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Observer Based Restricted Structure Generalized Predictive Control for quasi-LPV Nonlinear Systems

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Abstract: An observer based Restricted Structure Generalized Predictive Control (RS-GPC) algorithm is proposed. The novel feature is to assume the state-observer within the feedback loop is of reduced order. The aim is to inherit the natural robustness of low-order controllers and to provide a solution that may be easily simplified for real-time implementation. The nonlinear discrete-time, multivariable plant model is represented by a state-space system that may be in Linear Parameter Varying or State-Dependent forms. The controller gains are computed to minimize the type of cost-function that is found in traditional model predictive control but with some additional terms that enable gain magnitudes and the rate of change of control gains to be minimized. The cost-function also includes dynamically weighted tracking-error and control signal costing terms. The optimal controller includes a reduced order observer and a time-varying control gain matrix within the loop and background processing for the gain computations. Hard constraints may be imposed on the gain and rate of change of gain and on the control and output signals.

Keywords: Observer, predictive, restricted structure, optimal, Linear Parameter Varying, state-dependent

1 INTRODUCTION

There have been many attempts to derive "low-order" restricted structure optimal controllers using direct optimization of parameters, such as the early work of Vladimir Zakian (1973) and Lucien Polak (1982) and colleagues, and the more recent work of Hast (2015) and Grimble (2004). The Quantitative Feedback Theory (QFT) design method provides a low-order robust control solution (Houpis et al., 2006). Richalet (1993) also developed an approach to predictive control that attempts to simplify the solution. For applications, Sato (2010) used a classical PID controller structure with a predictive approach for weigh feeder control. Eielsen et al., (2013), compared five fixed structure low-order controllers for Nano positioning systems. However, most advanced controllers used in applications have a Kalman filtering separation structure; optimal or suboptimal.

Observer based control structures like the linear quadratic Gaussian problem solution where an optimal observer or *Kalman filter* is involved are very popular. The structure is also used in many forms of nonlinear control law, even when the "separation principle" does not apply. There are many benefits of an observer-based solution, such as the ability to run the estimator without closing the control loop during commissioning. However, there is a disadvantage with an optimal control approach. That is, the order of the controller to be implemented within the feedback loop is often high. It is equal to the number of states in the combined plant model, disturbances, reference model and any cost-function dynamic weightings.

The approach proposed here is to use a reduced-order observer or filter within the feedback loop that provides the state estimates $\hat{x}_e(t)$. This observer can be based on a reduced-order model of the plant, obtained by system identification or by using model order reduction. The state-estimate feedback gain is found to minimise a predictive control cost-index on-line. The low-order observer or filter is assumed to have a form that is pre-specified. The optimal low-order "observer based" feedback controller that is obtained and implemented within the feedback loop is therefore of a *Restricted Structure* (*RS*). It relegates the optimization computations to background processing.

Model Predictive Control (*MPC*) and so-called *Generalized Predictive Control* (*GPC*), is very successful in industry in its various model-based algorithm forms. The particular form of *MPC* algorithm is referred to as *Restricted Structure Generalized Predictive Control* (*RS-GPC*). The control is parameterized to have a state-estimate solution $u(t) = k_c(t)\hat{x}_e(t)$ where $\hat{x}_e(t)$ denotes the estimate of states from the low-order observer. It involves the online calculation of the gains $k_c(t)$, rather than the computation of the vector of future controls in "*implicit MPC.*" Hard and soft constraints may be applied to the state-estimate feedback-control gain terms, and to the rate of change of gains. There are options to simplify real-time implementation, such as storing the controller gains at different operating points and then using a scheduling algorithm as in "*explicit MPC.*"

Roadmap: The *qLPV* state-space model for the system is described in § 2 and the *RS* controller structure is introduced in § 3. The cost-function and the solution of the *RS-GPC* problem is considered in § 4 and an application is presented in § 5.

2 SYSTEM DESCRIPTION

The *discrete-time multivariable* plant model is represented by a *quasi Linear Parameter Varying* (*qLPV*), state-space model that can be open loop unstable. The system in Fig. 1 includes the plant, reference, noise and disturbance signals. The *qLPV* state-space model of the plant is assumed to be *pointwise*

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stabilizable and *detectable* and can be represented by an operator:

$$\left(\mathcal{W}_0 u_0\right)(t) = z^{-k}\left(\mathcal{W}_{0k} u_0\right)(t) \tag{1}$$

where the term $z^{-k}I$ denotes a diagonal matrix of delay elements with $k > 0$. The disturbance signal is a *qLPV* model driven by zero-mean white noise and a deterministic output disturbance component denoted $d(t)$. The reference $r(t)$ is deterministic, and the deterministic signals are known in the prediction horizon. The white measurement noise $v(t)$ has a constant covariance matrix $R_f = R_f^T \geq 0$ and the zero-mean white noise source $\xi(t)$ has an identity covariance matrix. Measurement noise $v(t)$ and driving noise $\xi(t)$ are also assumed zero-mean, independent, white noise.

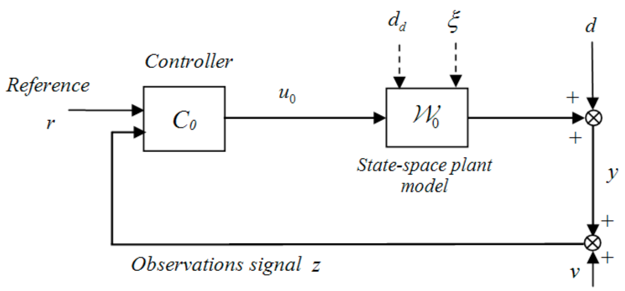

Fig. 1: Plant Model and RS-GPC Controller

2.1 State-Space qLPV/State-Dependent Plant

The *Linear Parameter Varying* (*LPV*), *State-Dependent* or *quasi-LPV* state-space plant and disturbance model is shown in Fig. 2. This model can be augmented to include cost-function output or error weighting terms. The *weighted error* can include any stable operator of the form as $e_p(t) = \mathcal{P}_c(z^{-1})e(t)$.

Augmented state model:

$$x(t+1) = \mathcal{A}_t x(t) + \mathcal{B}_t u_0(t-k) + \mathcal{D}_t \xi(t) + d_d(t) \tag{2}$$

Measured output: $y(t) = d(t) + \mathcal{C}_t x(t) + \mathcal{E}_t u_0(t-k)$ (3)

Observed output: $z(t) = d(t) + \mathcal{C}_t x(t) + \mathcal{E}_t u_0(t-k) + v(t)$ (4)

The through term matrix $\mathcal{E}_t$ is assumed full rank if the delay totals *k*-steps. The inferred output or weighted error signal:

$$e_p(t) = d_p(t) + \mathcal{C}_{pt} x(t) + \mathcal{E}_{pt} u_0(t-k) \tag{5}$$

where $\mathcal{A}_t, \mathcal{B}_t, \mathcal{C}_t, \mathcal{D}_t, \mathcal{E}_t, \mathcal{C}_{pt}, \mathcal{E}_{pt}$ are *state-dependent* or *qLPV* matrices. The known disturbance inputs at inputs and outputs are denoted $d(t)$, $d_d(t)$, $d_p(t)$ and signal $e_0(t) = r(t) - z(t)$.

The delay-free plant $\mathcal{W}_{0k} = \mathcal{E}_t + \mathcal{C}_t \Phi_t B_t$, where $\Phi_t = \left(zI - \mathcal{A}_t\right)^{-1}$

2.2 State-Space qLPV Prediction Model

Predicted states depend upon future inputs and current state. The future values of the states and outputs, at time *t*, may be obtained by repeated use of (2) to obtain the future state:

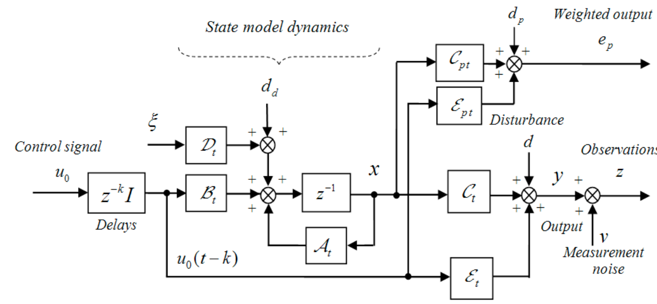

Fig. 2: Plant Model in qLPV State-Space Form

$$x(t+i) = d_{dd}(t+i-1) + \mathcal{A}_{t+i-1}\mathcal{A}_{t+i-2}\ldots\mathcal{A}_t x(t)$$
$$+\sum_{j=1}^{i}\mathcal{A}_{t+i-1}\mathcal{A}_{t+i-2}\ldots\mathcal{A}_{t+j}\left(\mathcal{B}_{t+j-1}u_0(t+j-1-k) + \mathcal{D}_{t+j-1}\xi(t+j-1)\right) \tag{6}$$

where the known disturbance term $d_{dd}(t-1) = 0$ for $i = 0$:

$$d_{dd}(t+i-1) = \sum_{j=1}^{i}\mathcal{A}_{t+i-1}\mathcal{A}_{t+i-2}\ldots\mathcal{A}_{t+j}d_d(t+j-1) \; for \;\; i > 0 \tag{7}$$

and if $j = i$ term $\mathcal{A}_{t+i-1}\mathcal{A}_{t+i-2}\ldots\mathcal{A}_{t+j}$ is defined as the identity. The future states may be obtained from (6) as:

$$x(t+i+k) = d_{dd}(t+i+k-1) + \mathcal{A}_{t+i+k-1}\mathcal{A}_{t+i+k-2}\ldots\mathcal{A}_{t+k}x(t+k)$$
$$+\sum_{j=1}^{i}\mathcal{A}_{t+i+k-1}\ldots\mathcal{A}_{t+j+k}\left(\mathcal{B}_{t+j+k-1}u_0(t+j-1) + \mathcal{D}_{t+j+k-1}\xi(t+j+k-1)\right) \tag{8}$$

Collecting the deterministic disturbance signal terms together:

$$d_{pd}(t+i+k) = d_p(t+i+k) + \mathcal{C}_{pt+i+k}\, d_{dd}(t+i+k-1) \tag{9}$$

The *weighted error* $e_p(t)$ at future times for $i \geq 1$:

$$e_p(t+i+k) = d_p(t+i+k) + \mathcal{C}_{pt+i+k}x(t+i+k) + \mathcal{E}_{pt+i+k}u_0(t+i) \tag{10}$$

Introduce the notation: $\mathcal{A}_{t+m}^{i-m} = \mathcal{A}_{t+i-1}\mathcal{A}_{t+i-2}\ldots\mathcal{A}_{t+m}$ for $i > m$, where $\mathcal{A}_{t+m}^0 = I$ for $i = m$ and $\mathcal{A}_t^i = \mathcal{A}_{t+i-1}\mathcal{A}_{t+i-2}\ldots\mathcal{A}_t$ for $i > 0$, where $\mathcal{A}_t^0 = I$ for $i = 0$, where $\mathcal{A}_{t+k+j}^{i-j} = \mathcal{A}_{t+i+k-1}\mathcal{A}_{t+i+k-2}\ldots\mathcal{A}_{t+j+k}$. Using this notation, the future tracking error:

$$e_p(t+i+k) = d_{pd}(t+i+k) + \mathcal{E}_{pt+i+k}u_0(t+i) + \mathcal{C}_{pt+i+k}\mathcal{A}_{t+k}^{i}x(t+k)$$
$$+\sum_{j=1}^{i}\mathcal{C}_{pt+i+k}\mathcal{A}_{t+k+j}^{i-j}\left(\mathcal{B}_{t+j+k-1}u_0(t+j-1) + \mathcal{D}_{t+j+k-1}\xi(t+j+k-1)\right) \tag{11}$$

2.3 Vector Matrix Prediction Equation

Introducing an obvious notation these signals they may be collected in the *N+1* vector form (*Ordys and Clarke*, 1993) as:

$$E_{Pt+k,N} = \mathcal{D}_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}x(t+k)$$
$$+(\mathcal{C}_{Pt+k,N}\mathcal{B}_{t+k,N} + \mathcal{E}_{Pt+k,N})U_{t,N}^0 + \mathcal{C}_{Pt+k,N}\mathcal{D}_{t+k,N}\Xi_{t+k,N} \tag{12}$$

For $N \geq 1$, the *N+1* square block matrices follow as:

$$\mathcal{C}_{Pt+k,N} = diag\{\mathcal{C}_{pt+k}, \mathcal{C}_{pt+1+k}, \ldots, \mathcal{C}_{pt+N+k}\}$$

$$\mathcal{E}_{Pt+k,N} = diag\{\mathcal{E}_{pt+k}, \mathcal{E}_{pt+1+k}, ..., \mathcal{E}_{pt+N+k}\} \text{ (N+1 square)}$$

$$\mathcal{A}_{t+k,N} = \begin{bmatrix} I \\ \mathcal{A}_{t+k}^{1} \\ \mathcal{A}_{t+k}^{2} \\ \vdots \\ \mathcal{A}_{t+k}^{N} \end{bmatrix}, \mathcal{B}_{t+k,N} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathcal{B}_{t+k} & 0 & \cdots & \vdots & 0 \\ \mathcal{A}_{t+k+1}^{1}\mathcal{B}_{t+k} & \mathcal{B}_{t+k+1} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ \mathcal{A}_{t+k+1}^{N-1}\mathcal{B}_{t+k} & \mathcal{A}_{t+k+2}^{N-2}\mathcal{B}_{t+k+1} & \cdots & \mathcal{B}_{t+k+N-1} & 0 \end{bmatrix}$$

$$\mathcal{D}_{t+k,N} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathcal{D}_{t+k} & 0 & \cdots & \vdots \\ \mathcal{A}_{t+k+1}\mathcal{D}_{t+k} & \mathcal{D}_{t+k+1} & \ddots & \\ \vdots & \vdots & \ddots & 0 \\ \mathcal{A}_{t+k+1}^{N-1}\mathcal{D}_{t+k} & \mathcal{A}_{t+k+2}^{N-2}\mathcal{D}_{t+k+1} & \cdots & \mathcal{D}_{t+k+N-1} \end{bmatrix},$$

$$\Xi_{t,N} = \begin{bmatrix} \xi(t) \\ \xi(t+1) \\ \vdots \\ \xi(t+N-1) \end{bmatrix}, \ E_{Pt+k,N} = \begin{bmatrix} e_p(t+k) \\ e_p(t+1+k) \\ \vdots \\ e_p(t+N+k) \end{bmatrix},$$

$$U_{t,N}^{0} = \begin{bmatrix} u_0(t) \\ u_0(t+1) \\ \vdots \\ u_0(t+N) \end{bmatrix}, \ U_{t,N}^{r} = \begin{bmatrix} u_r(t) \\ u_r(t+1) \\ \vdots \\ u_r(t+N) \end{bmatrix}, \ \mathcal{D}_{Pt,N} = \begin{bmatrix} d_{pd}(t) \\ d_{pd}(t+1) \\ \vdots \\ d_{pd}(t+N) \end{bmatrix} \quad (13)$$

Matrix $\Xi_{t,N}$ denotes a vector of future white noise inputs, $U_{t,N}^{0}$ denotes a block-vector of predicted controls and $U_{t,N}^{r}$ is a vector of future reference controls. From (8) the end-state:

$$x(t+N+k) = d_{dd}(t+N+k-1) + \mathcal{A}_{t+N+k-1}\mathcal{A}_{t+N+k-2}....\mathcal{A}_{t+k}x(t+k)$$
$$+\bar{\mathcal{B}}_{t+k,N}U_{t,N}^{0} + \bar{\mathcal{D}}_{t+k,N}\Xi_{t+k,N} \quad (14)$$

where $\bar{\mathcal{B}}_{t+k,N} = \left[\mathcal{A}_{t+1+k}^{N-1}\mathcal{B}_{t+k}, \ \cdots \ , \mathcal{B}_{t+N+k-1}, \ 0\right]$

and $\bar{\mathcal{D}}_{t+k,N} = \left[\mathcal{A}_{t+1+k}^{N-1}\mathcal{D}_{t+k}, \ \cdots \ , \mathcal{D}_{t+N+k-1}\right]$. The end-state error:

$$e_x(t+N+k) = r_x(t+N+k) - x(t+N+k) = r_x(t+N+k)$$
$$-d_{dd}(t+N+k-1) - \mathcal{A}_{t+k}^{N}x(t+k) - \bar{\mathcal{B}}_{t+k,N}U_{t,N}^{0} - \bar{\mathcal{D}}_{t+k,N}\Xi_{t+k,N} \quad (15)$$

The *k steps-ahead tracking error* $E_{Pt+k,N}$, includes any dynamic weighting and may be written, using (12), as:

$$E_{Pt+k,N} = \mathcal{D}_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}x(t+k) +$$
$$(\mathcal{C}_{Pt+k,N}\mathcal{B}_{t+k,N} + \mathcal{E}_{Pt+k,N})U_{t,N}^{0} + \mathcal{C}_{Pt+k,N}\mathcal{D}_{t+k,N}\Xi_{t+k,N}$$
$$= \mathcal{D}_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}x(t+k) + \mathcal{V}_{Pt+k,N}U_{t,N}^{0} + \mathcal{C}_{Pt+k,N}\mathcal{D}_{t+k,N}\Xi_{t+k,N} \quad (16)$$

where $\mathcal{V}_{Pt+k,N} = \mathcal{C}_{Pt+k,N}\mathcal{B}_{t+k,N} + \mathcal{E}_{Pt+k,N}$ (17)

2.4 Prediction Equations

The *i-steps* ahead prediction of the weighted error signal may be found by noting (11), assuming future controls are known. Let the prediction error $\hat{e}_p(t+i+k \mid t) = E\{\hat{e}_p(t+i+k) \mid t\}$ then:

$$\hat{e}_p(t+i+k \mid t) = d_{pd}(t+i+k) + \mathcal{E}_{pt+i+k}u_0(t+i)$$
$$+\mathcal{C}_{pt+i+k}\mathcal{A}_{t+k}^{i}\hat{x}(t+k \mid t) + \sum_{j=1}^{i}\mathcal{C}_{pt+i+k}\mathcal{A}_{t+k+j}^{i-j}\mathcal{B}_{t+j+k-1}u_0(t+j-1) \quad (18)$$

Noting (16), the vector of prediction errors $\hat{E}_{Pt+k,N}$ follows as:

$$\hat{E}_{P\,t+k,N} = \mathcal{D}_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k \mid t) + \mathcal{V}_{Pt+k,N}U_{t,N}^{0} \quad (19)$$

The estimation error $\tilde{E}_{Pt+k,N} = E_{Pt+k,N} - \hat{E}_{Pt+k,N}$ follows as:

$$\tilde{E}_{Pt+k,N} = \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}x(t+k) + \mathcal{C}_{Pt+k,N}\mathcal{D}_{t+k,N}\Xi_{t+k,N} - \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k \mid t)$$
$$= \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\tilde{x}(t+k|t) + \mathcal{C}_{Pt+k,N}\mathcal{D}_{t+k,N}\Xi_{t+k,N} \quad (20)$$

where *state estimation error* $\tilde{x}(t+k|t) = x(t+k) - \hat{x}(t+k \mid t)$. If the plant and model mismatch is ignored the *state estimation error* is independent of the choice of control. Recall $\hat{x}(t+k \mid t)$ and $\tilde{x}(t+k \mid t)$ are orthogonal and the expectation of the product of the future values of the control action (assumed known in the prediction equation), and the white noise driving signals, is null. It follows $\hat{E}_{Pt+k,N}$ in (19) and $\tilde{E}_{Pt+k,N}$ are orthogonal.

Time-Varying Kalman Estimator: The *Kalman filter* algorithm may be written in *Predictor* and *Corrector* form as:

$$\hat{x}(t+1 \mid t) = \mathcal{A}_t\hat{x}(t \mid t) + \mathcal{B}_t u_0(t-k) + d_d(t) \quad (21)$$

$$\hat{x}(t+1 \mid t+1) = \hat{x}(t+1 \mid t) + \mathcal{K}_{f\,t+1}\left(z(t+1) - \hat{z}(t+1 \mid t)\right) \quad (22)$$

$$\hat{z}(t+1 \mid t) = d(t+1) + \mathcal{C}_{t+1}\hat{x}(t+1 \mid t) + \mathcal{E}_{t+1}u_0(t+1-k) \quad (23)$$

Observe that $\hat{x}(t+k \mid t)$ only depends upon past values of the control signal. This is a full-order optimal estimator with different states to those of the reduced-order observer $\hat{x}_e(t \mid t)$.

3 OBSERVER BASED RS PREDICTIVE CONTROL

The *GPC* cost-index that motivates the criterion to be minimised, can contain a dynamic error weighting (Ordys and Clarke, 1993):

$$J = E\{\sum_{j=0}^{N} \mathrm{e}_p(t+j+k)^T \mathrm{e}_p(t+j+k) + \lambda_j^2 u_0(t+j)^T u_0(t+j)) \big| t\} \quad (24)$$

where $E\{.|t\}$ denotes the conditional expectation, conditioned on measurements up to time t and λ_j denotes a scalar control weighting. The future optimal control signal is to be calculated for the interval $\tau \in [t, \ t+N]$.

As mentioned above the state-space models generating the signals r_p and y_p may include any dynamic cost-function weighting $\mathcal{P}_c(z^{-1})$, such as a low-pass filter to penalise the low-frequency disturbances. The cost-function for the problem of interest can be defined to have a similar form but with some enhancements. A term to limit the gains of the controller (denoted $k_c(t)$) may be added into the cost-index, so that large gain magnitudes are penalized. The rate of gain variations $\Delta k_c(t)$ may also be costed, where:

$$\Delta k_c(t) = k_c(t) - k_c(t-1) \tag{25}$$

A terminal or end-state cost-term, which represents the error in the desired end-state, can also be included to allow robustness and stability properties to be modified. This end-state error:

$$e_x(t+i+k) = r_x(t+i+k) - x(t+i+k)$$

where $r_x(t+i+k)$ denotes an end-state setpoint. In some nonlinear applications, the steady-state control $U^r_{t,N}$, to provide a given output, is calculable, and this can be used in the criterion.

3.1 Optimal Control Cost-Function and Solution

The multi-step *RS-GPC* cost-function $J = E\{J_t|t\}$ is to be minimized that can be written in a concise vector-matrix form using the additional gain terms as follows:

$$J = E\{E^T_{Pt+k,N}E_{Pt+k,N} + (U^{0T}_{t,N} - U^{rT}_{t,N})\Lambda^2_N(U^0_{t,N} - U^r_{t,N}) + k^T_c(t)\Lambda^2_K k_c(t)$$
$$+\Delta k^T_c(t)\Lambda^2_D\Delta k_c(t) + e_x(t+N+k)^T P_s\, e_x(t+N+k)\big|t\} \tag{26}$$

The weightings and signals being penalized include:

- The signal $E_{Pt+k,N}$ represents the dynamically weighted future tracking error terms.
- The cost-weightings on the future inputs u_0 are defined as $\Lambda^2_N = diag\{\lambda^2_0, \lambda^2_1, ..., \lambda^2_N\}$.
- The cost-weightings on the controller gains $k_c(t)$ are defined as $\Lambda^2_K = diag\{\rho^2_0, \rho^2_1, ..., \rho^2_{N_k}\} > 0$.
- The weighting on the increment of the gains $\Delta k_c(t)$ is $\Lambda^2_D = diag\{\gamma^2_0, \gamma^2_1, ..., \gamma^2_{N_k}\}$ and on end-state error $P_s \geq 0$.

The integer N is the prediction horizon and N_k is the number of states in the low-order observer fed back through gains $k_c(t)$. Assume a *Kalman filter* is used for state-estimation and prediction in the optimization (background processing), then the vector of future errors can be replaced by the orthogonal predicted errors and estimation error terms. From (26) obtain:

$$J = E\{(\hat{E}_{Pt+k,N} + \tilde{E}_{Pt+k,N})^T(\hat{E}_{Pt+k,N} + \tilde{E}_{Pt+k,N}) + (U^{0T}_{t,N} - U^{rT}_{t,N})\Lambda^2_N(U^0_{t,N} - U^r_{t,N})$$
$$+k^T_c(t)\Lambda^2_K k_c(t) + \Delta k^T_c(t)\Lambda^2_D\Delta k_c(t)$$
$$+(\hat{e}_x(t+N+k) + \tilde{e}_x(t+N+k))^T P_s(\hat{e}_x(t+N+k) + \tilde{e}_x(t+N+k))\big|t\}$$

where $\Delta k_c(t) = k_c(t) - k_c(t-1)$. Let the known signal:

$$r_{dd}(t+k) = r_x(t+N+k) - d_{dd}(t+N+k-1) - \mathcal{A}^N_{t+k}\hat{x}(t+k\,|\,t) \tag{27}$$

From (15), the end-state error can be expanded as follows:

$$e_x(t+N+k) = r_x(t+N+k) - d_{dd}(t+N+k-1)$$
$$-\mathcal{A}^N_{t+k}\hat{x}(t+k\,|\,t) - \bar{\mathcal{B}}_{t+k,N}U^0_{t,N} - \bar{\mathcal{D}}_{t+k,N}\Xi_{t+k,N} - \mathcal{A}^N_{t+k}\tilde{x}(t+k\,|\,t)$$
$$= \left(r_{dd}(t+k) - \bar{\mathcal{B}}_{t+k,N}U^0_{t,N}\right) - \left(\bar{\mathcal{D}}_{t+k,N}\Xi_{t+k,N} + \mathcal{A}^N_{t+k}\tilde{x}(t+k\,|\,t)\right)$$

Exploiting orthogonality properties:

$$E\{e_x(t+N+k)^T P_s\, e_x(t+N+k)\big|t\}$$
$$= \left(r_{dd}(t+k) - \bar{\mathcal{B}}_{t+k,N}U^0_{t,N}\right)^T P_s\left(r_{dd}(t+k) - \bar{\mathcal{B}}_{t+k,N}U^0_{t,N}\right) + J_s(t) \tag{28}$$

The final term, that is not dependent on control action:

$$J_s(t) = E\left\{\left(\bar{\mathcal{D}}_{t+k,N}\Xi_{t+k,N} + \mathcal{A}^N_{t+k}\tilde{x}(t+k\,|\,t)\right)^T P_s\left(\bar{\mathcal{D}}_{t+k,N}\Xi_{t+k,N} + \mathcal{A}^N_{t+k}\tilde{x}(t+k\,|\,t)\right)\big|t\right\} \tag{29}$$

Cost-simplification: The terms in the criterion can be simplified by using the orthogonality of the estimate $\hat{E}_{Pt+k,N}$ and estimation error $\tilde{E}_{Pt+k,N}$. The vector form may be obtained, from (28), as:

$$J = \hat{E}^T_{Pt+k,N}\hat{E}_{Pt+k,N} + (U^{0T}_{t,N} - U^{rT}_{t,N})\Lambda^2_N(U^0_{t,N} - U^r_{t,N}) + k^T_c(t)\Lambda^2_K k_c(t)$$
$$+\Delta k^T_c(t)\Lambda^2_D\Delta k_c(t) + (r_{dd}(t+k) - \bar{\mathcal{B}}_{t+k,N}U^0_{t,N})^T P_s(r_{dd}(t+k) - \bar{\mathcal{B}}_{t+k,N}U^0_{t,N})$$
$$+J_s(t) + J_0(t) \tag{30}$$

where $J_0(t) = E\{\tilde{E}^T_{Pt+k,N}\tilde{E}_{Pt+k,N}\,|\,t\}$. From (20) the error $\tilde{E}_{Pt+k,N}$ and $J_0(t)$ are independent of the control and $\hat{x}(t+k\,|\,t)$ only depends upon past values of control. Let the deterministic signal:

$$\tilde{\mathcal{D}}_{Pt+k,N} = \mathcal{D}_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k\,|\,t) \tag{31}$$

The future error estimates, from (19), may now be written as:

$$\hat{E}_{P\,t+k,N} = \tilde{\mathcal{D}}_{Pt+k,N} + \mathcal{V}_{Pt+k,N}U^0_{t,N} \tag{32}$$

Cost-function expansion: The cost (30) may be expanded as:

$$J = U^{0T}_{t,N}\left(\mathcal{V}^T_{Pt+k,N}\tilde{\mathcal{D}}_{Pt+k,N} - \Lambda^2_N U^r_{t,N} - \bar{\mathcal{B}}^T_{t+k,N}P_s r_{dd}(t+k)\right)$$
$$+\left(\tilde{\mathcal{D}}^T_{Pt+k,N}\mathcal{V}_{Pt+k,N} - U^{rT}_{t,N}\Lambda^2_N - r^T_{dd}(t+k)P_s\bar{\mathcal{B}}_{t+k,N}\right)U^0_{t,N} + k^T_c(t)\Lambda^2_K k_c(t)$$
$$+\Delta k^T_c(t)\Lambda^2_D\Delta k_c(t) + U^{0T}_{t,N}\left(\mathcal{V}^T_{Pt+k,N}\mathcal{V}_{Pt+k,N} + \Lambda^2_N + \bar{\mathcal{B}}^T_{t+k,N}P_s\bar{\mathcal{B}}_{t+k,N}\right)U^0_{t,N}$$
$$= \tilde{\mathcal{D}}^T_{Pt+k,N}\tilde{\mathcal{D}}_{Pt+k,N} + r^T_{dd}(t+k)P_s r_{dd}(t+k) + U^{rT}_{t,N}\Lambda^2_N U^r_{t,N}$$
$$+J_s(t) + J_0(t) \tag{33}$$

3.2 Parameterizing an Observer Based RS Controller

The RS controller utilizes the low order observer to compute $\hat{x}_e(t|t)$ and the results will apply to the following cases:

- When a low-order observer is used within the feedback loop and the gains are computed in the background based on a separate full-order *Kalman filter*.
- Where a single full-order *Kalman filter* is used for both sets of computations; providing a benchmark solution.

Parameterized Control Action: The expression for the parameterized control involves the *m*-control inputs and the *q reduced-order* observer state-estimates. The gain vector:

$$K_c(t)=\begin{bmatrix} k_{c1}^T(t) \\ k_{c2}^T(t) \\ \vdots \\ k_{cq}^T(t) \end{bmatrix} \quad \text{and} \quad k_{cj}(t)=\begin{bmatrix} k_c^{j1} \\ k_c^{j2} \\ \vdots \\ k_c^{jq} \end{bmatrix} \tag{34}$$

Denote the vector of q low order observer state-estimates as:

$$\hat{x}_e(t)=\begin{bmatrix} \hat{x}_e^1 & \hat{x}_e^2 & \cdots & \hat{x}_e^q \end{bmatrix} \in R^q \tag{35}$$

The observer based control action, using (34) and (35):

$$u(t)=K_c(t)\hat{x}_e(t)$$
$$=\begin{bmatrix} k_c^{11} & k_c^{12}\ldots & k_c^{1q} \\ k_c^{21} & k_c^{22}\ldots & k_c^{2q} \\ & \vdots & \\ k_c^{m1} & k_c^{m2}\ldots & k_c^{mq} \end{bmatrix}\begin{bmatrix} \hat{x}_e^1 \\ \hat{x}_e^2 \\ \vdots \\ \hat{x}_e^q \end{bmatrix}=\begin{bmatrix} \hat{x}_e^1k_c^{11}+\hat{x}_e^2k_c^{12}+\ldots+\hat{x}_e^qk_c^{1q} \\ \hat{x}_e^1k_c^{21}+\hat{x}_e^2k_c^{22}+\ldots+\hat{x}_e^qk_c^{2q} \\ \vdots \\ \hat{x}_e^1k_c^{m1}+\hat{x}_e^2k_c^{m2}+\ldots+\hat{x}_e^qk_c^{mq} \end{bmatrix} \tag{36}$$

Gain Computational Form: For the optimal gain calculation, all the above gains need to be collected in a single vector. There are q gains in each channel in (36) that may be grouped as:

$$k_c(t)=\begin{bmatrix} k_{c1}^T & k_{c2}^T & \cdots & k_{cm}^T \end{bmatrix}^T \in R^{m\times q} \tag{37}$$

Also, introduce a block diagonal matrix $\hat{F}_x(t)$ as:

$$\hat{F}_x(t)=diag\left\{\hat{x}_e^T(t) \quad \hat{x}_e^T(t) \quad \cdots \quad \hat{x}_e^T(t)\right\} \tag{38}$$

The control signal $u(t)=\hat{F}_x(t)k_c(t)$ follows as:

$$u(t)=diag\left\{\hat{x}_e^T(t) \quad \hat{x}_e^T(t) \quad \cdots \quad \hat{x}_e^T(t)\right\}\begin{bmatrix} k_{c1} \\ k_{c2} \\ \vdots \\ k_{cm} \end{bmatrix}=\begin{bmatrix} \hat{x}_e^T(t)k_{c1} \\ \hat{x}_e^T(t)k_{c2} \\ \vdots \\ \hat{x}_e^T(t)k_{cm} \end{bmatrix} \tag{39}$$

3.3 Parameterizing the Vector of Future Controls

An implicit model predictive control like *GPC* is usually based on the *receding horizon* principle (Kwon and Pearson, 1977), where the optimal control is taken as the first element in the vector of future controls $U_{t,N}^0$. The equivalent assumption for *RS-GPC* is that $k_c(t)$ is assumed constant over the prediction horizon [*0, N*], and the computed $k_c(t)$ can then be used to compute the optimal control for time *t*. In the spirit of receding horizon control at the next sample time the process is repeated and a new value of the gain $k_c(t)$ is computed. The gains can be substituted in (39) to compute the vector of future controls.

The difference with other *MPC* observer based solutions is that the predicted control is computed using the chosen *RS*-controller. In the spirit of the receding horizon philosophy, the structure of the controller is assumed fixed over the prediction interval and only the initial control is implemented using the gain at that time. The vector of future controls $U_{t,N}$ is therefore given as:

$$U_{t,N}=\begin{bmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+N) \end{bmatrix}=\begin{bmatrix} \hat{F}_x(t) \\ \hat{F}_x(t+1) \\ \vdots \\ \hat{F}_x(t+N) \end{bmatrix}k_c(t) \tag{40}$$

where $\hat{F}_x(t+i)=diag\left\{\hat{x}_e^T(t+i) \quad \hat{x}_e^T(t+i) \quad \cdots \quad \hat{x}_e^T(t+i)\right\}$ (41)

and $\hat{x}_e^T(t+i)=\begin{bmatrix} \hat{x}_e^1(t+i) & \hat{x}_e^2(t+i) & \cdots & \hat{x}_e^q(t+i) \end{bmatrix}^T$.

The assumption is that the gains are constant throughout the prediction horizon and the vector of future controls may then be found given the predicted state-estimates. Let the matrix,

$$U_{fe}(t)=\begin{bmatrix} \hat{F}_x^T(t) & \hat{F}_x^T(t+1) & \cdots & \hat{F}_x^T(t+N) \end{bmatrix}^T \tag{42}$$

The i step-ahead control follows as: $u(t+i)=\hat{F}_x(t+i)k_c(t)$ or

$$u(t+i)=diag\left\{\hat{x}_e^T(t+i) \quad \hat{x}_e^T(t+i) \quad \cdots \quad \hat{x}_e^T(t+i)\right\}k_c(t)$$

The vector of future controls may now be written in terms as:

$$U_{t,N}=U_{fe}(t)k_c(t) \tag{43}$$

4 OPTIMIZATION OF THE PREDICTIVE CONTROLLER

The solution for the optimal *RS* controller gains may now be obtained. Using the parameterization (43) for the controller $U_{t,N}^0=U_{fe}(t)k_c(t)$ and the criterion (33) may be written as:

$$\begin{aligned} J &= k_c^TU_{fe}^T\left(\mathcal{V}_{Pt+k,N}^T\tilde{\mathcal{D}}_{Pt+k,N}-\Lambda_N^2U_{t,N}^r-\bar{\mathcal{B}}_{t+k,N}^TP_sr_{dd}(t+k)\right) \\ &+\left(\tilde{\mathcal{D}}_{Pt+k,N}^T\mathcal{V}_{Pt+k,N}-U_{t,N}^{rT}\Lambda_N^2-r_{dd}^T(t+k)P_s\bar{\mathcal{B}}_{t+k,N}\right)U_{fe}k_c \\ &+k_c^TU_{fe}^T\mathcal{X}_{t+k,N}U_{fe}k_c+k_c^T\Lambda_K^2k_c+\Delta k_c^T\Lambda_D^2\Delta k_c+U_{t,N}^{rT}\Lambda_N^2U_{t,N}^r \\ &+\tilde{\mathcal{D}}_{Pt+k,N}^T\tilde{\mathcal{D}}_{Pt+k,N}+r_{dd}^T(t+k)P_sr_{dd}(t+k)+J_s(t)+J_0(t) \end{aligned} \tag{44}$$

The time-varying matrix $\mathcal{X}_{t+k,N}$ in this expression satisfies:

$$\mathcal{X}_{t+k,N}=(\mathcal{V}_{Pt+k,N}^T\mathcal{V}_{Pt+k,N}+\Lambda_N^2+\bar{\mathcal{B}}_{t+k,N}^TP_s\bar{\mathcal{B}}_{t+k,N}) \tag{45}$$

Define $\tilde{\mathcal{D}}_{Pt+k,N}^0$ and substitute from (31) to obtain:

$$\tilde{\mathcal{D}}_{Pt+k,N}^0=U_{fe}^T\left(\mathcal{V}_{Pt+k,N}^T\tilde{\mathcal{D}}_{Pt+k,N}-\Lambda_N^2U_{t,N}^r-\bar{\mathcal{B}}_{t+k,N}^TP_sr_{dd}(t+k)\right)$$

To simplify let: $\mathcal{P}_{ct+k,N}=U_{fe}^T(t)\mathcal{V}_{Pt+k,N}^T$, $\bar{\mathcal{P}}_{ct+k,N}=U_{fe}^T(t)\bar{\mathcal{B}}_{t+k,N}^TP_s$

$$\mathcal{C}_{\phi t+k,N}=\mathcal{P}_{ct+k,N}\mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}=U_{fe}^T(t)\mathcal{V}_{Pt+k,N}^T\mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N} \tag{46}$$

Thence obtain:

$$\tilde{\mathcal{D}}_{Pt+k,N}^0=\mathcal{P}_{ct+k,N}\mathcal{D}_{Pt+k,N}+\mathcal{C}_{\phi t+k,N}\hat{x}(t+k\,|\,t)-U_{fe}^T(t)\Lambda_N^2U_{t,N}^r(t)-\bar{\mathcal{P}}_{ct+k,N}r_{dd}(t+k) \tag{47}$$

The cost-function (44) may now be written as:

$$\begin{aligned} J &= k_c^T(t)\tilde{\mathcal{D}}_{Pt+k,N}^0+\tilde{\mathcal{D}}_{Pt+k,N}^{0T}k_c(t)+k_c^T(t)U_{fe}^T(t)\mathcal{X}_{t+k,N}U_{fe}(t)k_c(t) \\ &+k_c^T(t)\Lambda_K^2k_c(t)+\Delta k_c^T(t)\Lambda_D^2\Delta k_c(t)+\tilde{\mathcal{D}}_{Pt+k,N}^T\tilde{\mathcal{D}}_{Pt+k,N} \\ &+r_{dd}^T(t+k)P_sr_{dd}(t+k)+U_{t,N}^{rT}\Lambda_N^2U_{t,N}^r+J_s(t)+J_0(t) \end{aligned} \tag{48}$$

This can be simplified by noting $\Delta k_c(t) = k_c(t) - k_c(t-1)$ where

$$\Delta k_c^T(t)\Lambda_D^2 \Delta k_c(t) = k_c^T(t)\Lambda_D^2 k_c(t) - k_c^T(t-1)\Lambda_D^2 k_c(t)$$
$$-k_c^T(t)\Lambda_D^2 k_c(t-1) + k_c^T(t-1)\Lambda_D^2 k_c(t-1)$$

The cost expression (48) may therefore be written as:

$$J = k_c^T(t)\tilde{\mathcal{D}}_{Pt+k,N}^0 + \tilde{\mathcal{D}}_{Pt+k,N}^{0T} k_c(t) - k_c^T(t-1)\Lambda_D^2 k_c(t) - k_c^T(t)\Lambda_D^2 k_c(t-1)$$
$$+k_c^T(t)\left(U_{fe}^T(t)\mathcal{X}_{t+k,N} U_{fe}(t) + \Lambda_K^2 + \Lambda_D^2\right)k_c(t) + \tilde{\mathcal{D}}_{Pt+k,N}^T \tilde{\mathcal{D}}_{Pt+k,N}$$
$$+k_c^T(t-1)\Lambda_D^2 k_c(t-1) + r_{dd}^T(t+k)P_s r_{dd}(t+k)$$
$$+U_{t,N}^{rT}\Lambda_N^2 U_{t,N}^r + J_s(t) + J_0(t) \tag{49}$$

To simplify (49), define the following time-varying matrix:

$$\mathcal{X}_{t+k,N_k} = U_{fe}^T(t)\mathcal{X}_{t+k,N} U_{fe}(t) + \Lambda_K^2 + \Lambda_D^2 \tag{50}$$

Cost-function: Substituting these results into (49):

$$J = k_c^T(t)\left(\tilde{\mathcal{D}}_{Pt+k,N}^0 - \Lambda_D^2 k_c(t-1)\right) + \left(\tilde{\mathcal{D}}_{Pt+k,N}^{0T} - k_c^T(t-1)\Lambda_D^2\right)k_c(t)$$
$$+k_c^T(t)\mathcal{X}_{t+k,N_k} k_c(t) + \tilde{\mathcal{D}}_{Pt+k,N}^T \tilde{\mathcal{D}}_{Pt+k,N} + k_c^T(t-1)\Lambda_D^2 k_c(t-1)$$
$$+r_{dd}^T(t+k)P_s r_{dd}(t+k) + U_{t,N}^{rT}\Lambda_N^2 U_{t,N}^r + J_s(t) + J_0(t) \tag{51}$$

The terms within the square brackets are independent of the control action. The necessary condition for optimality requires the gradient of the cost-function to be set to zero, to obtain the vector of future optimal controls. The *optimal gain*:

$$k_c(t) = -\mathcal{X}_{t+k,N_k}^{-1}\left(\tilde{\mathcal{D}}_{Pt+k,N}^0 - \Lambda_D^2 k_c(t-1)\right) \tag{52}$$

The sufficient condition requires that the *second-derivative* or *Hessian matrix* be positive definite $\mathcal{X}_{t+k,N_k} > 0$, which is ensured because the control gain costing is positive definite.

Asymptotic Behaviour: From (52) if the rate of change cost term $\Lambda_D^2 \to \infty$ the limiting gain $k_c(t) = k_c(t-1)$ becomes constant.

Minimum cost: Substituting in (51) for the gain $k_c(t)$ in (52) and simplifying, the minimum-cost becomes:

$$J_{min} = \tilde{\mathcal{D}}_{Pt+k,N}^T \tilde{\mathcal{D}}_{Pt+k,N} - \left(\tilde{\mathcal{D}}_{Pt+k,N}^{0T} - k_c^T(t-1)\Lambda_D^2\right)\mathcal{X}_{t+k,N_k}^{-1}\left(\tilde{\mathcal{D}}_{Pt+k,N}^0 - \Lambda_D^2 k_c(t-1)\right)$$
$$+k_c^T(t-1)\Lambda_D^2 k_c(t-1) + r_{dd}^T(t+k)P_s r_{dd}(t+k) + U_{t,N}^{rT}\Lambda_N^2 U_{t,N}^r + J_s(t) + J_0(t) \tag{53}$$

where $\tilde{\mathcal{D}}_{Pt+k,N}^0$ was defined in (47) and gain $k_c(t-1)$ is known.

Sub-optimality: The minimum-cost (53) may be compared with the minimum for conventional *GPC* to minimize (24), when the weightings Λ_K^2, Λ_D^2 and P_S tends to zero, so from (29) $J_s(t) \to 0$, $\mathcal{X}_{t+k,N_k} \to U_{fe}^T(t)(\mathcal{V}_{Pt+k,N}^T \mathcal{V}_{Pt+k,N} + \Lambda_N^2)U_{fe}(t)$ and $\tilde{\mathcal{D}}_{Pt+k,N}^0 \to \mathcal{P}_{ct+k,N}\tilde{\mathcal{D}}_{Pt+k,N}$. Assuming $U_{t,N}^r = 0$ and that $\mathcal{X}_{t+k,N}$ has an inverse the minimal cost, from (53) becomes:

$$J_{min} \to \tilde{\mathcal{D}}_{Pt+k,N}^T \tilde{\mathcal{D}}_{Pt+k,N} - \tilde{\mathcal{D}}_{Pt+k,N}^{0T}\mathcal{X}_{t+k,N_k}^{-1}\tilde{\mathcal{D}}_{Pt+k,N}^0 + J_0(t) \tag{54}$$

This minimum cost will be larger relative to the minimal-cost in the equivalent unrestricted *GPC* control problem, given as:

$$J_{min}^{GPC} = \tilde{\mathcal{D}}_{Pt+k,N}^T\left(I - \mathcal{V}_{Pt+k,N}(\mathcal{V}_{Pt+k,N}^T \mathcal{V}_{Pt+k,N} + \Lambda_N^2)^{-1}\mathcal{V}_{Pt+k,N}^T\right)\tilde{\mathcal{D}}_{Pt+k,N} + J_0(t) \tag{55}$$

The two costs (54) and (55) approach the same value in the special case when U_{fe} is square and full rank.

4.1 RS-GPC Main Theorem

Theorem 1: Observer Based RS-GPC Control

Consider the system and assumptions given at the start of §2. The *observer based restricted-structure generalized predictive controller* is required to minimize the cost-function:

$$J = E\{E_{Pt+k,N}^T E_{Pt+k,N} + (U_{t,N}^{0T} - U_{t,N}^{rT})\Lambda_N^2(U_{t,N}^0 - U_{t,N}^r) + k_c^T(t)\Lambda_K^2 k_c(t)$$
$$+\Delta k_c^T(t)\Lambda_D^2 \Delta k_c(t) + e_x(t+N+k)^T P_s\, e_x(t+N+k) \,|\, t\} \tag{56}$$

The *RS-GPC* control is parameterised as $u(t) = \hat{F}_x(t)k_c(t)$ where $\hat{F}_x(t)$ is given by (41). The optimal gains found to minimize (56) using a *receding horizon philosophy*:

$$k_c(t) = -\mathcal{X}_{t+k,N_k}^{-1}\left(\tilde{\mathcal{D}}_{Pt+k,N}^0 - \Lambda_D^2 k_c(t-1)\right) \tag{57}$$
$$= -\mathcal{X}_{t+k,N_k}^{-1}\Big(\mathcal{P}_{ct+k,N}\mathcal{D}_{Pt+k,N} + \mathcal{C}_{\phi t+k,N}\hat{x}(t+k\,|\,t)$$
$$-U_{fe}^T(t)\Lambda_N^2 U_{t,N}^r(t) - \bar{\mathcal{P}}_{ct+k,N} r_{dd}(t+k) - \Lambda_D^2 k_c(t-1)\Big) \tag{58}$$

The matrix $\mathcal{X}_{t+k,N_k}$ is given by (50) and $\mathcal{P}_{ct+k,N}$, $\mathcal{C}_{\phi t+k,N}$ and $\bar{\mathcal{P}}_{ct+k,N}$ by (46). The gain-vector $k_c(t)$ has the form:

$$k_c(t) = \Big[\underbrace{k_c^{11T}\; k_c^{12T} \cdots k_c^{1qT}}_{channel\ 1\ gains}\;\; \underbrace{k_c^{21T}\; k_c^{22T} \cdots k_c^{2qT}}_{channel\ 2\ gains}\;\; \cdots\;\; \underbrace{k_c^{m1T}\; k_c^{m2T} \cdots k_c^{mqT}}_{channel\ m\ gains}\Big]^T$$

The optimal control is given by (57) and parameterized vector of future controls for the prediction error computations:

$$U_{t,N} = U_{fe}(t)k_c(t) \tag{59}$$

where $U_{fe}(t) = \left[\hat{F}_x^T(t) \quad \hat{F}_x^T(t+1) \quad \cdots \quad \hat{F}_x^T(t+N)\right]^T$ •

Solution: These results were derived in the solution presented before the theorem. •

Comments on the control solution: The "denominator matrix" in the *RS-GPC* controller gain expression (58) is full rank because of the cost and system descriptions. It is interesting that the expression for the gain-vector is of a similar form to the usual *GPC* solution but the denominator matrix in (58) can be of much lower dimension than arises in *GPC* control. The size of this matrix depends upon the number of gains in the parameterised controller, which in turn depends upon the order of the observer used in the feedback loop. The weightings Λ_K^2 and Λ_D^2 ensures that the denominator matrix does not become singular.

4.2 Hard and Soft Constraints in RS-GPC

Soft constraints can be applied to the controller gains, or to the rate of change of gains, by changing the weighting terms Λ_K^2 and Λ_D^2 in (56). Hard-constraints may be applied in almost the same way as for *GPC* algorithms using *Quadratic Programming* (*QP*). Constraints may be applied to the magnitude of the controller gains, or to their rate of change. They may also be applied to the control signal inputs and outputs. The ability to bound the gain or gain variations is essential in some applications. Limiting controller gains has a direct effect on power demands and an indirect effect on nonlinear system behaviour and stability.

4.3 Low Order Observer Design

The *low-order* observer within the loop is not the same as the *Kalman-filter* in the background processing. It is chosen to be low-order to improve robustness and so that it can be approximated by a scheduled solution for real-time implementation. The design of the observer is important but not critical to the performance obtained. This is because the optimization process computes gains that optimize the solution whatever the choice of observer. Clearly if there is a perverse choice of observer, the optimal control will find it harder to provide a "best" solution. The observer should therefore be designed using a rational philosophy but the solution may not be very sensitive. The observer can be found by basing the plant model on just the low frequency plant modes or it can be model reduced and a low order *Kalman-filter* can be computed. If the state-space model is linear time-invariant, a *Luenberger-observer* can be designed. There is an interesting extension suggested by Morari and Maeder (2012) for nonlinear predictive controls where the plant model is augmented by a disturbance model, which accounts for model mismatch and disturbances. They noted, "In principle it would be possible to use an elaborate nonlinear model in the observer and a simple (locally linearized) model in the controller as long as the steady-state characteristics match." The opportunity to improve robustness and achieve offset free tracking in the *RS-GPC* controller is clear.

4.4 Cost-Function Tuning Variables

Tuning the controller should be as simple as traditional *MPC* (Grimble and Majecki, 2010). The weighting on the control signal provides a simple way to vary the speed of response. The initial choice of weighting on the error signal can be taken as the identity or if integral action is included then the weighting gain can be scaled to the identity. The remaining weightings $F_{CN}^1 = \Lambda_K^2$ and $F_{CN}^2 = \Lambda_D^2$ are on the magnitude of the gains and the rate of change of gains, and it is useful to limit these gains.

5 ROLLING MILL LOOPER CONTROL DESIGN

The interstand looper involves a roll on a torque-controlled arm located between the adjacent stands of a multi-stand hot rolling mill. It maintains a constant tension between the stands to correct any disturbances in the strip mass flow, during acceleration and deceleration. The plant I/O structure is shown in Fig. 3 and the problem is described in more detail in *Hearns et al.,* (1996).

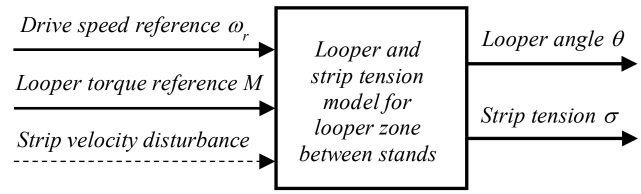

Fig. 3: Looper Control Plant Model

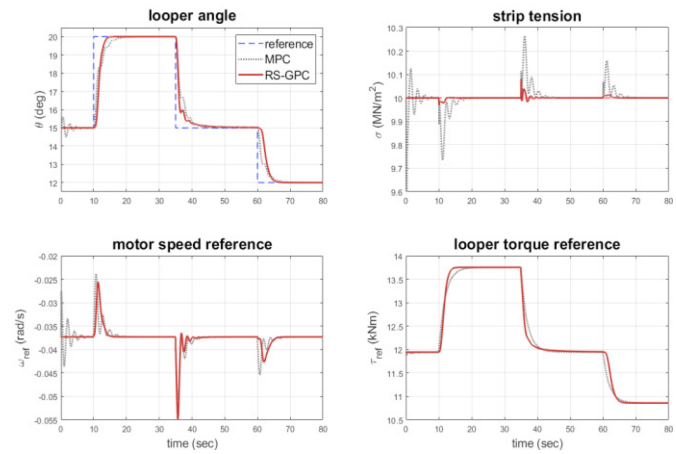

Fig. 4: Comparison Reduced-Order Observer Based RS-GPC with MPC Setpoint Following (Observer reduced by a single state)

The results in Fig. 4 compare model predictive control (MPC) and the RS-GPC algorithm, aimed at good looper-angle reference tracking. The observer is one state lower than full-order and is a *suboptimal Kalman Filter* within the loop. The Fig. 5 shows the observer based *RS-GPC* solution when the observer uses a plant model, which is five or six states lower than the full-order case and using the same cost-function. It is surprising that tracking performance is so good for this case but recall the background processing which involves the predictive control algorithm computes the "optimal" solution and attempts to compensate for the use of the low-order observer. The variances of tracking errors can be considered but they are not very meaningful, since the individual variances are not being minimized and MPC and RS cost-functions are very different.

	Looper Angle	Strip Tension
MPC	5776.2	15.91
RS-GPC	7110.7	0.25
RS -5s	6343.5	0.17
RS -6s	6620.1	0.24

The RS results are for the observer model reduced by one, five and six states, respectively. However, the RS criterion has the term $k_c^T(t)\Lambda_K^2 k_c(t)$ that depends on the number of states. The higher-order observer RS control includes five more weighting terms than the RS -6 states case so again it is difficult to compare results in this way.

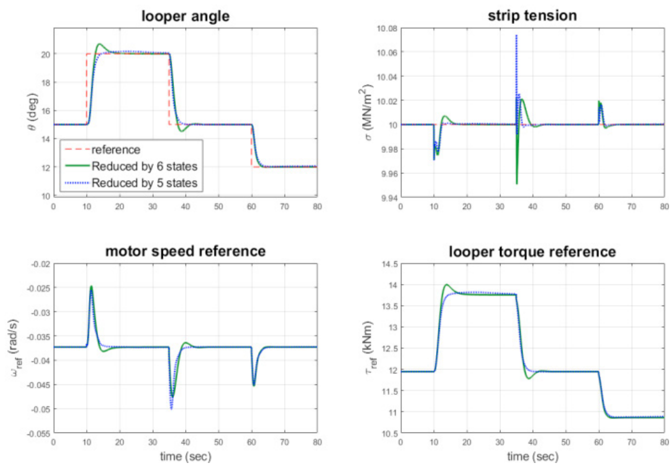

Fig. 5: Comparison When Observer States Reduced by 5 or 6 States

Note that if the low-order observer is made very fast by reducing the assumed measurement noise variance the RS results are all rather similar. As noted above the background processing can therefore ensure high performance is achieved providing the assumed low-order observer does not slow down the system too markedly.

6 Concluding Remarks

When a low order observer is to be used in a feedback control, there is not normally an exact analytic solution to the resulting optimal control problem. The main contribution is that the proposed predictive control solution here does not involve any approximation. The *RS-GPC* approach has potential in automotive applications such as Majecki et al. (2015, 2017) where the *RS* structure can be simplified for implementation. A low-order controller also seems to have a mother nature intended natural robustness. Moreover, predictive controls often exhibit an improvement in robustness over long prediction-horizons and this property has been observed in application examples.

The controller has natural feed-forward terms and provides some measure of time delay compensation. The use of an optimisation method to compute the parameterised controller gains provides the benefits of model-based control and reduces the tuning process to the selection of simple cost-function weightings. The advances on nonlinear predictive control (Findeisen et al. 2003, Mayne et al., 2005, 2014) suggests a number of extensions.

References

Eielsen, A. A., Vagia, M., Gravdahl, J. T., Pettersen, K. Y. (2013). Fixed-Structure, Low-Order Damping and Tracking Control Schemes for Nano positioning, *6th IFAC Symposium on Mechatronic Systems*, China.

Findeisen, R., Imsland, L., Allgower, F. and Foss B. A. (2003). State and Output Feedback Nonlinear Model Predictive Control: An Overview, *European Journal of Control*, Vol. 9, Issues 2–3, pp. 190-206

Grimble, M. J., (2004). Restricted structure predictive optimal control, *Optimal Control Applications and Methods,* Vol. 25, No. 3, pp. 107-145.

Grimble, M. J., and Majecki, P. (2010). State-Space Approach to Nonlinear Predictive Generalized Minimum Variance Control, *Int. Journal of Control*, Vol. 83, No. 8, pp. 1529-1547.

Hast, M., (2015). Design of Low-Order Controllers using Optimization Techniques, Ph.D. Thesis, Lund, Sweden.

Hearns G., Katebi, M. R., and Grimble M. J. (1996). Robust Control of a Hot Strip Mill Looper, *13th IFAC World Congress*, San Francisco, Vol. 29, No. 1, pp. 6239-6244.

Houpis, C. H., Rasmussen, S. J., Garcia-Sanz, M. (2006). Quantitative feedback theory: fundamentals and applications, CRC Press, Taylor & Francis.

Kwon W.H. and Pearson, A.E. (1977), A modified quadratic cost problem and feedback stabilization of a linear system, *IEEE Tran. on Automatic Control*, Vol. AC-22, No. 5, pp. 838-842.

Majecki, P, van der Molen, G. M., Grimble, M. J., Haskara I, Hu Y, Chang, C-F. (2015). Real-Time Predictive Control for SI Engines Using Linear Parameter-Varying Models, *5th IFAC Conference on Nonlinear Model Predictive Control*, Seville.

Majecki, P., Grimble, M. J., Haskara, I., Hu, Y., Chang, C. F., (2017). Total Engine Optimization and Control for SI Engines Using LPV Models, American Control Conference, Seattle, WA, pp. 24–26.

Morari, M. and Maeder, U. (2012). Nonlinear offset-free model predictive control, *Automatica,* Vol. 48, No. 9, pp. 2059-2067.

Mayne, D. Q. Seron, M.M. and Rakovića, S.V. (2005). Robust model predictive control of constrained linear systems with bounded disturbances, *Automatica*, Vol. 41, No. 2, pp. 219-224.

Mayne, D. Q. (2014), Model predictive control: Recent developments and future promise, *Automatica*, Vol. 50, No. 12, pp. 2967-2986.

Ordys, A. W., and Clarke, D. W. (1993). A state-space description for GPC controllers, *International Journal of Systems Science*, Vol. 23, No. 2.

Polak, E., Siegel, P., Wu, T., Nye, W. T. and Mayne, D. Q. (1982). DELIGHT.MIMO: An interactive optimization based multivariable control system design package, *IEEE Control Systems Magazine*, No. 4.

Richalet, J., (1993), Industrial applications of model based predictive control, *Automatica*, Vol. 29, No. 8, pp. 1251-1274.

Sato T, (2010). Design of a GPC-based PID controller for controlling a weigh feeder, *Control Engineering Practice*, 18, pp. 105-1103.

Zakian, V. and AI-Naib, U. (1973). Design of dynamical and control systems by the method of inequalities, *IEE Proceedings*, Pt. D, Vol. 120, No. 11, pp. 1421–1427.