Constructing consonant beliefs from multivariate data with scenario theory

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Abstract

A method for constructing consonant predictive beliefs for multivariate datasets is presented. We make use of recent results in scenario theory to construct a family of enclosing sets that are associated with a predictive lower probability of new data falling in each given set. We show that the sequence of lower bounds indexed by enclosing set yields a consonant belief function. The presented method does not rely on the construction of a likelihood function, therefore possibility distributions can be obtained without the need for normalization. We present a practical example in two dimensions for the sake of visualization, to demonstrate the practical procedure of obtaining the sequence of nested sets.

Problem statement and predictive beliefs

Given a bunch of *iid* samples $X_1, ..., X_n$, with $X_i \in \mathbb{R}^m$, what can be learnt about the unknown underlying distribution $\mathbb{P}_X$? What is the probability of observing a new sample in a given set?

- Samples are few — We cannot learn $\mathbb{P}_X$ exactly!
- Multivariate case — What about X_i interdependence?

We want to obtain a belief function Bel_X so that the inequality $\text{Bel}_X \leq \mathbb{P}_X$ holds at least $100\,(1-\beta)\%$ of the times.

(i) $\mathbb{P}^n\left(\text{Bel}_X \leq \mathbb{P}_X\right) \geq 1-\beta$

(ii) $\forall A \subseteq \mathbb{R}^m,\ \text{Bel}_X(A) \to \mathbb{P}_X(A),\ n \to \infty$

Scenario optimization

Let $z \in \mathcal{Z} \subseteq \mathbb{R}^d$ be a vector of (design) parameters. The scenario optimization consists in minimizing the convex cost function $f : \mathcal{Z} \to \mathbb{R}$:

$$\min_{z \in \mathcal{Z}} f(z) \quad \text{subject to: } z \in \bigcap_{i=1,\dots,n} \mathcal{Z}_{X_i}, \tag{1}$$

For example, let $z = (c_x, r)$, be the centre x-coordinate and the radius of a circle ($c_y = 0$). The area of the circle is minimized, see Figure (1):

$$\begin{aligned} \min \quad & \pi r^2 \\ \text{subject to:} \quad & (c_x - X_1)^2 + (0 - Y_1)^2 \leq r^2, \\ & \dots \\ & (c_x - X_n)^2 + (0 - Y_n)^2 \leq r^2 \end{aligned} \tag{2}$$

Definitions

Enclosing set of degree k: The optimal set $B_k \subseteq \mathbb{R}^m$, that strictly contains $n-k$ observations.

Lower probability of enclosing set B_k: The precise predictive probability of a given enclosing set of degree k, $\mathbb{P}_X(B_k)$, has a lower bound $\underline{p}_k$, with assigned one-sided coverage probability. B_k is determined by the program Eq.(1).

$$\mathbb{P}^n\left(\underline{p}_k \leq \mathbb{P}_X(B_k)\right) \geq 1-\beta, \tag{3}$$

Theorem

The lower bound $\underline{p}_k$ is a sequence of coherent predictive beliefs for any $k \in \mathbb{Z}_+$ such that $0 = k_0 < k_1 < \cdots < k_n = n$.

Computing the lower bound probability of the enclosing sets

$$\varphi(t) = \frac{\beta}{n+1}\sum_{j=k}^{n}\binom{j}{k}t^{j-k} - \binom{n}{k}t^{n-k}, \quad t \in [0,1] \tag{4}$$

$$\varphi(\hat{t}) = 0; \qquad \underline{p}_k = \hat{t}(n,k,\beta);$$

[1] Campi, M.C. and Garatti, S. (2018). *Wait-and-judge scenario optimization.* Mathematical Programming, 167(1), pp.155-189.

More references

[2] Garatti, S. and Campi, M.C. (2019). *Risk and complexity in scenario optimization.* Mathematical Programming, pp.1-37.

[3] Denœux, T. (1999). *Reasoning with imprecise belief structures.* International Journal of Approximate Reasoning, 20(1), 79-111.

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Minimum disk cover problem

Solving optimization program Eq.(2) with center y-coordinate = 0. This minimum disk is our first enclosing set $B_{k_1=3}$.

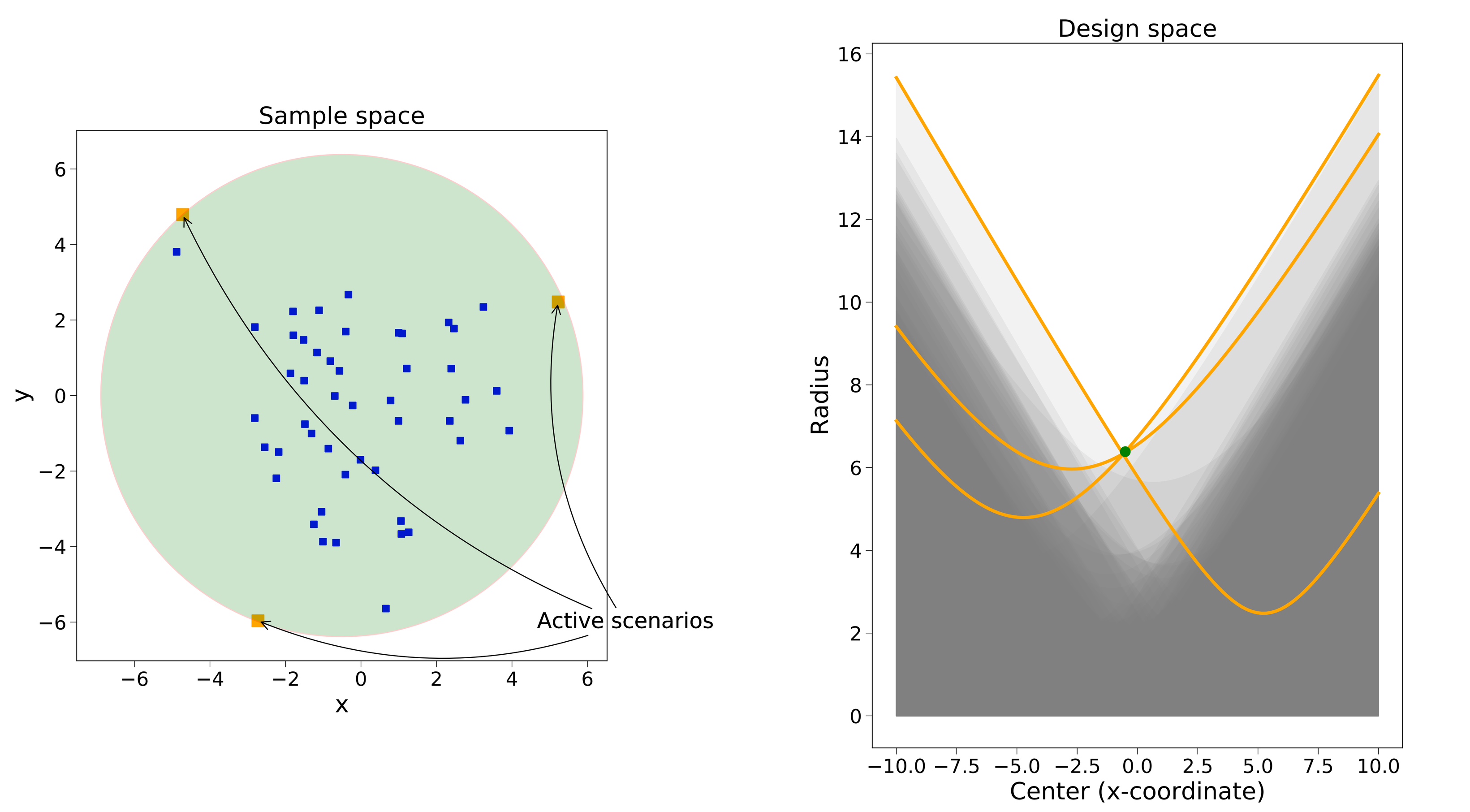

Figure 1: Left side: 50 *iid* samples (in blue) and smallest enclosing disk (in light green); the orange dots are the samples determining the optimum. Right side: space of acceptable designs and constraint functions corresponding to the three active scenarios in orange. The optimum is located on the deepest point lying on the constraint surface (green round dot).

Constructing predictive beliefs

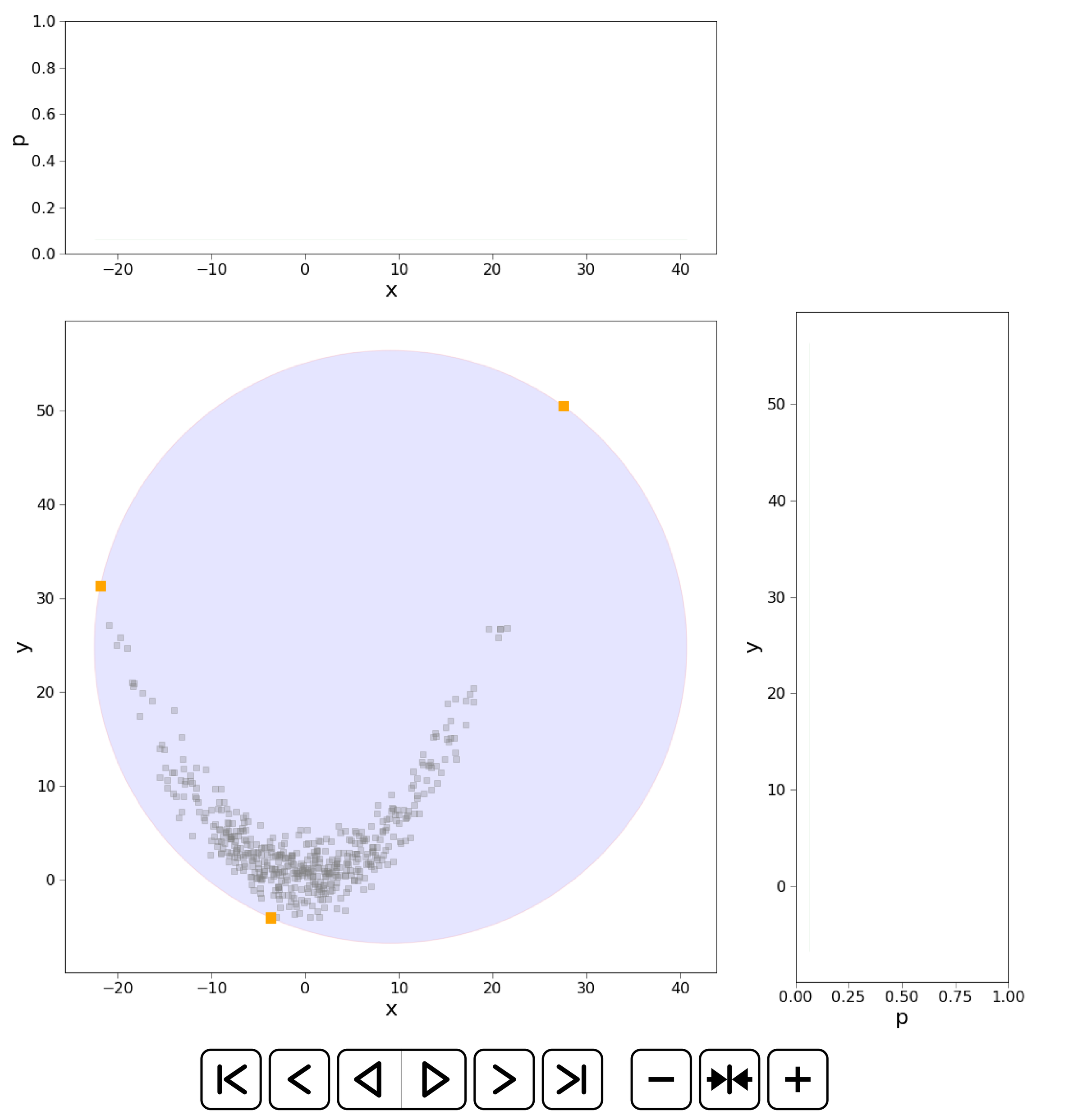

Figure 2: 500 *iid* samples and sequence of enclosing sets constructed by the algorithm. Each enclosing set is associated with a unique belief that determines the marginal plausibility contours shown in the side marginal plots.