt{Gr}}_G^\lambda, L) \to H^0(\overline{\mathtt{Gr}}_G^\lambda, \mathscr{O}_{(\overline{\mathtt{Gr}}_G^\lambda)^T} \otimes L).$$

The dual of the left term is an affine Demazure module, and the term on the right corresponds to sections over the T-fixed subscheme. After taking a limit over the poset of dominant weights, we recover the weak form of Frenkel-Kac described above. Moreover in this form, we gain information about the singularities of the affine Schubert varieties: if a projective variety X equipped with a T-action is smooth at a T-fixed point, then the T-fixed subscheme will be reduced. Thus from Zhu's result we obtain as a corollary that affine Schubert varieties $\overline{\mathtt{Gr}}_G^\lambda$ of type ADE are singular at all points $[t^\mu]$ when $\mu < \lambda$. Moreover, this implies that the affine Schubert varieties are singular at all strata associated with $\mu < \lambda$. This result was already known via different methods, see [MOV].

In this work we undertake an extension of this result. In the language of Kac-Moody theory, the Frenkel-Kac isomorphism applies to the simply-laced affine types as well as the twisted affine types (but not the non-simply-laced untwisted affines). So we may form twisted affine Grassmannians $\mathtt{Gr}_{\mathscr{G}}$ which are homogeneous spaces for twisted loop groups, and (usually) we may find line bundles $\mathscr{L}$ on these homogeneous spaces which are of "level" one. These twisted affine Grassmannians are associated to automorphisms σ; just as there is a natural T-action on $\mathtt{Gr}_G$ we have a natural T^σ-action on $\mathtt{Gr}_{\mathscr{G}}$, and we may examine the T^σ-fixed subscheme of a twisted affine Schubert variety $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$. The key construction is a global family of affine Schubert varieties whose generic fiber is an untwisted affine Schubert variety and whose special fiber is a twisted affine Schubert variety.

We now proceed with a discussion of the contents of each chapter.

In Chapter 2 we discuss the basics of affine Grassmannians in the untwisted setting. This material is standard by now, and can be found in several detailed references. We require not only the classical affine Grassmannians, but also a whole collection of "relative" or "global" affine Grassmannians, whose construction is due to foundational work of Beilinson and Drinfeld [BD]. Moreover we give a summary of the work of [Zh1], since it plays not only a motivational role, but the theorems found there for the untwisted affines are crucial for our proofs of the twisted affines.

The twisted affine Grassmannians are constructed using certain automorphisms of loop groups $G(\mathcal{K})$. In fact these automorphisms must be chosen with some care. In Chapter 3 we discuss the "absolutely special" automorphisms we use. This restricts the number of homogeneous spaces our result applies to; in particular we must exclude a homogeneous space for $A_{2n}^{(2)}$, for which our result cannot hold. The reason for this is simple; we cannot construct a line bundle of level one on this homogeneous space. We spend Chapter 3 developing notation and combinatorics associated to these absolutely special automorphisms.

In Chapter 4 we essentially repeat the constructions of Chapter 2, but in this case we work explicitly with the twisted affine Grassmannians. Chapter 4 is mostly devoted to results which have precise analogues with the results for untwisted affine Grassmannians in Chapter 2, and the chapters are written to coincide with each other as much as possible. Just as in Chapter 2, we must also construct several "global" versions of the twisted affine Grassmannians. In this setting there is a new feature; we produce flat schemes over curves whose generic fiber is an untwisted affine Schubert variety and whose special fiber is a twisted Schubert variety. This enables us to transfer data from the untwisted case to the twisted case, which is critical to our proof.

In Chapter 5 we undertake the (rather technical) construction of line bundles of level one on our global twisted affine Grassmannians. This construction uses recent results of Hong and Kumar (see [HK]) on twisted conformal blocks. In particular line bundles are first constructed on $\mathrm{Bun}_{\mathcal{G}}$ and are then pulled back to the global twisted affine Grassmannians.

After many pages of constructions, in Chapter 6 we finally arrive at our results. At this point we have a flat family over a curve, where the generic fiber is an untwisted affine Schubert variety and where the special fiber is a twisted affine Schubert variety. Moreover, we have a global line bundle $\mathcal{L}$ of level one on this flat family. Now flatness permits us to transfer Zhu's isomorphism in the untwisted case to the twisted case. Thus for many twisted affine Schubert varieties, we find that the space of global sections of the level one line bundle is isomorphic to the space of sections of the line one level bundle restricted to the T^{σ}-fixed subscheme. Theorem 6.1.5 is the main result of this work. As a corollary, we are also able to characterize the smooth locus of twisted affine Schubert varieties in most cases. Additionally, our result proves most cases of a conjecture of Haines-Richarz [HR].

Chapter 7 is a slight deviation from the main body of this work. As mentioned above, our proof of the restriction isomorphism the twisted case relies on the results and methods in the untwisted case. Thus Zhu's isomorphism for Schubert varieties in the untwisted case is a prerequisite for us. While Zhu was able to prove his isomorphism in all cases in types A and D, the result remains unfinished in type E. One particular case of relevance to our work concerns the restriction isomorphism for the affine Schubert variety associated to $\check{\omega}_4$ in type E_6. Thus the analysis of the T-fixed subscheme of $\overline{\mathtt{Gr}}_G^{\check{\omega}_4}$ remains incomplete, though we make progress in this last Chapter. In fact, there are further unproven cases in E_7 and E_8, but these algebras have no nontrivial diagram automorphisms, so they do not give rise to twisted affine algebras. Thus in Chapter 7 we present our unfinished efforts to prove the restriction isomorphism for $\overline{\mathtt{Gr}}_{E_6}^{\check{\omega}_4}$. With our techniques, this becomes a delicate but mostly doable problem in classical representation theory. Work in Chapter 7 narrows the problem of the restriction isomorphism for E_6 down to a question about the 0-weight space of the representation V_{ω_4} of E_6.

CHAPTER 2

Basics of Affine Grassmannians

In this chapter we review basic facts about affine Grassmannians and give a summary of Zhu's work on affine Demazure modules, which provides motivation and structure for the present work. All of the constructions and basic theorems provided in this chapter are very standard at this point, and are introduced without proof. This material is collected in the following sources: [MV], [Zh1], [Zh4], [BR] and [BD].

2.1 Basic Notation

Let G be a simply-connected semisimple algebraic group over $\mathbb{C}$. We choose a maximal torus and Borel subgroup $T \subset B \subset G$. We denote $X^*(T)$ the lattice of weights of T, and by $X_*(T)$ the lattice of coweights. Their natural pairing is denoted by $\langle , \rangle$. We let Φ denote the set of roots of G, and denote by Φ^+ the set of positive roots of G with respect to B. We let $\check{\Phi}$ denote the set of coroots, so $(\Phi, X^*(T), \check{\Phi}, X_*(T))$ is a root datum for G, and write W for the Weyl group of G. We denote by $\{\alpha_i \,|\, i \in I\}$ (respectively $\{\check{\alpha}_i \,|\, i \in I\}$ the set of simple roots in Φ (respectively coroots in $\check{\Phi}$), where I is the set of vertices of the associated Dynkin diagram of G. We choose a pinning $\{x_{\alpha_i}, y_{\alpha_i} \,|\, i \in I\}$ of G. Let $\mathcal{O} = \mathbb{C}[[t]]$ denote the formal power series in t and $\mathcal{K} = \mathbb{C}((t))$ denote the formal Laurent series in t. A coweight $\lambda \in X_*(T)$ naturally defines a point of $T(\mathcal{K}) \subset G(\mathcal{K})$ which we denote as t^λ. We write $\mathfrak{g}$ for $Lie(G)$, $\mathfrak{b}$ for $Lie(B)$ and $\mathfrak{t}$ for $Lie(T)$, though we may also use $\mathfrak{h}$.

2.2 Affine Grassmannians

For G a reductive algebraic group over $\mathbb{C}$, we may construct the loop group LG and the arc group L^+G as the following functors:

Definition 2.2.1. *LG represents the following functor:*

$$R \mapsto G(R((t)))$$

for R a $\mathbb{C}$-algebra.

L^+G represents the following functor:

$$R \mapsto G(R[[t]])$$

for R a $\mathbb{C}$-algebra.

The first is represented by an ind-scheme, the second represented by a scheme of infinite type over $\mathbb{C}$.

Definition 2.2.2. *We define the affine Grassmannian as the fpqc quotient*

$$\mathtt{Gr}_G = LG/L^+G.$$

Remark 2.2.3. *From this description we can see that the $\mathbb{C}$-points are $\mathtt{Gr}_G(\mathbb{C}) = G(\mathcal{K})/G(\mathcal{O})$.*

There is also the Beauville-Laszlo interpretation of the affine Grassmannian in terms of G-torsors on the formal disc. We denote by $\mathbb{D}_R = \operatorname{Spec} R[[t]]$ and $\mathbb{D}_R^* = \operatorname{Spec} R((t))$.

Theorem 2.2.4. *Let F be a G-torsor over $\mathbb{D} = \operatorname{Spec} \mathbb{C}[[t]]$, and denote by $\mathring{F}$ the trivial G-torsor on $\mathbb{D}$. We then have*

$$\mathtt{Gr}_G(R) = \left\{ (F, \beta) \,\middle|\, \beta : F|_{\mathbb{D}_R^*} \xrightarrow{\sim} \mathring{F}|_{\mathbb{D}_R^*} \right\}. \tag{2.1}$$

This description of the affine Grassmannian is due to Beauville-Laszlo [BL] in type A and due to [DS] for arbitrary types.

2.2.1 Stratification

We have the Cartan decomposition

$$G(\mathcal{K}) = \bigsqcup_{\lambda \in X_*(T)^+} G(\mathcal{O}) t^\lambda G(\mathcal{O}) \tag{2.2}$$

which induces a stratification of $\mathtt{Gr}_G$ by left $G(\mathcal{O})$-orbits. Moreover, each orbit has finitely many $T_{\mathbb{C}}$-fixed points (namely $t^{w\lambda}$ for all $w \in W$).

We define the affine Schubert varieties to be the closures of cells.

Definition 2.2.5. $\overline{\mathtt{Gr}}_G^\lambda = \overline{G(\mathcal{O}).[t^\lambda]}$

We can also easily describe the closure relations. For $\lambda, \mu \in X_*(T)$, let us write $\mu \leq \lambda$ if $\mathtt{Gr}_G^\mu \subseteq \overline{\mathtt{Gr}}_T^\lambda$.

Lemma 2.2.6. *We have $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a non-negative integral linear combination of simple coroots $\check{\alpha}_i$.*

2.2.2 Components

The components of $\mathtt{Gr}_G$ can be parametrized by the elements in $\pi_1(G)$, where $\pi_1(G) = X_*(T)/Q^\vee$. Denote by $\tilde{G}$ the simply-connected cover of G. Then we have an explicit decomposition

$$\mathtt{Gr}_G = \bigsqcup_{\gamma \in \pi_1(G)} \tilde{G}_{\mathcal{K}}/Ad_{t^{\lambda_\gamma}} \tilde{G}_{O}. \tag{2.3}$$

If G is simply-connected, then $\mathtt{Gr}_G$ has only one component, $\mathtt{Gr}_G = \mathtt{Gr}_G^\circ$.

2.3 Global Affine Grassmannians

We form "global" versions of the affine Grassmannian, with respect to a curve C. Let C be a fixed smooth curve over $\mathbb{C}$ and let $\mathring{F} = G \times C$ be the trivial G-torsor on C. Given an R point $p \in C(R)$, we write $\Gamma_p \subset \mathrm{Spec}(R) \times C$ for the graph of the inclusion $p \to C$.

Definition 2.3.1. *We define the global affine Grassmannian*

$$\mathtt{Gr}_{G,C}(R) := \left\{ (p, F, \beta) \middle| \begin{array}{l} p \in C(R) \\ \\ \beta : F|_{C_R \setminus \Gamma_p} \simeq \mathring{F}|_{C_R \setminus \Gamma_p} \end{array} \right\}. \tag{2.4}$$

When we wish to fix a closed point $p \in C(\mathbb{C})$, we write $\mathtt{Gr}_{G,p}$.

We have an isomorphism, for closed points $p \in C$,

$$\mathtt{Gr}_{G,p} \simeq \mathtt{Gr}_G \, .$$

This isomorphism may be described in the following (non-canonical) way; denote by $\mathcal{K}_p$ the field of fractions of the completed local ring at p, which we write as O_p. There is a non-canonical isomorphism $\mathscr{K}_p \simeq \mathscr{K}$ which is essentially is a choice of uniformizer. Then we certainly have an isomorphism $G(\mathcal{K}_p)/G(O_p) \simeq G(\mathcal{K})/G(O)$.

We produce two more generalizations of the global affine Grassmannian, both originally due to Beilinson and Drinfeld. Again, let F be a G-torsor.

Definition 2.3.2. *We define the Beilinson-Drinfeld Grassmannian:*

$$\mathtt{Gr}_{G,C^2}(R) := \left\{ (p_1, p_2, F, \beta) \left| \begin{array}{l} p_i \in C(R) \\ \beta : F|_{C_R \setminus \Gamma_{p_1} \cup \Gamma_{p_2}} \simeq \mathring{F}|_{C_R \setminus \Gamma_{p_1} \cup \Gamma_{p_2}} \end{array} \right. \right\}. \tag{2.5}$$

and the convolution Grassmannian

Definition 2.3.3.

$$\mathrm{Conv}_{G,C^2}(R) := \left\{ (p_1, p_2, F_1, F_2, \beta_1, \beta_2) \left| \begin{array}{l} p_i \in C(R) \\ \beta_2 : F_2|_{C_R \setminus \Gamma_{p_2}} \simeq F_1|_{C_R \setminus \Gamma_{p_2}} \\ \beta_1 : F_1|_{C_R \setminus \Gamma_{p_1}} \simeq \mathring{F}|_{C_R \setminus \Gamma_{p_1}} \end{array} \right. \right\}. \tag{2.6}$$

Notice that we have a natural map

$$\begin{gathered} \mathrm{Conv}_{G,C^2}(R) \to \mathtt{Gr}_{G,C^2}(R) \\ (p_1, p_2, F_1, F_2, \beta_1, \beta_2) \mapsto (p_1, p_2, F_2, \beta_1 \circ \beta_2). \end{gathered} \tag{2.7}$$

These and more general global affine Grassmannians were first carefully studied by Beilinson and Drinfeld, and they provide important examples of factorization spaces. In our work, subvarieties of these ind-varieties will be the main objects of interest. In fact, the Beilinson-Drinfeld Grassmannian $\mathtt{Gr}_{G,C^2}$ is the more central object, and the convolution Grassmannian is mainly used as a tool to prove things about $\mathtt{Gr}_{G,C^2}$; in this work we introduce it mostly for the sake of completeness.

2.3.1 Global Arc Groups

Given the curve C, we define a global version of the arc group as follows. If $p \in C(R)$, we denote by $\hat{\Gamma}_p$ the formal completion of Γ_p in $\mathrm{Spec}(R) \times C$.

Definition 2.3.4. *The global arc group is defined as follows:*

$$L^+G_C(R) := \left\{ (p, \beta) \left| \begin{array}{l} p \in C(R) \\ \beta : \mathring{F}|_{\hat{\Gamma}_p} \simeq \mathring{F}|_{\hat{\Gamma}_p} \end{array} \right. \right\}. \tag{2.8}$$

We have a left action

$$L^+G_C \times \mathtt{Gr}_{G,C} \to \mathtt{Gr}_{G,C}$$
$$((p_1,\beta),(p_2,F,\beta_2)) \mapsto (p_2,F',\beta_2) \tag{2.9}$$

where F' is obtained by trivializing F along $\hat{\Gamma}_{p_1}$, composing this trivialization with β_1, and regluing with β_2. One defines $L^+G_{C^2}$ analogously by admitting two points p_1, p_2 and β a trivialization of the trivial G-torsor along $\hat{\Gamma}_{p_1} \cup \hat{\Gamma}_{p_2}$.

2.3.2 Global Schubert Varieties

Just as in the case of "local" affine Grassmannians, we will define various affine Schubert varieties as closures of "L^+G"-orbits of certain analogues of torus fixed points, for the correct global construction of G_O (namely L^+G_C).

In the case of a constant group scheme $G \times C$, we first construct the appropriate fixed points (which will now be sections over C) s^λ for a coweight $\lambda \in X_*(T)$ first using $\mathtt{Gr}_{T,C}$ as follows:

Definition 2.3.5. *We define the section $s^\lambda_C(C) := (Id, F, \beta) \in \mathtt{Gr}_{T,C}$ with the following condition on F and β: for all weights μ of T, when we form the induced vector bundle $F \times_T \mathbb{C}_\mu$, we have identifications*

$$\beta_\mu : F \times_T \mathbb{C}_\mu \simeq O_{C\times C}(\langle\lambda,\mu\rangle\Gamma_{Id}).$$

Note that Γ_{Id} is the graph of $Id : C \to C$ in $C \times C$, or in other words is the diagonal.

This T-torsor naturally gives rise to a G-torsor using the inclusion $T \to G$.

Note moreover that we can recover something resembling the earlier data as follows. We describe the situation for a closed point. Let $i : \operatorname{Spec}\mathbb{C} \to C$ denote the inclusion of a $\mathbb{C}$-point of C, also written p. Let F and β be as defined. We have the pullback diagram

$$\begin{array}{ccc} C \times \operatorname{Spec}\mathbb{C} & \longrightarrow & C \\ \downarrow & & {\scriptstyle Id}\uparrow\!\downarrow \\ \operatorname{Spec}\mathbb{C} & \xrightarrow{\ i\ } & C \end{array}$$

Then i^*F is a T-torsor on $\operatorname{Spec}\mathbb{C} \times C$, $i^*(Id)$ is the section $i : \operatorname{Spec}\mathbb{C} \to C$ and $i^*\beta$ is a trivialization of F on $C \setminus p$ such that

$$i^*\beta_\mu : i^*F \times_T \mathbb{C}_\mu \simeq \mathscr{O}_{C\times \operatorname{Spec}\mathbb{C}}(\langle \lambda, \mu\rangle p). \tag{2.10}$$

A similar construction is used for the Beilinson-Drinfeld Grassmannian; here we define sections $s^{\lambda_1,\lambda_2}(C^2)$.

Definition 2.3.6. *We define sections of* $\mathtt{Gr}_{T,C^2}$ *as* $s^{\lambda_1,\lambda_2}(C^2) := ((pr_1, pr_2) \in Hom(C^2, C), F, \beta)$ *where for any weight* μ, β *provides pole conditions along the associated divisors:*

$$\beta_\mu : F \times_T \mathbb{C}_\mu \simeq O_{C^2\times C}(\langle \lambda_1, \mu\rangle \Gamma_{pr_1} + \langle \lambda_2, \mu\rangle \Gamma_{pr_2})).$$

Exactly as before we transform this into a G-torsor using the inclusion $T \to G$ and we may recover a more recognizable description of this section over a point p by pulling back along the inclusion $i : pt \to C$.

We can then define global affine Schubert varieties as flat closures of L^+G_C or $L^+G_{C^2}$ orbits on these sections.

Definition 2.3.7. *The global affine Schubert variety* $\overline{\mathtt{Gr}}^{\lambda}_{G,C}$ *is the flat closure of the* L^+G_C*-orbit on* s^λ.

Analogously we make the definition

Definition 2.3.8. *The Beilinson-Drinfeld affine Schubert variety* $\overline{\mathtt{Gr}}^{\lambda,\mu}_{G,C^2}$ *is the flat closure of the* $L^+G_{C^2}$*-orbit on* $s^{\lambda,\mu}$. *The convolution affine Schubert variety* $\overline{\mathrm{Conv}}^{\lambda,\mu}_{G,C^2}$ *is the flat closure of the* $L^+G_{C^2}$*-orbit on* $s^{\lambda,\mu}$.

Since we are taking the flat closure of orbits, these varieties are all flat over C or C^2 by definition. The more challenging task then becomes the description of the special fiber (and showing that the special fiber is reduced).

2.4 Summary of some work of Zhu

We summarize some of the contents of [Zh1] in the untwisted case, which this work generalizes to the twisted case. In the following, G will be a semisimple algebraic group of simply-laced type.

Theorem 2.4.1. $Pic(\mathtt{Gr}^0_G) = \mathbb{Z}$. *We denote the ample generator by* L.

Recall that Kac-Moody groups are constructed as central extensions of loop groups. There is a homomorphism $Pic(\mathtt{Gr}_G) \to \mathbb{Z}$ called the central charge; when $\mathscr{L} \in Pic(\mathtt{Gr}_G)$ is identified with a weight of the associated Kac-Moody algebra, the central charge is the restriction of the weight to the central $\mathbb{G}_m$.

Remark 2.4.2. *We will be interested in line bundles of central charge 1, and we call them line bundles of level 1. In the untwisted case, the ample generator* L *of* $Pic(\mathtt{Gr}_G)$ *is of level 1.*

Due to work of Beilinson and Drinfeld there exists a ("factorization") global line bundle $\mathscr{L}_{C^2}$ on $\mathtt{Gr}_{G,C^2}$ with restrictions $\mathscr{L}|_{\mathtt{Gr}_{G,p_1,p_2}} \simeq L \boxtimes L$ on $\mathtt{Gr}_{G,p_1,p_2} \simeq \mathtt{Gr}_G \times \mathtt{Gr}_G$ for $p_1 \neq p_2$ and $\mathscr{L}|_{G,p_1,p_1} \simeq L$ on $\mathtt{Gr}_{G,p_1,p_1} \simeq \mathtt{Gr}_G$ when $p_1 = p_2$. We write L for L and abuse notation by also writing L for the restriction $L|_{\overline{\mathtt{Gr}}^{\lambda}_G}$.

Using the flatness of $\overline{\mathrm{Conv}}^{\lambda_1,\lambda_2}_{G,C^2}$ over C^2 and the descriptions of the fibers, Zhu proves a geometric form of fusion:

Theorem 2.4.3. *(Zhu) As* $\mathfrak{g}$*-modules,*

$$H^0(\overline{\mathtt{Gr}}^{\lambda}_G, L^k) \otimes H^0(\overline{\mathtt{Gr}}^{\mu}_G, L^k) \simeq H^0(\overline{\mathtt{Gr}}^{\lambda+\mu}_G, L^k). \tag{2.11}$$

Fixing a point $o \in C$, the fusion theorem is used to prove the existence of a family $\overline{\mathtt{Gr}}^{\lambda,\mu}_{G,C}$ which is flat over C and whose fibers are $\overline{\mathtt{Gr}}^{\lambda,\mu}_{G,p} = \overline{\mathtt{Gr}}^{\lambda}_G \times \overline{\mathtt{Gr}}^{\mu}_G$ when $p \neq 0$ and $\overline{\mathtt{Gr}}^{\lambda,\mu}_{G,o} = \overline{\mathtt{Gr}}^{\lambda+\mu}_G$ when $p = o$. The nontrivial part of this proof is to demonstrate that the scheme structure on the special fiber is the reduced scheme structure of the affine Schubert variety.

Next, Zhu examines the T-fixed subscheme of $\overline{\mathtt{Gr}}^{\lambda}_G$. Denote by I^{λ} the ideal sheaf of the T-fixed subscheme of $\overline{\mathtt{Gr}}^{\lambda}_G$; in general $(\overline{\mathtt{Gr}}^{\lambda}_G)^T$ is a non-reduced scheme supported at a finite number of points (namely $[t^{w\mu}]$ for $w \in W$ and $\mu \leq \lambda$). Write $I^{\lambda}(1)$ for $I^{\lambda} \otimes L$ on $\overline{\mathtt{Gr}}^{\lambda}_G$. We then have a short exact sequence of sheaves

$$0 \to I^{\lambda}(1) \to L \to L \otimes \mathscr{O}_{(\overline{\mathtt{Gr}}^{\lambda}_G)^T} \to 0 \tag{2.12}$$

inducing a long exact sequence in cohomology:

$$\begin{aligned} 0 \to H^0(\overline{\mathtt{Gr}}^{\lambda}_G, I^{\lambda}(1)) \to H^0(\overline{\mathtt{Gr}}^{\lambda}_G, L) \to H^0(\overline{\mathtt{Gr}}^{\lambda}_G, \mathscr{O}_{(\overline{\mathtt{Gr}}^{\lambda}_G)^T} \otimes L) \\ \to H^1(\overline{\mathtt{Gr}}^{\lambda}_G, I^{\lambda}(1)) \to \dots \end{aligned} \tag{2.13}$$

The main theorem of [Zh1] is that in many cases, the map

$$H^0(\overline{\mathtt{Gr}}^{\lambda}_G, L) \to H^0(\overline{\mathtt{Gr}}^{\lambda}_G, \mathscr{O}_{(\overline{\mathtt{Gr}}^{\lambda}_G)^T} \otimes L) \tag{2.14}$$

is an isomorphism. There are several steps to this proof.

2.4.1 Step 1:

The map $H^0(\overline{\mathtt{Gr}}^{\lambda}_G, L) \to H^0(\overline{\mathtt{Gr}}^{\lambda}_G, \mathcal{O}_{(\overline{\mathtt{Gr}}^{\lambda}_G)^T} \otimes L)$ is shown to be surjective. It is sufficient to prove that $H^0(\mathtt{Gr}_G, L) \to H^0(\mathtt{Gr}_G, \mathcal{O}_{\mathtt{Gr}_T} \otimes L)$ is surjective, and this is done by showing that both of these modules (actually their duals) are modules over the homogeneous Heisenberg algebra $\hat{\mathfrak{t}}$.

2.4.2 Step 2:

Next Zhu shows that if we have the isomorphism

$$H^0(\overline{\mathtt{Gr}}^{\lambda}_G, L) \xrightarrow{\sim} H^0(\overline{\mathtt{Gr}}^{\lambda}_G, \mathcal{O}_{(\overline{\mathtt{Gr}}^{\lambda}_G)^T} \otimes L)$$

as well as

$$H^0(\overline{\mathtt{Gr}}^{\mu}_G, L) \xrightarrow{\sim} H^0(\overline{\mathtt{Gr}}^{\mu}_G, \mathcal{O}_{(\overline{\mathtt{Gr}}^{\mu}_G)^T} \otimes L)$$

for two different dominant coweights, then

$$H^0(\overline{\mathtt{Gr}}^{\lambda+\mu}_G, L) \xrightarrow{\sim} H^0(\overline{\mathtt{Gr}}^{\lambda+\mu}_G, \mathcal{O}_{(\overline{\mathtt{Gr}}^{\lambda+\mu}_G)^T} \otimes L).$$

2.4.3 Step 3:

Step 2 reduces the problem to showing that

$$H^0(\overline{\mathtt{Gr}}^{\check{\omega}_i}_G, L) \to H^0(\overline{\mathtt{Gr}}^{\check{\omega}_i}_G, \mathcal{O}_{(\overline{\mathtt{Gr}}^{\check{\omega}_i}_G)^T} \otimes L)$$

for all fundamental coweights $\check{\omega}_i \in X_*(T)$ in the types A,D and E.

In type A this demonstration is easy since all fundamental coweights are minuscule coweights, which provides $\overline{\mathtt{Gr}}^{\check{\omega}_i}_{A_n}$ with very strong geometric properties.

Type D is already much more complicated, requiring inductions and delicate information on the affine Demazure module $H^0(\overline{\mathtt{Gr}}^{\check{\omega}_i}_{D_n})$ when $\check{\omega}_i$ is not a miniscule coweight. However in [Zh1], Zhu succeeds in proving the restriction isomorphism for all fundamental coweights in type D.

However (here using the Bourbaki notation), Zhu is unable to prove the isomorphism for $\overline{\mathtt{Gr}}^{\check{\omega}_4}_{E_6}$, though he proves it for all other fundamental coweights for E_6. More fundamental coweights are missing for E_7 and E_8 as well. These are challenging problems in classical representation theory; see Chapter 7 for partial results to establishing the isomorphism for $\overline{\mathtt{Gr}}^{\check{\omega}_4}_{E_6}$.

CHAPTER 3

Absolutely Special Automorphisms

We wish to extend the results of Zhu in the untwisted case to analogous results in the twisted case. There is already a well-developed theory of general Kac-Moody algebras, their associated groups, and flag varieties, see [Ka], [Ku]. Moreover, the construction of the "twisted" affine algebras, groups and flag varieties bears a very close relationship with "untwisted" affine algebras, groups and flag varieties. This relationship arises due to the fact that the twisted algebras can be constructed by taking fixed points under certain automorphisms. This construction is closely connected with forms of algebras and groups over a ramified field extension. In this section we describe the automorphisms of relevance to us. The main references for this chapter are [Ka] and [HR2].

3.1 Twisted Affine Algebras

Recall the following construction in (Kac, 8.2). Let $\mathfrak{g}$ be a semisimple Lie algebra and let σ be an automorphism of $\mathfrak{g}$ satisfying $\sigma^m = 1$. Kac then constructs Lie algebras $\mathcal{L}(\mathfrak{g}, \sigma, m)$ given by the fixed points of the action of $\tilde{\sigma}$ where

$$\tilde{\sigma}(t^j \otimes x) = \xi^{-j} t^j \otimes \sigma(x) \tag{3.1}$$

for $x \in \mathfrak{g}$ and $\xi = exp(2\pi i/m)$.

Then, in the standard fashion, we can extend this Lie algebra:

Definition 3.1.1. $\hat{\mathcal{L}}(\mathfrak{g}, \sigma, m) := \mathcal{L}(\mathfrak{g}, \sigma, m) \oplus \mathbb{C}K' \oplus \mathbb{C}d'$.

Kac then shows that for the pairs where σ an automorphism and $\mathfrak{g}$ is a simple Lie algebra of type A, D, E, we can construct twisted loop algebras; the twisted affine Lie algebras arise precisely from this construction..

This construction may be repeated on the group level, using central extensions, thus producing twisted affine Kac-Moody groups. We do not really use this full construction since the affine Grassmannians one constructs for loop groups are isomorphic to the affine Grassmanians one constructs as quotients of affine Kac-Moody groups.

3.2 Absolutely Special Automorphisms

We now produce a new set of automorphisms σ of simple $\mathfrak{g}$ of type ADE which we call absolutely special. This collection of automorphisms of $\mathfrak{g}$ coincides almost exactly with those used in the construction of the twisted affine Kac-Moody algebras except in the case A_{2n} when our automorphism σ is distinct.

Let σ be an automorphism of order m on G preserving B and T. Let τ be a diagram automorphism preserving B, T and a pinning $\{x_{\alpha_i}, y_{\alpha_i} \mid i \in I\}$. Let r be the order of τ.

When $\mathfrak{g}$ is not $A_{2\ell}$, we take σ to be τ. When $\mathfrak{g}$ is $A_{2\ell}$, by [Ka, Theorem 8.6] there exists a unique automorphism σ of order $m = 4$ such that

$$\begin{cases} \sigma(e_i) = e_{\tau(i)}, & \text{if } i \neq \ell, \ell+1; \\ \sigma(e_i) = \mathrm{i} e_{\tau(i)}, & \text{if } i \in \{\ell, \ell+1\}; \\ \sigma(f_\theta) = f_\theta, & \end{cases} \tag{3.2}$$

where i is a square root of -1. One can check that

$$\begin{cases} \sigma(f_i) = f_{\tau(i)}, & \text{if } i \neq \ell, \ell+1; \\ \sigma(f_i) = -\mathrm{i} f_{\tau(i)}, & \text{if } i \in \{\ell, \ell+1\}; \\ \sigma(e_\theta) = e_\theta & \end{cases} . \tag{3.3}$$

In fact, $\sigma = \tau \circ \mathrm{i}^h$, where $h \in \mathfrak{t}$ such that

$$\alpha_i(h) = \begin{cases} 0, & \text{if } i \neq \ell, \ell+1 \\ 1, & \text{if } i = \ell, \ell+1 \end{cases} .$$

This automorphism induces a unique automorphism on G. We still call it σ.

We call these automorphisms on G or $\mathfrak{g}$ **absolutely special**. Throughout this paper, we will only consider absolutely special automorphisms.

The following table describes the fixed point Lie algebras for all absolutely special automorphisms:

where by convention C_1 is A_1 and $\ell \geq 3$ for $D_{\ell+1}$. When $(\mathfrak{g}, m) \neq (A_{2\ell}, 4)$, the fixed point Lie algebra $\mathfrak{g}^\sigma$ is well-known as listed in the above table. When $(\mathfrak{g}, m) = (A_{2\ell}, 4)$, the fixed Lie algebra $\mathfrak{g}^\sigma$ is of type C_ℓ, which can follow from the twisted Kac-Moody theory, cf. [Ka, §8].

$(\mathfrak{g}, m)$	$(A_{2\ell-1}, 2)$	$(A_{2\ell}, 4)$	$(D_{\ell+1}, 2)$	$(D_4, 3)$	$(E_6, 2)$
$\mathfrak{g}^\sigma$	C_ℓ	C_ℓ	B_ℓ	G_2	F_4

Table 3.1: The semisimple types, the orders of their absolutely special automorphisms, and the fixed subalgebra.

Remark 3.2.1. *Note that our "absolutely special" automorphisms coincide with diagram automorphisms in all cases except A_{2n}. This discrepancy will be discussed in the context of twisted affine Grassmannians in Chapter 4.*

3.2.1 Root Systems

We develop some notation which yields compatible descriptions of the twisted and untwisted root systems which are related to each other.

Recall that we follow the labelling of the vertices of the Dynkin diagram of $\mathfrak{g}$ in [Ka, Table Fin, p.53]. Set

$$\begin{cases} \beta_i = \alpha_i|_{\mathfrak{t}^\sigma}, \text{ for } i = 1, 2, \cdots, \ell, \quad \text{when } (\mathfrak{g}, m) = (A_{2\ell-1}, 2), \text{ or } (D_{\ell+1}, 2) \\ \beta_1 = \alpha_2|_{\mathfrak{t}^\sigma}, \beta_2 = \alpha_1|_{\mathfrak{t}^\sigma}, \quad \text{when } (\mathfrak{g}, m) = (D_4, 3) \\ \beta_1 = \alpha_6|_{\mathfrak{t}^\sigma}, \beta_2 = \alpha_3|_{\mathfrak{t}^\sigma}, \beta_3 = \alpha_2|_{\mathfrak{t}^\sigma}, \beta_4 = \alpha_1|_{\mathfrak{t}^\sigma}, \quad \text{when } (\mathfrak{g}, m) = (E_6, 2) \\ \beta_i = \alpha_i|_{\mathfrak{t}^\sigma}, \text{ for } i = 1, 2, \cdots, \ell - 1; \beta_\ell = (\alpha_\ell + \alpha_{\ell+1})|_{\mathfrak{t}^\sigma} = 2\alpha_\ell|_{\mathfrak{t}^\sigma}, \quad \text{when } (\mathfrak{g}, m) = (A_{2\ell}, 4). \end{cases} \tag{3.4}$$

Let I_σ be the set of all subscript indices of β_i. Then for each case, the set $\{\beta_j \mid j \in I_\sigma\}$ gives rise to the set of simple roots of $\mathfrak{g}^\sigma$. One can see easily that this labelling will coincide with the labelling of non simply-laced Dynkin diagrams in [Ka, Table Fin, p.53].

We now define a map $\eta : I \to I_\sigma$. When $(\mathfrak{g}, m) \neq (A_{2\ell}, 4)$, η is defined such that $\beta_{\eta(i)} = \alpha_i|_{\mathfrak{t}^\sigma}$ for any $i \in I$. When $(\mathfrak{g}, m) = (A_{2\ell}, 4)$, set

$$\eta(i) = \eta(2\ell + 1 - i) = i, \text{ for any } 1 \leq i \leq \ell.$$

Thus $\eta : I \to I_\sigma$ describes precisely how roots α_i correspond to roots β_i of $\mathfrak{g}^\sigma$.

Let $\{\check{\beta}_j \mid j \in I_\sigma\}$ be the set of simple coroots of $\mathfrak{g}^\sigma$. We can describe $\check{\beta}_j$ as follows:

$$\check{\beta}_j = \sum_{i \in \eta^{-1}(j)} \check{\alpha}_i. \tag{3.5}$$

Let $\{\lambda_j \mid j \in I_\sigma\}$ be the set of fundamental weights of $\mathfrak{g}^\sigma$, and let $\{\check{\lambda}_j \mid j \in I_\sigma\}$ be the set of fundamental coweights of $\mathfrak{g}^\sigma$. The fundamental weights can be described as follows:

$$\lambda_j = \omega_i|_{\mathfrak{t}^\sigma}, \qquad \text{for some } i \text{ with } \eta(i) = j. \tag{3.6}$$

In the case of fundamental coweights, we need to describe them separately. When $(\mathfrak{g}, m) \neq (A_{2\ell}, 4)$,

$$\check{\lambda}_j = \sum_{i \in \eta^{-1}(j)} \check{\omega}_i. \tag{3.7}$$

When $(\mathfrak{g}, m) = (A_{2\ell}, 4)$, we have

$$\check{\lambda}_j = \begin{cases} \check{\omega}_j + \check{\omega}_{2\ell+1-j}, & j = 1, 2, \cdots, \ell - 1 \\ \frac{1}{2}(\check{\omega}_\ell + \check{\omega}_{\ell+1}), & j = \ell \end{cases}. \tag{3.8}$$

One checks that these definitions are the correct ones: if we are not in type A_{2l} then it is clear that $\langle \check{\lambda}_j, \beta_i \rangle = \langle \sum_{k \in \eta^{-1}(j)} \check{\omega}_k, \alpha_i \rangle = \delta_{i,j}$ for $i, j \in I_\sigma$.

The verification in type A_{2l} is similarly routine.

In conclusion, given simple roots α_i, simple coroots $\check{\alpha}_i$, fundamental weights ω_i and fundamental coweights $\check{\omega}_i$ for a root datum of type ADE, we associate simple roots β_i, fundamental coroots $\check{\beta}_i$, fundamental weights λ_i and fundamental coweights $\check{\lambda}_i$ to the associated to the root system of $\mathfrak{g}^\sigma$. Moreover we have a map η which acts on the indices in the expected way.

CHAPTER 4

Basics of Twisted Affine Grassmannians

This chapter focuses on the construction of the twisted affine Grassmannians which are the focus of this work. Most of the constructions are analogous to those in Chapter 2, with the caveat that instead of taking a constant group scheme G (over $\mathbb{D}$ or C, for instance), we first construct a non-constant group scheme. Many of the results are very similar in form, and the structure of the chapter is similar. The main references for this chapter are [PR], [Zh2], and [Ri1].

4.1 Group Schemes and first construction

We provide two descriptions of twisted affine Grassmannians, and describe some of their basic properties.

Given an automorphism σ which arises from a diagram automorphism τ (here we include all absolutely special automorphisms and more, including τ for A_{2l}), we can construct a twisted affine Grassmannian in several ways.

We first construct some non-constant group schemes over the formal disc $\mathbb{D} = \operatorname{Spec} \mathcal{O}$.

Let $\pi_n : \mathbb{D} \to \mathbb{D}$ be the ramified cover of order $n = |\sigma|$, given by $t \mapsto t^n$. Given G a simple reductive group, we form the constant group scheme $G \times \mathbb{D}$. We let σ act on $\mathcal{O}$ by acting trivially on $\mathbb{C}$ and sending $t \mapsto \xi^{-1} t$ where ξ is a primitive nth root of unity and t is a chosen uniformizer. Thus σ may be interpreted as the generator of the Galois group $\operatorname{Gal}(\tilde{\mathbb{D}}/\mathbb{D})$, where we denote the first copy of $\mathbb{D}$ by $\tilde{\mathbb{D}}$. We make the following definition:

Definition 4.1.1. $\mathscr{G} = \operatorname{Res}_{\tilde{\mathbb{D}}/\mathbb{D}}(G \times \tilde{\mathbb{D}})^{\sigma}$.

Thus $\mathscr{G}$ represents the functor

$$R \mapsto G(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} R)^{\sigma}.$$

Moreover when G is simply conneced and semisimple, when we take the automorphisms σ or even τ in the case of A_{2l}, the group schemes $\mathscr{G}$ are parahoric group schemes.

We also define the twisted torus group scheme:

Definition 4.1.2. $\mathscr{T} = (\mathrm{Res}_{\tilde{\mathbb{D}}/\mathbb{D}}(T \times \tilde{\mathbb{D}})^{\sigma})^{\circ}$.

Note that $\mathrm{Res}_{\tilde{\mathbb{D}}/\mathbb{D}}(T \times \tilde{\mathbb{D}})^{\sigma}$ could be a disconnected scheme, and we take the neutral component.

Thus $\mathscr{G}$ (and $\mathscr{T}$) are non-constant group schemes over $\mathbb{D}$; the fiber over the generic point is isomorphic to $G \otimes \mathcal{K}$ (respectively $T \otimes \mathcal{K}$), and the fiber over the closed point is a non-reductive group $G(\mathbb{C}[t]/t^n)$ (respectively $T(\mathbb{C}[t]/t^n))^{\circ}$).

4.2 Twisted Affine Grassmannians

We can now construct the twisted affine Grassmannian as an fpqc quotient.

Definition 4.2.1. *We define the twisted affine Grassmannian as the fpqc quotient*

$$\mathtt{Gr}_{\mathscr{G}} = L\mathscr{G}/L^{+}\mathscr{G}.$$

Remark 4.2.2. *From this description we can see that the* $\mathbb{C}$*-points are* $\mathtt{Gr}_{\mathscr{G}}(\mathbb{C}) = G(\mathcal{K})^{\sigma}/G(\mathcal{O})^{\sigma}$.

We can also mimic the Beauville-Laszlo construction for twisted affine Grassmannians.

Theorem 4.2.3. *Let* $\mathscr{F}$ *be a* $\mathscr{G}$*-torsor over* $\mathbb{D}$*, and denote by* $\mathring{\mathscr{F}}$ *the trivial* $\mathscr{G}$*-torsor. We then have*

$$\mathtt{Gr}_{\mathscr{G}}(R) = \left\{ (\mathcal{F}, \beta) \middle| \beta : \mathcal{F}|_{\mathbb{D}_R^*} \xrightarrow{\sim} \mathring{\mathcal{F}}|_{\mathbb{D}_R^*} \right\}. \tag{4.1}$$

Proof. See [Zh2]. □

4.2.1 Stratification

Recall that in the untwisted case, we have the Bruhat decomposition

$$G(\mathcal{K}) = \sqcup_{\lambda \in X_*(T)^+} G(\mathcal{O}) t^{\lambda} G(\mathcal{O})$$

which induces a stratification of $\mathtt{Gr}_G$ by left $G(\mathcal{O})$-orbits. Moreover, each orbit has finitely many $T_{\mathbb{C}}$-fixed points $t^{w\lambda}$ for $w \in W$, $\lambda \in X_*(T)$.

We have a similar structure on $\mathtt{Gr}_{\mathscr{G}}$ which we explain as follows. First, we have the norm map

$$\begin{aligned} T(\mathcal{K}) &\to T(\mathcal{K})^{\sigma} \\ t^{\lambda} &\mapsto n^{\lambda} = \prod_{i=0}^{n-1} \sigma^{i}(t^{\lambda}). \end{aligned} \tag{4.2}$$

On the other hand, we have an identification

$$T(\mathcal{K})^\sigma / T(\mathcal{O})^\sigma = X_*(T)_\sigma \tag{4.3}$$

where $X_*(T)_\sigma$ is the lattice of coinvariants for the σ action on $X_*(T)$. Thus to $\lambda \in X_*(T)$, we assign two elements: $n^\lambda \in T(\mathcal{K})^\sigma$ and $\overline{\lambda} \in X_*(T)_\sigma$. This produces a bijection between these two sets, where the bijection sends $\overline{\lambda} \mapsto n^\lambda$.

After the inclusion $T(\mathcal{K})^\sigma \to G(\mathcal{K})^\sigma$, we can view the n^λ as points in $G(\mathcal{K})^\sigma$, defining cosets $[n^\lambda]$ in $\mathtt{Gr}_{\mathscr{G}}$. We define the affine Schubert varieties to be the reduced closures of the left $G(\mathcal{O})^\sigma$-orbits on $[n^\lambda]$; so

Definition 4.2.4. $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}} = \overline{G(\mathcal{O})^\sigma.[n^\lambda]}$.

Thus we have the following stratification of the twisted affine Grassmannian.

Theorem 4.2.5. *We have a stratification of the twisted affine Grassmannian in terms of $G(\mathcal{O})^\sigma$-orbits: $\mathtt{Gr}_{\mathscr{G}}(\mathbb{C}) = \bigsqcup_{\bar{\lambda} \in X_*(T)_\sigma^+} \mathtt{Gr}_{\mathscr{G}}^{\bar{\lambda}}$.*

Proof. See [Ri1]. □

For any $\bar{\lambda}, \bar{\mu} \in X_*(T)_\sigma^+$, we write $\bar{\mu} \preceq \bar{\lambda}$ if $\mathtt{Gr}_{\mathscr{G}}^{\bar{\mu}} \subseteq \overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$. For any $i \in I$, let $\bar{\check{\alpha}}_i$ denote the image of $\check{\alpha}_i$ in $X_*(T)_\sigma$. For any $j \in I_\sigma$, set

$$\gamma_j = \bar{\check{\alpha}}_i, \quad \text{if } j = \eta(i). \tag{4.4}$$

It is clear that γ_j is well-defined.

The following lemma follows from [Ri1, Corollary 2.10].

Lemma 4.2.6. *$\bar{\mu} \preceq \bar{\lambda}$ if and only if $\bar{\lambda} - \bar{\mu}$ is a non-negative integral linear combination of $\{ \gamma_j \mid j \in I_\sigma \}$.*

4.2.2 Components

As previously noted, there exists a natural bijection

$$T(\mathcal{K})^\sigma / T(\mathcal{O})^\sigma \simeq X_*(T)_\sigma, \tag{4.5}$$

where $X_*(T)_\sigma$ denotes the set of σ-coinvariants in $X_*(T)$. Any $\bar{\lambda} \in X_*(T)_\sigma$ corresponds to the coset $n^\lambda T(\mathcal{O})^\sigma$, where λ is a representative of $\bar{\lambda}$. By Theorem [PR, Theorem 0.1], the components of $\mathtt{Gr}_{\mathscr{G}}$ can be parametrized by elements in $\pi_1(G)_\sigma$, where $\pi_1(G) \simeq X_*(T)/\check{Q}$, and $(X_*(T)/\check{Q})_\sigma$ is the the set of coinvariants of σ in $X_*(T)/\check{Q}$.

When G is of adjoint type, we describe $(X_*(T)/\check{Q})_\sigma$ in the following table.

(G,m)	$(A_{2\ell-1},2)$	$(A_{2\ell},4)$	$(D_{2\ell+1},2)$	$(D_{2\ell},2)$	$(D_4,3)$	$(E_6,2)$
$X_*(T)/\check{Q}$	$\mathbb{Z}_{2\ell}$	$\mathbb{Z}_{2\ell+1}$	$\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_3$
$(X_*(T)/\check{Q})_\sigma$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0

Table 4.1: Coinvariants of fundamental groups. This also provides the number of components of $\mathtt{Gr}_{\mathscr{G}}$ for G adjoint.

We now assume G is of adjoint type. From the perspective of the geometric Satake, we can determine the minimal elements in $X_*(T)^+_\sigma$, in other words the minimal Schubert variety in each connected component of $\mathtt{Gr}_{\mathscr{G}}$. From the table (4.2.2), we see that when $(G,m)=(A_{2\ell-1},2)$, $\mathtt{Gr}_{\mathscr{G}}$ has two components, where $\mathtt{Gr}_{\mathscr{G}}^{\overline{\check{\omega}}_1}$ is the minimal Schubert variety in the non-neutral component, since $\overline{\check{\omega}}_1$ gives the miniscule dominant weight of $H\simeq \mathrm{Sp}_{2\ell}$. When $(G,m)=(D_{\ell+1},2)$, $\mathtt{Gr}_{\mathscr{G}}$ also has two components and $\mathtt{Gr}_{\mathscr{G}}^{\overline{\check{\omega}}_\ell}$ is the minimal Schubert variety in the non-neutral component, since $\overline{\check{\omega}}_\ell$ is the miniscule dominant weight of $H\simeq \mathrm{Spin}_{2\ell+1}$. Otherwise, $\mathtt{Gr}_{\mathscr{G}}$ has only one component. In fact, when $(G,m)=(A_{2\ell},4)$, $H\simeq \mathrm{SO}_{2\ell+1}$, in which case the weight lattice $X_*(T)_\sigma$ coincides with the root lattice of H.

Let S denote the following set

$$S=\begin{cases}\{0\} & \text{if } (G,r)\neq (A_{2\ell-1},2),(D_{\ell+1},2)\\ \{0,\check{\omega}_1\} & \text{if } (G,r)=(A_{2\ell-1},2)\\ \{0,\check{\omega}_\ell\} & \text{if } (G,r)=(D_{\ell+1},2)\end{cases}. \tag{4.6}$$

For any $\kappa\in S$, let $\mathtt{Gr}_{\mathscr{G},\kappa}$ be the component of $\mathtt{Gr}_{\mathscr{G}}$ containing the Schubert variety $\mathtt{Gr}_{\mathscr{G}}^{\bar{\kappa}}$, or equivalently containing the point $e_{\bar{\kappa}}$. Then,

$$\mathtt{Gr}_{\mathscr{G}}=\sqcup_{\kappa\in S}\mathtt{Gr}_{\mathscr{G},\kappa}\,.$$

We can now describe our choice of automorphism in the A_{2n} case in a satisfactory way, using a little Bruhat-Tits theory.

According to the table above, the twisted affine Grassmannian in types A and D are disjoint unions of homogeneous spaces for the $G(\mathcal{K})^\tau$ action. These homogeneous spaces have descriptions coming from

Bruhat-Tits theory and local Dynkin diagrams.

The twisted affine Dynkin diagrams are found in Kac [Ka]. They are as follows:

$\mathfrak{g}$	Dynkin Diagram
$A_2^{(2)}$	$\underset{\alpha_0}{\overset{2}{\circ}} \Lleftarrow \underset{\alpha_1}{\overset{1}{\circ}}$
$A_{2l}^{(2)} (l \geq 2)$	$\underset{\alpha_0}{\overset{2}{\circ}} \Leftarrow \underset{\alpha_1}{\overset{2}{\circ}} - \cdots - \underset{\alpha_{l-1}}{\overset{2}{\circ}} \Leftarrow \underset{\alpha_l}{\overset{1}{\circ}}$
$A_{2l-1}^{(2)} (l \geq 3)$	$\underset{\alpha_1}{\overset{1}{\circ}} - \underset{\alpha_2}{\overset{\overset{\alpha_0\ \overset{1}{\circ}}{\mid}\ 2}{\circ}} - \underset{\alpha_3}{\overset{2}{\circ}} - \cdots - \underset{\alpha_{l-1}}{\overset{2}{\circ}} \Leftarrow \underset{\alpha_l}{\overset{1}{\circ}}$
$D_{l+1}^{(2)} (l \geq 2)$	$\underset{\alpha_0}{\overset{1}{\circ}} \Leftarrow \underset{\alpha_1}{\overset{1}{\circ}} - \cdots - \underset{\alpha_{l-1}}{\overset{1}{\circ}} \Rightarrow \underset{\alpha_l}{\overset{1}{\circ}}$
$E_6^{(2)}$	$\underset{\alpha_0}{\overset{1}{\circ}} - \underset{\alpha_1}{\overset{2}{\circ}} - \underset{\alpha_2}{\overset{3}{\circ}} \Leftarrow \underset{\alpha_3}{\overset{2}{\circ}} - \underset{\alpha_4}{\overset{1}{\circ}}$
$D_4^{(3)}$	$\underset{\alpha_0}{\overset{1}{\circ}} - \underset{\alpha_1}{\overset{2}{\circ}} \Lleftarrow \underset{\alpha_2}{\overset{1}{\circ}}$

Table 4.2: Twisted affine algebras and their Dynkin diagrams.

According to Bruhat-Tits theory, associated to each node of the twisted affine Dynkin diagram is a maximal parahoric subgroup, which we denote P_i for the index i. Moreover, certain indices are special, and give rise to "special" maximal parahorics. We list them as follows (also see [Tits]):

For $A_2^{(2)}$, P_0 and P_1 correspond to special parahorics. For $A_{2l}^{(2)}$, P_0 and P_l correspond to special parahorics. For $A_{2l-1}^{(2)}$, P_0 and P_1 correspond to special parahorics. For $D_{l+1}^{(2)}$, P_0 and P_l are special parahorics. For $E_6^{(2)}$, P_0 is a special parahoric. Lastly, for $D_4^{(3)}$, P_0 is a special parahoric.

Let G be adjoint and let $\tilde{G}$ be the simply connected cover. For types $A_{2n-1}^{(2)}$, $D_4^{(3)}$, $D_n^{(2)}$ ($n \geq 5$) and $E_6^{(2)}$ and when G is adjoint, we can describe the components in two ways: for $\gamma_i \in (X_*(T)/Q^\vee)_\sigma$ we have

$$\mathtt{Gr}_{\mathscr{G}} = \bigsqcup_{\gamma_i} \tilde{G}(\mathcal{K})^\sigma / Ad_{t^{\gamma_i}} \tilde{G}(\mathcal{O})^\sigma$$

but also as

$$\mathtt{Gr}_{\mathscr{G}} = \bigsqcup \tilde{G}(\mathcal{K})^\sigma / P_i$$

with the P_i ranging over the special parahorics.

Thus to understand twisted affine Schubert varieties, we would have to distinguish at every moment which component we were examining.

However, note that in the cases of $A_{2l-1}^{(2)}$ and $D_{2l}^{(2)}$, while there are two different special parahorics, they are related by a diagram automorphism and are thus isomorphic. This implies that the two components $L\mathscr{G}/P_0$ and $L\mathscr{G}/P_l$ are also related by a diagram automorphism, so essentially we need only consider the geometry of the neutral component.

This description fails for $A_{2n}^{(2)}$; there are two special parahorics but only one component of the twisted affine Grassmannian. In fact in this case, $\tilde{G}(\mathcal{K})^\sigma/P_0$ and $\tilde{G}(\mathcal{K})^\sigma/P_l$ are two distinct, non-isomorphic homogeneous spaces, only one of which has a line bundle of central charge 1.

Thus we cannot extend our results to the homogeneous space $L\mathscr{G}/P_l$, but we can extend our results to $L\mathscr{G}/P_0$. This latter homogeneous space is isomorphic (at the level of $\mathbb{C}$-points) to $G(\mathcal{K})^\sigma/G(O)^\sigma$, where now σ is the absolutely special automorphism of order 4. This topic is treated in Haines-Richarz [HR]. This explains our notation of "absolutely special" automorphisms and explains why we are not simply using the diagram automorphism τ in the $A_{2l}^{(2)}$ case.

4.3 Global Twisted Affine Grassmannians

As in the untwisted case, we may globalize the twisted affine Grassmannian using torsors of non-trivial group schemes over curves. We first construct the global versions of our non-constant group schemes. Let $\tilde{C} \to C$ be a surjective morphism of complete smooth algebraic curves over $\mathbb{C}$ of degree n, with ramification points $p_i \in C$.

We produce non-constant group schemes over C by using Weil restriction again.

Definition 4.3.1. *We define* $\mathcal{G}_C := \mathrm{Res}_{\tilde{C}/C}(G \times \tilde{C})^\sigma$. *When the curve* C *is understood from context we will write merely* $\mathcal{G}$.

In an identical fashion to what was done above, if T is a chosen maximal torus in G, we wish to form the global twisted torus $\mathcal{T}$.

Definition 4.3.2. *We define* $\mathcal{T}_C := (\mathrm{Res}_{\tilde{C}/C}(T \times \tilde{C})^\sigma)^\circ$. *When the curve* C *is understood from context we will write merely* $\mathcal{T}$.

The group scheme $\mathcal{G}$ has the following properties:

1. For any $y \in C$, when y is unramified in C, the fiber $\mathcal{G}_{|_y}$ over y is isomorphic to G; the restriction $\mathcal{G}_y$ to the formal disc $\mathbb{D}_y$ around y is isomorphic to the constant group scheme $G_{\mathbb{D}_y}$ over $\mathbb{D}_y$.

2. When y is ramified, $\mathcal{G}_{|_y}$ is isomorphic to $G(\mathbb{C}[t]/t^r)^\sigma$, which is not reductive; the restriction $\mathcal{G}_y$ to $\mathbb{D}_y$ is isomorphic to $\mathscr{G}$.

We can make a global version of the affine Grassmannian using the non-constant group scheme $\mathcal{G}$. As before, let $\mathcal{G}_C$ be the group scheme associated to the ramified covering $\tilde{C} \to C$. Let $p \in C(R)$, and denote by $\mathcal{F}$ a $\mathcal{G}$-torsor over C. We denote by $\mathring{\mathcal{F}}$ the trivial $\mathcal{G}$-torsor over C. We then make the global twisted affine Grassmannian according to the same recipe as we defined the global affine Grassmannian.

Definition 4.3.3. *We define*

$$\mathtt{Gr}_{\mathcal{G},C}(R) := \left\{ (p, \mathcal{F}, \beta) \middle| \begin{matrix} p \in C(R) \\ \beta : \mathcal{F}|_{C_R \setminus \Gamma_p} \simeq \mathring{\mathcal{F}}|_{C_R \setminus \Gamma_p} \end{matrix} \right\}. \tag{4.7}$$

When we wish to fix a point $p \in C(R)$, we write $\mathtt{Gr}_{\mathcal{G},p}(R)$.

For closed points $p \in C$, when p is unramified in $\pi : \tilde{C} \to C$, we have an isomorphism

$$\mathtt{Gr}_{\mathcal{G},p} \simeq \mathtt{Gr}_G, \tag{4.8}$$

and when $p \in C$ is ramified in $\pi : \tilde{C} \to C$, we have

$$\mathtt{Gr}_{\mathcal{G},p} \simeq \mathtt{Gr}_{\mathscr{G}}. \tag{4.9}$$

Thus the global twisted affine Grassmannian allows us to compare twisted and untwisted affine Grassmannians.

We produce two more generalizations of the global twisted affine Grassmannian, in perfect analogy with Chapter 2.

Definition 4.3.4. *We define the Beilinson-Drinfeld Grassmannian:*

$$\mathtt{Gr}_{\mathcal{G},C^2}(R) := \left\{ (p_1, p_2, \mathcal{F}, \beta) \middle| \begin{matrix} p_i \in C(R) \\ \beta : \mathcal{F}|_{C_R \setminus \Gamma_{p_1} \cup \Gamma_{p_2}} \simeq \mathring{\mathcal{F}}|_{C_R \setminus \Gamma_{p_1} \cup \Gamma_{p_2}} \end{matrix} \right\}. \tag{4.10}$$

and the convolution Grassmannian

Definition 4.3.5.

$$\mathrm{Conv}_{\mathcal{G},C}(R) := \left\{ (p_1, p_2, \mathcal{F}_1, \mathcal{F}_2, \beta_1, \beta_2) \,\middle|\, \begin{array}{l} p_i \in C(R) \\ \beta_2 : \mathcal{F}_2|_{C_R \setminus \Gamma_{p_2}} \simeq \mathcal{F}_1|_{C_R \setminus \Gamma_{p_2}} \\ \beta_1 : \mathcal{F}_1|_{C_R \setminus \Gamma_{p_1}} \simeq \mathring{\mathcal{F}}|_{C_R \setminus \Gamma_{p_1}} \end{array} \right\}. \tag{4.11}$$

Notice that we have a natural map

$$\begin{aligned} \mathrm{Conv}_{\mathcal{G},C^2}(R) &\to \mathtt{Gr}_{\mathcal{G},C^2}(R) \\ (p_1, p_2, \mathcal{F}_1, \mathcal{F}_2, \beta_1, \beta_2) &\mapsto (p_1, p_2, \mathcal{F}_2, \beta_1 \circ \beta_2). \end{aligned} \tag{4.12}$$

We must also construct appropriate global analogues of $G(O)$, which will enable us to define affine Schubert varieties in a later section.

Definition 4.3.6. *We define*

$$L^+\mathcal{G}_C(R) := \left\{ (p, \beta) \,\middle|\, \begin{array}{l} p \in C(R) \\ \beta : \mathring{\mathcal{F}}|_{\hat{\Gamma}_p} \simeq \mathring{\mathcal{F}}|_{\hat{\Gamma}_p} \end{array} \right\}. \tag{4.13}$$

where $\hat{\Gamma}_p$ is the formal completion of the subscheme $p \to C$. Any $\mathcal{G}$-torsor restricted to a formal disc is trivializable (perhaps after base change), and β is a trivialization of the trivial torsor. Note that if we fix $p \in C$ as an unramified closed point, $L^+\mathcal{G}_p(\mathbb{C}) = G(O)$ and if p is a ramification point, $L^+\mathcal{G}_p(\mathbb{C}) = G(O)^\sigma$. In the case of the Beilinson-Drinfeld Grassmannian, we simply add a point:

Definition 4.3.7. *We define*

$$L^+\mathcal{G}_{C^2}(R) := \left\{ (p_1, p_2, \beta) \,\middle|\, \begin{array}{l} p_i \in C(R) \\ \beta : \mathring{\mathcal{F}}|_{\hat{p}_1 \cup \hat{p}_2} \simeq \mathring{\mathcal{F}}|_{\hat{p}_1 \cup \hat{p}_2} \end{array} \right\}. \tag{4.14}$$

We have a left action of $L^+\mathcal{G}_{C^2}$ on $\mathtt{Gr}_{\mathcal{G},C^2}$ defined as follows:

$$(p_1, p_2, \beta) \times (p_3, p_4, \mathcal{F}, \beta_2) \mapsto (p_3, p_4, \mathcal{F}', \beta_2) \tag{4.15}$$

where $\mathcal{F}'$ is the modification of $\mathcal{F}$ at the two points p_1, p_2 formed by pulling back $\mathcal{F}$ to the cover

$C_R \setminus \hat{\Gamma}_{p_1} \cup \hat{\Gamma}_{p_2}$, twisting $\mathcal{F}|_{\hat{\Gamma}_{p_i}}$ by β, and then forming $\mathcal{F}'$ by descent.

4.3.1 Global Twisted Schubert Varieties

The following constructions are analogous to those in Chapter 2, where these objects were constructed for the constant group scheme $G \times C$.

Let σ be an absolutely special automorphism of G, and let $\pi : \tilde{C} \to C$ be a ramified morphism of degree $|\sigma|$.

We define a section $s^\lambda : \tilde{C} \to \mathtt{Gr}_{\mathcal{T},C}$ as follows. We have

Definition 4.3.8. *$s^\lambda(\tilde{C}) := (Id \in Hom(\tilde{C}, \tilde{C}), \mathcal{F}, \beta)$ such that for all weights μ, we have the induced homomorphism*

$$\beta_\mu : \mathcal{F} \times_T \mathbb{C}_\mu \simeq \mathcal{O}_{\tilde{C}\times\tilde{C}}(\sum_i \langle \sigma^i \lambda, \mu \rangle \Gamma_{\sigma^i})$$

for $\mathcal{F}$ a (Γ, T)-torsor on $\tilde{C}$.

Note that this is in fact a section over $\tilde{C}$, and not over C. It descends to a section over C when $\langle \lambda, \mu \rangle = \langle \sigma^i \lambda, \mu \rangle$ for all μ (so that λ is defined "over the base field").

For points $p \in \tilde{C}$, we can understand the section as follows: $s^\lambda(p) = t^\lambda$ which corresponds to $\lambda \in X_*(T)$ if p is not fixed by σ and $s^\lambda = t^{\overline{\lambda}}$ which corresponds to $\overline{\lambda} \in X_*(T)_\sigma$ if p is fixed by σ.

Technically s^λ is a section of $\mathtt{Gr}_{\mathcal{T},C}$ but we can include it into $\mathtt{Gr}_{\mathcal{G},C}$; by abuse of notation we will denote this section by s^λ as well. We can then define $\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},C}$ as the minimal closure of the $L^+\mathcal{G}_{\tilde{C}}$-orbit on s^λ.

We have the following facts about $\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},C}$.

Theorem 4.3.9. *([Zh2]): $\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},C}$ is flat over $\tilde{C}$.*

Proof. This is essentially by construction. □

Theorem 4.3.10. *([Zh2]): The fibers of $\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},\tilde{C}}$ have the following description: if $p \in \tilde{C}$ is fixed by σ, then $\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},p} \simeq \overline{\mathtt{Gr}}^\lambda_{G}$. On the other hand, if p is not fixed by σ, then $\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},p} \simeq \overline{\mathtt{Gr}}^{\overline{\lambda}}_{\mathscr{G}}$.*

Proof. Even in the case where $\mathcal{G}$ is a constant group scheme, the proof of this fact is highly nontrivial. It is easier to see that the *reduced* scheme structure on the special fiber is the desired one. More difficult to show is that $\overline{\mathtt{Gr}}^\lambda_{\mathcal{G},p}$ is a reduced scheme. This requires a proof using fusion, the convolution affine Grassmannian, and an analysis of the partial Bott-Samelson which is the special fiber of the convolution Grassmannian. □

This flat scheme will allow us to translate information from the generic fiber (an untwisted affine Schubert variety) to the special fiber (a twisted affine Schubert variety).

CHAPTER 5

Construction of Line Bundles of Level 1

In this chapter we deal with line bundles on twisted affine Grassmannians. This proceeds in several steps. First, we define a line bundle $\mathscr{L}$ on $\mathtt{Gr}_{\mathscr{G}}$ which is level one on each component of $\mathtt{Gr}_{\mathscr{G}}$; this is completed in Proposition 5.1.7. Our next objective is a "global" line bundle $\mathcal{L}$ on $\mathtt{Gr}_{\mathcal{G},C}$ which restricts to the "local" line bundle of level one on each fiber. To construct this line bundle we first build a line bundle on $\mathrm{Bun}_{\mathcal{G}}$ and pull it back to $\mathtt{Gr}_{\mathscr{G},C}$; this is accomplished in Theorem 5.1.17. References for this Chapter are [Ka], [Ku] and [Fa] for generalities, and most pertinently [HK].

5.1 Conformal Blocks, Line Bundles

We must first collect some facts about line bundles on the twisted affine Grassmannians $\mathtt{Gr}_{\mathscr{G}}$.

We define $P(\sigma, c)$ to be the set of dominant weights of $\mathfrak{g}^\sigma$ of level c. This set parametrizes highest-weight integrable representations of $\tilde{\mathfrak{g}}^\sigma$, which we denote by $\mathscr{H}_c(\lambda)$.

Lemma 5.1.1. *For an absolutely special automorphism σ, we have*

$$P(\sigma,1)=\begin{cases}\{0\} \text{ if } (\sigma,m)\neq(A_{2\ell-1},2),(D_{\ell+1},2)\\ \{0,\lambda_1\} \qquad \text{if } (\sigma,m)=(A_{2\ell-1},2)\\ \{0,\lambda_\ell\} \qquad \text{if } (\sigma,m)=(D_{\ell+1},2)\end{cases}.$$

Proof. We first consider the case when $(\mathfrak{g}, m) \neq (A_{2\ell}, 4)$. We can read from [HK, Lemma 2.1], for any $\lambda \in (\mathfrak{t}^\sigma)^*$, $\lambda \in P(\sigma, 1)$ if and only if

$$\langle\lambda,\check{\beta}_i\rangle\in\mathbb{Z}_{\geq 0} \quad \text{for any } i\in I_\sigma,$$

and $\langle\lambda,\check{\theta}_0\rangle \leq 1$, where θ_0 is the highest short root of $\mathfrak{g}^\sigma$ and $\check{\theta}_0$ is the coroot of θ_0, and hence $\check{\theta}_0$ is the highest coroot of $\mathfrak{g}^\sigma$. In this case, $\lambda \in P(\sigma, 1)$ if and only if $\lambda = 0$ or a miniscule dominant weight of $\mathfrak{g}^\sigma$ (cf. [BH, Lemma 2.13]). Following the labellings in [Ka, Table Fin,p53], when $\mathfrak{g}^\sigma$ is of type C_ℓ, λ_1 is the only miniscule weight; when $\mathfrak{g}^\sigma$ is of type B_ℓ, λ_ℓ is the only miniscule weight. Any other non simply-laced Lie algebra has no miniscule weight. This finishes the argument of the lemma when $(\mathfrak{g}, m) \neq (A_{2\ell}, 4)$.

Now, we assume that $(\mathfrak{g}, m) = (A_{2\ell}, 4)$. In this case, it is more convenient to choose a different set of simple roots for $\mathfrak{g}^\sigma$, rather than the one described in (3.4). Namely, we can also choose

$$\{\alpha_i|_{\mathfrak{t}^\sigma} \mid i = 1, 2, \cdots, \ell - 1\} \cup \{-\theta|_{\mathfrak{t}^\sigma}\}$$

as the set of simple roots of $\mathfrak{g}^\sigma$. With this set of simple root, we can also read from [HK, Lemma 2.1], for any $\lambda \in (\mathfrak{t}^\sigma)^*$, $\lambda \in P(\sigma, 1)$ if and only if $\lambda = 0$.

□

Remark 5.1.2. *It is not true that* $0 \in P(\sigma, 1)$ *for any automorphism* σ*. For example,* $0 \notin P(\tau, 1)$*, when* $\mathfrak{g} = A_{2\ell}$ *and* τ *is a diagram automorphism; instead* $0 \in P(\tau, 2)$*.*

We define the following map

$$\iota : X_*(T) \to (\mathfrak{t}^\sigma)^*, \tag{5.1}$$

such that for any $\lambda \in X_*(T)$, $\iota(\lambda)(h) = (\lambda, h)$, where we regard λ as an element in $\mathfrak{t}$ and $(,)$ is the normalized Killing form on $\mathfrak{t}$. It is clear that $\iota(0) = 0$. This map naturally descends to a map $X_*(T)_\sigma \to (\mathfrak{t}^\sigma)^*$. By abuse of notation, we still call it ι.

Recall some terminology introduced in Section 3.2. I_σ is the set parametrizing simple roots of $\mathfrak{g}^\sigma$, and we also defined a map $\eta : I \to I_\sigma$. The set $\{\check{\lambda}_j \mid j \in I_\sigma\}$ is the set of fundamental coweights of $\mathfrak{g}^\sigma$, and $\{\lambda_j \mid j \in I_\sigma\}$ is the set of fundamental weights of $\mathfrak{g}^\sigma$. We also recall that $\check{\alpha}_i$ is a simple coroot of $\mathfrak{g}$ for each $i \in I$, and γ_j is the image of $\check{\alpha}_i$ in $X_*(T)_\sigma$. The following lemma already appears in [Ha, Lemma 3.2] in a slightly different setting.

Lemma 5.1.3. *For any* $j \in I_\sigma$*, we have*

$$\iota(\gamma_j) = \begin{cases} \beta_j, & \text{if } (\mathfrak{g}, m) \neq (A_{2\ell}, 4), \text{ or }, (\mathfrak{g}, m) = (A_{2\ell}, 4) \text{ and } j \neq \ell \\ \frac{1}{2}\beta_\ell, & \text{if } (\mathfrak{g}, m) = (A_{2\ell}, 4) \text{ and } j = \ell \end{cases}.$$

Proof. By the definition of ι, for any $\gamma_j = \overline{\check{\alpha}}_i$ with $j = \eta(i)$, and $k \in I_\sigma$ we have the following equalities:

$$\langle \check{\lambda}_k, \gamma_j \rangle = \langle \check{\lambda}_k, \iota(\overline{\check{\alpha}}_i) \rangle = (\check{\lambda}_k, \check{\alpha}_i) = \langle \check{\lambda}_k, \alpha_i \rangle.$$

Then, this lemma follows from the description of fundamental coweights of $\mathfrak{g}^\sigma$ in (3.7) and (3.8). □

Recall the set S defined in (4.6).

Lemma 5.1.4. *For any $i \in I$, we have $\iota(\check{\omega}_i) = \lambda_{\eta(i)}$. As a consequence, ι maps $X_*(T)^+_\sigma$ bijectively into the set of dominant weights of $\mathfrak{g}^\sigma$. Furthermore, ι maps S bijectively into $P(\sigma, 1)$.*

Proof. For any $i \in I$ and $j \in I_\sigma$, we have

$$\langle \iota(\check{\omega}_i), \check{\beta}_j \rangle = (\check{\omega}_i, \check{\beta}_j) = (\check{\omega}_i, \sum_{a \in \eta^{-1}(i)} \check{\alpha}_a) = \delta_{\eta(i),j}.$$

Hence, $\iota(\check{\omega}_i) = \lambda_{\eta(i)}$.

In view of Lemma 5.1.1, ι maps S bijectively into $P(\sigma, 1)$. □

Remark 5.1.5. *In view of Lemma 5.1.3 and Lemma 5.1.4, when $(G, m) \neq (A_{2\ell}, 4)$, the root systems of $\mathfrak{g}^\sigma$ and $H := (\check{G})^\tau$ can be naturally identified, where H is discussed in Chapter 4. Namely, $\{\overline{\check{\omega}}_i \mid i \in I\}$ is a set of fundmental weights of H corresponding to $\{\lambda_j \mid j \in I_\sigma\}$ of $\mathfrak{g}^\sigma$, and the set of simple roots $\{\gamma_j \mid j \in I_\sigma\}$ corresponds to $\{\beta_j \mid j \in I_\sigma\}$ of $\mathfrak{g}^\sigma$.*

For any $g \in G(\mathcal{K})^\sigma$, we can define a Lie algebra automorphism

$$\widehat{\mathrm{Ad}}_g(x[f]) := \mathrm{Ad}_g(x[f]) + \frac{1}{m}\mathrm{Res}_{t=0}(g^{-1}dg, x[f])K, \tag{5.2}$$

for any $x[f] \in \mathfrak{g}(\mathcal{K})^\sigma$, where $(,)$ is the normalized Killing form on $\mathfrak{g}$. By Lemma 5.1.4, $\iota(\kappa) \in P(\sigma, 1)$ for any $\kappa \in S$. Thus, $c\iota(\kappa) \in P(\sigma, c)$ for any level $c \geq 1$.

Set

$$\mathscr{H}_c := \oplus_{\kappa \in S} \mathscr{H}_c(c\iota(\kappa)). \tag{5.3}$$

Let $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathcal{K} \oplus \mathbb{C}K' \oplus \mathbb{C}d'$ be the untwisted Kac-Moody algebra associated to $\mathfrak{g}$, where K' is the canonical center and d' is the scaling element. We may define an automorphism σ on $\tilde{\mathfrak{g}}$ as follows,

$$\sigma(x[f(t)]) = \sigma(x)[f(\epsilon t)], \quad \sigma(K') = K', \quad \sigma(d') = d',$$

for any $x[f] \in \mathfrak{g}\otimes\mathcal{K}$. Then the fixed point Lie algebra $\tilde{\mathfrak{g}}^\sigma$ is exactly the twisted Kac-Moody alegbra $\tilde{L}(\mathfrak{g},\sigma)$ containing $\hat{L}(\mathfrak{g},\sigma)$ as the derived algebra. Following from [Ka, Theorem 8.7,§8], in this realization the canonical center K in $\tilde{L}(\mathfrak{g},\sigma)$ is equal to mK', and the scaling element d in $\tilde{\mathfrak{g}}$ is equal to d' when $\tilde{\mathfrak{g}}^\sigma$ is not $A^{(2)}_{2\ell}$, and $d = 2d'$when $\tilde{\mathfrak{g}}^\sigma = A^{(2)}_{2\ell}$.

For any $g \in G(\mathcal{K})$, one can define an automorphism $\widehat{\mathrm{Ad}}_g$ on $\tilde{\mathfrak{g}}$ as in [Ku, Section 13.2.3]. From the formula *loc.cit*, it is clear that if $g \in G(\mathcal{K})^\sigma$, then $\widehat{\mathrm{Ad}}_g$ commutes with σ. In particular, it follows that $\widehat{\mathrm{Ad}}_g$ restricts to an automorphism on $\tilde{L}(\mathfrak{g},\sigma)$. One may observe easily that, restricting further to $\widehat{L}(\mathfrak{g},\sigma)$, this is exactly the automorphism defined in (5.2).

By demanding that $d\cdot v_\kappa = 0$ for each $\kappa\in S$, the action $\hat{L}(\mathfrak{g},\sigma)$ on $\mathscr{H}$ extends uniquely to an action of $\tilde{L}(\mathfrak{g},\sigma)$.

Lemma 5.1.6. *For any $g \in G(\mathcal{K})^\sigma$, there exists an intertwining operator $\rho_g : \mathscr{H}_c \simeq \mathscr{H}_c$ such that*

$$\rho_g(x[f]\cdot v) = \widehat{\mathrm{Ad}}_g(x[f])\cdot\rho_g(v), \tag{5.4}$$

for any $x[f] \in \mathfrak{g}(\mathcal{K})^\sigma$ and $v \in \mathscr{H}_c$. In particular, for any $\kappa \in S$,

$$\widehat{\mathrm{Ad}}_{n^{-\kappa}}(\mathscr{H}_c(0)) = \mathscr{H}_c(c\iota(\kappa)), \;\; \textit{and} \;\; \widehat{\mathrm{Ad}}_{n^{-\kappa}}(\mathscr{H}_c(c\iota(\kappa))) = \mathscr{H}_c(0). \tag{5.5}$$

Proof. Let G' be the simply-connected cover of G, and let $p : G'(\mathcal{K})^\sigma \to G(\mathcal{K})^\sigma$ be the induced map. Then,

$$G(\mathcal{K})^\sigma = \sqcup_{\kappa\in S}\, n^{-\kappa}\overline{G'(\mathcal{K})^\sigma}, \tag{5.6}$$

where $\overline{G'(\mathcal{K})^\sigma} = p(G'(\mathcal{K})^\sigma)$. By twisted analogue of Faltings Lemma (cf. [HK, Proposition 10.2]), for any element $g \in \overline{G'(\mathcal{K})^\sigma}$, there exists an operator ρ_g which maps $\mathscr{H}_c(c\iota(\kappa))$ to $\mathscr{H}_c(c\iota(\kappa))$ with the desired property (5.4), for any $\kappa\in S$. By decomposition (5.6), it suffices to show that, for nonzero κ, $n^{-\kappa}$ satisfies property (5.5).

Assume $\kappa \neq 0$ in S. From the table (4.2.2), the group $(X_*(T)/\check{Q})_\sigma$ is at most of order 2. Therefore,

$n^{-2\kappa} \in \overline{G'(\mathcal{K})^\sigma}$. For each $\mathscr{H}_c(c\iota(\kappa))$, we denote the action by $\pi_{c,\kappa} : \hat{L}(\mathfrak{g}, \sigma) \to \mathrm{End}(\mathscr{H}_c(c\iota(\kappa)))$. Then the property (5.4) for $n^{-2\kappa}$, is equivalent to the existence of an isomorphism of representations,

$$\rho_{n^{-2\kappa}} : (\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa}) \simeq (\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa} \circ \widehat{\mathrm{Ad}}_{n^{-2\kappa}}). \tag{5.7}$$

Let v_κ be the highest weight vector in $\mathscr{H}_c(c\iota(\kappa))$. Then v_κ is of $\mathfrak{t}^\sigma$-weight $c\iota(\kappa)$. We regard $\check{\beta}_i$ as elements in $\mathfrak{t}^\sigma$. By formula (5.2),

$$\widehat{\mathrm{Ad}}_{n^{-\kappa}}(\check{\beta}_i) = \check{\beta}_i - (\kappa, \check{\beta}_i)c = \check{\beta}_i - \langle \iota(\kappa), \check{\beta}_i \rangle c.$$

Hence, v_κ is of $\mathfrak{t}^\sigma$-weight 0 and a highest weight vector in the representation

$$(\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa} \circ \widehat{\mathrm{Ad}}_{n^{-\kappa}}).$$

By Schur lemma, there exists an intertwining operator $\rho_{0\kappa}$,

$$\rho_{0\kappa} : (\mathscr{H}_c(0), \pi_{c,0}) \simeq (\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa} \circ \widehat{\mathrm{Ad}}_{n^{-\kappa}}). \tag{5.8}$$

We also can regard $\rho_{0\kappa}$ as the following intertwining operator

$$\rho_{0\kappa} : (\mathscr{H}_c(0), \pi_{c,0} \circ \widehat{\mathrm{Ad}}_{n^{-\kappa}}) \simeq (\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa} \circ \widehat{\mathrm{Ad}}_{n^{-2\kappa}}) \tag{5.9}$$

Combining isomorphisms (5.7),(5.9), we get

$$(\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa}) \xrightarrow{\rho_{n^{-2\kappa}}} (\mathscr{H}_c(c\iota(\kappa)), \pi_{c,\kappa} \circ \widehat{\mathrm{Ad}}_{n^{-2\kappa}}) \xrightarrow{(\rho_{0\kappa})^{-1}} (\mathscr{H}_c(0), \pi_{c,0} \circ \widehat{\mathrm{Ad}}_{n^{-\kappa}}).$$

We define $\rho_{n^{-\kappa}}$ to be the following operator

$$\rho_{n^{-\kappa}} = (\rho_{0\kappa}, (\rho_{0\kappa})^{-1} \circ \rho_{n^{-2\kappa}}) : \mathscr{H}_c(0) \oplus \mathscr{H}_c(c\iota(\kappa)) \simeq \mathscr{H}_c(0) \oplus \mathscr{H}_c(c\iota(\kappa)).$$

The map $\rho_{n^{-\kappa}}$ satisfies property (5.4). □

As discussed in Section the components of $\mathtt{Gr}_{\mathscr{G}}$ are parametrized by elements in $(X_*(T)/\check{Q})_\sigma$. Moreover, $\mathtt{Gr}_{\mathscr{G}} = \sqcup_{\kappa \in S} \mathtt{Gr}_{\mathscr{G},\kappa}$, where S is defined in (4.6).

Let $\mathscr{G}'$ be the parahoric group scheme $\mathrm{Res}_{\tilde{O}/O}(G'_{\tilde{O}})^{\sigma}$, and let $L^+\mathscr{G}'$ (resp. $L\mathscr{G}'$) denote the jet group scheme (resp. loop group scheme) of $\mathscr{G}'$. The group $L\mathscr{G}$ acts on $L\mathscr{G}'$ by conjugation. Set

$$L^+\mathscr{G}'_{\kappa} := \mathrm{Ad}_{n^{-\kappa}}(L^+\mathscr{G}').$$

Then, $L^+\mathscr{G}'_{\kappa}$ is a subgroup scheme of $L\mathscr{G}'$. We have

$$\mathtt{Gr}_{\mathscr{G},\kappa} \simeq L\mathscr{G}'/L^+\mathscr{G}'. \tag{5.10}$$

By the twisted analogue of Faltings lemma (cf. [HK, Proposition 10.2]), there exists a group homomorphism $L\mathscr{G}' \to \mathrm{PGL}(\mathscr{H}_1(0))$. Consider the central extension

$$1 \to \mathbb{G}_m \to \mathrm{GL}(\mathscr{H}_1(0)) \to \mathrm{PGL}(\mathscr{H}_1(0)) \to 1. \tag{5.11}$$

The pull-back of (5.11) to $L\mathscr{G}'$ defines the following canonical central extension of $L\mathscr{G}'$:

$$1 \to \mathbb{G}_m \to \widehat{L\mathscr{G}'} \to L\mathscr{G}' \to 1. \tag{5.12}$$

It is known that $L\hat{\mathscr{G}}'$ is a Kac-Moody group of twisted type (up to a scaling multiplicative group) in the sense of Kumar and Mathieu, see [PR]. Let $L^+\hat{\mathscr{G}}'_{\kappa}$ denote the preimage of $L^+\mathscr{G}'_{\kappa}$ in $L\hat{\mathscr{G}}'$ via the projection map $L\hat{\mathscr{G}}' \to L\mathscr{G}'$. As the same proof as in [BH, Lemma 2.19], $L^+\hat{\mathscr{G}}'_{\kappa}$ is a parabolic subgroup in $L\hat{\mathscr{G}}'$, moreover

$$\mathtt{Gr}_{\mathscr{G},\kappa} \simeq L\hat{\mathscr{G}}'/L^+\hat{\mathscr{G}}'_{\kappa}, \tag{5.13}$$

i.e. $\mathtt{Gr}_{\mathscr{G},\kappa}$ is a partial flag variety of the Kac-Moody group $L\hat{\mathscr{G}}'$.

Proposition 5.1.7. *There exists a line bundle $\mathscr{L}$ on $\mathtt{Gr}_{\mathscr{G}}$ such that $\mathscr{L}$ is of level one on each component of $\mathtt{Gr}_{\mathscr{G}}$.*

Proof. We first consider the simply-connected cover G' of G. By [HK, Theorem 10.7 (1)], there exists a canonical splitting of $\widehat{L\mathscr{G}'} \to L\mathscr{G}'$ in the central extension (5.11) over $L^+\mathscr{G}'$. We may define a line bundle $\mathscr{L}$ on $\mathtt{Gr}_{\mathscr{G}'} = \widehat{L\mathscr{G}'}/\widehat{L^+\mathscr{G}'}$ via the character $\widehat{L^+\mathscr{G}'} := \mathbb{G}_m \times L^+\mathscr{G}' \to \mathbb{G}_m$ defined via the first projection. In fact, as the argument in [LS, Lemma 4.1], this line bundle is the ample generator of $\mathrm{Pic}(\mathtt{Gr}_{\mathscr{G}'})$ of level 1. This finishes the proof of part (1).

We now consider the case when G is of adjoint type. Since the neutral component $\mathtt{Gr}_{\mathscr{G},\circ}$ is isomorphic to $\mathtt{Gr}_{\mathscr{G}'}$, we get the level one line bundle on $\mathtt{Gr}_{\mathscr{G},\circ}$ induced from the one on $\mathtt{Gr}_{\mathscr{G}'}$. For any other component $\mathtt{Gr}_{\mathscr{G},\kappa}$, by (5.13) we have an isomorphism $\mathtt{Gr}_{\mathscr{G},\circ} \simeq \mathtt{Gr}_{\mathscr{G},\kappa}$. Therefore, this gives rise to the level one line bundle on $\mathtt{Gr}_{\mathscr{G},\kappa}$.

□

The line bundle $\mathscr{L}$ on $\mathtt{Gr}_{\mathscr{G}}$ naturally has a $L\hat{\mathscr{G}}'$-equivariant structure, since $\mathscr{L}$ admits a unique $L\hat{\mathscr{G}}'$-equivariant structure on each component of $\mathtt{Gr}_{\mathscr{G}}$ as a partial flag variety of $L\hat{\mathscr{G}}'$. Now, by the standard Borel-Weil-Bott theorem for Kac-Moody group (cf. [Ku]), we get the following theorem.

Theorem 5.1.8. *As representations of $\hat{L}(\mathfrak{g},\sigma)$, we have $H^0(\mathtt{Gr}_{\mathscr{G}},\mathscr{L}^c)^\vee \simeq \mathscr{H}_c$, where $\mathscr{L}^c$ is the c-power of $\mathscr{L}$.*

Let v_0 be the highest weight vector in $\mathscr{H}_0$. For any $\bar{\lambda} \in X_*(T)_\sigma$, we define

$$v_{\bar{\lambda}} := \rho_{n^\lambda}(v_0), \tag{5.14}$$

where ρ_{n^λ} in defined in Lemma 5.1.6. Then $v_{\bar{\lambda}}$ is independent of the choice of the representative λ in $X_*(T)$ and is well-defined up to a nonzero scalar.

Lemma 5.1.9. *The $\mathfrak{t}^\sigma$-weight of the vector $v_{\bar{\lambda}}$ is $-c\iota(\bar{\lambda})$.*

Proof. For any $h \in \mathfrak{t}^\sigma$, by Lemma 5.1.6,

$$h \cdot v_{\bar{\lambda}} = h \cdot \rho_{n^\lambda}(v_0) = \rho_{n^\lambda}(\widehat{\mathrm{Ad}}_{n^{-\lambda}}(h)v_0).$$

By the formula (5.2), we have

$$\widehat{\mathrm{Ad}}_{n^{-\lambda}}(h) = h - \langle\lambda, h\rangle K.$$

It follows that

$$h \cdot v_{\bar{\lambda}} = -\langle\lambda, h\rangle c v_{\bar{\lambda}} = -c\iota(\lambda)(h)v_{\bar{\lambda}}.$$

This concludes the proof of the lemma. □

Definition 5.1.10. *For any dominant $\bar{\lambda} \in X_*(T)^+_\sigma$, we define the twisted affine Demazure module $D(c, \bar{\lambda})$ as the following $\mathfrak{g}[t]^\sigma$-module,*

$$D(c, \bar{\lambda}) := U(\mathfrak{g}[t]^\sigma)v_{\bar{\lambda}}.$$

In view of Lemma 5.1.9, $D(c, \bar{\lambda})$ contains an irreducible representation $V(-c\iota(\lambda))$ of $\mathfrak{g}^\sigma$ of lowest weight $-c\iota(\lambda)$. The following theorem follows from [Ku, Theorem 8.2.2 (a)].

Theorem 5.1.11. *As $\mathfrak{g}[t]^\sigma$-modules, $H^0(\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \mathscr{L}^c)^\vee \simeq D(c, \bar{\lambda})$.*

5.1.1 Construction of level one line bundles on $\mathrm{Bun}_{\mathcal{G}}$

Note 5.1.12. *In this chapter and going forward, we are free to choose the curve C to be $\mathbb{P}^1_{\mathbb{C}}$. In this setting, the ramified cover $\tilde{C} \to C$ of degree 1,2 or 3 can be chosen to be ramified at two points called o and ∞.*

In this subsection, we consider the parahoric Bruhat-Tits group scheme $\mathcal{G} := \mathrm{Res}_{\tilde{C}/C}(G \times \tilde{C})^\Gamma$ over C as in the setting of Section 4.

Let $\mathrm{Bun}_{\mathcal{G}}$ be the moduli stack of $\mathcal{G}$-torsors on C. It is known that $\mathrm{Bun}_{\mathcal{G}}$ is a smooth Artin stack (cf. [He]). By [He, Theorem 3], the Picard group $\mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})$ of $\mathrm{Bun}_{\mathcal{G}}$ is isomorphic to $\mathbb{Z}$, since the group $X^*(\mathcal{G}|_y)$ of characters for $\mathcal{G}|_y$ is trivial for any $y \in C$. In this subsection, we will construct the ample generator $\mathcal{L} \in \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})$ when G is simply-connected, and we will construct a level one line bundle on every component of $\mathtt{Gr}_{\mathcal{G},C}$ when G is of adjoint type.

By Lemma 5.1.1, we have $0 \in P(\sigma, 1)$ for any absolutely special automorphism σ. Recall that $\mathscr{H}_1(0)$ is the basic representation of level one associated to $0 \in P(\sigma, 1)$.

We now define the following space of twisted covacua of level one,

$$\mathscr{V}_{C,\sigma}(0) := \frac{\mathscr{H}_1(0)}{\mathfrak{g}[t^{-1}]^\sigma \cdot \mathscr{H}_1(0)}, \tag{5.15}$$

where $\mathfrak{g}[t^{-1}]^\sigma$ is the Lie subalgebra of $\hat{L}(\mathfrak{g}, \sigma)$.

Lemma 5.1.13. *The dimension of the vector space $\mathscr{V}_{C,\sigma}(0)$ is 1.*

Proof. Let v_0 be the highest weight vector in $\mathscr{H}_1(0)$. Then

$$\mathscr{H}_1(0) = U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma) \cdot v_0 = U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma)(t^{-1}\mathfrak{g}[t^{-1}])^\sigma v_0 \oplus \mathbb{C}v_0,$$

where $U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma)$ denotes the universal enveloping algebra of $(t^{-1}\mathfrak{g}[t^{-1}])^\sigma$. We can write $\mathfrak{g}[t^{-1}]^\sigma =$

$\mathfrak{g}^\sigma \oplus (t^{-1}\mathfrak{g}[t^{-1}])^\sigma$. Hence,

$$\mathfrak{g}[t^{-1}]^\sigma \cdot \mathscr{H}_1(0) = \mathfrak{g}^\sigma \cdot U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma)(t^{-1}\mathfrak{g}[t^{-1}])^\sigma v_0 + U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma)(t^{-1}\mathfrak{g}[t^{-1}])^\sigma v_0 \tag{5.16}$$

$$= U((t^{-1}\mathfrak{g}[t^{-1}])^\sigma)(t^{-1}\mathfrak{g}[t^{-1}])^\sigma v_0, \tag{5.17}$$

where the first equality holds since $\mathfrak{g}^\sigma \cdot v_0 = 0$, and the second equality holds since $\mathfrak{g}^\sigma$ normalizes $(t^{-1}\mathfrak{g}[t^{-1}])^\sigma$ under the Lie bracket. Therefore, $\dim \mathscr{V}_{C,\sigma}(0) = 1$. □

Let G' be the simply-connected cover of G. Recall the Heinloth uniformization theorem for $\mathcal{G}' := \mathrm{Res}_{\tilde{C}/C}(G' \times \tilde{C})^\Gamma$ over the affine line $C\backslash o$ (cf. [He]),

$$\mathrm{Bun}_{\mathcal{G}'} \simeq G'[t^{-1}]^\sigma \backslash \mathtt{Gr}_{\mathscr{G}'},$$

where $\mathtt{Gr}_{\mathscr{G}'}$ denotes the affine Grassmannian of $\mathscr{G}' := \mathrm{Res}_{\tilde{O}/O}(G'_{\tilde{O}})^\sigma$, and $G'[t^{-1}]^\sigma \backslash \mathtt{Gr}_{\mathscr{G}'}$ denotes the fppf quotient.

Theorem 5.1.14. *The line bundle $\mathscr{L}$ descends to a line bundle $\mathcal{L}$ on* $\mathrm{Bun}_{\mathcal{G}'}$.

Proof. Let $\mathscr{L}$ be the level one line bundle on $\mathtt{Gr}_{\mathscr{G}'}$ constructed from Proposition 5.1.7. To show that the line bundle $\mathscr{L}$ can descend to $\mathrm{Bun}_{\mathcal{G}'}$, as in the argument in [So], it suffices to show that there is a $G'[t^{-1}]^\sigma$-linearization on $\mathscr{L}$. This is equivalent to the splitting of the central extension (5.12) over $G'[t^{-1}]^\sigma$. We use the same argument as in [So, Proposition 3.3], since the vector space $\mathscr{V}_{C,\sigma}(0)$ is nonvanishing by Lemma 5.1.13, the central extension (5.12) splits over $G'[t^{-1}]^\sigma$. □

We consider the projection map $\mathrm{pr} : \mathtt{Gr}_{\mathcal{G}',C} \to \mathrm{Bun}_{\mathcal{G}'}$. By abuse of notation, we still denote by $\mathcal{L}$ the line bundle on $\mathtt{Gr}_{\mathcal{G}',C}$ pulling-back from $\mathcal{L}$ on $\mathrm{Bun}_{\mathcal{G}'}$.

Corollary 5.1.15. *The restriction of the line bundle $\mathcal{L}$ to the fiber $\mathtt{Gr}_{\mathcal{G}',p}$ is the ample generator of* $\mathrm{Pic}(\mathtt{Gr}_{\mathcal{G}',p})$*, for any $p \in C$.*

Proof. It follows from Theorem 5.1.14 and [Zh2, Proposition 4.1]. □

The following theorem is interesting by itself, but will not be used in this paper.

Theorem 5.1.16. *There is a natural isomorphism*

$$H^0(\mathrm{Bun}_{\mathcal{G}'}, \mathcal{L}) \simeq \mathscr{V}_{C,\sigma}(0)^\vee,$$

where $\mathscr{V}_{C,\sigma}(0)^\vee$ *denotes the dual of* $\mathscr{V}_{C,\sigma}(0)$*. In particular,*

$$\dim H^0(\mathrm{Bun}_{\mathcal{G}'}, \mathcal{L}) = 1.$$

Proof. The theorem follows from the same argument as in [HK, Theorem 12.1]. □

Now, we would like to construct the line bundle $\mathcal{L}$ of level one on $\mathtt{Gr}_{\mathcal{G},C}$, where $\mathcal{G} = \mathrm{Res}_{\tilde{C}/C}(G_{\tilde{C}})^\sigma$ with G of adjoint type.

Theorem 5.1.17. *There exists a line bundle* $\mathcal{L}$ *on* $\mathtt{Gr}_{\mathcal{G},C}$ *such that the restriction of* $\mathcal{L}$ *to the fiber* $\mathtt{Gr}_{\mathcal{G},p}$ *is the level one line bundle on* $\mathtt{Gr}_{\mathcal{G},p}$*, for any* $p \in C$*.*

Proof. Let X be a component of $\mathtt{Gr}_{\mathcal{G},C}$. Fix any point $x \in X$, $x \in \mathtt{Gr}_{\mathcal{G},p}$ for a unique $p \in \tilde{C}$. If $p = o$, then X contains at least one component of $\mathtt{Gr}_{\mathscr{G}}$. If $p \neq o, \infty$, then x is a point in an affine Schubert variety $\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},p}$ for some $\lambda \in X_*(T)^+$. By Theorem 4.3.10, $\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},p}$ admits a flat degeneration to $\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$. If $p = \infty$, x is a point in a twisted affine Schubert variety $\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathcal{G},\infty}$. Similarly, there is a flat family connecting $\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathcal{G},\infty}$ and $\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$. This concludes that X must contain at least one component of $\mathtt{Gr}_{\mathscr{G}}$. In other words, $\mathtt{Gr}_{\mathscr{G}}$ has as many or more components than $\mathtt{Gr}_{\mathcal{G},C}$.

Recall that the components of $\mathtt{Gr}_{\mathscr{G}}$ are parametrized by $(X_*(T)/\check{Q})_\sigma$. On the other hand, by [He, Theorem 2], $\pi_0(\mathrm{Bun}_{\mathcal{G}})$ can also be identified with $(X_*(T)/\check{Q})_\sigma$. We have a natural projection $\mathrm{pr} : \mathtt{Gr}_{\mathcal{G},C} \to \mathrm{Bun}_{\mathcal{G}}$. In view of Heinloth's uniformization theorem [He, Theorem 3], the map pr is surjective. Hence $\mathtt{Gr}_{\mathcal{G},C}$ has as many or more components than $\mathrm{Bun}_{\mathcal{G}}$. It forces that $\mathtt{Gr}_{\mathscr{G}}$, $\mathtt{Gr}_{\mathcal{G},C}$ and $\mathrm{Bun}_{\mathcal{G}}$ have the same number of components. In particular, it follows that there is a natural bijection bertween components of $\mathtt{Gr}_{\mathscr{G}}$ and $\mathtt{Gr}_{\mathcal{G},C}$.

It is well-known that the neutral component $\mathtt{Gr}_{\mathcal{G},C,\circ}$ of $\mathtt{Gr}_{\mathcal{G},C}$ is isomorphic to $\mathtt{Gr}_{\mathcal{G}',C}$. Thus, we naturally get the level one line bundle $\mathcal{L}$ on the neutral component $\mathtt{Gr}_{\mathcal{G},C,\circ}$. Recall the set S in (4.6) that parametrizes the components of $\mathtt{Gr}_{\mathscr{G}}$. For any nonzero $\kappa \in S$ (if it exists), the component $\mathtt{Gr}_{\mathscr{G},\kappa}$ of $\mathtt{Gr}_{\mathscr{G}}$ contains $e_{\bar{\kappa}}$. Thus, the associated component $\mathtt{Gr}_{\mathcal{G},C,\kappa}$ is exactly the one containing s^κ. The component $\mathtt{Gr}_{\mathscr{G},\kappa}$ is isomorphic to

$$L\mathcal{G}_C/\mathrm{Ad}_{s^\kappa}(L^+\mathcal{G}_C),$$

where is s^κ is a C-point in $\mathtt{Gr}_{\mathcal{G},C}$ as defined in Chapter 4. Then there exists a natural isomorphism

$$\mathtt{Gr}_{\mathcal{G}',C} = L\mathcal{G}'_C / L^+\mathcal{G}'_C \simeq L\mathcal{G}'_C / \mathrm{Ad}_{s^\kappa}(L^+\mathcal{G}'_C),$$

given by $gL^+\mathcal{G}'_C \mapsto \mathrm{Ad}_{s^{-\kappa}}(g)\mathrm{Ad}_{s^\kappa}(L^+\mathcal{G}'_C)$. Therefore, the line bundle of level one on the non-neutral component can be realized as the pull-back from the line bundle $\mathcal{L}$ on the neutral component $\mathtt{Gr}_{\mathcal{G},C,\circ}$ via this isomorphism. □

CHAPTER 6

Flat Degenerations and Smoothness

In this chapter we may reap the fruits of our labors. Having constructed flat families and global line bundles of level one, we proceed by using Zhu's results in the untwisted setting (this amounts to cohomological vanishing at a generic fiber of the flat family) and use it to show cohomological vanishing at the special fiber. The main theorem of this Chapter and indeed of this work is Theorem 6.1.5, which proves the restriction isomorphism for twisted affine Schubert varieties in all the cases where we have the related theorem in the untwisted case. A nice corollary is Theorem 6.1.11, which describes the smooth locus of many twisted affine Schubert varieties.

6.1 Flat Degeneration and Smoothness Results

We first recall a theorem in [Zh1, Theorem 1.3.4].

Theorem 6.1.1. *The natural morphism* $\mathtt{Gr}_T \to \mathtt{Gr}_G$ *identifies* $\mathtt{Gr}_T$ *as the* T*-fixed point ind-subscheme* $(\mathtt{Gr}_G)^T$ *of* $\mathtt{Gr}_G$.

We now prove an analogue of Theorem 6.1.1 in the setting of absolutely special parahoric group schemes.

Theorem 6.1.2. *The natural morphism* $\mathtt{Gr}_{\mathscr{T}} \to \mathtt{Gr}_{\mathscr{G}}$ *identifies* $\mathtt{Gr}_{\mathscr{T}}$ *as the* T^σ*-fixed point ind-subscheme* $(\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}$ *of* $\mathtt{Gr}_{\mathscr{G}}$.

Proof. Let $L^{--}G$ be the ind-group scheme represented by the following functor, for any $\mathbb{C}$-algebra R,

$$L^{--}G(R) := \ker(\mathrm{ev}_\infty : G(R[t^{-1}]) \to G(R)),$$

where ev_∞ is the evaluation map sending t^{-1} to 0. Let $L^{--}\mathscr{G}$ be the ind-group scheme which represents the following functor , for any $\mathbb{C}$-algebra R,

$$L^{--}\mathscr{G}(R) := \ker(\mathrm{ev}_\infty : G(R[t^{-1}])^\sigma \to G(R)^\sigma).$$

We can similarly define $L^{--}T$ and $L^{--}\mathscr{T}$.

By the similar argument as in [Zh4, Lemma 2.3.5] or [HR2, Lemma 3.1], we have an open embedding

$$L^{--}\mathscr{G} \hookrightarrow \mathtt{Gr}_{\mathscr{G}}$$

given by $g \mapsto ge_0$, where e_0 is the base point in $\mathtt{Gr}_{\mathscr{G}}$. Let I be the Iwahori subgroup of $L^+\mathscr{G}$, which is the preimage of B^σ via the evaluation map $\mathrm{ev} : L^+\mathscr{G} \to G^\sigma$ for a σ-stable Borel subgroup B in G. We have the following decomposition

$$\mathtt{Gr}_{\mathscr{G}} = \bigsqcup_{\bar{\lambda} \in X_*(T)_\sigma} Ie_{\bar{\lambda}}. \tag{6.1}$$

For each $\bar{\lambda} \in X_*(T)_\sigma$, we choose a representative $\lambda \in X_*(T)$. The twisted Iwahori Schubert cell

$$Ie_{\bar{\lambda}} = n^\lambda \mathrm{Ad}_{n^{-\lambda}}(I)e_0$$

is contained in $n^\lambda L^{--}\mathscr{G}e_0$. Then by the decomposition (6.1), $\bigcup_{\bar{\lambda} \in X_*(T)_*} n^\lambda L^{--}\mathscr{G}e_0$ is an open covering of $\mathtt{Gr}_{\mathscr{G}}$. We may naturally regard $\mathtt{Gr}_{\mathscr{T}}$ as an ind-subscheme of $\mathtt{Gr}_{\mathscr{G}}$. Hence, we may regard e_0 as the base point in $\mathtt{Gr}_{\mathscr{T}}$. Under this convention,

$$\bigcup_{\lambda \in X_*(T)_\sigma} n^\lambda L^{--}\mathscr{T}e_0 = \bigcup_{\lambda \in X_*(T)_\sigma} L^{--}\mathscr{T}n^\lambda e_0$$

is an open covering of $\mathtt{Gr}_{\mathscr{T}}$. Therefore, it suffices to show that for each $\bar{\lambda} \in X_*(T)_\sigma$,

$$(n^\lambda L^{--}\mathscr{G}e_0)^{T^\sigma} \simeq n^\lambda L^{--}\mathscr{T}e_0.$$

Further, it suffices to show that $(L^{--}\mathscr{G})^{T^\sigma} \simeq L^{--}\mathscr{T}$, where the action of T^σ on $L^{--}\mathscr{G}$ is by conjugation. From the proof of [HR2, Proposition 3.4], one may see that $(L^{--}G)^{T^\sigma} \simeq L^{--}T$. This actually implies that $(L^{--}\mathscr{G})^{T^\sigma} \simeq L^{--}\mathscr{T}$. Hence, this finishes the proof of the theorem. □

An immediate consequence of Theorem 6.1.2 is the following corollary.

Corollary 6.1.3. *The T^σ-fixed $\mathbb{C}$-point set in $\mathtt{Gr}_{\mathscr{G}}$ is $\{e_{\bar{\lambda}} \,|\, \lambda \in X_*(T)_\sigma\}$.*

6.1.1 A duality isomorphism for twisted Schubert varieties

Let $\mathtt{Gr}_G$ be the affine Grassmannian of G, and let L be the line bundle on $\mathtt{Gr}_G$ that is of level one on every component of $\mathtt{Gr}_G$. For any $\lambda \in X_*(T)$, let $\overline{\mathtt{Gr}}_G^\lambda$ denote the closure of $G(O)$-orbit at $L_\lambda := t^\lambda G(O) \in \mathtt{Gr}_G$. Let $(\overline{\mathtt{Gr}}_G^\lambda)^T$ denote the T-fixed point subscheme of $\overline{\mathtt{Gr}}_G^\lambda$. Zhu [Zh1] proved that

Theorem 6.1.4. *When G is simply-laced and not of type E, the restriction map* $H^0(\overline{\mathtt{Gr}}_G^\lambda, \mathrm{L}) \to H^0((\overline{\mathtt{Gr}}_G^\lambda)^T, \mathrm{L}|_{(\overline{\mathtt{Gr}}_G^\lambda)^T})$ *is an isomorphism.*

Our goal is to prove an analagous isomorphism for the twisted affine Grassmannians, and our tool for doing this will be the flat family $\overline{\mathtt{Gr}}_{\mathcal{G}}^\lambda$.

From Theorem 6.1.2, we have the identification $\mathtt{Gr}_{\mathscr{T}} \xrightarrow{\simeq} \mathtt{Gr}_{\mathscr{G}}^{T^\sigma}$. Let $\mathscr{I}^{\bar{\lambda}}$ denote the ideal sheaf of the T^σ-fixed subscheme $(\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}})^{T^\sigma}$ of $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$. Then we have a short exact sequence of sheaves

$$0 \to \mathscr{I}^{\bar{\lambda}} \to \mathscr{O}_{\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}} \to \mathscr{O}_{(\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}})^{T^\sigma}} \to 0. \tag{6.2}$$

Recall that $\mathscr{L}$ is the line bundle on $\mathtt{Gr}_{\mathscr{G}}$ which is of level one on every component. Tensoring the above short exact sequence with $\mathscr{L}$ and taking the functor of global sections, we obtain the following exact sequence

$$0 \to H^0(\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}, \mathscr{I}^{\bar{\lambda}} \otimes \mathscr{L}) \to H^0(\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}, \mathscr{L}) \xrightarrow{r} H^0((\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}})^{T^\sigma}}) \to \cdots, \tag{6.3}$$

where r is the restriction map.

Theorem 6.1.5. *When $\mathscr{G}$ is not of type $E_6^{(2)}$, the restriction map*

$$H^0(\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}, \mathscr{L}) \xrightarrow{r} H^0((\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}})^{T^\sigma}})$$

is an isomorphism.

This theorem will follow from the following proposition and Lemma 6.1.8. In fact, this theorem holds for many twisted affine Schubert varieties of $E_6^{(2)}$, see Remark 6.1.9. The following proposition does not exclude $E_6^{(2)}$.

Proposition 6.1.6. *The map r is a surjection.*

Proof. It is well-known that any twisted affine Schubert varietiety $\overline{\mathtt{Gr}}_{\mathscr{G}}^{\bar{\lambda}}$ is a usual Schubert variety in a partial affine flag variety of Kac-Moody group. See the identification (5.10) and an argument for untwisted case in

[BH, Proposition 2.21]. By [Ku, Theorem 8.2.2 (d)], we have that for any $\bar\lambda \geq \bar\mu$ in $X_*(T)^+_\sigma$, the following restriction map

$$H^0(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}) \to H^0(\overline{\mathtt{Gr}}^{\bar\mu}_{\mathscr{G}}, \mathscr{L}) \tag{6.4}$$

is surjective, and

$$H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}) = \varprojlim H^0(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}|_{\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}}). \tag{6.5}$$

We also have the following surjective map

$$H^0((\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}) \to H^0((\overline{\mathtt{Gr}}^{\bar\mu}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}) \tag{6.6}$$

for all $\bar\lambda \geq \bar\mu$, since these T^σ-fixed closed subschemes are affine and the morphism $(\overline{\mathtt{Gr}}^{\bar\mu}_{\mathscr{G}})^{T^\sigma} \to (\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}$ is a closed embedding. Moreover,

$$H^0((\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}}) = \varprojlim H^0((\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}}).$$

Therefore, for any $\bar\lambda \in X_*(T)^+_\sigma$ we have the following surjective maps

$$H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}) \to H^0(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}), \quad H^0((\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}) \to H^0((\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}).$$

Then to prove the map

$$H^0(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}) \to H^0((\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}})$$

is surjective, it is sufficient to prove that the map

$$H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}) \to H^0((\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}}) \tag{6.7}$$

is surjective, since we will have the following commutative diagram, for all $\bar\lambda$,

$$\begin{array}{ccc} H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}) & \longrightarrow & H^0((\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}}) \\ \downarrow & & \downarrow \\ H^0(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}|_{\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}}) & \xrightarrow{r} & H^0((\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}}). \end{array} \tag{6.8}$$

By Theorem 6.1.2, we have $\mathtt{Gr}_{\mathscr{T}} \simeq (\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}$. Therefore, the surjectivity of the map (6.7) follows from the following Lemma 6.1.7. ◻

We first make a digression on Heisenberg algebras and their representations. The subspace $\hat{\mathfrak{t}}^\sigma := (\mathfrak{t}_{\mathcal{K}})^\sigma \oplus \mathbb{C}K \hookrightarrow \hat{L}(\mathfrak{g}, \sigma)$ is a Lie subalgebra. In fact, $\hat{\mathfrak{t}}^\sigma$ is an extended (completed) Heisenberg algebra with center $\mathfrak{t}^\sigma \oplus \mathbb{C}K$. Therefore, any integrable irreducible highest weight representation of $\hat{\mathfrak{t}}^\sigma$ is parametrized by an element $\mu \in (\mathfrak{t}^\sigma)^*$ and the level c, i.e. K acts by the scalar c on this representation. We denote this representation by $\pi_{\mu,c}$. By the standard construction,

$$\pi_{\mu,c} = \mathrm{ind}^{\hat{\mathfrak{t}}^\sigma}_{(\mathfrak{t}_{\mathcal{O}})^\sigma \oplus \mathbb{C}K} \mathbb{C}_{\mu,c}, \tag{6.9}$$

where ind is the induced representation in the sense of universal enveloping algebras, and $\mathbb{C}_{\mu,c}$ is the 1-dimensional module over $(\mathfrak{t}_{\mathcal{O}})^\sigma \oplus \mathbb{C}K$ where the action of $(\mathfrak{t}_{\mathcal{O}})^\sigma$ factors through $\mathfrak{t}^\sigma$.

Lemma 6.1.7. *The restriction map $H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}^c) \to H^0(\mathtt{Gr}_{\mathscr{T}}, \mathscr{L}^c|_{\mathtt{Gr}_{\mathscr{T}}})$ is surjective.*

Proof. Proving surjectivity here is equivalent to proving injectivity for the dual modules,

$$0 \to H^0(\mathtt{Gr}_{\mathscr{T}}, \mathscr{L}^c|_{\mathtt{Gr}_{\mathscr{T}}})^\vee \to H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}^c)^\vee.$$

Note that both of these spaces are modules for the Heisenberg algebra $\hat{\mathfrak{t}}^\sigma$; the morphism is a $\hat{\mathfrak{t}}^\sigma$-morphism. Since $\mathscr{T}$ is discrete, we naturally have the following decomposition

$$H^0(\mathtt{Gr}_{\mathscr{T}}, \mathscr{L}^c|_{\mathtt{Gr}_{\mathscr{T}}}) \simeq \bigoplus_{\bar{\lambda} \in X_*(T)_\sigma} \mathscr{O}_{\mathtt{Gr}_{\mathscr{T}}, e_{\bar{\lambda}}} \otimes \mathscr{L}^c|_{e_{\bar{\lambda}}},$$

where $\mathscr{O}_{\mathtt{Gr}_{\mathscr{T}}, e_{\bar{\lambda}}}$ is the structure sheaf of the component of $\mathtt{Gr}_{\mathscr{T}}$ containing $e_{\bar{\lambda}}$. We also notice that, the identify component of $\mathtt{Gr}_{\mathscr{T}}$ is naturally the formal group with Lie algebra $(\mathfrak{t}_{\mathcal{K}})^\sigma/(\mathfrak{t}_{\mathcal{O}})^\sigma$. In view of the construction (6.9), we have

$$H^0(\mathtt{Gr}_{\mathscr{T}}, \mathscr{L}^c|_{\mathtt{Gr}_{\mathscr{T}}})^\vee = \bigoplus_{\bar{\lambda} \in X_*(T)_\sigma} \pi_{-c\iota(\bar{\lambda}),c};$$

where the map $\iota : X_*(T)_\sigma \to (\mathfrak{t}^\sigma)^*$ is defined in (5.1). Since each $\pi_{-c\iota(\bar{\lambda}),c}$ is irreducible, and generated by a $-c\iota(\bar{\lambda})$-weight vector $w_{-c\iota(\bar{\lambda})}$, it suffices to show that the morphism

$$\pi_{-c\iota(\bar{\lambda}),c} \to H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}^c)^\vee$$

sends $w_{-c\iota(\bar\lambda)}$ to a nonzero vector.

By Theorem 5.1.8, we may define a Plücker embedding

$$\phi : \mathtt{Gr}_{\mathscr{G}} \to \mathbb{P}(\mathscr{H}_c)$$

given by $ge_0 \mapsto [\rho_g(v_0)]$ for any $ge_0 \in \mathtt{Gr}_{\mathscr{G}}$, where ρ_g is defined in Lemma 5.1.6, and $[\rho_g(v_0)]$ represents the line in $\mathscr{H}_c$ that contains $\rho_g(v_0)$. Then we may pick a linear form $f_{\bar\lambda}$ on $\mathscr{H}_c$ which is nonzero on $[v_{\bar\lambda}]$, and which is 0 on other weight vectors, where $v_{\bar\lambda}$ is defined in (5.14). The restriction $f_{\bar\lambda}|_{\phi(\mathtt{Gr}_{\mathscr{G}})}$ produces a nontrivial element in $H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L})$, since $\phi(e_{\bar\lambda}) = v_{\bar\lambda}$.

Observe that the map $\pi_{-c\iota(\bar\lambda),c} \to H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}^c)^\vee$ sends $w_{-c\iota(\bar\lambda)}$ to a nonzero scalar of $v_{\bar\lambda}$. Thus the map $\pi_{-c\iota(\bar\lambda),c} \to H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}^c)^\vee$ is nontrivial and thus injective. □

By Lemma 6.1.7, we obtain the following short exact sequence

$$0 \to H^0(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{I}^{\bar\lambda} \otimes \mathscr{L}) \to H^0(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}) \xrightarrow{r} H^0(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L} \otimes \mathscr{O}_{(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}})}) \to 0.$$

Thus, the obstruction to the map r being an isomorphism is the vanishing of the first term $H^0(\overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{I}^{\bar\lambda} \otimes \mathscr{L})$.

Let I^λ denote the ideal sheaf of the T-fixed subscheme on $\overline{\mathtt{Gr}}^{\lambda}_{G}$. We will show that the vanishing of the first term can be deduced from the vanishing of $H^0(\overline{\mathtt{Gr}}^{\lambda}_{G}, \mathrm{I}^\lambda \otimes \mathrm{L})$.

Recall that $\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C}$ is a global Schubert variety defined in Chapter 4. The constant group scheme $T^\sigma \times C$ over C is naturally a closed subgroup scheme of $\mathcal{T}$. Hence T^σ acts on $\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C}$ naturally. Let $(\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C})^{T^\sigma}$ be the T^σ-fixed subscheme of $\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C}$, and let $\mathcal{I}^\lambda$ be the ideal sheaf of $(\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C})^{T^\sigma}$. Then, $\mathcal{I}^\lambda|_p$ is the ideal sheaf of $(\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C}|_p)^{T^\sigma}$. Recall that,

$$\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},o} = \overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \quad \overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},\infty} \simeq \overline{\mathtt{Gr}}^{\bar\lambda}_{\mathscr{G}}, \quad \overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},p\neq o,\infty} \simeq \overline{\mathtt{Gr}}^{\lambda}_{G}.$$

In particular, we have

$$\mathcal{I}^\lambda|_o = \mathscr{I}^\lambda, \quad \mathcal{I}^\lambda|_\infty \simeq \mathscr{I}^\lambda, \quad \mathcal{I}^\lambda|_{p\neq o,\infty} \simeq \mathrm{I}^\lambda.$$

Lemma 6.1.8. *Assume that G is not of type E_6. Then the ideal $\mathcal{I}^\lambda$ is flat over C.*

Proof. Consider $\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C\setminus\{o,\infty\}}$ and the T^σ-fixed subscheme $(\overline{\mathtt{Gr}}^{\lambda}_{\mathcal{G},C\setminus\{o,\infty\}})^{T^\sigma}$. We denote by Z^λ the flat closure of $(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C\backslash\{o,\infty\}})^{T^\sigma}$ in $\mathrm{Gr}_{\mathcal{G},C}$. Since Z is the closure of a T^σ-fixed subscheme, we see that $Z^\lambda|_o \subset \overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}|_o$, and $Z^\lambda|_\infty \subset \overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}|_\infty$.

To show I^λ is flat over C, it is sufficient to show that $(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C})^{T^\sigma}$ is flat over C. This is equivalent to showing $Z^\lambda = (\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C})^{T^\sigma}$. In particular, it suffices to show the fibers $Z^\lambda|_o$ and $Z^\lambda|_\infty$ are isomorphic to $(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}$. Since the fiber $Z^\lambda|_\infty$ at ∞ is similar to the fiber $Z^\lambda|_o$ at o, it suffices to show that $Z^\lambda|_o = (\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}$. Note that both of these are finite schemes, we can compare the dimensions of their structure sheaves as follows:

$$\begin{aligned}
\dim \mathscr{O}_{(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}} \geq \dim \mathscr{O}_{Z^\lambda|_o} &= \dim \mathscr{O}_{(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},p\neq o,\infty})^{T^\sigma}} \\
&= \dim \mathscr{O}_{(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},p\neq o,\infty})^{T}} \\
&= \dim H^0(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G}}, \mathcal{L}|_{p\neq o,\infty}) \\
&= \dim H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{L}) \\
&\geq \dim \mathscr{O}_{(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}})^{T^\sigma}},
\end{aligned}$$

where the first equality follows from the flatness of Z^λ over C, the third equality follows from Theorem 6.1.4, the fourth equality follows since $\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},C}$ is flat over C (cf. Theorem 4.3.10), and the last inequality follows from Proposition 6.1.6 . From this comparison, it follows that $\dim \mathscr{O}_{Z^\lambda|_o} = \dim \mathscr{O}_{(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},p\neq o})^{T^\sigma}}$. Hence, $\mathscr{O}_{Z^\lambda|_o} = \mathscr{O}_{(\overline{\mathrm{Gr}}^{\lambda}_{\mathcal{G},p\neq 0})^{T^\sigma}}$. This concludes the proof of the lemma. ◻

Proof of Theorem 6.1.5. By Lemma 6.1.8, if $H^0(\overline{\mathrm{Gr}}^{\lambda}_{G}, \mathrm{I}^\lambda \otimes \mathrm{L}) = 0$, then $H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{I}^{\bar\lambda} \otimes \mathscr{L}) = 0$. When G is not of type E_6, from [Zh1, Section 2.2] it is known that $H^0(\overline{\mathrm{Gr}}^{\lambda}_{G}, \mathrm{I}^\lambda \otimes \mathrm{L}) = 0$ for any $\lambda \in X_*(T)$. Hence, when $\mathscr{G}$ is not of type $E_6^{(2)}$, $H^0(\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}, \mathscr{I}^{\bar\lambda} \otimes \mathscr{L}) = 0$ for any $\bar\lambda \in X_*(T)^+_\sigma$. Therefore, the theorem follows from Lemma 6.1.7 and the long exact sequence (6.3). ◻

Remark 6.1.9. *In [Zh1], Zhu also proved Theorem 6.1.4 for many cases of affine Schubert varieties when G is type E. In particular, when G is type E_6 and for any λ which is a non-negative summation of fundamental coweights $\check\omega_1, \check\omega_2, \check\omega_4, \check\omega_5, \check\omega_6$ following the labelling in [Ka, Table Fin, p.53], Theorem 6.1.4 holds. Therefore, it follows that when $\bar\lambda \in X_*(T)_\sigma$ is a non-negative summation of $\bar{\check\omega}_1, \bar{\check\omega}_2, \bar{\check\omega}_6 \in X_*(T)^+_\sigma$, our Lemma 6.1.8 and Theorem 6.1.5 hold. Note that $\bar{\check\omega}_1 = \bar{\check\omega}_5$ and $\bar{\check\omega}_2 = \bar{\check\omega}_4$. To fully prove the case of E_6, by the method in [Zh1], it suffices to prove Theorem 6.1.4 when $\lambda = \check\omega_3$. Due to the complexity of this method for exceptional groups, this case is still open.*

As an application of Theorem 6.1.5, we get a geometric Frenkel-Kac isomorphism for twisted affine algebras.

Theorem 6.1.10. *For any absolutely special $\mathscr{G}$, the restriction map*

$$H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}) \to H^0(\mathtt{Gr}_{\mathscr{T}}, \mathscr{L}|_{\mathtt{Gr}_{\mathscr{T}}})$$

is an isomorphism, via the embedding $\mathtt{Gr}_{\mathscr{T}} \to \mathtt{Gr}_{\mathscr{G}}$.

Proof. By Theorem 6.1.2, it suffices to show that the restriction map $r : H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}) \to H^0(\mathtt{Gr}_{\mathscr{T}}, \mathscr{L}|_{(\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}})$ is an isomorphism. In view of (6.5) and (6.6) and as a consequence of Theorem 6.1.5, the restriction map r is an isomorphism when $\mathscr{G}$ is not $E_6^{(2)}$.

When $\mathscr{G}$ is of type $E_6^{(2)}$, the element $\bar{\check{\omega}}_1 \in X_*(T)^+_\sigma$ corresponds to the highest root of $H := (\check{G})^\vee$, see Chapter 4. Thus, for any $\bar{\lambda} \in X_*(T)_\sigma$, there exists $k \in \mathbb{N}$ such that $\bar{\lambda} \leq k\bar{\check{\omega}}_1$. It follows that

$$H^0(\mathtt{Gr}_{\mathscr{G}}, \mathscr{L}) = \varprojlim_k H^0(\overline{\mathtt{Gr}}^{k\bar{\check{\omega}}_1}_{\mathscr{G}}, \mathscr{L}|_{\overline{\mathtt{Gr}}^{k\bar{\check{\omega}}_1}_{\mathscr{G}}}),$$

and

$$H^0((\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\mathtt{Gr}_{\mathscr{G}})^{T^\sigma}}) = \varprojlim_k H^0((\overline{\mathtt{Gr}}^{k\bar{\check{\omega}}_1}_{\mathscr{G}})^{T^\sigma}, \mathscr{L}|_{(\overline{\mathtt{Gr}}^{k\bar{\check{\omega}}_1}_{\mathscr{G}})^{T^\sigma}}).$$

Now, by Remark 6.1.9, we see that the restriction map r is also an isomorphism when $\mathscr{G}$ is $E_6^{(2)}$.

□

6.1.2 Application: Smooth locus of twisted affine Schubert varieties

We now wish to investigate the smooth locus of the Schubert variety $\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$.

Theorem 6.1.11. *Assume that $\mathscr{G}$ is of type $A^{(2)}_{2\ell-1}, D^{(2)}_{\ell+1}, D^{(3)}_4$. For any $\lambda \in X_*(T)^+_\sigma$, the smooth locus of $\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ is precisely the open Schubert cell $\mathtt{Gr}^{\bar{\lambda}}_{\mathscr{G}}$.*

Proof. For any $\bar{\mu} \in X_*(T)^+_\sigma$, if $e_{\bar{\mu}} = n^\mu e_0$ is a smooth point in $\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$, then by [Zh1, Lemma 2.3.3] $\dim \mathscr{O}_{(\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}})^{T^\sigma}, e_{\bar{\mu}}} = 1$.

By Theorem 5.1.11, we have $H^0(\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}}, \mathscr{L})^\vee \simeq D(1, \bar{\lambda})$, where $D(1, \bar{\lambda})$ is the Demazure module defined in Definition 5.1.10. Then by Theorem 6.1.5, we have

$$\dim D(1, \bar{\lambda})_{-\iota(\bar{\mu})} = \operatorname{length} \mathscr{O}_{(\overline{\mathtt{Gr}}^{\bar{\lambda}}_{\mathscr{G}})^{T^\sigma}, e_{\bar{\mu}}},$$

where $D(1,\bar\lambda)_{-\iota(\bar\mu)}$ is the $-\iota(\bar\mu)$-weight space in $D(1,\bar\lambda)$. We will prove that for any $\bar\mu \lneqq \bar\lambda$, $\dim D(1,\bar\lambda)_{-\iota(\bar\mu)} \geq 2$, which would imply that $e_{\bar\mu}$ is not a smooth point in $\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}$. From the surjectivity of (6.4), we have an embedding $D(1,\bar\mu) \hookrightarrow D(1,\bar\lambda)$. On the the other hand, $V(-\iota(\bar\lambda)) \hookrightarrow D(1,\bar\lambda)$, where $V(-\iota(\bar\lambda))$ is the irreducible representation of $\mathfrak{g}^\sigma$ of lowest weight $-\iota(\bar\lambda)$. In view of Lemma 4.2.6, Lemma 5.1.3 and Lemma 5.1.4, when G is not of type $A_{2\ell}$, the relation $\bar\mu \lneqq \bar\lambda$ implies that $\iota(\bar\mu) \lneqq \iota(\bar\lambda)$. Hence, $V(-\iota(\bar\lambda))_{-\iota(\bar\mu)} \neq 0$. Furthermore, as subspaces in $D(1,\bar\lambda)$,

$$D(1,\bar\mu) \cap V(-\iota(\bar\lambda)) = 0.$$

It follows that $\dim D(1,\bar\lambda)_{-\iota(\bar\mu)} \geq 2$. This concludes the proof of the theorem. □

From the proof of the above theorem, we see that our technique does not fully apply to the Schubert variety $\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}$ when $\mathscr{G}$ is of type $A^{(2)}_{2\ell}$. Also, since the duality theorem is not fully established yet for $E^{(2)}_6$, we can only get some partial result in this case.

Recall the group $H = (\check{G})^\tau$ mentioned in Chapter 4. By the ramified geometric Satake, $(X_*(T)_\sigma, X_*(T)^+_\sigma, \gamma_j, j \in I_\sigma)$ can be regarded as the weight lattice, the set of dominant weights, and simple roots of H. When $(G,m) = (E_6, 2)$, H is F_4; when $(G,m) = (A_{2\ell}, 4)$, H is B_ℓ of adjoint type. We follow the labelling of Dynkin diagram in [Ka, Table Fin, p 53]. Let $\{ \varpi_j \mid j \in I_\sigma \}$ be the set of fundamental dominant weights of H.

Theorem 6.1.12. *1. Let $\mathscr{G}$ be of type $E^{(2)}_6$. If $\bar\lambda$ is a non-negative linear combination of the fundmental weights $\varpi_1, \varpi_2, \varpi_4$ of H, then the smooth locus of $\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}$ is the open cell $\mathrm{Gr}^{\bar\lambda}_{\mathscr{G}}$.*

2. Let $\mathscr{G}$ be of type $A^{(2)}_{2\ell}$. For any $\bar\lambda, \bar\mu \in X_*(T)^+_\sigma$ with $\bar\mu \lneqq \bar\lambda$, the Schubert cell $\mathrm{Gr}^{\bar\mu}_{\mathscr{G}}$ is contained in the singular locus of $\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}$, except if $\bar\mu \prec \bar\lambda$ is a cover relation such that the simple short root γ_ℓ appears in $\bar\lambda - \bar\mu$.

Proof. Part (1) of the theorem follows from Remark 5.1.5 and Remark 6.1.9, where under the map $\eta : I \to I_\sigma$, $\varpi_1 = \overline{\check\omega}_6, \varpi_3 = \overline{\check\omega}_2, \varpi_4 = \overline{\check\omega}_1$.

For part (2) of the theorem, we will prove this part by several steps. Let c_ℓ be the coefficient of γ_ℓ in $\bar\lambda - \bar\mu$.

Step 1. Observe from the proof of Theorem 6.1.11, when the coefficient c_ℓ is even, we have $\dim D(1,\bar\lambda)_{-\iota(\bar\mu)} \geq 2$. Thus, $e_{\bar\mu}$ is singular in $\overline{\mathrm{Gr}}^{\bar\lambda}_{\mathscr{G}}$.

Step 2. Assume that the coefficient $c_\ell > 1$. There exists sequence of dominant elements in $X_*(T)^+_\sigma$,

$$\bar\mu = \bar\lambda_k \prec \bar\lambda_{k-1} \prec \cdots \prec \bar\lambda_1 \prec \bar\lambda_0 = \bar\lambda, \tag{6.10}$$

such that each $\prec$ is a cover relation. Then, by a theorem of Stembridge [St, Theorem 2.8], for each i, $\bar{\lambda}_i - \bar{\lambda}_{i+1}$ is a positive root of H, for any $0 \leq i \leq k-1$, and the coefficient of γ_ℓ in each $\bar{\lambda}_i - \bar{\lambda}_{i+1}$ is either 0 or 1. Let j be the least integer such that the coefficient of γ_ℓ in $\bar{\lambda}_{j-1} - \bar{\lambda}_j$ is 1. Such j exists, since $c_\ell \neq 1$. Then the coefficient of γ_ℓ in $\bar{\lambda}_j - \bar{\mu}$ is even. By Step 1, we have $\dim D(1, \bar{\lambda}_j)_{-\iota(\bar{\mu})} \geq 2$. On the other hand, we have the inclusion $D(1, \bar{\lambda}_j) \subset D(1, \bar{\lambda})$. It follows that $\dim D(1, \bar{\lambda})_{-\iota(\bar{\mu})} \geq 2$. Hence, the variety $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ is singular at the point $e_{\bar{\mu}}$.

Step 3. We now assume that the coefficient $c_\ell = 1$. By assumption, $\bar{\mu} \prec \bar{\lambda}$ is not a cover relation. Then, in the sequence of cover relations in (6.10), either the coefficient of γ_ℓ in $\bar{\lambda}_{k-1} - \bar{\lambda}_k$ is 0, or the coefficient of γ_ℓ in $\bar{\lambda}_0 - \bar{\lambda}_1$ is 0. If the coefficient of γ_ℓ in $\bar{\lambda}_{k-1} - \bar{\lambda}_k$ is 0, by Step 1 $\dim D(1, \bar{\lambda}_{k-1})_{-\iota(\bar{\mu})} \geq 2$, implying that $\dim D(1, \bar{\lambda})_{-\iota(\bar{\mu})} \geq 2$. Hence $e_{\bar{\mu}}$ is singular in $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$. If the coefficient of γ_ℓ in $\bar{\lambda}_0 - \bar{\lambda}_1$ is 0, then by Step 1 again, $e_{\bar{\lambda}_1}$ is a singular point in $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$. Since the singular locus of $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ is closed, the point $e_{\bar{\mu}}$ is also singular. □

Let $\mathscr{G}$ be of type $A^{(2)}_{2\ell}$. We now explicitly describe the cover relation $\bar{\mu} \prec \bar{\lambda}$ such that γ_ℓ appears in $\bar{\lambda} - \bar{\mu}$. Note that $X_*(T)_\sigma$ is a root lattice of $H \simeq SO_{2n+1}$. In fact, $X_*(T)_\sigma$ is spanned by $\varpi_1, \varpi_2, \cdots, \varpi_{\ell-1}, 2\varpi_\ell$. Reading more carefully from [St, Theorem 2.8], we can see that, $\bar{\mu} \prec \bar{\lambda}$ is a cover relation such that γ_ℓ appears in $\bar{\lambda} - \bar{\mu}$, if and only if one of the followings holds:

1. $\bar{\lambda} - \bar{\mu} = \gamma_\ell$.

2. $\bar{\lambda} - \bar{\mu} = \sum_{j=i}^{\ell} \gamma_j$ and $\mu = \sum_{k=1}^{i-1} a_k \varpi_k$ with all $a_k > 0$, for some $1 \leq i \leq \ell - 1$.

Remark 6.1.13. *1. When $\mathscr{G}$ is a special but not absolutely special parahoric group scheme of type $A^{(2)}_{2\ell}$, i.e. when σ is the diagram automorphism and G is of type $A_{2\ell}$, there is a counter-example that $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ is smooth but $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}} \neq \mathrm{Gr}^{\bar{\lambda}}_{\mathscr{G}}$, cf. [HR, Section 5.1] or [Zh3, page 3]. This phenomenon is somewhat related to Remark 5.1.2.*

2. One can define the affine Grassmannian $\mathrm{Gr}_{\mathscr{G}}$ and twisted affine Schubert varieties $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ of the absolutely special parahoric group scheme $\mathscr{G}$ with the base field k *of characteristic p. In [HR, Section 6], when $p \neq r$, Haines and Richarz reduced the question of the smooth locus of the $\overline{\mathrm{Gr}}^{\bar{\lambda}}_{\mathscr{G}}$ over characteristic p to characteristic zero case. In fact, by the work of Lourenço [Lo], one may construct a global twisted affine Schubert variety over $\mathbb{Z}$ so that the base change to the field* k *of characteristic p (including $p = r$) is the given twisted affine Schubert variety defined over* k. *Then the argument of Haines and Richarz* *can still apply to the case of characteristic* $p = r$. *Therefore, Theorem 6.1.11 also holds for any base field* k.

CHAPTER 7

Partial Results on E_6

In this chapter we make partial progress towards completing Zhu's localization result for E_6.

7.1 Partial Results on E6

We return to the untwisted case for type E_6. In this chapter, in contrast to the previous chapters in which we used the notation of Kac (especially for affine and twisted affine Dynkin diagrams) we now use Bourbaki notation.

In this notation, Zhu has proven that

$$H^0(\mathtt{Gr}_{E_6}^{\check{\omega}_i}, L) \simeq H^0((\mathtt{Gr}_{E_6}^{\check{\omega}_i})^T, L) \tag{7.1}$$

in all cases except $\check{\omega}_4$. Proving the restriction isomorphism in this case would prove the result for all coweights for type E_6. Moreover, if we could prove the localization theorem for E_6, we would be able to use our theorems from Chapter 6 to obtain results about the twisted affine Schubert varieties in this case, which is the last remaining twisted case.

Mostly we work on the dual side; namely affine Demazure modules. It is convenient to begin with a description of the $\mathfrak{g}$-module structure, along with the d-weight of each such irreducible module. The Demazure module $H^0(\mathtt{Gr}_{E_6}^{\check{\omega}_4}, L)^*$ decomposes into a direct sum of $\mathfrak{g}$ modules which may have different weights with respect to the derivation part of the Cartan $\mathbb{C}d$. We write $V_{\lambda,j}$ for the irreducible $\mathfrak{g}$-module of highest T-weight λ and with d-weight j.

We will write $V_{\lambda,j}(\mu)$ for the μ-weight space of this irreducible, where $\mu \in X_*(T)$.

Theorem 7.1.1. $H^0(\mathtt{Gr}_{E_6}^{\check{\omega}_4}, L)^* = V_0 \oplus V_{\omega_2,-1} \oplus V_{\omega_1+\omega_6,-2} \oplus V_{\omega_2,-2} \oplus V_{\omega_4,-3}$.

Proof. See [Kl]. To be pedantic we may say we are considering $\mathfrak{g} \oplus \mathbb{C}d$ modules. □

The problem is to show that each weight space appears with the correct dimension when we only examine the restriction to the T-fixed subscheme.

There are two theorems that we can use to reduce this problem.

Theorem 7.1.2. *The surjection* $H^0(\mathtt{Gr}_{E_6}^{\check{\omega}_4}, L) \to H^0(\mathtt{Gr}_{E_6}^{\check{\omega}_1+\check{\omega}_6}, L)$ *induces an injection on the dual spaces.*

Corollary 7.1.3. $V_0 \oplus V_{\omega_2,-1} \oplus V_{\omega_1+\omega_6,-2} \subset H^0((\mathtt{Gr}_{E_6}^{\check{\omega}_4})^T, L \otimes \mathscr{O}_{(\overline{\mathtt{Gr}}_{E_6}^{\check{\omega}_4})^T})^*$.

Proof. This follows from the theorem and the fact that the restriction isomorphism holds for the coweight $\check{\omega}_1 + \check{\omega}_6$, see Zhu. □

Note moreover that we need only consider dominant weight spaces of these representations, as the second space can be given a W-module structure such that the morphism

$$H^0(\mathtt{Gr}_{E_6}^{\check{\omega}_4}, L) \to H^0((\mathtt{Gr}_{E_6}^{\check{\omega}_4}), L)$$

is W-equivariant.

Lemma 7.1.4. *Assume that* $V_\mu(\lambda)$ *has a basis consisting of elements* $f_{\alpha_{i_1}} f_{\alpha_{i_2}} \dots f_{\alpha_{i_n}} w v_\mu$ *where* $f_{\alpha_{i_j}}$ *are the Chevalley generators corresponding to a Levi* M *for which the restriction isomorphism is known. Then* $H^0(\overline{\mathtt{Gr}}_G^{\iota(\mu)}, I^{\iota(\mu)}(1))^*(\lambda) = 0$.

Proof. We follow the proof in [Zh1]. Let $\{u_d^*\}$ be dual to the described basis, and take σ a weight vector in $H^0(\overline{\mathtt{Gr}}_G^{\iota(\mu)}, I^{\iota(\mu)}(1))^*(\lambda) = 0$. Then we can write $\sigma = \sum_d \sigma_d u_d^*$. Viewing $u_d^* \in H^0(G/P, \mathscr{O}(\mu))$ via the Plucker embedding, we see that u_d^* does not vanish on the open subspace $U_{-\alpha_{i_1}} \dots U_{-\alpha_{i_n}} w_d$, whereas $u_d' *$ does vanish uniformly on this open set. By the assumption of the lemma, when we restrict σ to $U_{-\alpha_{i_1}} \dots U_{-\alpha_{i_n}} w_d$ it must vanish uniformly. Thus $\sigma_d = 0$ for all d, and $\sigma = 0$. □

Fact 7.1.5. *All Levi subquotients of* E_6 *are of type A or D, for which the restriction isomorphism is known, due to [Zh1].*

With these preparations complete, we may begin to examine individual dominant weight spaces in $V_{\omega_4,-3}$ and $V_{\omega_2,-2}$. Our examination of $V_{\omega_4,-3}$ rests almost entirely on Lemma 7.1.4, but our examination of $V_{\omega_2,-2}$ relies on facts about the "Heisenberg" raising operators which will be introduced later.

Note 7.1.6. *For ease of notation, going forward we will write* $D_{\omega_4}(\lambda)$ *for* $H^0(\mathtt{Gr}_{E_6}^{\check{\omega}_4}, L)^*(\lambda)$ *and* $D_{\omega_4}^T(\lambda)$ *for* $H^0((\mathtt{Gr}_{E_6}^{\check{\omega}_4})^T, L \otimes \mathscr{O}_{(\overline{\mathtt{Gr}}_{E_6}^{\check{\omega}_4})^T})^*(\lambda)$.

7.2 $V_{\omega_4,-3}$

By W-equivariance, we need only examine the dominant weight spaces. The dominant character of V_{ω_4} is $e^{\omega_4} + 4e^{\omega_1+\omega_6} + 15e^{\omega_2} + 45e^0$. We must show that $D^T_{\omega_4}$ has the same weight spaces with the same dimensions.

Said differently,

$$dimD_{\omega_4}(\omega_4 - 3\delta) = 1$$

$$dimD_{\omega_4}(\omega_1 + \omega_6 - 3\delta) = 4,$$

$$dimD_{\omega_4}(\omega_2 - 3\delta) = 15$$

and

$$dimD_{\omega_4}(0 - 3\delta) = 45.$$

Our task is to show that $dimD^T_{\omega_4}(\mu)$ is equal to $dimD_{\omega_4}(\mu)$ for all the μ listed above.

7.2.1 $V_{\omega_4,-3}(\omega_4)$

This is a one-dimensional weight space. Let $\sigma_{\check{\omega}_4}$ be the linear form on $\bigoplus_{\gamma\in\pi_1(G)} L(\Lambda + \iota\omega_{i_\gamma})$ which is not zero on $t^{\check{\omega}_4}v_{\Lambda_0}$ and zero on the other weight spaces. This form restricts to a form on the T-fixed subscheme supported at $t^{\check{\omega}_4}$. Thus $D^T_{\omega_4}(\omega_4 - 3\delta)$ is 1-dimensional.

7.2.2 $V_{\omega_4,-3}(\omega_1 + \omega_6)$

In terms of simple roots, we have $\omega_4 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_2$ and $\omega_1 + \omega_6 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6$. Thus the difference $\omega_4 - (\omega_1 + \omega_6) = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$; in other words, this difference is supported on the Levi of type D_4 with simple roots $\alpha_2, \alpha_3, \alpha_4, \alpha_5$.

Thus we have a basis of $V_{\omega_4}(\omega_1 + \omega_6)$ of the form $f_{\alpha_{i_1}} \dots f_{\alpha_{i_n}} v_{\omega_4}$ where the $f_{\alpha_{i_j}} \in L$ of type D_4. Thus the Lemma 7.1.4 applies, so $D^T_{\omega_4}(\omega_1 + \omega_6 - 3\delta)$ is 4-dimensional.

7.2.3 $V_{\omega_4,-3}(\omega_2)$

We will again use Lemma 7.1.4; thus we must provide a basis as in the assumption of the Lemma. This case is significantly more challenging and requires some brute force. The difference $\omega_4 - (\omega_1 + \omega_6) = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$.

We actually consider all expressions of the form $f_{\alpha_{i_1}} \dots f_{\alpha_{i_{10}}} \omega_4$ such that this vector is of weight ω_2; this provides a spanning set of vectors in $V_{\omega_4}(\omega_2)$ (with many relations!). It turns out that the vast majority of such expressions admit the following description:

$$\underbrace{f_{\alpha_{i_1}} \dots f_{\alpha_{i_k}}}_{\text{In a Levi}} \underbrace{f_{\alpha_{i_{k+1}}} \dots f_{\alpha_{i_{10}}} v_{\omega_4}}_{\text{Extremal weight vect.}} \tag{7.2}$$

Thus they are precisely of the form necessary for Lemma 7.1.4.

Definition 7.2.1. *We call vectors of the form* $\underbrace{f_{\alpha_{i_1}} \dots f_{\alpha_{i_k}}}_{\textit{In a Levi}} \underbrace{f_{\alpha_{i_{k+1}}} \dots f_{\alpha_{i_{10}}} v_{\omega_4}}_{\textit{Extremal weight vect.}}$ *Levi/extremal vectors.*

We provide the following diagram, which describes which lowering operators we may apply to a weight vector and whether or not the result is an extremal weight vector.

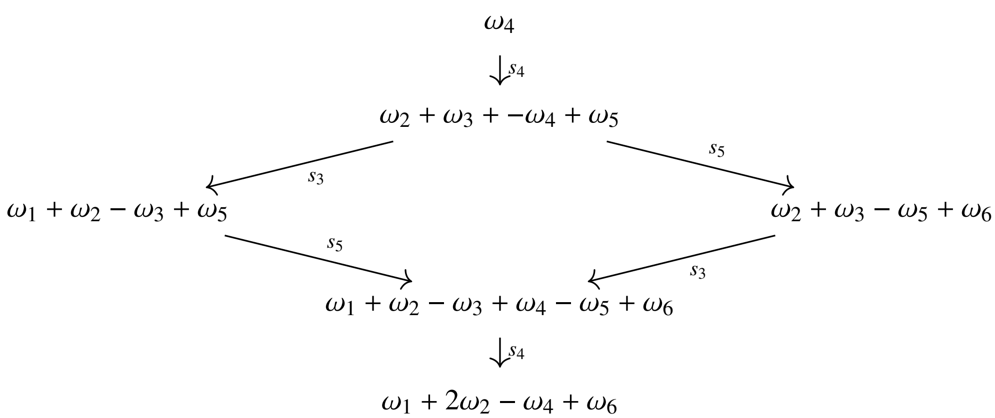

Figure 7.1: This is the poset of extremal weights μ such that $\omega_2 < \mu \leq \omega_4$ and such that the support of $\mu - \omega_2$ is not a proper subdiagram.

We have *not* included arrows corresponding to reflections s_{α_i} when the following two conditions are satisfied: the pairing $\langle \alpha_i, \mu \rangle = 1$ and $s_{\alpha_i}\mu - \omega_2$ is supported on a proper subdiagram; all of these vectors are of the type Levi/extremal, or in other words satisfy the assumptions of Lemma 7.1.4.

Thus the "worst possible" case, from the perspective of producing Levi/extremal vectors, has the first five lowering operators as follows: $f_{\alpha_2} f_{\alpha_4} f_{\alpha_5} f_{\alpha_3} f_{\alpha_4} v_{\omega_4}$. All other nontrivial applications of 5 lowering operators will result in a Levi/extremal vector.

Thus every element $f_{\alpha_{i_1}} \dots f_{\alpha_{i_{10}}} v_{\omega_4}$ is of the form

$$\underbrace{f_{\alpha_{i_1}} \dots f_{\alpha_{i_k}}}_{\text{In a Levi}} \underbrace{f_{\alpha_{i_{k+1}}} \dots f_{\alpha_{i_{10}}} v_{\omega_4}}_{\text{Extremal weight vect.}}$$

except those of which the first five lowering operators are precisely $f_{\alpha_2} f_{\alpha_4} f_{\alpha_5} f_{\alpha_3} f_{\alpha_4} v_{\omega_4}$; what remains are the lowering operators f_1, f_3, f_4, f_5, f_6. By the same logic, the next lowering operator must be f_4, since any other lowering operator would commute with f_4, returning us to the Levi/extremal weight situation.

Thus we are left with is considering the following element:

$$f_{\alpha_6} f_{\alpha_1} f_{\alpha_3} f_{\alpha_5} f_{\alpha_4} f_{\alpha_2} f_{\alpha_4} f_{\alpha_5} f_{\alpha_3} f_{\alpha_4} v_{\omega_4}.$$

After applying a commutation relation $f_{\alpha_4} f_{\alpha_2} = f_{\alpha_2} f_{\alpha_4} + [f_{\alpha_2}, f_{\alpha_4}]$, this is equal to

$$f_{\alpha_6} f_{\alpha_1} f_{\alpha_3} f_{\alpha_5} f_{\alpha_2} f_{\alpha_4} f_{\alpha_4} f_{\alpha_5} f_{\alpha_3} f_{\alpha_4} v_{\omega_4}$$

$$+ f_{\alpha_6} f_{\alpha_1} f_{\alpha_3} f_{\alpha_5} [f_{\alpha_4}, f_{\alpha_2}] f_{\alpha_4} f_{\alpha_5} f_{\alpha_3} f_{\alpha_4} v_{\omega_4}.$$

The first expression is 0. The second expression is Levi/extremal, since the weight of $f_{\alpha_4} f_{\alpha_5} f_{\alpha_3} f_{\alpha_4} v_{\omega_4}$ is $\omega_1 + 2\omega_2 - \omega_4 + \omega_6$, and the pairing $\langle \check{\alpha}_2 + \check{\alpha}_4, \omega_1 + 2\omega_2 - \omega_4 + \omega_6 \rangle = 1$, so

$$s_{\alpha_2+\alpha_4} f_{\alpha_4} f_{\alpha_5} f_{\alpha_3} f_{\alpha_4} v_{\omega_4} = [f_{\alpha_2}, f_{\alpha_4}] f_{\alpha_4} f_{\alpha_5} f_{\alpha_3} f_{\alpha_4} v_{\omega_4}. \tag{7.3}$$

Thus $dim D^T_{\omega_4}(\omega_2 - 3\delta) = 15$.

7.2.4 $V_{\omega_4,-3}(0)$

The 0 weight space of V_{ω_4} decomposes as a direct sum of two irreducible representations for the Weyl group, one 15 dimensional and the other 30 dimensional, see [Ree].

We can produce several Levi/extremal vectors; for instance, $\alpha_1 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$ is an extremal weight as seen by

$$\alpha_1 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6 = s_2 s_4 s_3 s_5 s_1 s_6 s_2 s_4 s_3 s_5 s_4 \omega_4. \tag{7.4}$$

Moreover if we branch V_{ω_4} along the Levi spanned by $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ (whose derived group is of type A_5), we find that the 0 weight space of the representation with highest weight $\alpha_1 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$ is nontrivial. Thus there must be at least one global section σ of weight 0 which vanishes globally, or in other words $D^T_{\omega_4}(0 - 3\delta)$ is non-empty. By the W-equivariance of this last module, we see that there are three options.

1. The chosen Levi/extremal weight vector lies entirely inside the representation of dimension 30

2. The chosen Levi/extremal weight vector lies entirely inside the representation of dimension 15

3. The chosen Levi/extremal weight vector consists of summands from the two irreducibles.

This is the current state of our knowledge; our methods are sufficient to show that $D^T_{\omega_4}(0-3\delta)$ is either 15 dimensional, 30 dimensional or 45 dimensional (recall that for the restriction isomorphism to hold, this space must be 45 dimensional).

7.3 $V_{\omega_2,-2}$

Recall that the subspace of D_{ω_4} with d-weight -2 decomposes into a direct sum of $\mathfrak{g}$-modules $V_{\omega_2} \oplus V_{\omega_1+\omega_6}$. Moreover, $V_{\omega_1+\omega_6,-2} \subset D^T_{\omega_4}$ by Theorem 7.1.2. Thus we must show that $V_{\omega_2,-2} \subset D^T_{\omega_4}$. The dominant character of V_{ω_2} is $e^{\omega_2} + 6e^0$; this is the adjoint representation of $\mathfrak{g}$.

Said slightly differently, we have

$$dimD_{\omega_4}(0-2\delta) = 26$$

and

$$dimD_{\omega_4}(\omega_2-2\delta) = 6.$$

Moreover we already know that

$$dimD^T_{\omega_4}(0-2\delta) \geq 20$$

and

$$dimD^T_{\omega_4}(\omega_2-2\delta) \geq 5,$$

both by Lemma 7.1.2.

7.3.1 Heisenberg Action

First we consider the "0-string" of the full basic representation $\mathscr{H}$; this is the direct sum $\mathscr{H}(0-n\delta)$ for all $n \geq 0$. The finite Weyl group of E_6 acts on each of these weight spaces, so $\mathscr{H}(0-n\delta)$ is a direct sum of irreducible representations of the Weyl group. We describe $\mathscr{H}(0)$, $\mathscr{H}(0-\delta)$, $\mathscr{H}(0-2\delta)$, $\mathscr{H}(0-3\delta)$.

By [Ka] Chapter 12, the q-dimension here is $1 + 6q + 27q^2 + 98q^3 + \ldots$. Indeed we can go further and decompose these spaces in terms of tensor powers of the Cartan subalgebra $\mathfrak{h}$. In particular, $\mathscr{H}(0)$ is trivial, $\mathscr{H}(0-\delta)$ is spanned by $\mathfrak{h}[t^{-1}]v_{\Lambda_0}$, $\mathscr{H}(0-2\delta)$ is spanned by $\mathfrak{h}[t^{-1}]\mathfrak{h}[t^{-1}]v_{\Lambda_0} \oplus \mathfrak{h}[t^{-2}v_{\Lambda_0}] \simeq S^2\mathfrak{h} \oplus S^1\mathfrak{h}$ and lastly $\mathscr{H}(0-3\delta)$ is spanned by $\mathfrak{h}[t^{-1}]\mathfrak{h}[t^{-1}]\mathfrak{h}[t^{-1}]v_{\Lambda_0} \oplus \mathfrak{h}[t^{-1}]\mathfrak{h}[t^{-2}]v_{\Lambda_0} \oplus \mathfrak{h}[t^{-3}]v_{\Lambda_0}$. This is isomorphic to $S^3\mathfrak{h} \oplus T^2\mathfrak{h} \oplus \mathfrak{h}$.

We now wish to further decompose these spaces into irreducible representations for the finite Weyl group.

$\mathscr{H}(0)$ and $\mathscr{H}(0-\delta) \simeq \mathfrak{h}$ are both already irreducible Weyl modules.

$\mathscr{H}(0-2\delta)$ decomposes as follows: $\mathfrak{h}[t^{-2}]v_{\Lambda_0}$ is irreducible just as above. The 21 dimensional space isomorphic to $S^2\mathfrak{h}$ is not irreducible; there is a 1-dimensional W-invariant subspace given by (the dual of) the Killing form $(,)$ restricted to $\mathfrak{h}$. Thus $\mathscr{H}(0-2\delta) \simeq \text{Triv} \oplus \text{Comp} \oplus \text{Adj}$ where Comp is an irreducible 20-dimensional W-module.

Lastly we decompose $\mathscr{H}(0-3\delta)$. The factor isomorphic to $\mathfrak{h}$ is irreducible. The factor isomorphic to $T^2\mathfrak{h}$ decomposes as usual into symmetric and exterior powers, generated by elements of the form $h_1[t^{-1}]h_2[t^{-2}] \pm h_2[t^{-1}]h_1[t^{-2}]$. Finally we consider the 56-dimensional subspace isomorphic to $S^3\mathfrak{h}$. We know that there is a 30-dimensional irreducible subspace since $dimV_{\omega_4}(0)$ is a direct sum of a 30 dimensional irreducible W-module and a 15 dimensional irreducible W-module, and $V_{\omega_4,3} \subset \mathscr{H}$ as $\mathfrak{g}$-modules. We are not sure how the remaining 26-dimensional space decomposes, but we have the following lemma:

Lemma 7.3.1. *The 26-dimensional complement to the 30 dimensional irreducible representation in $S^3\mathfrak{h}$ cannot contain a 15-dimensional irreducible W-module.*

Proof. Assume to the contrary that there is a 15-dimensional irreducible W-module. The dimensions of irreducible W modules are 1,6,15,20,24, etc, from the character table. Thus if it does decompose, it must decompose into a 15-dimensional module and then either a 6-dimensional irreducible and 5 1-dimensional irreducibles or a 15-dimensional and 11 1-dimensional irreducibles.

Both of these options are impossible for the following reasons. The only two one-dimensional representations of W are the trivial representation and the sign representation.

The trivial representation cannot appear in $S^3\mathfrak{h}$ since this would give a W-invariant degree 3 polynomial. Using the Chevalley isomorphism $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$, we see that this would imply the existence of a degree 3 invariant polynomial on $\mathfrak{g}$ (after dualizing). This is impossible, because the possible degrees of invariant polynomials of E_6 are 2,5,6,8,9,12; this list is the set of exponents +1 which can be found in [Bou].

The other option is that all of these 1-dimensional irreducibles are the sign representation. However,

$V_{\Lambda_0}(-3\delta) = V_{\omega_4} \oplus k_1 V_{\omega_1+\omega_6} \oplus k_2 V_{\omega_2} \oplus k_3 V_0$; in other words no other representations can appear in the $\mathfrak{g}$-module decomposition of $\mathscr{H}(-3\delta)$, since $-3\delta + \omega_4$ is an extremal weight of V_{Λ_0}. None of the 0-weight spaces of these representations has any factor isomorphic to the sign representation of W. Thus the sign representation may not appear either, and the decomposition with an irreducible factor of dimension 15 is impossible. □

Corollary 7.3.2. *The 15-dimensional irreducible W-module with weight* $0 - 3\delta$ *is isomorphic to* $\bigwedge^2 \mathfrak{h}$ *and is generated by* $(h_1[t^{-1}]h_2[t^{-2}] - h_2[t^{-1}]h_1[t^{-2}])v_{\Lambda_0}$.

7.3.2 $V_{\omega_2,-2}(0)$

We return to the Demazure module D_{ω_4}. The weight space $D_{\omega_4}(0 - 2\delta)$ decomposes into the direct sum $V_{\omega_1+\omega_6}(0) \oplus V_{\omega_2}(0)$. The factor $V_{\omega_1+\omega_6}(0)$ already appears in $D^T_{\omega_4}$ by Theorem 7.1.2. The remaining 6-dimensional irreducible W-module must be the one spanned by $\mathfrak{h}[t^{-2}]v_{\Lambda_0}$. The missing 6-dimensional space $\mathfrak{h}[t^{-2}]$ is in the image of the raising operators $\mathfrak{h}[t]$ applied to $h_1[t^{-1}]h_2[t^{-2}] - h_2[t^{-1}]h_1[t^{-2}]$. Thus we see that showing that proving that $dim D_{\omega_4}(0 - 3\delta) = dim D^T_{\omega_4}(0 - 3\delta)$ implies that $D_{\omega_4}(0 - 2\delta) = D^T_{\omega_4}(0 - 2\delta)$ by Corollary 7.3.2.

7.3.3 $V_{\omega_2,-2}(\omega_2)$

We must find a basis vector $v \in V_{\omega_2,-2}(\omega_2)$. By W-equivariance, it is sufficient to find the lowest weight vector $v \in V_{\omega_2-2\delta}(w_0\omega_2)$ which is in the image of the raising operators $e_0, e_1 \dots e_5, e_6$. For convenience we translate to the dominant chamber; thus we examine vectors of the form $f_{i_1} \dots f_{i_k} v_{\omega_4-3\delta}$ of weight $\omega_2 - 2\delta$ and is also annihilated by $e_1, \dots e_6$ (this is the condition that we have a highest-weight vector). Moreover we may consider vectors precisely of the following form; exactly one f_{i_j} is $e_\theta[t]$, and the rest are lowering operators $f_1, \dots f_6$.

We first claim that f_{i_1} is the only possible term which can be $e_\theta[t]$. If $f_{i_j} = e_\theta[t]$ and $j \neq 1$, then $f_{i_j} f_{i_{j+1}} \dots f_{i_k} v_{\omega_4-3\delta}$ has T-weight μ such that $\theta < \mu$ in the root lattice. This implies either that $f_{i_j} f_{i_{j+1}} \dots f_{i_k} v_{\omega_4-3\delta} = 0$ or that $f_{i_j} f_{i_{j+1}} \dots f_{i_k} v_{\omega_4-3\delta} \in V_{\omega_1+\omega_6,-2}$, in which case $f_{i_1} \dots f_{i_k} v_{\omega_4-3\delta} \in V_{\omega_1+\omega_6,-2}$ by irreducibility.

Thus to find a basis vector in the desired weight space we need only consider elements of the form $e_\theta[t] f_{i_2} \dots f_{i_k} v_{\omega_4-3\delta}$, and such that the T-weight of $f_{i_2} \dots f_{i_k} v_{\omega_4-3\delta}$ is 0.

We now examine commutation relations here.

For instance, we have

$$e_\theta[t]f_{i_2}\dots f_{i_k}v_{\omega_4-3\delta} = f_{i_2}e_\theta[t]f_{i_3}\dots f_{i_k}v_{\omega_4-3\delta}$$
$$ke_{\theta-\alpha_{i_2}}[t]f_{i_3}\dots f_{i_k}v_{\omega_4-3\delta}.$$

For the same reasons as above, the first term is either 0 or lies in $V_{\omega_1+\omega_6,-2}$, and so cannot contribute to a basis of $V_{\omega_2,-2}(\omega_2)$. Thus we need only consider elements of the form $e_{\theta-\alpha_{i_2}}[t]f_{i_3}\dots f_{i_k}v_{\omega_4-3\delta}$. We may repeat this procedure, obtaining more generally

$$e_{\theta-\beta}[t]f_{i_l}\dots f_{i_k}v_{\omega_4-3\delta} = f_{i_l}e_{\theta-\beta}[t]f_{i_{l+1}}\dots f_{i_k}v_{\omega_4-3\delta}$$
$$[e_{\theta-\beta}[t], f_{i_l}]f_{i_{l+1}}\dots f_{i_k}v_{\omega_4-3\delta}.$$

The first term cannot contribute to the $V_{\omega_2,-2}(\omega_2)$ weight space since it is either 0 or in $V_{\omega_1+\omega_6,-2}$. The second term $[e_{\theta-\beta}[t], f_{i_l}]f_{i_{l+1}}\dots f_{i_k}v_{\omega_4-3\delta}$ is either equal to $ke_{\theta-\beta-\alpha_{i_l}}[t]f_{i_{l+1}}\dots f_{i_k}v_{\omega_4-3\delta}$ if $\theta-\beta-\alpha_{i_l}$ is a root of Φ_{E_6} and 0 otherwise.

We repeat this algorithm until we have a spanning set of vectors of $V_{\omega_2,-2}(\omega_2)$ of the form $h[t]f_{i_m}\dots f_{i_k}v_{\omega_4-3\delta}$, where $h \in \mathfrak{h}$. Thus $f_{i_m}\dots f_{i_k}v_{\omega_4-3\delta}$ has $\mathfrak{h}\oplus\mathbb{C}d$-weight $\omega_2 - 3\delta$ or is the 0 vector. However, by our examination of the $V_{\omega_4,-3}(\omega_2)$ weight space, we know that this entire weight space is in $D^T_{\omega_4}$. Since $D^T_{\omega_4}$ is $\mathfrak{h}[t]$-stable, we find that $V_{\omega_2,-2}(\omega_2) \subset D^T_{\omega_4}$, or in other words $D^T_{\omega_4}(\omega_2 - 2\delta)$ is 6-dimensional (5 dimensions coming from $V_{\omega_1+\omega_6,-2}(\omega_2)$ and the remaining dimension being provided by $V_{\omega_2,-2}(\omega_2)$). This concludes the case of this weight space.

7.3.4 Conclusion

To finish the restriction isomorphism for E_6, we needed to show the following dimensions:

$$dim D^T_{\omega_4}(\omega_4 - 3\delta) = 1$$

$$dim D^T_{\omega_4}(\omega_1 + \omega_6 - 3\delta) = 4,$$

$$dimD^T_{\omega_4}(\omega_2 - 3\delta) = 15$$

$$dimD^T_{\omega_4}(0 - 3\delta) = 45$$

$$dimD^T_{\omega_4}(0 - 2\delta) = 26$$

(with a 20-dimensional subspace already accounted for by $V_{\omega_1+\omega_6,-2}(0)$) and

$$dimD^T_{\omega_4}(\omega_2 - 2\delta) = 6$$

(with a 5-dimensional subspace already accounted for by $V_{\omega_1+\omega_6,-2}(\omega_2)$).

We proved the first, second, third, and sixth of these dimensions directly and showed that the 5th statement would follow from the 4th statement. Thus to conclude this proof, one must show that $dimD^T_{\omega_4}(0 - 3\delta) = 45$. As mentioned above, by W-equivariance and the existence of at least one Levi/extremal vector, we know that this dimension must either be 15, 30 or 45. Hopefully this task can be finished at some future point in time.

REFERENCES

[BL] A. Beauville and Y. Laszlo. Conformal blocks and generalized theta functions. Communications in Mathematical Physics, 164(2) 385-419 1994. https://doi.org/cmp/1104270837

[BR] P. Baumann and S. Riche, Notes on the geometric Satake equivalence, https://arxiv.org/abs/1703.07288.

[BD] A. Beilinson and V. Drinfeld, Quantization of Hitchin?s integrable system and Hecke eigensheaves, available at www.math.uchicago.edu/mitya/langlands.

[BH] M. Besson and J. Hong, A combinatorial study of affine Schubert varieties in affine Grassmannian. arXiv:1906.09341. To appear in Transformation group.

[BT] D. Bernard and J. Thierry-Mieg. Level one representations of the simple affine Kac-Moody algebras in their homogeneous gradations. Comm. Math. Phys. 111 (1987), no. 2, 181-246.

[Bou] Bourbaki, N., Groupes et Algèbres de Lie, Chapitres 4, 5, et 6, Masson (1981).

[CIK] V. Chari, B. Ion and D. Kus, Weyl modules for the hyperspecial current algebra. Int. Math. Res. Not. IMRN 2015, no. 15, 6470-6515.

[DS] Drinfeld, V. and C. Simpson. *B*-structures on *G*-bundles and local triviality. Mathematical Research Letters 2 (1995): 823-829.

[EM] S. Evens and I. Mirković, Characteristic cycles for the loop Grassmannian and nilpotent orbits. Duke Math. J. 97 (1999), no. 1, 109-126.

[Fa] Faltings, G. Algebraic loop groups and moduli spaces of bundles. J. Eur. Math. Soc. 5, 41?68 (2003). https://doi.org/10.1007/s10097-002-0045-x

[FK] G. Fourier and D. Kus, Demazure modules and Weyl modules: the twisted current case. Trans. Amer. Math. Soc. 365 (2013), no. 11, 6037-6064.

[FL] G. Fourier and P. Littelmann, Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions. Adv. Math. 211 (2007), no. 2, 566-593.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman. Vertex operator algebras and the Monster. Pure and Applied Mathematics, 134. Academic Press, Inc., Boston, MA, 1988.

[Ha] T. Haines, Dualities for root systems with automorphisms and applications to non-split groups. Represent. Theory 22 (2018), 1-26.

[He] J. Heinloth, Uniformization of G-bundles. Math. Annalen, 347(3): 499-528, 2010.

[HK] J. Hong and S. Kumar, Conformal blocks for Galois covers of algebraic curves. arXiv:1807.00118.

[HR] T. J. Haines and T. Richarz, Smoothness of Schubert varieties in twisted affine Grassmannians. arXiv:1809.08464. To appear in Duke Mathematical Journal.

[HR2] T. J. Haines and T. Richarz, The test function conjecture for parahoric local models. arXiv:1801.07094. To appear in JAMS.

[Ka] V. Kac, Infinite-dimensional Lie algebras. Third edition. Cambridge University Press, Cambridge, 1990.

[Ku] S. Kumar, Kac-Moody Groups, their Flag Varieties and Representation Theory. Progress in Mathematics, 204. Birkhäuser Boston, Inc., Boston, MA, 2002.

[Kl] Michael Kleber, Combinatorial structure of finite-dimensional representations of Yangians: the simply-laced case, International Mathematics Research Notices, Volume 1997, Issue 4, 1997, Pages 187?201, https://doi.org/10.1155/S1073792897000135

[KV] D. Kus and R. Venkatesh, Twisted demazure modules, fusion product decomposition and twisted Q-systems. Represent. Theory 20 (2016), 94-127.

[Lo] J. N. P. Lourenço, Grassmanniennes affines tordues sur les entiers. arXiv:1912.11918.

[LS] Y. Laszlo and C. Sorger. The line bundles on the moduli of parabolic G-bundles over curves and their sections. Ann. Sci. École Norm. Sup. (4) 30 (1997), no. 4, 499-525.

[MOV] A. Malkin, V. Ostrik and M. Vybornov, The minimal degeneration singularities in the affine Grassmannians, Duke Math. J. 126 (2005), 233-249.

[MV] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. Ann. of Math. (2) 166 (2007), no. 1, 95-143.

[PR] G. Pappas and M. Rapoport. Twisted loop groups and their flag varieties, Adv.Math.219,118-198 (2008).

[Ree] M. Reeder, Zero weight spaces and the Springer correspondence, Indag. Math.9(1998), 431?441.

[Ri1] T. Richarz, Schubert varieties in twisted affine flag varieties and local models. J. Algebra 375 (2013), 121-147.

[Ri2] T. Richarz, Local models and Schubert varieteis in twited affine Grassmannians. Diploma, Bonn university.

[So] C. Sorger, On moduli of G-bundles of a curve for exceptional G. Ann. Sci. Éc. Norm. Supér., IV. Sŕ. 32, 127-133 (1999).

[St] J. Stembridge, The Partial Order of Dominant Weights. Advances in Mathematics, Volume 136, Issue 2, June 1998. Pages 340-364.

[Tits] Tits, J. et al. ?Reductive groups over local fields.? (1979).

[Zh1] X. Zhu, Affine Demazure modules and T-fixed point subschemes in the affine Grassmannian, Advances in Mathematics, Volume 221, Issue 2, 2009, Pages 570-600.

[Zh2] X. Zhu, On the coherence conjecture of Pappas and Rapoport. Ann. of Math. (2) 180 (2014), no. 1, 1-85.

[Zh3] X. Zhu, The geometric Satake correspondence for ramified groups. Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 2, 409-451.

[Zh4] X. Zhu, An introduction to affine Grassmannians and the geometric Satake equiv- alence. Geometry of moduli spaces and representation theory, 59-154, IAS/Park City Math. Ser., 24, Amer. Math. Soc., Providence, RI, 2017.