Deformations of cluster mutations and invariant presymplectic forms

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Abstract

We consider deformations of sequences of cluster mutations in finite type cluster algebras, which destroy the Laurent property but preserve the presymplectic structure defined by the exchange matrix. The simplest example is the Lyness 5-cycle, arising from the cluster algebra of type A_2: this deforms to the Lyness family of integrable symplectic maps in the plane. For types A_3 and A_4 we find suitable conditions such that the deformation produces a two-parameter family of Liouville integrable maps (in dimensions two and four, respectively). We also perform Laurentification for these maps, by lifting them to a higher-dimensional space of tau functions with a cluster algebra structure, where the Laurent property is restored. More general types of deformed mutations associated with affine Dynkin quivers are shown to correspond to four-dimensional symplectic maps arising as reductions of the discrete sine-Gordon equation.

1 Lyness maps and Zamolodchikov periodicity

It was observed by Lyness in 1942 [28] that the recurrence

$$x_{n+2}x_n = x_{n+1} + 1 \tag{1.1}$$

generates the sequence

$$x_0, x_1, \frac{x_1+1}{x_0}, \frac{x_0+x_1+1}{x_0x_1}, \frac{x_0+1}{x_1}, x_0, x_1, \ldots, \tag{1.2}$$

which repeats with period five. The Lyness 5-cycle also arises in Coxeter's frieze patterns [3], or as a simple example of Zamolodchikov periodicity in integrable quantum field theories [37], which can be understood in terms of the associahedron K_4 and the cluster algebra defined by the A_2 Dynkin quiver [10], and this leads to a connection with Abel's pentagon identity for the dilogarithm [29]. The birational map of the plane corresponding to the recurrence (1.1), that is

$$(x, y) \mapsto \left(y, \frac{y+1}{x} \right), \tag{1.3}$$

also appears in the theory of the Cremona group: as conjectured by Usnich and proved by Blanc [1], the birational transformations of the plane that preserve the symplectic form

$$\omega = \frac{1}{xy}\, \mathrm{d}x \wedge \mathrm{d}y, \tag{1.4}$$

are generated by $SL(2, \mathbb{Z})$, the torus and transformation (1.3).

More generally, the birational map

$$\varphi : \quad (x, y) \mapsto \left(y, \frac{ay+b}{x} \right), \tag{1.5}$$

with two parameters a, b is also referred to as the Lyness map. By rescaling $(x, y) \to (ax, ay)$, the parameter $a \neq 0$ can be removed, so that this is really a one-parameter family, which is described in [6] as "the simplest singular map of the plane." There are also analogous recurrences in higher dimensions, given by the family

$$x_{n+N} x_n = \sum_{j=1}^{N-1} x_{n+j} + b,$$

which have been shown to admit $\left\lfloor \frac{N}{2} \right\rfloor$ independent first integrals for each order N [34].

Unlike the special case $b = a^2$, which can be rescaled to (1.3), in general the orbits of (1.5) do not all have the same period, and generic orbits are not periodic over an infinite field (e.g. $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$). Moreover, while the iterates in (1.2) are Laurent polynomials in the initial values x_0, x_1 with integer coefficients, which is one of the characteristic features of the cluster variables in a cluster algebra, the iterates of (1.5) are not Laurent polynomials unless $b = a^2$. However, the general map does preserve the same symplectic form (1.4), and there is a conserved quantity $K = K(x, y)$ given by

$$K = \frac{xy(x+y) + a(x^2+y^2) + (a^2+b)(x+y) + ab}{xy}. \tag{1.6}$$

Thus the Lyness map (1.5) is integrable in the Liouville sense, and can be considered as a deformation of the periodic map (1.3) which arises from mutations in a finite type cluster algebra. The purpose of this work is to consider how other integrable maps can be obtained from deformations of cluster mutations. The Zamolodchikov periodicity of

Y-systems or T-systems associated with finite type root systems has been extended and generalized in various ways (see [14, 26, 30] and references), but as far as we are aware the deformations we consider are new.

Following the framework of cluster algebras, we start from a quiver Q (without 1- or 2-cycles) associated with a skew-symmetric *exchange matrix* $B = (b_{ij}) \in \mathrm{Mat}_N(\mathbb{Z})$ and an N-tuple of *cluster variables* $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. Here we consider the cluster variables x_i taking values in a field $\mathbb{F}$; the main cases of interest are $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, but for some of our later analysis it will be convenient to consider $x_i \in \mathbb{Q} \subset \mathbb{Q}_p$. The initial seed is denoted $(B, \mathbf{x})$. Now, for each integer $k \in [1, N]$ we define a mutation μ_k which produces a new seed $(B', \mathbf{x}') = \mu_k(B, \mathbf{x})$, where $B' = (b'_{ij})$ with

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \mathrm{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise,} \end{cases} \tag{1.7}$$

and $\mathbf{x}' = (x'_j)$ with

$$x'_j = \begin{cases} x_k^{-1}\, f_k(M_k^+, M_k^-) & \text{for } j = k \\ x_j & \text{for } j \neq k. \end{cases} \tag{1.8}$$

Here, $[a]_+ = \max(a, 0)$, $f_k : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is a differentiable function and

$$M_k^+ := \prod_{i=1}^N x_i^{[b_{ki}]_+} \ , \ M_k^- := \prod_{i=1}^N x_i^{[-b_{ki}]_+} \ .$$

For $f_k(M_k^+, M_k^-) = M_k^+ + M_k^-$, the first relation in (1.8) becomes the usual exchange relation $x'_k x_k = M_k^+ + M_k^-$ for cluster mutations in a coefficient-free cluster algebra. In this case, we know that there is a log-canonical presymplectic form compatible with cluster mutations [9, 15, 22]. We extend this result to include more general types of mutations.

Lemma 1.1. *Let Q be a quiver associated with the exchange matrix $B = (b_{ij})$ and $(B', \mathbf{x}') = \mu_k(B, \mathbf{x})$, as defined by* (1.7) *and* (1.8)*. Then*

$$\sum_{i<j} \frac{b'_{ij}}{x'_i x'_j} \mathrm{d}x'_i \wedge \mathrm{d}x'_j = \sum_{i<j} \frac{b_{ij}}{x_i x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j \tag{1.9}$$

if and only if

$$f_k(M_k^+, M_k^-) = M_k^+ g_k\left(\frac{M_k^-}{M_k^+}\right) \ , \tag{1.10}$$

for an arbitrary differentiable function $g_k : \mathbb{F} \to \mathbb{F}$.

Remark 1.2. Equivalently, the function f_k can be written in the form

$$f_k(M_k^+, M_k^-) = M_k^- \tilde{g}_k\left(\frac{M_k^+}{M_k^-}\right) \ ,$$

for $\tilde{g}_k$ arbitrary.

Proof: Using $\sum'$ to denote a sum over indices with index k omitted, we have

$$\begin{aligned}\omega &= \sum_{i<j} \frac{b_{ij}}{x_i x_j}\, \mathrm{d}x_i \wedge \mathrm{d}x_j \\ &= \tfrac{1}{2}\left(\textstyle\sum'_{i,j} b_{ij} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j + \sum'_i b_{ik} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_k + \sum'_j b_{kj} \mathrm{d}\log x_k \wedge \mathrm{d}\log x_j\right) \\ &= \tfrac{1}{2}\textstyle\sum'_{i,j} b_{ij} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j + \sum'_i b_{ik} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_k,\end{aligned}$$

and similarly

$$\begin{aligned}\omega' &= \sum_{i<j} \frac{b'_{ij}}{x'_i x'_j}\, \mathrm{d}x'_i \wedge \mathrm{d}x'_j \\ &= \tfrac{1}{2}\textstyle\sum'_{i,j} b'_{ij} \mathrm{d}\log x'_i \wedge \mathrm{d}\log x'_j + \sum'_i b'_{ik} \mathrm{d}\log x'_i \wedge \mathrm{d}\log x'_k \\ &= \tfrac{1}{2}\textstyle\sum'_{i,j} (b_{ij} + \mathrm{sgn}(b_{ik})[b_{ik}b_{kj}]_+) \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j - \sum'_i b_{ik} \mathrm{d}\log x_i \wedge (-\mathrm{d}\log x_k + \mathrm{d}\log f_k).\end{aligned}$$

Hence if we consider the sets

$$\beta_k^+ = \{i \in \{1, \dots N\} : b_{ki} > 0\}, \qquad \beta_k^- = \{i \in \{1, \dots N\} : b_{ki} < 0\},$$

then noting that $[b_{ik}b_{kj}]_+ = 0$ unless either $i \in \beta_k^+$, $j \in \beta_k^-$ or vice versa, and defining

$$\mathrm{d}S_k^\pm := \pm \mathrm{d}\log M_k^\pm = \sum_{i \in \beta_k^\pm} b_{ki} \mathrm{d}\log x_i,$$

we have

$$\begin{aligned}\omega' - \omega &= \tfrac{1}{2}\textstyle\sum'_{i,j} \mathrm{sgn}(b_{ik})[b_{ik}b_{kj}]_+ \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j - \sum'_i b_{ik} \mathrm{d}\log x_i \wedge \mathrm{d}\log f_k \\ &= \tfrac{1}{2}\left(\sum_{\substack{i \in \beta_k^- \\ j \in \beta_k^+}} b_{ik}b_{kj} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j - \sum_{\substack{i \in \beta_k^+ \\ j \in \beta_k^-}} b_{ik}b_{kj} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j\right) \\ &\quad + \textstyle\sum'_i b_{ki} \mathrm{d}\log x_i \wedge \left(\dfrac{M_k^+}{f_k}\dfrac{\partial f_k}{\partial M_k^+} \mathrm{d}\log M_k^+ + \dfrac{M_k^-}{f_k}\dfrac{\partial f_k}{\partial M_k^-} \mathrm{d}\log M_k^-\right) \\ &= -\sum_{\substack{i \in \beta_k^- \\ j \in \beta_k^+}} b_{ki}b_{kj} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j \\ &\quad + (\mathrm{d}S_k^+ + \mathrm{d}S_k^-) \wedge \left(\frac{M_k^+}{f_k}\frac{\partial f_k}{\partial M_k^+} \mathrm{d}S_k^+ - \frac{M_k^-}{f_k}\frac{\partial f_k}{\partial M_k^-} \mathrm{d}S_k^-\right) \\ &= \left(\frac{M_k^+}{f_k}\frac{\partial f_k}{\partial M_k^+} + \frac{M_k^-}{f_k}\frac{\partial f_k}{\partial M_k^-} - 1\right) \mathrm{d}S_k^- \wedge \mathrm{d}S_k^+.\end{aligned}$$

Hence $\omega' = \omega$ iff $f_k = f_k(M_k^+, M_k^-)$ satisfies the linear partial differential equation

$$M_k^+ \frac{\partial f_k}{\partial M_k^+} + M_k^- \frac{\partial f_k}{\partial M_k^-} = f_k,$$

of which the general solution is given by (1.10) with g_k arbitrary. □

According to Lemma 1.1, if the exchange matrix B remains invariant under a sequence of mutations of the form (1.10) then the map that is generated by the same sequence of cluster mutations will preserve a presymplectic form, i.e. the following theorem holds.

Theorem 1.3. *Let $\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_\ell}$, for $i_j \in \{1, \ldots, N\}$, $j \in \mathbb{N}$, be a sequence of mutations defined from* (1.7) *and* (1.8)*, with each function f_{i_j} being of the form* (1.10)*, such that*

$$\mu_{i_\ell} \cdots \mu_{i_2} \mu_{i_1}(B, \mathbf{x}) = (B, \tilde{\mathbf{x}}).$$

Then the map $\varphi : \mathbf{x} \mapsto \tilde{\mathbf{x}}$ preserves the two-form

$$\omega = \sum_{i<j}^{N} \frac{b_{ij}}{x_i x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j. \tag{1.11}$$

Remark 1.4. The preceding result admits a slight generalization to the case of cluster algebras (or quivers Q) with periodicity under mutations. In the most general setting, as described by Nakanishi [29], these are defined by an exchange matrix with the property that $\mu_{i_\ell} \cdots \mu_{i_2} \mu_{i_1}(B) = \hat{\rho}(B)$, where $\hat{\rho}$ is some permutation of $(1, 2, \ldots, N)$ acting on the indices (equivalently, on the nodes of the quiver Q). The particular case $\mu_m \cdots \mu_2 \mu_1(B) = \rho^m(B)$, for the cyclic permutation $\rho : (1, 2, \ldots, N) \mapsto (N, 1, 2, \ldots, N-1)$ was called cluster mutation-periodicity with period m by Fordy and Marsh [13], who gave a complete classification of the case $m = 1$. A straightforward adaptation of the above argument shows that if B is periodic, then the map $\varphi = \hat{\rho}^{-1} \mu_{i_\ell} \cdots \mu_{i_2} \mu_{i_1}$ leaves B invariant and preserves the corresponding log-canonical presymplectic form (1.11), in the sense that $\varphi^*(\omega) = \omega$. Lemma 2.3 in [12] covers the special case of this result for ordinary cluster mutations when B is cluster mutation-periodic with period 1, so $\varphi = \rho^{-1} \mu_1$ and the map can be written as a single recurrence relation. We shall consider an example of this with a generalized mutation in section 3. The slightly different (but closely related) problem of when an ordinary difference equation preserves a log-canonical Poisson bracket was considered in [7].

In the next section our aim is to generalize the example of the Lyness map (1.5), corresponding to the root system A_2, to other finite type root systems of type A, by taking mutations defined by affine functions f_k with additional parameters that destroy the Laurent property but preserve the two-form (1.11). Section 3 contains more general choices of mutations, starting from affine Dynkin diagrams, where the factors g_k in (1.10) involve Möbius transformations, which lead to travelling wave reductions of the discrete sine-Gordon equation. We end with a few final remarks.

2 Deformations of type A periodic maps

In this section, extra parameters are included in the regular exchange relation by taking $g_k(x) = b_k x + a_k$, since

$$f_k(M_k^+, M_k^-) = M_k^+ g_k\left(\frac{M_k^-}{M_k^+}\right) = a_k M_k^+ + b_k M_k^- \ . \tag{2.1}$$

Hence, according to Theorem 1.3, quivers which are periodic under a particular sequence of mutations (or more generally, are periodic up to a permutation) give rise to parametric cluster maps that preserve the presymplectic form (1.11). If the corresponding exchange matrix is non-singular the parametric cluster maps are symplectic. We begin by examining the case of A_2 in more detail, and then apply this approach to study the integrability of parametric cluster maps associated with the A_3 and A_4 quivers.

2.1 Deformed mutations for A_2 quiver

The exchange matrix of type A_2 is

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In this case, B corresponds to a cluster mutation-periodic quiver with period 1 and $M_1^+ = x_2$, $M_1^- = 1$. So, by the modification of Theorem 1.3 as in Remark 1.4, taking $\rho : (1,2) \mapsto (2,1)$, for any differentiable function $\tilde{g} : \mathbb{F} \to \mathbb{F}$ the map $\varphi = \rho^{-1}\mu_1$ given by

$$\varphi : (x_1, x_2) \mapsto \left(x_2, \frac{1}{x_1}\tilde{g}(x_2)\right) \ , \tag{2.2}$$

is symplectic with respect to $\omega = \frac{1}{x_1 x_2} \mathrm{d}x_1 \wedge \mathrm{d}x_2$. (Compared with (1.10) we have $f_1(x,1) = x g_1(1/x) = \tilde{g}(x)$: in general, replacing $g_k(x) \to x g_k(1/x)$ corresponds to sending $B \to -B$, which is equivalent to replacing the corresponding quiver $Q \to Q^{opp}$, the same quiver with all arrows reversed; see also Remark 1.2.)

With $(x, y) = (x_1, x_2)$ and $\tilde{g}(x) = ax + b$, we reproduce the Lyness map (1.5). Starting from the periodic map (1.3), and relabelling the initial data as (x_0, x_1), any cyclic symmetric function of the iterates x_0, x_1, x_2, x_3, x_4 in the periodic orbit (1.2) gives a first integral. So in the periodic case there are two independent integrals, namely

$$\begin{aligned} K_1 &= \sum_{j=0}^{4} x_j = -3 + \prod_{j=0}^{4} x_j = \frac{x_0^2 x_1 + x_0 x_1^2 + x_0^2 + x_1^2 + 2(x_0 + x_1) + 1}{x_0 x_1}, \\ K_2 &= \sum_{j=0}^{4} x_j x_{j+1} \\ &= \frac{x_0 x_1 (x_0^2 x_1^2 + x_0^3 + x_1^3 + x_0^2 + x_1^2 + x_0 + x_1 + 2) + x_0^3 + x_1^3 + 2(x_0^2 + x_1^2) + x_0 + x_1}{x_0^2 x_1^2}. \end{aligned}$$

Both of the latter are sums of Laurent monomials, so in the case of the map with parameters, first integrals can be sought by taking arbitrary linear combinations of the same monomials and solving the resulting conditions on the coefficients. Thus in the case of (1.5), the first integral (1.6) can be considered as a deformation of K_1 above; but a first integral composed of the Laurent monomials in K_2 only exists when $b = a^2$ and the map is periodic, corresponding to the undeformed situation.

Although the Laurent phenomenon does not persist for the iterates of the Lyness recurrence

$$x_{n+2}x_n = ax_{n+1} + b \tag{2.3}$$

when $b \neq a^2$, it was pointed out in [12] that there is a connection to a cluster algebra via a lift to a space of higher dimension, defined by the substitution

$$x_n = \frac{\tau_{n+5}\tau_n}{\tau_{n+3}\tau_{n+2}},$$

which leads to the Somos-7 recurrence

$$\tau_{n+7}\tau_n = a\,\tau_{n+6}\tau_{n+1} + b\,\tau_{n+4}\tau_{n+3}. \tag{2.4}$$

As explained in [13], Somos-type recurrences such as the above, with a sum of two monomials on the right-hand side, can be generated by mutations in a cluster algebra. In the case of (2.4), it is a cluster algebra of rank 7, extended by the addition of the parameters a, b as frozen variables.

The rest of this section is devoted to the analogous constructions for A_3 and A_4.

2.2 A_3 quiver with parameters

For the A_3 quiver with exchange matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

as in Figure 1, we take $f_k(M_k^+, M_k^-) = a_k M_k^+ + b_k M_k^-$. In this case,

$$\varphi(B, \mathbf{x}) := \mu_3\mu_2\mu_1(B, \mathbf{x}) = \big(B, \varphi(\mathbf{x})\big),$$

where the composition $\varphi = \mu_3\mu_2\mu_1$ acts on the cluster variables $\mathbf{x} = (x_1, x_2, x_3)$ according to

$$\begin{array}{lll} \mu_1: & (x_1, x_2, x_3) \mapsto (x_1', x_2, x_3), & x_1'x_1 = b_1 + a_1x_2, \\ \mu_2: & (x_1', x_2, x_3) \mapsto (x_1', x_2', x_3), & x_2'x_2 = b_2 + a_2x_1'x_3, \\ \mu_3: & (x_1', x_2', x_3) \mapsto (x_1', x_2', x_3'), & x_3'x_3 = b_3 + a_3x_2'. \end{array} \tag{2.5}$$

Since $\varphi(B) = B$, so the exchange matrix B remains invariant under this sequence of mutations, by Theorem 1.3 the map φ preserves the corresponding log-canonical two-form, that is

$$\varphi^*(\omega) = \omega,$$

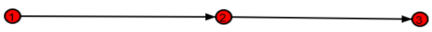


Figure 1: The A_3 quiver.

where

$$\omega = \frac{1}{x_1 x_2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{1}{x_2 x_3} \mathrm{d}x_2 \wedge \mathrm{d}x_3 \ .$$

The original coefficient-free cluster algebra is given by setting $a_i = 1 = b_i$ for $i = 1, 2, 3$, and in that case the map φ is periodic with period 6, that is $\varphi^6(\mathbf{x}) = \mathbf{x}$. Moreover, one can write down three independent first integrals for the periodic map, by taking appropriate symmetric functions along each orbit, such as $\sum_{i=0}^{5} (\varphi^*)^i(x_j)$, $\prod_{i=0}^{5} (\varphi^*)^i(x_j)$, etc.

However, before considering the deformed case (2.5), there are two ways to simplify the calculations. First of all, assuming the case of generic parameter values $a_i b_i \neq 0$ for all i, we apply the scaling action of the three-dimensional algebraic torus $(\mathbb{F}^*)^3$, given by $x_i \to \lambda_i \, x_i$, $\lambda_i \neq 0$, and use this to remove three parameters, so that we obtain

$$a_1 \to 1, \quad b_1 \to 1, \quad a_2 \to d, \quad b_2 \to c, \quad a_3 \to 1, \quad b_3 \to e,$$

where c, d, e are arbitrary. Having simplified the space of parameters, the map φ is equivalent to iteration of the system of recurrences

$$\begin{aligned} x_{1,n+1} x_{1,n} &= x_{2,n} + 1, \\ x_{2,n+1} x_{2,n} &= d x_{1,n+1} x_{3,n} + c, \\ x_{3,n+1} x_{3,n} &= x_{2,n+1} + e. \end{aligned} \tag{2.6}$$

Secondly, because we are in an odd-dimensional situation where B necessarily has determinant zero, so that ω is degenerate, so following [12] (cf. Theorem 2.6 therein) we can use

$$\ker B = < (1, 0, 1)^T >, \qquad \mathrm{im}\, B = (\ker B)^{\perp} = < (0, 1, 0)^T, (-1, 0, 1)^T >$$

to generate the one-parameter scaling group $(x_1, x_2, x_3) \to (\lambda x_1, x_2, \lambda x_3)$ and the projection π onto its monomial invariants,

$$\pi : \qquad y = x_2, \qquad w = \frac{x_3}{x_1}.$$

On the y, w-plane, φ induces the reduced map

$$\hat{\varphi} : \qquad \begin{pmatrix} y \\ w \end{pmatrix} \mapsto \begin{pmatrix} \big(d(y+1)w + c\big)/y \\ (dw + c)/(yw) + (e - c)/\big(w(y+1)\big) \end{pmatrix}, \tag{2.7}$$

which is symplectic, preserving the nondegenerate two-form

$$\hat{\omega} = \mathrm{d}\log y \wedge \mathrm{d}\log w, \qquad \pi^*\hat{\omega} = \omega. \tag{2.8}$$

In the original case where all parameters are 1, the reduced map (2.7) with $c = d = e = 1$ has period 3, because $x_{2,n+3} = x_{2,n}$ and $x_{3,n+3}/x_{1,n+3} = x_{3,n}/x_{1,n}$ for all n. Thus in that case there are two functionally independent first integrals in the plane, which can be taken as

$$\begin{array}{rcccl} K_1 & = & \prod_{i=0}^{2}(\hat{\varphi}^*)^i(y) & = & \frac{(yw+w+1)(y+w+1)}{yw} = -2 + \sum_{i=0}^{2}(\hat{\varphi}^*)^i(y), \\ K_2 & = & \sum_{i=0}^{2}(\hat{\varphi}^*)^i(w) & = & \frac{yw^3+yw^2+y^2w+w^2+2w+1}{yw(w+1)} \end{array} \tag{2.9}$$

(while the product $\prod_{i=0}^{2}(\hat{\varphi}^*)^i(w) = 1$, so does not give a nontrivial integral).

Next, we modify K_1 and K_2 by inserting constant coefficients in front of each of their terms, which are all Laurent monomials in K_1, while for K_2 we can replace the term $w+1$ in the denominator by an arbitrary linear function of w. If we require that (at least) one of these modified integrals should be preserved by the map $\hat{\varphi}$, then this puts a finite number of constraints on the coefficients and parameters c, d, e, which are necessary and sufficient for the deformed symplectic map to be Liouville integrable. Thus we obtain the following result.

Theorem 2.1. *The condition*

$$c = e$$

is necessary and sufficient for the symplectic map (2.7) to admit a deformation of the first integral K_1, given by

$$K_1 = \frac{(yw + w + d)(y + dw + c)}{yw}, \tag{2.10}$$

hence $\hat{\varphi}$ is integrable whenever this condition holds. Requiring that a deformation of K_2 should be preserved imposes the stronger conditions

$$c = d^2 = e,$$

in which case both

$$K_2 = \frac{w^3y + d(y+1)w^2 + (y^2 + 2d^2)w + d^3}{yw(w+d)} \tag{2.11}$$

and K_1 given by (2.10) with $c = d^2$ are preserved, and all the orbits of $\hat{\varphi}$ are periodic with period 3.

Proof: Starting from a general sum of monomials

$$K_1 = y + \alpha\, w + \beta\, \frac{w}{y} + \frac{\gamma}{y} + \frac{\delta}{w} + \frac{\epsilon}{yw} + \mathrm{const}$$

(where we have fixed the scale by assuming that the first term has coefficient 1, and there is the freedom to add an arbitrary constant), we apply the map (2.7) and require

that $\hat{\varphi}^*(K_1) = K_1$. Comparing the rational functions one each side of the latter equation imposes the requirement $c = e$ and fixes $\alpha = \beta = d$, $\gamma = c+d^2$, $\delta = d$, $\epsilon = cd$; then choosing to add the constant $c + 1$ means that K_1 can be factored as in (2.10). Applying the same approach to K_2 requires the additional constraint $c = d^2$, restricting to the one-parameter family of period 3 maps

$$\hat{\varphi} : \qquad \begin{pmatrix} y \\ w \end{pmatrix} \mapsto \begin{pmatrix} \big(d(y+1)w + d^2\big)/y \\ d(w+d)/(yw) \end{pmatrix},$$

which have two independent first integrals given by (2.10) with $c = d^2$ and (2.11). □

Remark 2.2. When $c = e$, the integrable symplectic map

$$\hat{\varphi} : \qquad \begin{pmatrix} y \\ w \end{pmatrix} \mapsto \begin{pmatrix} \big(d(y+1)w + c\big)/y \\ (dw + c)/(yw) \end{pmatrix}, \tag{2.12}$$

preserves the pencil of biquadratic curves defined by (2.10), which means that there is a map of QRT type [5, 31] preserving the same pencil, given by the composition of the horizontal and vertical switch on each curve in the pencil, namely

$$\hat{\psi} : \qquad \begin{pmatrix} y \\ w \end{pmatrix} \mapsto \begin{pmatrix} \bar{y} \\ \bar{w} \end{pmatrix}, \qquad \bar{y}y = \frac{(dw + c)(w + d)}{w}, \quad \bar{w}w = \frac{\bar{y} + c}{\bar{y} + 1}. \tag{2.13}$$

From general considerations about automorphisms of elliptic curves, since they each correspond to translation by a point, these two maps should commute with one another, and indeed it is straightforward to verify that

$$\hat{\psi} \circ \hat{\varphi} = \hat{\varphi} \circ \hat{\psi}.$$

However, it appears that generically the two maps correspond to translation by two independent points of infinite order, so (over $\mathbb{Q}$, say) this should generate a family of curves with Mordell-Weil group of rank at least 2. (As a special case, when $c = d = 1$ the map $\hat{\psi}$ has period 2 for any initial data, corresponding to translation by a 2-torsion point, whereas the period 3 map $\hat{\varphi}$ corresponds to addition of a 3-torsion point; so the points are independent, albeit not of infinite order in this case.)

We now treat the singularity pattern of the iterates of (2.12), in order to obtain its Laurentification in the sense of [17], i.e. a lift to a map with the Laurent property in a space of higher dimension, in which the new variables can be regarded as tau functions. Rather than a standard singularity confinement analysis, we study orbits defined over $\mathbb{Q}$, and consider a p-adic analogue of confinement, as in [24]. The possible singularity patterns can then be obtained using the empirical approach introduced in [20], simply by inspecting the prime factorization of a few terms along a particular orbit.

Thus we choose some particular values for the coefficients and initial data: taking $c = 2$, $d = 3$ and $(y_0, w_0) = (1, 1)$, we find the first few iterates are

$$(8, 5), (\tfrac{137}{8}, \tfrac{17}{40}), (\tfrac{1607}{1096}, \tfrac{1048}{2329}), (\tfrac{800200}{220159}, \tfrac{1068874}{210517}), (\tfrac{3210496223}{160740175}, \tfrac{728705399}{780395050}), (\tfrac{7129742296469}{2344013756975}, \tfrac{2735651842025}{10626437852503}),$$

so that the values of y_n for $n = 1, 2, 3, \ldots$ factorize as

$$2^3, \tfrac{137}{2^3}, \tfrac{1607}{2^3 \cdot 137}, \tfrac{2^3 \cdot 5^2 \cdot 4001}{137 \cdot 1607}, \tfrac{11 \cdot 17 \cdot 113 \cdot 137 \cdot 1109}{5^2 \cdot 1607 \cdot 4001}, \tfrac{13 \cdot 19 \cdot 43 \cdot 1607 \cdot 417727}{5^2 \cdot 11 \cdot 17 \cdot 113 \cdot 1109 \cdot 4001}, \ldots,$$

while the factorizations of the corresponding values of w_n are

$$5, \tfrac{17}{2^3 \cdot 5}, \tfrac{2^3 \cdot 131}{17 \cdot 137}, \tfrac{2 \cdot 47 \cdot 83 \cdot 137}{131 \cdot 1607}, \tfrac{467 \cdot 971 \cdot 1607}{2 \cdot 5^2 \cdot 47 \cdot 83 \cdot 4001}, \tfrac{5^2 \cdot 4001 \cdot 27349681}{11 \cdot 17 \cdot 113 \cdot 467 \cdot 971 \cdot 1109}, \ldots,$$

and so on. For the primes $p = 113, 137, 1607, 4001$, the values of the p-adic norm $|y_n|_p$ follow the pattern $1, p^{-1}, p, p, p^{-1}, 1$, with the corresponding values of $|w_n|_p$ being $1, 1, p, p^{-1}, 1, 1$, while for the primes $p = 2$ and 5 there are instances of the same patterns but with $p \to p^3$ and $p \to p^2$, respectively. (For some of these primes, the whole pattern is not visible above, but it can easily be verified by computing the next few terms, which are omitted here.) In w_n there are also other primes that do not appear in y_n, e.g. $p = 17, 47, 83, 131, 467, 971$, and for these the pattern of $|w_n|_p$ is $1, p^{-1}, p, 1$. This immediately suggests that y_n, w_n can be written using two different tau functions σ_n, τ_n, as

$$\tilde{\pi} : \qquad y_n = \frac{\tau_{n-2}\tau_{n+1}}{\tau_{n-1}\tau_n}, \quad w_n = \frac{\sigma_{n+1}\tau_{n-1}}{\sigma_n \tau_n}, \tag{2.14}$$

so that the first type of p-adic singularity corresponds to $\tau_n \equiv 0 \bmod p$ for some n, and the second occurs when $\sigma_n \equiv 0 \bmod p$.

Our next goal is to show that the tau functions in (2.14) satisfy a system of bilinear equations, namely

$$\begin{aligned} \sigma_{n+2}\tau_{n-2} &= d\,\sigma_{n+1}\tau_{n-1} + c\,\sigma_n\tau_n, \\ \sigma_n\tau_{n+2} &= \sigma_{n+2}\tau_n + d\,\sigma_{n+1}\tau_{n+1} \end{aligned} \tag{2.15}$$

(we expect that these could be viewed as a reduction of coupled discrete Hirota equations [4, 36]), and to prove that this system has the Laurent property. The first equation in (2.15) is straightforward to obtain, as it arises directly from substituting the tau function expressions (2.14) into the second component of (2.12), rewritten in the form of a recurrence, but the second bilinear equation requires more work. If we look at the singularity pattern in the original three-dimensional system (2.6) with $e = c$, then we see that

$$x_{1,n} = \rho_n \frac{\sigma_{n+1}}{\tau_n}, \qquad x_{3,n} = \rho_n \frac{\sigma_n}{\tau_{n-1}},$$

with a new prefactor ρ_n appearing, while $x_{2,n} = y_n$ is already accounted for. Substituting in these formulae to rewrite the system (2.6) in terms of ρ_n, σ_n, τ_n yields

$$\begin{aligned} \rho_n\rho_{n+1}\,\sigma_{n+1}\sigma_n &= \tau_{n+1}\tau_{n-2} + \tau_n\tau_{n-1}, \\ \tau_{n+2}\tau_{n-2} &= \rho_n\rho_{n+1}\,d\,\sigma_{n+1}^2 + c\,\tau_n^2, \\ \rho_n\rho_{n+1}\,\sigma_{n+2}\sigma_{n+1} &= \tau_{n+2}\tau_{n-1} + c\,\tau_{n+1}\tau_n. \end{aligned} \tag{2.16}$$

For the above system, the initial values are $\rho_0, \sigma_0, \sigma_1, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1$, and in principle one could use this to give a direct proof that the sequences (σ_n) and (τ_n) are Laurent polynomials in the initial data, although the sequence ρ_n is not. However, note that the product

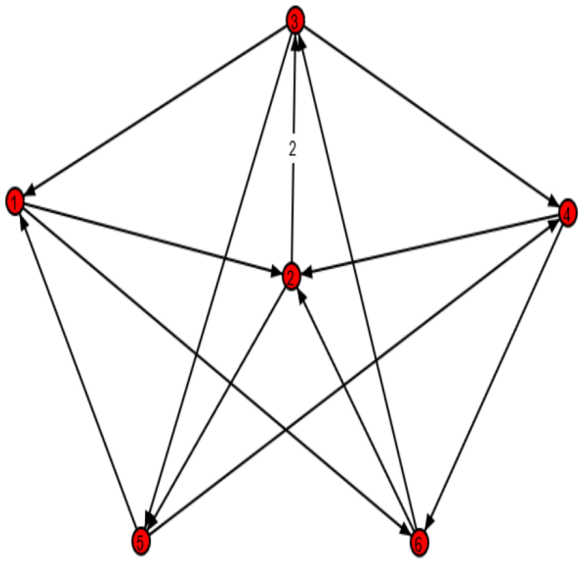


Figure 2: The initial quiver Q associated with the exchange matrix (2.18).

$\rho_n\rho_{n+1}$ can be eliminated from any two of the equations in (2.16), so doing this for each pair gives a set of three equations of degree 3, and then eliminating τ_{n+2} from any two of the latter results in the first equation in (2.15), while eliminating τ_{n+2} instead produces the relation

$$\sigma_n\tau_{n+2}\tau_{n-2} = d\,\sigma_{n+1}(\tau_{n+1}\tau_{n-2} + \tau_n\tau_{n-1}) + c\,\sigma_n\tau_n^2.$$

Finally, the second relation in (2.15) follows by combining the first relation with the above to eliminate τ_{n-2}.

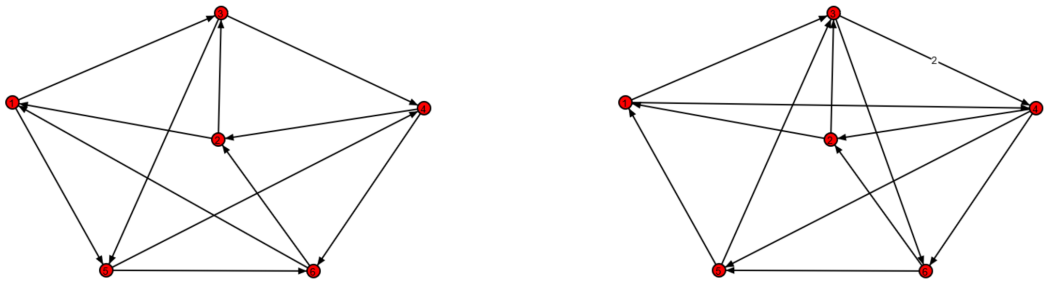
Immediate evidence for the Laurent property can be seen by iterating the system (2.15) for $c = 2$, $d = 3$ with all initial values $\tau_{-2} = \tau_{-1} = \tau_0 = \tau_1 = \sigma_0 = \sigma_1 = 1$, corresponding to the initial values $y_0 = w_0 = 1$ in the orbit considered above. The first few terms are the integers

$$\begin{array}{lllllllll} (\tau_n)_{n\geq 1}: & 1, & 8, & 137, & 1607, & 100025, & 23434279, & 4436678467, & 1750170148834, \\ (\sigma_n)_{n\geq 1}: & 1, & 5, & 17, & 131, & 7802, & 453457, & 27349681, & 18332191183, \end{array}$$

and so on. It is also easy to verify directly that the first few terms τ_2, σ_1, etc. obtained by iteration of (2.15) are Laurent polynomials in the initial data with coefficients belonging to $\mathbb{Z}[c, d]$.

To make further progress, it is helpful to consider the initial data for (2.15) as a set of cluster variables $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6) = (\tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1)$, and calculate the pullback of the symplectic form (2.8) by the map $\tilde{\pi}$ defined by the tau function expressions (2.14), that is

$$\tilde{\omega} = \tilde{\pi}^*\hat{\omega} = \sum_{i<j} b^*_{ij}\mathrm{d}\log\tilde{x}_i \wedge \mathrm{d}\log\tilde{x}_j, \tag{2.17}$$



(a) The quiver $\tilde{\mu}_1(Q)$. (b) The quiver $\tilde{\mu}_5\tilde{\mu}_1(Q)$.

Figure 3: The effect of two mutations on the quiver corresponding to (2.18).

where $B^* = (b^*_{ij})$ is the skew-symmetric matrix

$$B^* = \begin{pmatrix} 0 & 1 & -1 & 0 & -1 & 1 \\ -1 & 0 & 2 & -1 & 1 & -1 \\ 1 & -2 & 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \end{pmatrix}. \tag{2.18}$$

The quiver corresponding to this matrix is shown in Figure 2. It is not hard to see that, when $c = 1 = d$, the bilinear equations (2.15) for $n = 0$ are generated by applying a mutation at node 1, denoted by $\tilde{\mu}_1$ (to distinguish it from mutations in the original A_3 quiver), followed by mutation $\tilde{\mu}_5$: see Figure 3. To prove the Laurent property for the case of arbitrary coefficients, it is necessary to extend the quiver with two extra frozen nodes.

Theorem 2.3. *The sequences of tau functions (σ_n) and (τ_n) for the integrable map (2.12) consist of elements of the Laurent polynomial ring $\mathbb{Z}_{>0}[c, d, \tau_{-2}^{\pm 1}, \tau_{-1}^{\pm 1}, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \sigma_0^{\pm 1}, \sigma_1^{\pm 1}]$, being generated by a sequence of mutations in a cluster algebra defined by the quiver in Figure 2 with the addition of two frozen nodes.*

Proof: In order to include the coefficients, we define an extended cluster $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_8) = (\tau_{-2}, \ldots, \tau_1, \sigma_0, \sigma_1, c, d)$, where $\tilde{x}_7 = c$ and $\tilde{x}_8 = d$ are frozen variables, and take an extended exchange matrix

$$\tilde{B}^* = \begin{pmatrix} 0 & 1 & -1 & 0 & -1 & 1 \\ -1 & 0 & 2 & -1 & 1 & -1 \\ 1 & -2 & 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 & 0 \end{pmatrix}, \tag{2.19}$$

where two more rows have been appended to (2.18). (The diagram of the quiver with the additional arrows to/from the frozen nodes does not look quite so clear compared with Figure 2, so it has been omitted.) Applying the mutation $\tilde{\mu}_1$ gives the exchange relation

$$\sigma_2\tau_{-2} = d\,\sigma_1\tau_{-1} + c\,\sigma_0\tau_0,$$

and produces a new cluster $(\sigma_2, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, c, d)$ and a new matrix $\tilde{\mu}_1(\tilde{B}^*)$ corresponding to the quiver in Figure 3(a) with appropriate arrows to/from the frozen nodes 7 and 8. Next, by applying the mutation $\tilde{\mu}_5$, the exchange relation is

$$\tau_2\sigma_0 = d\,\sigma_1\tau_1 + \sigma_2\tau_0,$$

with the new cluster being $(\sigma_2, \tau_{-1}, \tau_0, \tau_1, \tau_2, \sigma_1, c, d)$, and the new exchange matrix $\tilde{\mu}_5\tilde{\mu}_1(\tilde{B}^*)$ corresponding to the quiver in Figure 3(b) with suitable extra arrows added to take the coefficients into account. Continuing in a similar way, we find a sequence of mutations to successively generate $\sigma_3, \tau_3, \sigma_4, \tau_4$, and so on, such that overall after applying the composition of 12 mutations given by

$$\tilde{\mu}_{463524136251} := \tilde{\mu}_4\tilde{\mu}_6\tilde{\mu}_3\tilde{\mu}_5\tilde{\mu}_2\tilde{\mu}_4\tilde{\mu}_1\tilde{\mu}_3\tilde{\mu}_6\tilde{\mu}_2\tilde{\mu}_5\tilde{\mu}_1 \tag{2.20}$$

(in order from right to left), the quiver returns to its starting position; so we have

$$\tilde{\mu}_{463524136251}(\tilde{B}^*) = \tilde{B}^*, \qquad \tilde{\mu}_{463524136251}(\tilde{\mathbf{x}}) = (\tau_4, \tau_5, \tau_6, \tau_7, \sigma_6, \sigma_7, c, d),$$

with the index of each of the tau functions increased by 6. Hence by induction both sequences (σ_n), (τ_n) are generated by repeatedly applying this composition of mutations, and the Laurent property follows from the fact that the tau functions are all elements of the cluster algebra, for which it is also known that the Laurent polynomials in the initial data have positive integer coefficients [16, 27]. □

Remark 2.4. Preliminary calculations suggest that the iterates of the QRT map (2.13), which commutes with $\hat{\varphi}$, have a different singularity structure, corresponding to a tau function substitution of the form

$$y_n = \frac{\eta_n}{\sigma_n\tau_{n-1}}, \quad w_n = \frac{\sigma_{n+1}\tau_{n-1}}{\sigma_n\tau_n},$$

where η_n has weight two. It would be interesting to see whether this has a cluster algebra interpretation.

2.3 A_4 quiver with parameters

For the exchange matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

corresponding to the quiver of type A_4, once again we start from functions of the form $f_k(M_k^+, M_k^-) = a_k M_k^+ + b_k M_k^-$, with arbitrary coefficients such that $a_k b_k \neq 0$. By rescaling $x_j \to \lambda_j\, x_j$ with $\lambda_j \in \mathbb{F}^*$, we can set four of the parameters to 1, so that it is sufficient to consider a four-parameter family of mutations, given by

$$\begin{array}{lll} \mu_1: & (x_1, x_2, x_3, x_4) \mapsto (x_1', x_2, x_3, x_4), & x_1' x_1 = b_1 + a_1 x_2, \\ \mu_2: & (x_1', x_2, x_3, x_4) \mapsto (x_1', x_2', x_3, x_4), & x_2' x_2 = 1 + x_1' x_3, \\ \mu_3: & (x_1', x_2', x_3, x_4) \mapsto (x_1', x_2', x_3', x_4), & x_3' x_3 = 1 + x_2' x_4, \\ \mu_4: & (x_1', x_2', x_3', x_4) \mapsto (x_1', x_2', x_3', x_4'), & x_4' x_4 = b_4 + a_4 x_3'. \end{array} \tag{2.21}$$

Then, defining the action of $\varphi = \mu_4\mu_3\mu_2\mu_1$ on the cluster $\mathbf{x} = (x_1, x_2, x_3, x_4)$ as above,

$$\varphi(B, \mathbf{x}) := \mu_4\mu_3\mu_2\mu_1(B, \mathbf{x}) = \big(B, \varphi(\mathbf{x})\big),$$

so the nondegenerate exchange matrix B remains invariant under this sequence of mutations, and according to Theorem 1.3 the map

$$\mathbf{x} \mapsto \varphi(\mathbf{x})$$

is symplectic with respect to

$$\omega = \frac{1}{x_1 x_2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{1}{x_2 x_3} \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \frac{1}{x_3 x_4} \mathrm{d}x_3 \wedge \mathrm{d}x_4\ . \tag{2.22}$$

Equivalently, by computing the inverse matrix $P = B^{-1} = (p_{ij})$, the map φ preserves the nondegenerate Poisson bracket given by $\{\, x_i, x_j \,\} = p_{ij}\, x_i x_j$, which has the explicit form

$$\{\, x_2, x_1 \,\} = x_2 x_1, \qquad \{\, x_4, x_1 \,\} = x_4 x_1, \qquad \{\, x_4, x_3 \,\} = x_4 x_3, \tag{2.23}$$

with all other brackets being zero.

In the original case of the undeformed quiver, corresponding to $a_1 = a_4 = b_1 = b_4 = 1$ in (2.21), the map φ is completely periodic with period 7, and admits four independent integrals in dimension four. Here we focus on

$$I_1 = \sum_{j=0}^{6} (\varphi^*)^j(x_1), \qquad I_2 = \prod_{j=0}^{6} (\varphi^*)^j(x_1), \tag{2.24}$$

since in the undeformed case these Poisson commute with respect to the bracket (2.23), that is

$$\{\, I_1, I_2 \,\} = 0. \tag{2.25}$$

Being a sum/product of cluster variables in the (finite) A_4 cluster algebra, both of these integrals are Laurent polynomials in terms of the initial cluster $\mathbf{x}$, so to deform them we can just take arbitrary linear combinations of the Laurent monomials that appear.

Theorem 2.5. *The conditions*

$$b_1 = 1 = b_4 \tag{2.26}$$

on the parameters a_i, b_i (for $i = 1, 4$) in (2.21) are necessary and sufficient for the first integrals defined by (2.24) in the periodic case to deform to a pair of rational conserved quantities for the symplectic map $\varphi = \mu_4\mu_3\mu_2\mu_1$ that are in involution, i.e. they satisfy (2.25) with respect to the Poisson bracket (2.23). Hence the resulting two-parameter family of maps φ is Liouville integrable, with the two functionally independent commuting integrals

$$\begin{aligned} I_1 =& \frac{1}{x_1x_2x_3x_4}\Big(a_1a_4x_1x_2 + a_1{a_4}^2x_1x_2x_3 + a_1x_1x_2x_3 + a_1a_4x_1x_2{x_3}^2 + a_1a_4x_1x_4 \\ &+ a_1a_4x_1{x_2}^2x_4 + a_1a_4x_3x_4 + a_1a_4{x_1}^2x_3x_4 + a_4x_2x_3x_4 + {a_1}^2a_4x_2x_3x_4 + a_4{x_1}^2x_2x_3x_4 \\ &+ a_1a_4{x_2}^2x_3x_4 + a_1a_4x_1{x_3}^2x_4 + a_1a_4x_1x_2{x_4}^2 + a_1x_1x_2x_3{x_4}^2\Big), \\ I_2 =& \frac{(a_1+x_2)(x_1+x_3)\,(a_4+x_3)\,(x_2+x_4)\,(x_1x_2 + a_4x_1x_2x_3 + x_1x_4 + x_3x_4 + a_1x_2x_3x_4)}{x_1{x_2}^2{x_3}^2x_4}. \end{aligned}$$

Proof: The calculation of the conditions on the coefficients of the monomials appearing in the deformed versions of the integrals (2.24) is direct, and leads to the above forms of I_1, I_2 together with the requirement that b_1 and b_4 should both equal 1. An explicit calculation of their Poisson bracket then shows that the deformed integrals are also in involution, as required for Liouville integrability. □

Table 1: Prime factors in an orbit of the integrable deformed A_4 map with $a_1 = 2, a_4 = 3$.

n	0	1	2	3	4	5	6	7	8	9	10	11
x_1	1	3	3	3	7	$\frac{2^2}{7}$	$\frac{151}{2^2\cdot 5}$	$\frac{5\cdot 11\cdot 61}{7\cdot 151}$	$\frac{7\cdot 251}{11\cdot 61}$	$\frac{3\cdot 11\cdot 571}{5^2\cdot 251}$	$\frac{3\cdot 5^2\cdot 7\cdot 5653}{11\cdot 19\cdot 23\cdot 571}$	$\frac{3\cdot 19\cdot 23\cdot 54403}{7\cdot 137\cdot 5653}$
x_2	1	2^2	2^2	$2\cdot 5$	$\frac{3}{2}$	$\frac{2\cdot 29}{5\cdot 7}$	$\frac{643}{2^3\cdot 7}$	$\frac{2^3\cdot 3\cdot 23}{151}$	$\frac{5233}{5^2\cdot 61}$	$\frac{2\cdot 61613}{19\cdot 23\cdot 251}$	$\frac{1031\cdot 5519}{11\cdot 137\cdot 571}$	$\frac{2\cdot 11\cdot 569\cdot 42043}{5^2\cdot 353\cdot 5653}$
x_3	1	5	13	2	$\frac{13}{5}$	$\frac{3^2\cdot 13}{7^2}$	$\frac{2\cdot 71}{11}$	$\frac{11\cdot 17\cdot 89}{5^2\cdot 151}$	$\frac{79\cdot 3529}{11\cdot 19\cdot 23\cdot 61}$	$\frac{1431173}{7\cdot 137\cdot 251}$	$\frac{7\cdot 73\cdot 51539}{5^2\cdot 353\cdot 571}$	$\frac{13\cdot 17\cdot 43\cdot 237379}{7\cdot 5653\cdot 7507}$
x_4	1	2^4	$\frac{5}{2}$	$\frac{2\cdot 7}{5}$	$\frac{2\cdot 11}{7}$	$\frac{2^3\cdot 5^2}{7\cdot 11}$	$\frac{7\cdot 19\cdot 23}{2^3\cdot 5^2}$	$\frac{2^6\cdot 7\cdot 137}{19\cdot 23\cdot 151}$	$\frac{2\cdot 5^2\cdot 151\cdot 353}{7\cdot 11\cdot 61\cdot 137}$	$\frac{2\cdot 11\cdot 61\cdot 7507}{5^2\cdot 251\cdot 353}$	$\frac{19\cdot 101\cdot 251\cdot 359}{11\cdot 571\cdot 7507}$	$\frac{2^8\cdot 11\cdot 571\cdot 109943}{7\cdot 19\cdot 101\cdot 359\cdot 5653}$

To determine the singularity structure of the integrable map φ we consider a particular rational orbit with parameters $a_1 = 2, a_4 = 3$ and all initial x_j equal to 1 (see Table 1). Applying the empirical p-adic method from [20] once more, we observe that in the numerators of x_2 and x_3 there are certain primes that do not appear elsewhere, e.g. there are isolated values of n where $|x_{2,n}|_p = p^{-1}$ for $p = 29, 643, 5233, 61613$, and similarly there are isolated n where $|x_{3,n}|_p = p^{-1}$ for $p = 17, 71, 79, 89, 3529, 1431173$. On the other hand, for $p = 61, 151, 251, 571$ there are particular values of n where $|x_{1,n}|_p = |x_{2,n}|_p = |x_{3,n}|_p = |x_{4,n}|_p = p$ and also $|x_{1,n-1}|_p = p^{-1}$, $|x_{4,n+1}|_p = p^{-1}$. Also for $p = 137, 353, 7507$ there is a

pattern where p first appears in the numerator of x_4, then in its denominator at the next step, then successively in the denominators of x_3, x_2, x_1, before appearing in the numerator of x_1, then disappearing at the 7th step (some of the factorizations required to see this are omitted from Table 1 for reasons of space); the product of primes $19 \cdot 23$ exhibits the same pattern, although these primes also appear separately elsewhere. These four singularity patterns in the iterates of φ suggest introducing four tau functions $\eta_n, \theta_n, \sigma_n, \tau_n$, where the first two have weight two and the last two have weight one, such that

$$\tilde{\pi}: \qquad x_{1,n} = \frac{\sigma_n \tau_{n+1}}{\sigma_{n+1}\tau_n}, \quad x_{2,n} = \frac{\eta_n}{\sigma_{n+2}\tau_n}, \quad x_{3,n} = \frac{\theta_n}{\sigma_{n+3}\tau_n}, \quad x_{4,n} = \frac{\sigma_{n+5}\tau_{n-1}}{\sigma_{n+4}\tau_n}, \tag{2.27}$$

and direct substitution into the recurrence versions of (2.21) with $b_1 = 1 = b_4$, replacing $x_j \to x_{j,n}$, $x'_j \to x_{j,n+1}$, gives the system

$$\begin{aligned} \tau_{n+2}\sigma_n &= \tau_n\sigma_{n+2} + a_1\,\eta_n, \\ \eta_{n+1}\eta_n &= \sigma_{n+1}\tau_{n+2}\theta_n + \sigma_{n+2}\sigma_{n+3}\tau_n\tau_{n+1}, \\ \theta_{n+1}\theta_n &= \sigma_{n+5}\tau_{n-1}\eta_{n+1} + \sigma_{n+3}\sigma_{n+4}\tau_n\tau_{n+1}, \\ \sigma_{n+6}\tau_{n-1} &= \sigma_{n+4}\tau_{n+1} + a_4\,\theta_{n+1}. \end{aligned} \tag{2.28}$$

Initial evidence that this system has the Laurent property is provided by setting $\sigma_0 = \cdots = \sigma_5 = \eta_0 = \theta_0 = \tau_{-1} = \tau_0 = \tau_1 = 1$, corresponding to all initial $x_{j,0} = 1$, $j = 1, 2, 3, 4$ as in Table 1, and iterating the above with $a_1 = 2$, $a_4 = 3$, which produces integer-valued tau functions as in Table 2.

Table 2: Tau functions for the same orbit of the deformed A_4 map as in Table 1.

n	0	1	2	3	4	5	6	7	8	9
τ_{n+1}	1	3	9	27	189	1728	97848	2608848	64408608	3516556032
η_n	1	4	12	90	648	37584	19999872	3399542784	1546939772928	1748502507552768
θ_n	1	5	39	288	8424	454896	212004864	74543597568	59937513504768	487379529497051136
σ_{n+5}	1	16	120	1008	9504	172800	24164352	1272692736	140540313600	15780710449152

If the initial data for (2.28) is regarded as a cluster, that is

$$(\tilde{x}_1, \ldots, \tilde{x}_{11}) = (\sigma_0, \ldots, \sigma_5, \eta_0, \theta_0, \tau_{-1}, \tau_0, \tau_1),$$

then the pullback of the symplectic form (2.22) under the map $\tilde{\pi}$ defined by (2.27) is

$$\tilde{\omega} = \tilde{\pi}^*\omega = \sum_{i<j} b^*_{ij}\, \mathrm{d}\log \tilde{x}_i \wedge \mathrm{d}\log \tilde{x}_j,$$

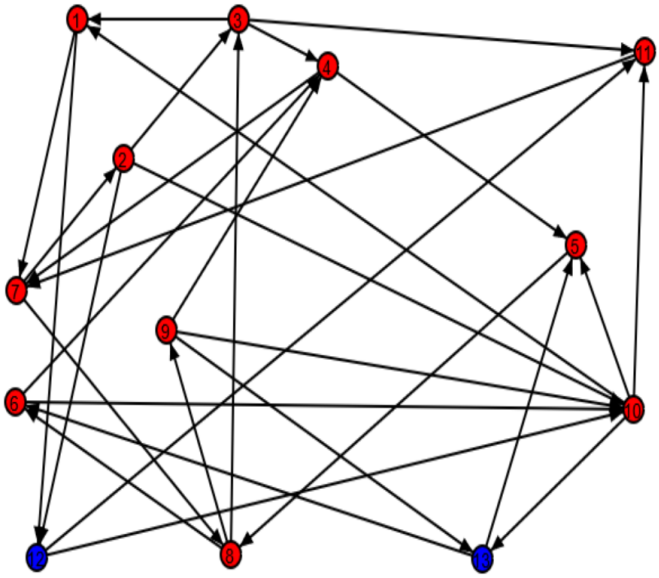


Figure 4: The initial quiver associated with the extended exchange matrix (2.30).

where $B^* = (b^*_{ij})$ is the exchange matrix

$$B^* = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ & & & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 \\ & & & & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ & & & & & 0 & 0 & -1 & 0 & 1 & 0 \\ & & & & & & 0 & 1 & 0 & 0 & -1 \\ & & & * & & & & 0 & 1 & 0 & 0 \\ & & & & & & & & 0 & 1 & 0 \\ & & & & & & & & & 0 & 1 \\ & & & & & & & & & & 0 \end{pmatrix} \tag{2.29}$$

(since the matrix is skew-symmetric, for brevity we put an asterisk to represent the terms below the diagonal). As in the A_3 case, this is sufficient to generate a sequence of mutations for the tau functions in the original undeformed system, but in order to include the parameters a_1, a_4 it it necessary to add these as frozen variables.

Theorem 2.6. *The sequences of tau functions (τ_n), (η_n), (θ_n), (σ_n) for the integrable map $\varphi = \mu_4\mu_3\mu_2\mu_1$ defined by (2.21) with $b_1 = b_4 = 1$ consist of elements of the Laurent polynomial ring $\mathbb{Z}_{>0}[a_1, a_4, \sigma_0^{\pm 1}, \sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \sigma_3^{\pm 1}, \sigma_4^{\pm 1}, \sigma_5^{\pm 1}, \eta_0^{\pm 1}, \theta_0^{\pm 1}, \tau_{-1}^{\pm 1}, \tau_0^{\pm 1}, \tau_1^{\pm 1}]$, being generated by a sequence of mutations in a cluster algebra defined by the exchange matrix (2.29) with the addition of two frozen variables, corresponding to the quiver shown in Figure 4.*

Proof: We take an extended cluster

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_{13}) = (\sigma_0, \ldots, \sigma_5, \eta_0, \theta_0, \tau_{-1}, \tau_0, \tau_1, a_1, a_4),$$

with the coefficients a_1, a_4 corresponding to additional frozen nodes in the quiver associated with $\tilde{B}^* = (b^*_{ij})$, the extended exchange matrix given by

$$\tilde{B}^* = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 \end{pmatrix} \tag{2.30}$$

(here we have shown the full matrix so that the exponents of all the exchange relations are visible in each column). The initial quiver is shown in Figure 4. Mutating at node 1 gives the exchange relation

$$\tilde{\mu}_1 : \quad \tau_2\sigma_0 = \tau_0\sigma_2 + a_1\,\eta_0,$$

producing the new cluster $\tilde{\mu}_1(\tilde{\mathbf{x}}) = (\tau_2, \sigma_1, \ldots, \sigma_5, \eta_0, \theta_0, \tau_{-1}, \tau_0, \tau_1, a_1, a_4)$, and subsequently applying mutations $\tilde{\mu}_7, \tilde{\mu}_8, \tilde{\mu}_9$ successively generates exchange relations corresponding to the other three equations in (2.28) for $n = 0$, with the result being the cluster $\tilde{\mu}_9\tilde{\mu}_8\tilde{\mu}_7\tilde{\mu}_1(\tilde{\mathbf{x}}) = (\tau_2, \sigma_1, \ldots, \sigma_5, \eta_1, \theta_1, \sigma_6, \tau_0, \tau_1, a_1, a_4)$. To generate each new instance of the four equations in (2.28) with the index n increased by 1, it is necessary to apply a similar block of four mutations. Let us define the following composition of four mutations by

$$\hat{\mu}_{ij} := \tilde{\mu}_i\tilde{\mu}_8\tilde{\mu}_7\tilde{\mu}_j,$$

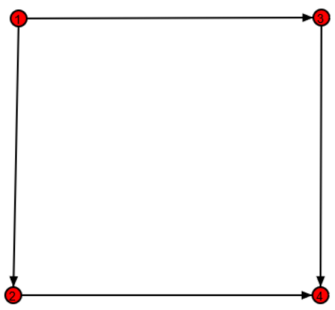
and to index mutations we use $\overline{10}, \overline{11}$ to distinguish nodes 10 and 11 from nodes with single-digit labels. Then if we take a particular composition of 36 mutations given by 9 of these blocks of four, namely

$$\hat{\tilde{\mu}} := \hat{\mu}_{6\overline{11}}\,\hat{\mu}_{5\overline{10}}\,\hat{\mu}_{49}\,\hat{\mu}_{36}\,\hat{\mu}_{25}\,\hat{\mu}_{14}\,\hat{\mu}_{\overline{11}3}\,\hat{\mu}_{\overline{10}2}\,\hat{\mu}_{91} = \tilde{\mu}_{687\overline{11}587\overline{10}4879387628751874\overline{11}873\overline{10}8729871}$$

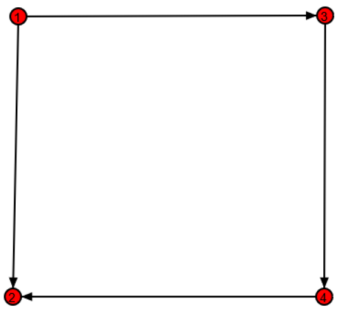
(where in the second expression the notation from (2.20) has been reused), then the quiver returns to its starting position; so we have

$$\hat{\tilde{\mu}}(\tilde{B}^*) = \tilde{B}^*, \qquad \hat{\tilde{\mu}}(\tilde{\mathbf{x}}) = (\sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \eta_9, \theta_9, \tau_8, \tau_9, \tau_{10}, a_1, a_4),$$

with the index of each of the tau functions increased by 9. Thus by repeatedly applying these 9 blocks of four mutations, all of the tau functions for the integrable map are produced from clusters in the cluster algebra defined by (2.30). □



(a) The quiver $\tilde{A}_{2,2}$.



(b) The quiver $\tilde{A}_{3,1}$.

Figure 5: Two orientations of the $A_3^{(1)}$ Dynkin diagram.

3 Reductions of the discrete sine-Gordon equation

In this section we consider two examples of four-dimensional maps that arise as reductions of the lattice sine-Gordon equation introduced in [18], that is

$$a_1(x_{n,m}x_{n+1,m+1} - x_{n+1,m}x_{n,m+1}) + a_2x_{n,m}x_{n+1,m}x_{n,m+1}x_{n+1,m+1} = a_3 \ , \tag{3.1}$$

where a_j, $j = 1, 2, 3$ are arbitrary parameters. Travelling waves of (3.1) are obtained by imposing periodicity under shifts by N steps in one lattice direction together with M steps in the other direction, so that

$$u_{n+N,m+M} = u_{n,m} \implies u_{n,m} = x_k, \quad k = Mn - Nm;$$

this is called the (N, M) reduction.

The two examples we consider below each correspond to particular orientations of the affine $A_3^{(1)}$ Dynkin diagram, as in Figure 5 (where the notation $\tilde{A}_{p,q}$ means there are p clockwise arrows and q anticlockwise arrows).

3.1 (2,2) periodic reduction of the lattice sine-Gordon equation

Let us consider the quiver with exchange matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix};$$

this is mutation equivalent to $\tilde{A}_{2,2}$ as in Figure 5(a), which corresponds to the exchange matrix $\mu_3(B)$. Then for $k = 1, 2, 3, 4$ we take the function

$$g_k(x) = \frac{a_1x + a_3}{a_2x + a_1},$$

for arbitrary parameters a_1, a_2, a_3, so that the exchange relation (1.8) contains the function

$$f_k(M_k^+, M_k^-) = M_k^+ g_k\left(\frac{M_k^-}{M_k^+}\right) = M_k^+ \frac{a_1 M_k^- + a_3 M_k^+}{a_2 M_k^- + a_1 M_k^+} \ .$$

Next, we consider a sequence of mutations which leaves matrix B invariant, specifically

$$\varphi(B, \mathbf{x}) := \mu_3\mu_1\mu_4\mu_2(B, \mathbf{x}) = (B, \tilde{\mathbf{x}}), \text{ where } \tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$$

and

$$\tilde{x}_2 = \frac{1}{x_2}\left(\frac{a_1x_1x_3 + a_3}{a_2x_1x_3 + a_1}\right) \ , \ \tilde{x}_4 = \frac{1}{x_4}\left(\frac{a_1x_1x_3 + a_3}{a_2x_1x_3 + a_1}\right) \ ,$$
$$\tilde{x}_1 = \frac{1}{x_1}\left(\frac{a_1\tilde{x}_2\tilde{x}_4 + a_3}{a_2\tilde{x}_2\tilde{x}_4 + a_1}\right) \ , \ \tilde{x}_3 = \frac{1}{x_3}\left(\frac{a_1\tilde{x}_2\tilde{x}_4 + a_3}{a_2\tilde{x}_2\tilde{x}_4 + a_1}\right) \ .$$

So, according to Theorem 1.3, the map $\varphi : \mathbf{x} \mapsto \tilde{\mathbf{x}}$ preserves the two form

$$\omega = \frac{1}{x_1x_2}\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{1}{x_1x_4}\mathrm{d}x_1 \wedge \mathrm{d}x_4 - \frac{1}{x_2x_3}\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \frac{1}{x_3x_4}\mathrm{d}x_3 \wedge \mathrm{d}x_4 \ .$$

In this case, the map φ corresponds to the $(2, 2)$ periodic reduction of the lattice sine-Gordon equation (3.1) (see Figure 6).

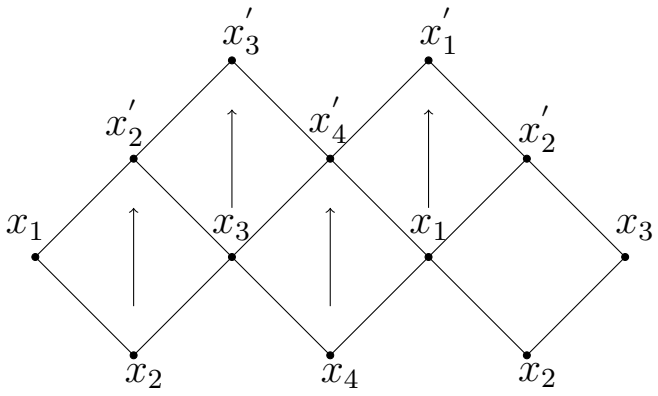


Figure 6: The $(2, 2)$ staircase periodic reduction of the quadrilateral equation (3.1)

The matrix B (and hence ω) is degenerate, of rank two. To obtain a symplectic map, we take a pair of monomials corresponding to an integer basis for

$$\mathrm{im}\, B =< (1, 0, 1, 0)^T, (0, 1, 0, 1)^T >,$$

namely

$$\pi : \qquad y_1 = x_1x_3, \quad y_2 = x_2x_4.$$

Under the projection π defined above, ω is the pullback of the symplectic form

$$\hat{\omega} = \frac{1}{y_1 y_2}\, \mathrm{d}y_1 \wedge \mathrm{d}y_2,$$

which is preserved by the induced map

$$\hat{\varphi}: \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}, \quad \tilde{y}_2 = \frac{1}{y_2}\left(\frac{a_1 y_1 + a_3}{a_2 y_1 + a_1}\right)^2, \tilde{y}_1 = \frac{1}{y_1}\left(\frac{a_1 \tilde{y}_2 + a_3}{a_2 \tilde{y}_2 + a_1}\right)^2. \tag{3.2}$$

The above map has the first integral

$$K = \frac{a_2^2 y_1^2 y_2^2 + 2a_1 a_2 (y_1^2 y_2 + y_1 y_2^2) + a_1^2 (y_1^2 + y_2^2) + 2a_1 a_3 (y_1 + y_2) + a_3^2}{y_1 y_2},$$

so it is Liouville integrable. In fact it is of QRT type: the level sets $K = \text{const}$ are symmetric biquadratic curves, and $\hat{\varphi} = \iota_h \circ \iota_v = (\iota \circ \iota_v)^2$ where the involutions ι_h, ι_v correspond to the horizontal and vertical switches on each level set, and $\iota : y_1 \leftrightarrow y_2$. For Laurentification of symmetric QRT maps, see [17].

In four dimensions, the other degrees of freedom in the original map φ have essentially trivial dynamics, since

$$\frac{\tilde{x}_1}{\tilde{x}_3} = \left(\frac{x_1}{x_3}\right)^{-1}, \qquad \frac{\tilde{x}_2}{\tilde{x}_4} = \left(\frac{x_2}{x_4}\right)^{-1}.$$

3.2 (4,-1) periodic reduction of the lattice sine-Gordon equation

We consider the quiver with exchange matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}.$$

The matrix B is non-degenerate and satisfies $\mu_1(B) = \rho(B)$ for the cyclic permutation $\rho : (1,2,3,4) \mapsto (4,1,2,3)$, so it defines a cluster mutation-periodic quiver with period 1 [13]. Following the example in subsection 3.1, we consider

$$g_1(x) = x\left(\frac{a_1 + a_3 x}{a_2 + a_1 x}\right).$$

Here, $M_1^+ = x_2 x_4$, $M_1^- = 1$ and

$$f_1(M_1^+, M_1^-) = M_1^+ g_1\left(\frac{M_1^-}{M_1^+}\right) = \frac{a_1 x_2 x_4 + a_3}{a_2 x_2 x_4 + a_1}.$$

Hence, the appropriate analogue of Theorem 1.3 (see Remark 1.4) implies that the map $\varphi = \rho^{-1}\mu_1$ given by

$$\varphi : (x_1, x_2, x_3, x_4) \mapsto \left(x_2, x_3, x_4, \frac{1}{x_1}\Big(\frac{a_1 x_2 x_4 + a_3}{a_2 x_2 x_4 + a_1}\Big)\right) \tag{3.3}$$

preserves the symplectic form

$$\omega = \frac{1}{x_1 x_2}\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{1}{x_1 x_4}\mathrm{d}x_1 \wedge \mathrm{d}x_4 + \frac{1}{x_2 x_3}\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \frac{1}{x_3 x_4}\mathrm{d}x_3 \wedge \mathrm{d}x_4 \ .$$

The map (3.3) is associated with the $(4, -1)$ periodic reduction of the lattice sine-Gordon equation (3.1), and can be rewritten in recurrence form as

$$a_1(x_n x_{n+4} - x_{n+1}x_{n+3}) + a_2 x_n x_{n+1} x_{n+3} x_{n+4} = a_3 \ .$$

Closed-form expressions for integrals of periodic reductions of the sine-Gordon equation were presented in [23] and their involutivity was proved in [35].

4 Concluding remarks

We have considered autonomous recurrences or maps obtained by including additional constant parameters in sequences of cluster mutations that generate completely periodic dynamics, and have shown that it is possible to preserve the presymplectic structure defined by the exchange matrix, and also (by imposing suitable constraints on the parameters) obtain Liouville integrable maps. Our starting point for showing Liouville integrability has been the fact that the original periodic maps admit first integrals defined by cyclic symmetric functions of variables along a period of the orbit. Only the examples of A_2, A_3 and A_4 have been dealt with here, but it would be instructive to make a more systematic study of such functions from the viewpoint of the associated Poisson algebra in order to extend these results to cluster algebras defined by other finite type Dynkin diagrams. We have also treated more general types of mutations, involving Möbius transformations, and shown that for some particular affine type exchange matrices these lead to reductions of the discrete sine-Gordon equation.

The parameters a_k, b_k appearing in our deformed mutations have been assumed constant, but Theorem 1.3 applies equally well to non-autonomous recurrences/maps. In particular, taking

$$a_k = \frac{y_k}{1 + y_k}, \quad b_k = \frac{1}{1 + y_k}$$

in (2.1) leads to the expression for a mutation μ_k in a cluster algebra with coefficients [11], which themselves mutate according to

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j \left(1 + y_k^{-\mathrm{sgn}(b_{jk})}\right)^{-b_{jk}} & \text{otherwise.} \end{cases}$$

The dynamics of the coefficients generates the associated Y-system [26]. In [21], it is shown that non-autonomous dynamics also arises from autonomous Y-systems in the case where the exchange matrix is degenerate: one of the simplest examples is provided by the Y-system

$$y_{n+7}y_n = \frac{(1+y_{n+6})(1+y_{n+1})}{(1+y_{n+4}^{-1})(1+y_{n+3}^{-1})}$$

corresponding to the Somos-7 recurrence (2.4), solved in terms of the q-Painlevé V equation

$$x_{n+2}x_n = x_{n+1} + \alpha_n\,\mathfrak{q}^n, \qquad \alpha_{n+6} = \alpha_n, \tag{4.1}$$

which is a non-autonomous version of the Lyness recurrence. The fact that the period of α_n is 6 is important, since if $\mathfrak{q} = 1$ and α_n is periodic with a period that is not a divisor of 6, then (4.1) appears to exhibit chaotic dynamics [2].

As another example based on the A_2 exchange matrix, taking $g_1(x) = \frac{ax+b}{cx+d}$ and letting the coefficients a, b, c, d depend on the index n gives the sequence of symplectic maps

$$\varphi_n(x,y) = \left(y, \frac{a_n y + b_n}{x(c_n y + d_n)}\right)$$

that corresponds to the non-autonomous nonlinear recurrence

$$x_{n+2} = \frac{a_n x_{n+1} + b_n}{x_n(c_n x_{n+1} + d_n)}\ .$$

Invariants of this recurrence when the coefficients are periodic were presented in [8] and have also been studied in the framework of QRT (and non-QRT) maps with periodic coefficients [32, 33].

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