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Sabiou M. Inoua

Vernon L. Smith

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Sabiou M. Inoua and Vernon L. Smith

Chapman University

Abstract. We offer an information theory of market price formation, formalizing and elaborating on an old, implicit, classical tradition of supply and demand based on buyers' and sellers' monetary valuations of commodities (formally their reservation prices) and competition as a multilateral higgling and bargaining process. The early laboratory market experiments, as it turns out with hindsight, established the remarkable stability, efficiency, and robustness of the old view of competitive price discovery, and not the neoclassical price theory (based on individual utility and profit maximization for given prices). Herein, we present a partial-equilibrium version of the theory in which wealth is implicitly constant, and reservation values are fixed, as in the early experiments, formulating an information interpretation à la Shannon that corresponds with modern notions of the pricing system as an information signaling system. Competitive equilibrium price, we show, conveys maximum information about the distribution of traders' valuations. We illustrate the theory as it applies to a few market conditions (notably a non-clearing market case) and institutions (posted-price market, English auction, double auction, sealed-bid call market).

1 Introduction: Classical Market Price Formation and Experiments

Neoclassical marginal analysis postulated consumers who choose quantities, conditional on given prices, to maximize utility defined over a continuous commodity space subject to a budget constraint in a premarket exercise, with producers choosing quantities, given prices, to minimize cost. The premarket requirement of given prices and the requirement that the "law of one price" hold in the market, precluded any possibility of articulating a price formation or discovery process. Hence, Jevons (1871 [1888]) believed that buyers and sellers must have complete (or perfect) information on his model to reach equilibrium, a strong knowledge assumption that is the ancestor of more sophisticated modern versions (such as the common-knowledge axiom in game theory); Walras (1874 [1954]) pushed deeper than Jevons, by envisioning a market sufficiently well organized to provide a central procedure for announcing prices, thereby accommodating demand and supply theory based on optimal quantities at exogenously set prices to be bought or sold; centrally announced tentative prices are then adjusted in increments proportioned to excess supply (or demand) until a market clearing price is identified.[1]

The early market experiments implemented neither Jevons's requirement of complete information, nor the Walrasian tatonnement procedure in a continuous commodity space; rather the experiments implemented a discrete commodity space using private buyers' willingness to pay

[1] Walras's mechanism is not among the many pricing institutions found in practice: "From 1919 until its abandonment in 2015, due to recurrent charges of price manipulation, the London gold price was determined ("fixed") using a procedure that implemented Walras's *tâtonnement*—to our knowledge the only such market application, wherein it ultimately failed." (Inoua & Smith, 2020c)

(WTP) or reservation values, and sellers' minimum willingness to accept (WTA) reservation costs, with justifying references to Marshall (1890). (Chamberlin, 1948; V. L. Smith, 1962, 1964, 1965)

Neoclassical theory was rooted in the postulate that prices were given (and corresponding optimal demand and supply deduced) before they could have been found in markets, thus imposing pre-market price-taking action and the law of one price on the subsequent expression of market demand and supply. A proper theory would aim to derive such features, if correct, as results from more elementary assumptions. Moreover, the neoclassical utility framework adds nothing consequential if commodities are discrete with consumers predominantly purchasing at most one unit of each item—shopping baskets almost all have at most one quart of milk, and/or one pound of butter, and so on. Consequently, individual marginal utility is total utility and exhaustively specified by WTP value.[2]

As we argued elsewhere (Inoua & Smith, 2020a, 2020b, 2020c), the old, classical, view of market competition as a collective, multilateral, higgling and bargaining process, offers a helpful foundation on which to build a modern theory of market price formation, despite any shortcomings of the original classical formulation (notably its insistence on long-run, natural value). Formally, classical competition involves buyer-buyer outbidding, seller-seller underselling, and buyer-seller haggling. Underlying this classical tradition, based on Adam Smith's sketch in *Wealth of Nations* (1776 [1904], Book I, Ch. VII) and the clarifications and additions made by his French, English, and Italian followers, is an implicit conception of supply and demand based on traders' monetary

[2] Markets solve deeper consumer inventory problems by finding small and large packages of divisible items that simplify and discretize quantity choice depending on consumer family size and shopping frequency. Hence, consumer prices are formed directly out of the values most relevant to the immediate experience of consumers.

valuations of commodities: buyers and sellers come to market with revealable maximum WTP reservation values (as in auctions), and minimum WTA reservation costs. The collective market interaction of buyers trying to buy cheap, and sellers trying to sell dear, aggregates demand and supply as an integral part of the process of finding prices. Neither aggregation nor price discovery can be separated in classical market processes. Only at the close of the market can we properly identify quantities demanded and supplied as functions of price; and only at the close can we refer to the dispersed information in the crowd of buyers and sellers as having been aggregated into prices as external coordinating signals.

The implicit general methodology underlying this old conception of markets (again, putting aside its limitations) is the just-mentioned realistic approach to economic motives based on observable variables (formally, the traders' reservation "prices"), but also on the principle that the interesting economic regularities are unintended properties emerging from market interactions (as conveys Adam Smith's famous "invisible hand" metaphor). As regards price theory, this paper's subject, the fundamental market interaction leading to emergent regularities is of course competition itself; and the core emergent property of competitive price discovery is informational, as emphasized F. A. Hayek (1937, 1945, 1968 [2002]). That is, the competitive price system synthesizes a sum of dispersed information about consumers' needs and preferences and producers' means of production and costs.

This paper's goal is to develop a mathematical theory of the old competitive price mechanism and derive from it an emergent informational characterization of the price mechanism. The discussion comes full circle, in this regard, because a core concept needed to this effect was hinted at in the early experiments. We first motivate it.

2 Motivation: The Competition and Information Principle

Competitive price dynamics, as the analysis of early lab price dynamics data suggest (V. L. Smith, 1962, 1965), can be characterized as a minimization of the famous area under the supply and demand curves, the integral function:

$$V(p) = \int_0^p S(x)dx + \int_p^\infty D(x)dx. \tag{1}$$

With hindsight, the centrality of this "hypothesis found to be most successful in these experiments" has not been fully appreciated. (V. L. Smith, 1962, p. 112)

As reinterpreted and generalized below, the minimum principle turns out to be a fundamental principle of price theory; it is a natural formulation of classical competitive dynamics, provided we treat supply and demand as aggregate value and cost distributions across individuals (rather than as individual outcomes of utility and profit maximization for given price), and provided we recognize that the famous area minimized yields the *potential* market surplus function available in a market (rather than the *actual* surplus extracted through trade, a dual concept introduced below). Then the minimization hypothesis means the potential gains from trade, the total surplus not yet realized (the money still on the table to speak in the vernacular) shrinks transaction after transaction as the traders compete to grab the most of it (and hence the traders' total realized gains, or actual surplus, increases transaction after transaction, in a dual manner).[3] We first pause to motivate this hypothesis as a natural formulation of competitive dynamics, postponing the details of the proofs until the next sections.

The simplest formulation of the minimum principle follows if we assume smooth supply and demand curves (which we assume in this section to motivate the general treatment in the next section, postponing the rigorous smooth model treatment itself in Subsection 3.4). Then it is easily seen that the function (1) is an integral of excess supply:

$$\frac{dV}{dp} = S(p) - D(p). \tag{2}$$

Therefore, the integral function is minimized at a market-clearing price (which in the smooth case is unique as we shall see in Subsection 3.4) as long as the excess supply function is upward sloping (which is necessarily the case). Moreover, by the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dp}\frac{dp}{dt} = [S(p) - D(p)]\frac{dp}{dt}.$$

Thus, the law of supply and demand amounts to writing

$$\frac{dV}{dt} \leq 0. \tag{3}$$

It turns out that the dynamic inequality (3) is a specific (smooth) manifestation of a general law—a competition principle—that applies more generally for discontinuous supply and demand curves (as is usually the case as we shall see shortly) and even when this discontinuity is such as to preclude existence of a market-clearing price: *the minimum of the potential surplus function*

[3] Originally, the hypothesis, called the "excess-rent hypothesis", was formulated as a best empirical descriptor of competitive equilibrium convergence. (V. L. Smith, 1962) Here we are offering a more natural, more general, formulation of the hypothesis.

is the general competitive equilibrium concept, and there is always at least one such minima for any supply and demand configuration.

By definition of supply and demand as cumulative distribution functions of consumers' (utility) valuations (or use-values, or values for short) and producers' cost valuations (or costs), the smooth treatment of supply and demand curves is reasonable only as an approximation of a large market (see Subsection 3.2). More generally, the following formula for the potential surplus function is not hard to derive (see Lemma 1 in Subsection 3.2 below):

$$V(p) = \sum_{v \geq p} |v - p| + \sum_{c \leq p} |c - p|, \tag{4}$$

in which the notation means summation across all values $v \geq p$ and all costs $c \leq p,$ due to the profitability condition of trade. (A proper normalization is needed for a large-market case: one simply thinks in average terms.) For the usual case of discrete values and costs $\{v_i : i = 1,...,n\}$ and $\{c_j : j = 1,...,m\},$ which we refer to generically as reservation "prices" $\{r_u : u = 1,...,m+n\},$ then (4) reads

$$V(p) = \sum_{u=1}^{m+n} |r_u - p| a_u(p), \tag{5}$$

where we indicate whether a unit u is affordable (or profitable) at a standing price $p,$ namely:

$$a_i(p) = I(v_i \geq p), i = 1,...,n, \tag{6}$$

$$a_j(p) = I(c_j \leq p), j = 1,...,m. \tag{7}$$

We use throughout this paper, the indicator function notation:

$$I(A) = \begin{cases} 1 \text{ if } A \text{ is true,} \\ 0 \text{ if } A \text{ is false.} \end{cases}$$

In the form (5), the potential surplus (trader profit) function measures the overall distance between the standing price and the distribution of values and costs. To emphasize this interpretation, we shall refer to it at times as the *price-value distance function* (where value here means more generally all the traders' valuations, including the sellers' cost valuations). By definition, any point that minimizes the price-value distance function is a best summary of the value and cost distribution: namely a *generalized median* of the distribution of traders' valuations. (Without the profitability condition, we would have a simple median of the values and costs.) We shall call the set of these generalized medians, the value center.

The basic affordability or profitability function (6)-(7) plays an intrinsic role in the formulation of competition. If a competitive price move $p_t \to p_{t+1}$ involves outbidding (underselling) among the pairs (u,l) until one of the buyers (sellers) can no longer afford the standing price, then by definition $a_u(p_{t+1})a_l(p_{t+1})=0$, meaning either $a_u(p_{t+1})=0$ or $a_l(p_{t+1})=0$, depending on which unit holder is excluded from the price race. More generally, if a price move $p_t \to p_{t+1}$ is driven by multilateral buyer-buyer or seller-seller competition involving k units $\{u_1,...,u_k\}$, and at least one of the unit holders is excluded from the race (meaning pushed beyond the limit price they can stand), then we have by definition:

$$a_{u_1}(p_{t+1})a_{u_2}(p_{t+1})\ldots a_{u_k}(p_{t+1})=0. \tag{8}$$

As to buyer-seller haggling involving a pair (u,l) and causing a price move $p_t \to p_{t+1}$, the rivalry consists, not in excluding the other party from trading, but, to the contrary, in a party conceding a more favorable price to the counterparty by forgoing some surplus; that is, buyer-seller haggling leads to the seller cutting the price or the buyer offering a higher price, hence a reduction of the initial potential surplus $(p_t - c_j)$ or $(v_i - p_t)$ to the lower $(p_{t+1} - c_j)$ or $(v_i - p_{t+1})$. More generally, a buyer-seller haggling involves a concession of potential surplus of the form:

$$|r_u - p_{t+1}| \leq |r_u - p_t|. \tag{9}$$

The two implications (8) and (9) of competition are summarized in V^k, which reflects all possible competitive interactions, where k is the maximum number of units that compete simultaneously in a given period as allowed by the trading institution:[4]

$$V^k(p) = \sum_{k_1+\ldots+k_{m+n}=k} \frac{k!}{k_1!k_2!\ldots k_{m+n}!} \prod_{u=1}^{m+n} a_u(p)|r_u - p|^{k_u}. \tag{10}$$

By the principle of competition, we mean throughout the assumption that the market considered is predominantly driven by competition in the sense that for every price move $p_t \to p_{t+1}$ there is an integer $k_t \geq 2$ such that $V^{k_t}(p_{t+1}) \leq V^{k_t}(p_t)$. But since $V^{k_t}(p_{t+1}) \leq V^{k_t}(p_t)$ is equivalent to $V(p_{t+1}) \leq V(p_t)$, we adopt this latter, simpler but equivalent, formulation, and we will not repeat the rather complex multinomial formula (10), although it is implicitly assumed throughout.

The competition principle thus formulated leads naturally to an interpretation of price discovery as an information aggregation process, a core function of the price mechanism according to the famous intuition by Hayek (1937, 1945), which stimulated theoretical contributions, within a neoclassical framework, of a general informational characterization of price formation, notably the Hurwicz program (Hurwicz, 1960; Hurwicz, Radner, & Reiter, 1975), whose conceptual apparatus was adopted in experimental economics (V. L. Smith, 1982). In this paper we suggest a classical

[4] We have used the identity $(a_u)^k = a_u, k \geq 1$, shared by all indicator functions.

informational characterization of competitive price discovery, which we refer to as the *principle of maximum information* (PMI). An aspect of this informational characterization of competition was already hinted at earlier, when we formulated the competition principle as a minimization of the price-value distance function (5): thus, competitive price evolves towards a best, robust, summary (a generalized median) of the traders' valuations.[5]

The PMI is more naturally formulated in the language of information theory (Shannon, 1948). (We motivate a general formulation here, adopting a price concept common to all trading institutions—transaction price—and specializing the principle for the particularly informationally rich double-auction market institution in Subsection 4.6.) Think of a market as an information processing machine wherein the trade messages reveal hidden information about the underlying supply and demand (the private traders' valuations), thereby conveying such information into public prices (Figure 1).

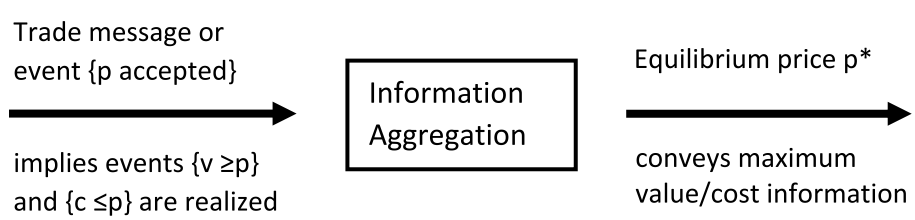

Figure 1. The Market as an Information Aggregation Machinery.

A transaction concluded at a price p reveals existence of a cost $c \leq p$ and a value $v \geq p,$ which are private information. In probabilistic terms, the trade is formally the realization of the event

[5] This interpretation of the price mechanism links value theory to the robust statistics literature and suggests a fundamental reason in favor of median-based statistics (based on least-absolute errors) in economics compared to the more popular mean-based approach (based on least-square errors). More on this in Footnote 20.

$\{p \text{ accepted}\}$, implying realization of the two events $\{v \geq p\}$ and $\{c \leq p\}$. Thus every trade that takes place in a market provides uncertainty-reducing knowledge of net economic worth. Value, v, measures resource value added to society; cost c measures resource value foregone by society. Hence, acceptance implies a contract price such that $v \geq p \geq c$, signaling net gain for society. According to Shannon's theory, the amount of information the realization of an event reveals is measured (up to a multiplicative constant that fixes the unit of information) by the log-probability of the event (and this important formula follows from the simple axiom of additivity of information revealed by the joint realization of independent events). In standard information theory, the total information (or entropy) is obtained by integrating (averaging) over a (whole) probability distribution of possible realizations. For our purpose, however, the total information a trade reveals about the distribution of values and costs is obtained by integrating across the relevant range of the cost and value distributions revealed by the trade event.

Let $\text{prob}(c \leq p)$ be the probability that a unit, randomly drawn from the total $m+n$ units, is a supply unit with cost $c \leq p$; and similarly let $\text{prob}(v \geq p)$ be the probability that a unit is a demand unit with value $v \geq p$. Then the information content of a trade event occurring at a price p is measured by the integral function: [6]

$$I(p) = -\int_{c_{\min}}^{p} \log \text{prob}(c \leq x)dx - \int_{p}^{v_{\max}} \log \text{prob}(v \geq x)dx, \tag{11}$$

[6] The choice of cumulative probability in information measurement, natural for our purpose, has also been defended as more general than the standard one based on simple probability (Rao, Chen, Vemuri, & Wang, 2004). Dynamic refinements of Shannon entropy also have been suggested that are based on the less general probability density concept but that involve similar integration ranges than the ones we adopted (Di Crescenzo & Longobardi, 2002; Ebrahimi, 1996).

where $c_{\min} = \min\{c_j\}$ and $v_{\max} = \max\{v_i\}$.[7] By definition of supply and demand as distribution functions, we can rewrite the information function as

$$\frac{dI}{dp} = \log \frac{D(p)}{S(p)}. \tag{12}$$

$$I(p) = -\int_{c_{\min}}^{p} \log \frac{S(x)}{m+n} dx - \int_{p}^{v_{\max}} \log \frac{D(x)}{m+n} dx. \tag{13}$$

The competition principle is equivalent to maximization of this information function. This follows easily for the smooth-market model:

$$\frac{dI}{dt} = \left\{ \log \frac{D(p)}{S(p)} \right\} \frac{dp}{dt} \geq 0. \tag{14}$$

Having briefly motivated the competition principle and its welfare and informational interpretation, we now derive the theory more formally (starting from scratch to avoid any ambiguity). Although we derive the PMI (and its various aspects) from the competition principle as a matter of exposition, the informational principle is more general and more fundamental, as we will emphasize throughout this paper (notably in the Applications, Section 4, and in the Conclusion).

[7] By an abuse of notation, *I* stands for both the indicator function and the information function being introduced: the risk of confusion is minor, however.

3 Theory

3.1 Setup: Notations, Definitions, Identities

Consider the market for a good or service involving the reservation values and costs of all individual actors combined:[8]

$$\mathbf{r} = [v_1, ..., v_n; c_1, ..., c_m],$$

which we write generically as $\mathbf{r} = [r_u]$. An example, is the first reported market experiment, Table 1:

Costs	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25
Values	3.25	3.00	2.75	2.50	2.25	2.00	1.75	1.50	1.25	1.00	0.75

Table 1. Example of value and cost distributions (V. Smith, 1962, Chart 1).

The values and costs are represented by their cumulative distribution functions (interpreted as market demand and supply functions) defined by:

$$D(v) = \#\{i : v_i \geq v\} = \sum_{i=1}^{n} I(v_i \geq v). \tag{15}$$

$$S(c) = \#\{j : c_j \leq c\} = \sum_{j=1}^{m} I(c_j \leq c). \tag{16}$$

[8] Multiple units may belong to the same trader.

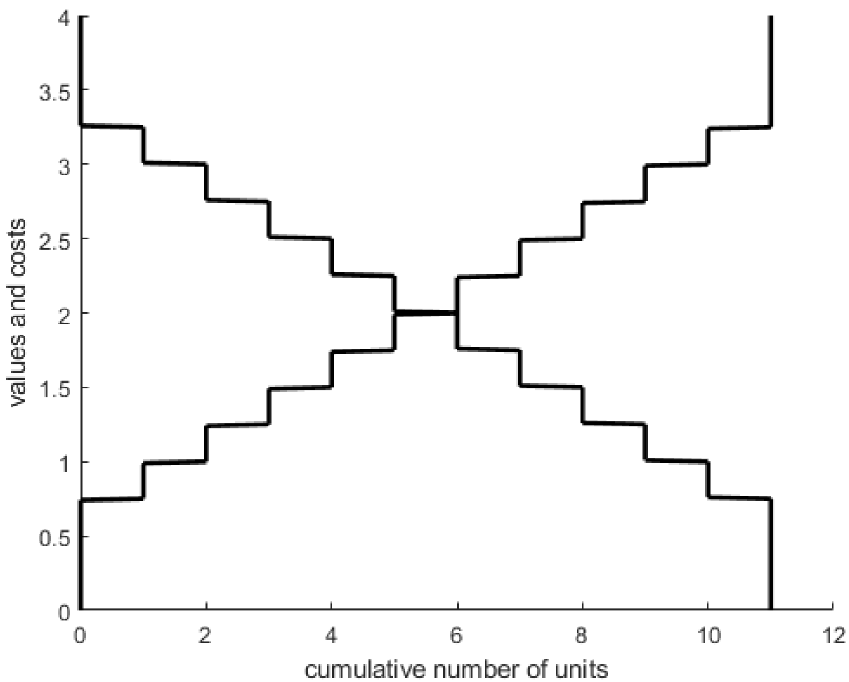

Figure 2. Cumulative Value and Cost Distributions in Table 1.

For any arbitrary (nonnegative) real number x (say a transaction price), the gap between the value and cost distributions (or market excess demand function) is defined as

$$Z(x) = D(x) - S(x). \tag{17}$$

Let the quantity allocation vector be given by the trade indicators, referenced collectively as

$$\mathbf{q} = [q_1, ..., q_n; q_{n+1}, ..., q_{m+n}],$$

where $q_i = 1$ if unit i is bought and $q_i = 0$ otherwise; $q_j = -1$ if unit j is sold and $q_j = 0$ otherwise. To each unit is associated a transaction price, where nontraded units can be assigned arbitrary prices (say the final transaction price): let these transaction prices be referred to as

$$\mathbf{p} = [p_1, ..., p_n; p_{n+1}, ..., p_{m+n}].$$

Total sales and total purchases are identical, both in terms of number of units traded and in terms of monetary value. Thus, we have the accounting identity of sales and purchases:

$$\sum_{u=1}^{m+n} q_u = 0. \tag{18}$$

$$\sum_{u=1}^{m+n} p_u q_u = 0. \tag{19}$$

The total surplus generated through trade is by definition the sum of buyers' and sellers' profits, $\sum_u (r_u - p_u) q_u$, which, owing to identity (20), reducing to $\sum_u r_u q_u$, hence is independent of the price distribution per se [which, for example, we could have simplified into a uniform transaction price vector $[p, \ldots, p]$ and still get the same exact total surplus, invoking identity (18)]. Thus, define the surplus function, which gives the total gain from trade as a function of the allocation vector, to be the scalar product of reservation valuations and the allocation vector:

$$\Lambda(\mathbf{q}) = \sum_{u=1}^{m+n} r_u q_u = \mathbf{r} \cdot \mathbf{q}. \tag{21}$$

Since by construction $\mathrm{mean}(\mathbf{q}) = 0$, we have $\mathrm{cov}(\mathbf{r}, \mathbf{q}) = \mathbf{r} \cdot \mathbf{q} / (n+m)$, hence surplus function is the covariance between the allocation vector and the traders' valuations:

$$\Lambda(\mathbf{q}) = (m+n)\,\mathrm{cov}(\mathbf{r}, \mathbf{q}). \tag{22}$$

Let the distance function between price and value be[9]

$$V(p) = \sum_{i=1}^{n} \left| v_i - p \right| I(v_i \geq p) + \sum_{j=1}^{m} \left| c_j - p \right| I(c_j \leq p). \tag{23}$$

Define the *center of value* to be the set of points that minimize the price-value distance function:

$$C = \arg\min_{x \in \mathbb{R}} V(x). \tag{24}$$

[9] Recall that, the word value, as used here, stands for the distribution of traders' valuations (including sellers' cost valuations).

Assuming no unit is traded at a loss, the transaction price domain (in which the commodity's transaction price is bound to lie) is necessarily,

$$\Theta = [\min(c_j), \max(v_i)], \tag{25}$$

which we assume nonempty, to avoid trivial no-trade markets.[10] Thus, we assume $n \geq 1$, $m \geq 1$, and $\min(c_j) \leq \max(v_i)$. We also write simply $\Theta = [c_{\min}, v_{\max}]$.

Let the information that a transaction price conveys about the value and cost distributions be measured by the function introduced above (Section 2), which, by additivity of integration, simplifies to:[11]

$$I(p) = -\int_{c_{\min}}^{v_{\max}} \log \frac{D(x)}{m+n} dx + \int_{c_{\min}}^{p} \log \frac{D(x)}{S(x)} dx. \tag{26}$$

This function is well-defined since $D, S > 0$ on the transaction price set $\Theta = [c_{\min}, v_{\max}]$.

Let the price dynamics be formulated in terms of the *standing transaction price*, and written recursively as

$$p_{t+1} = H(p_t), \tag{27}$$

[10] We note, however, that this condition suppresses non-trade information that is of potential value in fostering innovation or responding to other changes in the data of an economy. For example, suppose we have a no-trade good with ask price = min (c)> max (v) = bid price. But if min(c) is little above max (v) this condition may induce a cost lowering innovation allowing trade, or an importer may see an opportunity to better the no trade domestic ask price. Hayek perceptively noted that "economic theory…starts from the assumption of a 'given' supply of scarce goods. But…which things are goods and how scarce—these are precisely the things which competition has to discover." (Hayek, 1984, p. 256) Hence, non-trade prices may be informative signals that induce adaptive change.

[11] Following standard custom in information theory, we take the base-2 logarithm, hence measure information in bits in the simulations below. Of course, this is merely a convention.

where H is a continuous function. One can define p_t to be the price of the tth transaction, thus interpreting t as a transaction number, incrementing by a unit whenever a transaction occurs. It is convenient to write the price trajectory more explicitly as $\{H^t(p_0) : t \geq 0\}$. [12]

Empirically, price dynamics is not always solely driven by market competition, but it may be constrained by the specific trading institutions (Section 4) and even be controlled by a regulation.[13] We shall distinguish a price dynamics purely driven by traders' competition from a constrained price dynamics: the theoretical reference is of course the unconstrained case, which we defined precisely as a case where there is no artificial bound on the dynamics of the potential surplus function, which can therefore reach its lowest possible minimum. We call any deviation from this purely competitive equilibrium state, a *friction*.

In summary, we adopt the following definitions and assumptions:

Definition 1 *(Market). Let a market be abstractly identified with the data* $[\mathbf{r}, H]$*, namely the distribution of values and costs* $\mathbf{r} = [v_1, ..., v_n;\ c_1, ..., c_m]$ *and the price motion* $\{H^t(p_0) : t \geq 0\}$.

Definition 2 *(Competitive market). Let* $[\mathbf{r}, H]$ *be a market. We say that a price move* $p_t \to p_{t+1} = H(p_t)$ *is competitive (driven by traders' competition) if* $V(p_{t+1}) \leq V(p_t)$*. The market is*

[12] The superscript notation stands for composition of the function H with itself (rather than an exponent): that is, $H^0(x) = x$ and $H^t(x) = H^{t-1}(H(x))$ denote iterations of H.

[13] A regulatory constraint on price can be written generically as $g(p) \leq 0$, where in practice g is usually an affine function for State intervention: thus a price control setting a floor $p \leq p_{\min}$ corresponds to $g(p) = p - p_{\min}$, whereas a price ceiling constraint $p \geq p_{\max}$ imposed on price corresponds to[13]

[13] A regulatory $g(p) = p_{\max} - p$.

competitive if its whole price trajectory $\{H^t(p_0) : t \geq 0\}$ *is a succession of competitive price moves. We say that the market reaches competitive equilibrium (C.E.) if it attains stationary potential surplus.*

Definition 3 *(Pure competition). Consider a market* $[\mathbf{r}, H]$. *We say that its price dynamics is purely competitive (or that competition is unconstrained or frictionless) if the minimum potential surplus is attainable under the trading rules inducing* H: *that is, if for any arbitrary* x_0,

$$\min_{t \geq 0}\{V(H^t(x_0))\} = \min_{x \in \mathbb{R}} V(x). \tag{28}$$

We assume the commodity is bought for final use and thus exclude re-trading (and speculation by the same token) and assume given $[\mathbf{r}, H]$. Besides this general assumption fixing the framework, the price theory under development rests on one core principle:

Assumption *(Competition Principle). The market is competitive (in the sense of Definition 2).*

In mathematical jargon, the competition principle amounts to saying that the potential surplus function is a Lyapunov function. The main mathematical ingredient is the so-called *invariance principle*, an extension of Lyapunov's classic stability theorem due to Barbashin, Krasovskii, and LaSalle.[14] The core idea behind the invariance principle (or invariant set theorem) is a generalization to sets of the concept of equilibrium and stable points, with convergence to a point attractor becoming convergence to a set attractor; intuitively, a dynamical system converges to a set if it ends up being within (or on the boundary) of the set: thus, the system may be endlessly moving, but within the set. A set is *invariant* with respect to a dynamical system if the system, once it

[14] Lyapunov (1892, 1892 [1992]), Krasovskii (1959 [1963]), LaSalle (1960), Barbashin and Krasovskii (1961).

reaches that set, will always stays in it: in other words, an invariant set is a generalization of the concept of equilibrium point. A dynamical system that possesses a Lyapunov function will converge to an invariant set on which the Lyapunov function attains a stationary value. That is the essence of the invariance principle. (See Appendix for the formal statement.)

3.2 Implications of the Theory

First, we emphasize the following implication of the competition principle that has an interest in itself (rather than as a new ingredient for the sequel). In discrete time, the law of supply and demand is usually ambiguously formulated by saying that price change and excess demand have the same sign. Which excess demand? At which price? The initial one? The final one? The standard assumption is the initial excess demand, as suggested by the direction of causation: $(p_{t+1}-p_t)Z(p_t)\geq 0$; but particularly for a long enough time interval, it is clear that intermediary supply-demand imbalances along the way cannot be summarized by the initial one.

Proposition 1 *(Law of Supply and Demand). From the competition principle* $V(p_{t+1})\leq V(p_t)$ *derives a natural formulation of the law of supply and demand in discrete time: the sign of a price change* $p_t \rightarrow p_{t+1}$ *is the same as the sign of the average excess demand across the price path.*

Proof. Since V is an integral of excess supply (Lemma 1 below), the competition principle reads

$$\int_{p_t}^{p_{t+1}}[D(x)-S(x)]dx\geq 0. \tag{29}$$

This is a natural formulation of the law of supply and demand as follows. Define the average excess demand function across the motion $p_t \rightarrow p_{t+1}$ as

$$z(p_t, p_{t+1}) = \begin{cases} Z(p_t), & p_t = p_{t+1}, \\ \dfrac{1}{p_{t+1} - p_t} \displaystyle\int_{p_t}^{p_{t+1}} Z(x)dx, & p_t \neq p_{t+1}. \end{cases} \tag{30}$$

Thus, the competition principle is equivalent to the statement $(p_{t+1} - p_t) z(p_t, p_{t+1}) \geq 0$. ■

Proposition 2 *(Competitive Dynamics). A competitive price converges to a set of minimum price-value distance (or potential surplus) across the trajectory followed: that is,* $p_t \to \{x : V(x) = \mu\}$*,* as $t \to \infty$*, where* $\mu = \min\{V(H^{\tau}(p_0)) : \tau \geq 0\}$.

Proof. This is a direct consequence of the invariance principle (Mathematical Appendix).[15] The main idea is intuitively clear: since V is non-increasing, the sequence $\{V(H^t(p_0)) : t \geq 0\}$ is a sequence of nonincreasing numbers that converges to a limit μ. ■

Remark 1. The competitive attractor $\{x : V(x) = \mu\}$ in general may depend on the whole price trajectory $\{H^t(p_0)\}$. To say anything more about price dynamics requires more details about H, which in practice means the specifics of the market institution under study (the rules organizing trade in the market) and other exogenous constraining factors. Theorem 1 below, the central result of this paper, applies to unconstrained competition (in the sense of Definition 3).[16]

[15] In fact, the invariance principle allows for a finer specification of the competitive attractor: notably, only the invariant subsets of $V^{-1}(\mu)$ are relevant. For our purpose, however, Proposition 2 suffices.

[16] Theorem 1 might be extended, we believe, to a more general case of constrained competition, with appropriate proviso such as a so-called "constraint qualification", as developed in the theory of convex optimization.

Theorem 1 *(Purely Competitive Dynamics). A purely competitive market, from any initial condition, converges to the value center, which is an optimal robust summary of the distribution of traders' valuations, and reflects its maximum information:* $p_t \to C = \arg\min_{x\in\mathbb{R}} V(x) = \arg\max_{x\in\Theta} I(x)$*. This global attractor of competitive price corresponds to maximum surplus. Quantity traded also reveals maximum information about the reservation prices in that the competitive allocation* $\mathbf{q}^*$*, which maximizes surplus, is an allocation with maximum covariance with the reservation prices:* $\Lambda(\mathbf{q}^*) = \max\{\text{cov}(\mathbf{r}, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^{m+n}\}$.

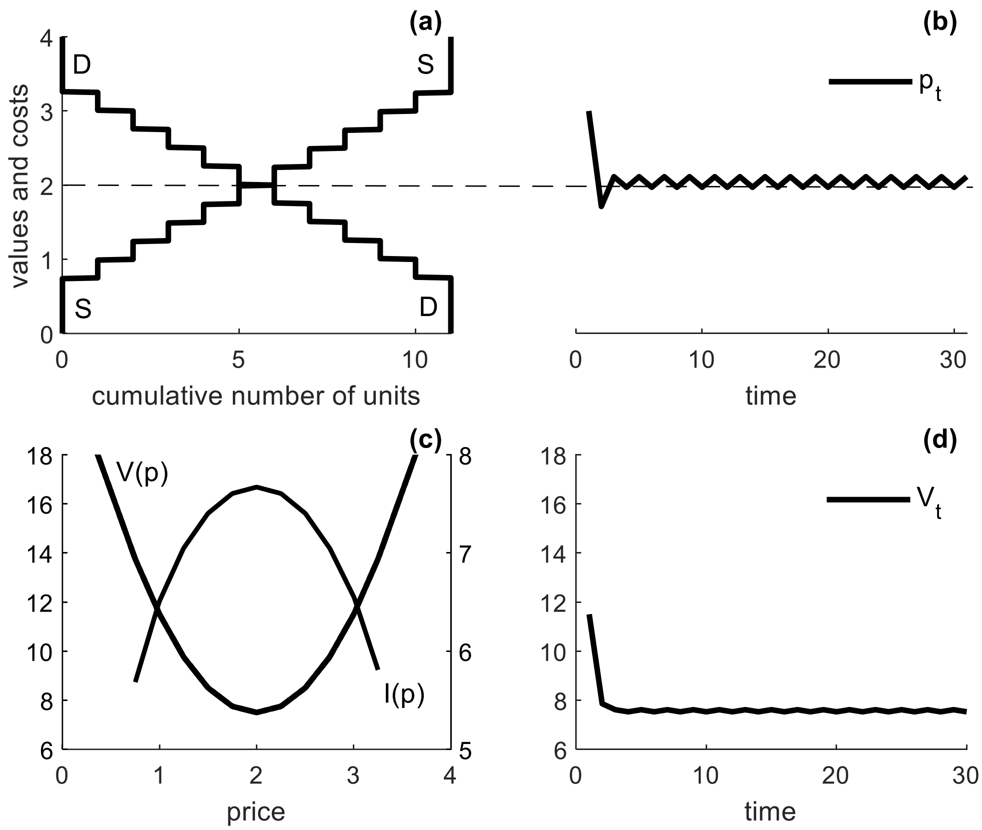

Figure 3. Simulation of a Lab Market Dynamics (V. L. Smith, 1962, Chart 1). (a) Supply and Demand (Table 1 and Figure 2). (b) Price dynamics. (c) Potential surplus function *V* (left scale) and information function *I* (in bits, right scale). (d) Dynamics of *V*.

Proof. The convergence $p_t \to C$ is a consequence of Proposition 2 and the definition of pure competition (Definition 3), or $V^{-1}(\mu) \subseteq C$. The rest of the proof follows from Lemmas 1-3 below: that $C = \arg\max I(x)$ is proven in Lemmas 3; that $\min V = \max \Lambda$ follows from the important price-quantity duality in Lemma 3.■

Lemma 1 *(Properties of the Potential Surplus Function). The price-value distance function is an integral of excess supply and a convex Lyapunov function for a competitive market. Besides continuity and convexity, more specifically, the function satisfies the following properties:*

$$V(p) = \sum_{i=1}^{n} (v_i - p) a_i(p) + \sum_{j=1}^{m} (p - c_j) a_j(p). \tag{31}$$

$$V(p) = V(0) + \int_0^p [S(x) - D(x)] dx. \tag{32}$$

$$V(p) \to \infty, |p| \to \infty. \tag{33}$$

$$V(p_{t+1}) \leq V(p_t). \tag{34}$$

$$V \geq 0. \tag{35}$$

Proof. We stated (31), (34), and (35), true by construction or assumption, merely for completeness. The following formulas follow from graphical inspection of the relevant areas (Figure 4):

$$\int_0^p a_j(x) dx = (p - c_j) a_j(p),$$

$$\int_p^\infty a_i(x) dx = (v_i - p) a_i(p).$$

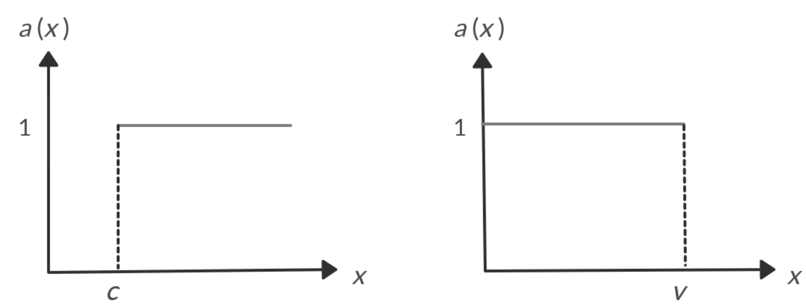

Figure 4. The basic profitability/affordability indicator function. Left: $a(x) = I(c \leq x)$. Right: $a(x) = I(v \geq x)$.

Thus, by summation, we get as claimed in Section 2 (Motivation):

$$V(p) = \sum_{i=1}^{n} (v_i - p) a_i(p) + \sum_{j=1}^{m} (p - c_j) a_j(p) = \int_p^\infty D(x) dx + \int_0^p S(x) dx.$$

Formula (32) then follows by additivity of integration across intervals:

$$V(0) = \sum_{i=1}^{n} v_i = \int_0^\infty D(x) dx = \int_0^p D(x) dx + \int_p^\infty D(x) dx. \tag{36}$$

Continuity (standard): $|V(x)-V(y)|=|\int_x^y Z|\leq \int_x^y |Z| \leq L|x-y|$ since $|Z|\leq L$ for some $L\geq 0$; hence $V(x)\to V(y)$ as $x\to y$, proving continuity of V. Convexity of $|c_j-x|I(c_j\leq x)$ and $|v_i-x|I(v_i\geq x)$ (obvious graphically and easy to prove) is preserved by summation, hence V is convex.[17] Property (33):[18] $\lim_{x\to-\infty}|c_j-x|I(c_j\leq x)=0$, $\lim_{x\to\infty}|c_j-x|I(c_j\leq x)=\infty$, $\lim_{x\to\infty}|v_i-x|I(v_i\geq x)=0$, and $\lim_{x\to+\infty}|v_i-p|I(v_i\geq p)=0$, hence, by summation, $\lim_{x\to\pm\infty} V(x)\to+\infty$. ■

Remark 2. The competitive attractor C is graphically identified as the conjunction of the supply and demand curves when these latter are treated as if continuous (solid) curves (hence ignoring the discontinuities). More rigorously, the following properties characterize it.

Lemma 2 *(Properties of the Value Center). (a) The value center* $C=\arg\min V(x)$ *is nonempty for any market* $[\mathbf{r},H]$. *(b) It generalizes the market-clearing equilibrium concept:* $C=[r^+,r^-]$, *where* $r^+=\sup\{x:Z(x)>0\}$ *and* $r^-=\inf\{x:Z(x)<0\}$; if $Z(p^*)=0$ then $p^*\in C$. *(c) It is in the trade price range:* $C\subseteq\Theta=[c_{\min},v_{\max}]$. *It coincides with the information-maximizing price set:* $C=\arg\max_{x\in\Theta} I(x)$.

Proof. Graph-based proof: By definition $C=\arg\max\int_0^p Z$; since Z is a nonincreasing step function, two cases are possible, as depicted in Figure 5; thus by definition of $r^\pm$, the area is maximum for p in $[r^+,r^-]$. That C generalizes the clearing set $Z^{-1}(0)$, which may be empty, is also clear

[17] To save space, we adopt a simplified integral notation in the proof, omitting dx, for example.

[18] In mathematical jargon, this property is sometimes referred to as “radial unboundedness”.

graphically. More formal proof: Take $\varepsilon > 0$. By definition of r^+, $Z \geq 0$ on $[r^+ - \varepsilon, r^+]$ and $Z \leq 0$ on $[r^+, r^+ + \varepsilon]$, thus $V(r^+ - \varepsilon) - V(r^+) = \int_{r^+-\varepsilon}^{r^+} Z \geq 0$ and $V(r^+ + \varepsilon) - V(r^+) = \int_{r^+}^{r^+ + \varepsilon} Z$ hence r^+ is a local minimum, and so is also any point in $[r^+, r^-]$; and similarly r^- is also a local minimum. By convexity of V, local minima of this latter are also global minima, hence $C = [r^+, r^-]$. If ≤ 0, $Z(p^*) = 0$, then $p^* \geq r^+$ by definition of r^+, and $p^* \leq r^-$ by definition of r^-, hence $p^* \in C$.[19] (c) For $x < c_{\min}$, $S(x) = 0$ and $Z(x) \geq Z(c_{\min}) = D(c_{\min}) \geq D(v_{\max}) > 0$, hence $x < r^+$. Similarly if $x > v_{\max}$ then $x > r-$. Thus $C \subseteq [c_{\min}, v_{\max}]$. (d) Define $\rho = \log(D/S)$ on $[c_{\min}, v_{\max}]$. Clearly $Z = S[\exp(\rho) - 1]$ and $\operatorname{sign}(Z) = \operatorname{sign}(\rho)$, $\{x : Z(x) < 0\} = \{x : \rho(x) < 0\}$, and $\{x : Z(x) > 0\}$ $= \{x : \rho(x) > 0\}$. Thus the same argument used in identifying $[r^+, r^-] = \arg\max \int_0^p Z$ will lead to the conclusion $\arg\max I(x) = \arg\max \int_0^p \rho = [r^+, r^-]$. ■

[19] On can go further in the analysis of C as generalizing the clearing set $Z^{-1}(0)$ through the concept of generalized inverse. (See, e.g., Embrechts & Hofert, 2013.)

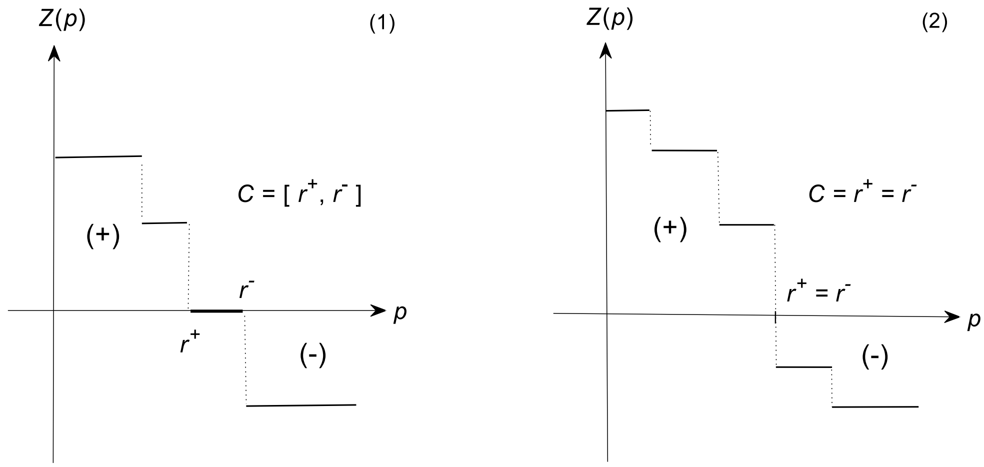

Figure 5. The center of value is graphically the set of prices that maximize the area below excess demand *Z*. It generalizes the concept of market-clearing price: formally *C* = [r^+, r^-], where the endpoints are, the critical points at which *Z* changes sign.

The following core lemma characterizes the market mechanism as a linear program.[20]

Lemma 3 *(Fundamental Price-Quantity Duality). A purely competitive market mechanism can be characterized as a linear program where primal and dual are equivalent to:*

$$\min\{V(p)\} \qquad \text{(primal)}$$
$$\max\{\Lambda(\mathbf{q})\}. \qquad \text{(dual)}$$

If the price-quantity pair $(p^*, \mathbf{q}^*)$ *solves the two programs, then* $V(p^*) = \Lambda(\mathbf{q}^*)$.

Proof. Given $\mathbf{r} = [v_1, \ldots, v_n;\ c_1, \ldots, c_m] = [r_1, \ldots, r_n;\ r_{n+1}, \ldots, r_m]$, we can write

$$V(p) = \sum_{u=1}^{n} |r_u - p| I(r_u \geq p) + \sum_{u=n+1}^{m} |r_u - p| I(r_u \leq p).$$

[20] A similar linear-programming formulation is common in the robust statistics literature (Abdelmalek, 1974; Gutenbrunner & Jurecková, 1992; Koenker, 2005; Koenker & Bassett Jr, 1978; Koenker & Hallock, 2001; Wagner, 1959).

Following standard procedure in linear programming, we decompose the relevant variables into positive and negative parts, corresponding to price $p = p_1 - p_2$ (we must not impose $p_2 = 0$ beforehand) and the potential unit surpluses $s_{1u} = \max\{0, r_u - p\}$, $s_{2u} = \max\{p - r_u, 0\}$, so that by construction $r_u - p = s_{1u} - s_{2u}$ and $V(p) = \sum_{u=1}^{n} \nu_{1u} + \sum_{u=n+1}^{m} s_{2u}$. Let $\mathbf{1}_k = [1,...,1]$ and $\mathbf{0}_k = [0, ...,0]$ be the $1 \times k$ vectors of 1 and 0, and let $\mathbf{I}_k$ be the $k \times k$ identity matrix. Let the vectors $\mathbf{s}_1 = [s_{11},..., s_{1m+n}]$, $\mathbf{s}_2 = [s_{21}, ..., s_{2m+n}]$, $\mathbf{s} = [p_1\ p_2, \mathbf{s}_1\ \mathbf{s}_2]^{\mathrm{T}}$, $\mathbf{e} = [0\ \ 0\ \ \mathbf{1}_n\ \ \mathbf{0}_m\ \mathbf{1}_m\ \ \mathbf{0}_n]$, and the $(m+n) \times [2(m+n)+2]$ matrix $\mathbf{J} = [\mathbf{1}_{m+n}^{\mathrm{T}}\ -\mathbf{1}_{m+n}^{\mathrm{T}}\ \ \mathbf{I}_{m+n}\ -\mathbf{I}_{m+n}]$, so that $V(p) = \mathbf{es}$ and the constraint $\mathbf{r} - p\mathbf{1}_{m+n}^{\mathrm{T}} = \mathbf{s}_1 - \mathbf{s}_2$ is equivalent to $\mathbf{Js} = \mathbf{r}^{\mathrm{T}}$. Thus the optimization $\min\{V(p) : p \in \mathbb{R}\}$ is equivalent to the linear program:

$$\min\{\mathbf{es} : \mathbf{Js} = \mathbf{r}^{\mathrm{T}}, \mathbf{s} \geq \mathbf{0}\}. \tag{37}$$

The dual program is:

$$\max\{\mathbf{rq} : \mathbf{J}^{\mathrm{T}}\mathbf{q} \leq \mathbf{e}^{\mathrm{T}}, \mathbf{q} \in \mathbb{R}^{m+n}\}. \tag{38}$$

The constraint in the dual program is equivalent (if written explicitly) to the requirements: $\mathbf{1}_{m+n}\mathbf{q} \leq 0, -\mathbf{1}_{m+n}\mathbf{q} \geq 0$, $\mathbf{q} \leq [\mathbf{1}_n\ \ \mathbf{0}_m]^{\mathrm{T}}$, $\mathbf{q} \geq [-\mathbf{1}_m\ \ \mathbf{0}_n]^{\mathrm{T}}$, which means:

$$\mathbf{1}_{m+n}\mathbf{q} = \sum_{u=1}^{m+n} q_u = 0; \tag{39}$$

$$q_u \in \begin{cases} [0,1], u = 1,...,n, \\ [-1,0], u = n+1,...,m+n. \end{cases} \tag{40}$$

Assume the pair $(\mathbf{s}^*, \mathbf{q}^*)$ solves the two programs. Then the so-called complementary slackness condition implies (after some identification):

$$q_i^* = \begin{cases} +1, v_i > p^*, \\ 0, v_i < p^*, \end{cases} \quad (41)$$

$$q_j^* = \begin{cases} -1, c_j < p^*, \\ 0, c_j > p^*, \end{cases} \quad (42)$$

Hence, the dual program is surplus maximization. We know that the first program has an optimal solution and we even characterized the optimal set (the value center C). It follows from the strong duality theorem of linear programming theory that $V(p^*) = \mathbf{es}^* = \mathbf{rq}^* = \Lambda(\mathbf{q}^*)$. ■

The maximum surplus is of course the famous area below the supply and demand curves (Figure 6), given by the formula:

$$\Lambda(\mathbf{q}^*) = \int_0^\infty \min\{S(x), D(x)\} dx. \quad (43)$$

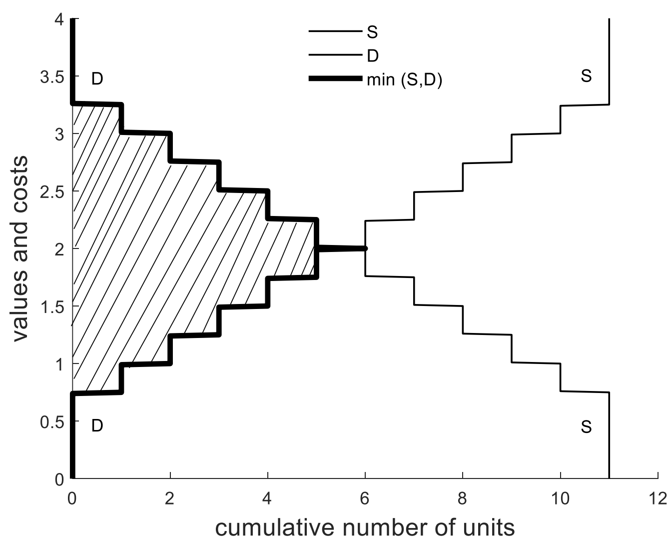

Figure 6. Maximum Attainable Surplus is the Hatched Area.

Proof. We know that $C = [r^+, r^-]$ from Lemma 2. For any p^* in C, and by definition of $r^\pm$, $D \geq S$ in $[0, p^*]$ and $D \leq S$ in $[p^*, \infty]$. Hence $\int_0^\infty \min(S,D) = \int_0^{p^*} \min(S,D) + \int_{p^*}^\infty \min(S,D) = \int_0^{p^*} S + \int_{p^*}^\infty D = V(p^*) = \Lambda(\mathbf{q}^*)$. ■

In experimental contexts, the maximum surplus is the maximum money that the subjects could take collectively from the experimenter. It has thus become customary in experimental economics to measure the efficiency of a market by the fraction (or percentage) of the maximum attainable surplus that the subject-traders extract (Plott & Smith, 1978).[21] The efficiency the market reaches at time t is therefore:

$$\text{Efficiency}(\mathbf{q}_t) = \frac{\Lambda(\mathbf{q}_t)}{\max \Lambda(\mathbf{x})} = \frac{\mathbf{r} \cdot \mathbf{q}_t}{\Lambda(\mathbf{q}^*)}. \tag{44}$$

The overwhelming experimental finding is $\text{Efficiency}(\mathbf{q}_t) \to 100\%$ (or nearly so) for non-retradable goods.

3.3 Competition: Dual Formulation

Competition among traders has an implication in terms of price dynamics, on which we focused mostly so far, but also in quantity allocation emphasized in the duality result (Lemma 3). In fact, we could have chosen to take as axiom, the dual formulation of the competition principle. Moreover, the study of trading institutions is perhaps more explicitly done in the dual formulation of price theory, which this section briefly presents. A trading institution, described formally in V. L. Smith (1982, p. 925), specifies, among other things, the types of messages (price or quantity quotes: bids, asks, acceptance, notably) traders are allowed to send, and an *allocation rule*, which states the allocation of commodity units each trader receives given the standing message vector sent by all traders.

Formally, let the allocation dynamics be written recursively as

[21] Efficiency in experimental markets was defined by Plott and Smith (1978). "These markets are perfectly efficient if and only if the maximum amount of money is extracted by the participants from the experimenter." (1978, p. 139)

$$\mathbf{q}_{t+1} = R(\mathbf{q}_t). \tag{45}$$

We can rephrase competitive value theory in terms of the allocation rule R, stating the dual version of each implication point in the discussion centered on price. Thus, the dual formulation of the competition principle reads:

Competition Principle (Dual Formulation). *A move $\mathbf{q}_t \to \mathbf{q}_{t+1}$ of the allocation vector is driven by competition if the following condition holds:*

$$\operatorname{cov}(\mathbf{r}, \mathbf{q}_{t+1}) \geq \operatorname{cov}(\mathbf{r}, \mathbf{q}_t). \tag{46}$$

In other words, a trading institution is competitive if its allocation rule R is such that $\operatorname{cov}(\mathbf{r}, R(\mathbf{q}_t)) \geq \operatorname{cov}(\mathbf{r}, \mathbf{q}_t)$ for any admissible allocation $\mathbf{q}$. The trading rule is purely competitive (it imposes no friction on competitive discovery) if the maximum covariance is attainable under the trading rule.

The two dual formulations of pure competition agree by Lemma 3: a market reaches a purely competitive equilibrium allocation $\mathbf{q}^*$ if and only if the final standing transaction price p^* is in the value center, since:

$$\Lambda(\mathbf{q}^*) = \max \Lambda(\mathbf{q}) = \min V(p) = V(p^*).$$

One can show that the covariance condition (46) imposes a constraint on the trade sequence: (46) holds notably when higher-value buyers (lower-cost sellers) tend to outbid (undersell) lower-value buyers (higher-cost sellers) and hence succeed to trade units to the exclusion of these latter: competitive trading tends to obey a trading priority rule whereby trades occur in sequence of increasing costs and decreasing values. This need only be the case probabilistically, however.

One can see the general idea behind these implications of the dual formulation of the competition principle particularly in view of the following identity:

$$\frac{1}{2}\sum_{u,h=1}^{m+n}(r_u - r_h)(q_u - q_h) = (m+n)\sum_{u=1}^{m+n} r_u q_u - \sum_{u=1}^{m+n} r_u \sum_{i=1}^{m+n} q_u = (m+n)\operatorname{cov}(\mathbf{r},\mathbf{q}) = \Lambda(\mathbf{q}).$$

From this identity it is not hard to see that the covariance is increasing if trade occurs by increasing (decreasing) order of values (costs), for example.

We close this brief discussion on the dual view of competition and institution with a more specific theoretical reading of a few bid/ask acceptance or trade priority rules common in DA-run markets: to avoid any ambiguity, we need to make explicit the following definition, implicit in the very notion of competition: we say that bid/ask submissions are competitive if they preserve the ranking of the reservation prices. If so, it is consistent with the hypothesis that institutional evolution is directly responsive to the Max-WTP/Min-WTA efficient price discovery process.

Proposition 3 (Competitive Trade Priority Rules)*. Consider a trading institution allowing traders to compete through bid/ask submissions and whereby trade is to occur by priority of higher bids/lower asks, at a standing ask or bid. Then the trading institution is purely competitive (that is, it will lead to a final transaction price in the value center) provided the maximum trade volume is attainable under it. The same is true more generally of any trading rule* $R = R(\mathbf{m})$, *based on traders' messages* $\mathbf{m} = \mathbf{m}(\lambda)$, *which organizes trade among the value and cost distribution* $\mathbf{r} =$

$[r_u]$ *by priority of a (one-dimensional) trade-message identifying parameter* $\lambda = [\lambda_u]$ *(bid/ask or message submission time) submitted competitively.*[22]

Proof. Consider the allocation vector $\mathbf{q} = [1,...,1,0,...,0;-1,...,-1,0,...,0]$, where the nonzero entries amount to the maximum number of mutually tradable units. By the so-called rearrangement inequality, the maximum possible surplus $\sum_u r_u q_u$ is achieved when the units are allocation (or rationed) according to the ranks of the reservation prices, namely when $(r_u - r_h)(q_u - q_h) \geq 0$. This allocation is a purely competitive one. It is achieved more generally by any allocation rule $q_u = q_u(\lambda_u)$ such that $(q_u - q_h)(\lambda_u - \lambda_h) \geq 0$ and $(r_u - r_h)(\lambda_u - \lambda_h) \geq 0$. By Lemma 3, maximum actual surplus allocation corresponds to minimum potential surplus, or the value center. ■

3.4 Large Market Model

In the old view of competition, the number of buyers or sellers per se is not a crucial parameter, as was clear from the ongoing theory and as will be illustrated more specifically in the next section on applications of the theory. For mathematical simplicity, however, and merely for that, it is useful at times to assume a textbook-style large market, whose behavior can be assumed smooth by the law of large numbers. This section develops this specific large-market model. Formally, we study the limit behavior of a market $\{[v_1,...,v_n;\ c_1,...,c_m];H\}$ when $m+n \to \infty$. To this end, the theory is better restated in terms of per-unit average functions, by normalizing all

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[22] A second example is a trading rule according to which traders compete to submit trade orders the earliest possible and trade is to occur by priority of submission time (first-in, first-out rule).

relevant functions. (In fact, we could have formulated the theory from the beginning in normalized form, stating everything relatively to the market's size, $m+n$.)

Throughout this section, therefore, we think in terms of per-unit average functions, while keeping the same notation. Thus, we redefine the supply, demand, and excess demand functions as follows:

$$D(v) = \frac{\#\{i : v_i \geq v\}}{m+n}. \tag{47}$$

$$S(c) = \frac{\#\{j : c_j \leq c\}}{m+n}. \tag{48}$$

$$Z(p) = D(p) - S(p). \tag{49}$$

The potential surplus (or price-value distance) function per unit can be written as

$$V(p) = D(p)s_1(p) + S(p)s_2(p), \tag{50}$$

where, by definition, we write

$$s_1(p) = \frac{1}{D(p)} \sum_{i=1}^{n} (v_i - p) I(v_i \geq p), \tag{51}$$

$$s_2(p) = \frac{1}{S(p)} \sum_{j=1}^{m} (p - c_j) I(c_j \leq p). \tag{52}$$

The integral formula (1) for the potential surplus yields (by identification) the known formulas:[23]

$$s_1(p) = \frac{1}{S(p)} \int_0^p S(x)dx, \tag{53}$$

$$s_2(p) = \frac{1}{D(p)} \int_p^{\infty} D(x)dx. \tag{54}$$

[23] These functions are familiar in probability theory: the first function is sometimes referred to as the "mean excess function" or "mean residual life" function. Assume p in $\Theta = [c_{\min}, v_{\max}]$ to avoid zero as denominator.

In normalized form, the surplus (namely average surplus per unit) is simply:

$$\Lambda(\mathbf{q}) = \mathrm{cov}(\mathbf{r}, \mathbf{q}). \tag{55}$$

As emphasized in Lemma 3 (price-quantity duality), there is an intrinsic connection between surplus maximization and competitive allocation; in fact, we could have chosen to take as an axiom, the dual formulation of the competition principle: $\mathrm{cov}(\mathbf{r}, \mathbf{q}_{t+1}) \geq \mathrm{cov}(\mathbf{r}, \mathbf{q}_t), t \geq 0$. Due notably to buyer-buyer and seller-seller rivalry (or type 2 competition), the covariance between the allocation vector and the reservation prices is increasing through competition, and one can show that units tend to trade by priority of higher values and lower costs. In contrast, absence of type 2 competition is indicated by the condition:

$$\mathrm{cov}(v, q_1 \mid v \geq p) = \mathrm{cov}(c, q_2 \mid c \leq p) = 0, \tag{56}$$

where q_1 and q_2 are the restrictions of the trade indicator $\mathbf{q}$ to demand and supply units, respectively. We will also consider explicitly *short-side rationing*, which means that the total quantity traded is the minimum between the quantity demanded and that which is supplied:

$$Q = \min(S, D). \tag{57}$$

Intuitively, as we said, a market involving a sufficiently large number of diverse reservation prices behave smoothly.[24] More formally:

[24] Throughout, “smooth” means “differentiable with continuous derivative”. The intuition behind this concept of large market goes back to an observation of Cournot (1838 [1897], p. 50) on aggregate demand, which is smooth by the law of large numbers. As he turned to the supply side, Cournot would also inspire a second, more influential, notion of a large market, a model of “perfect competition”, which mathematical economists would translate abstractly (in the language of measure theory) as a market with “continuum of traders” (Aumann, 1964).

Definition 4 *(Large Market). We say that a market* $[\mathbf{r}, H]$ *is large if its average behavior is smooth: that is,* the function H is smooth, and *the costs and values are distributed according to continuous density functions supported respectively on* $[c_{\min}, c_{\max}]$ *and* $[v_{\min}, v_{\max}]$.[25]

We will write price dynamics for a large market as

$$\frac{dp}{dt} = H(p). \tag{58}$$

As already emphasized (Section 2), the law of supply and demand reads simply:

$$\frac{dV}{dt} = [S(p) - D(p)]H(p) \leq 0. \tag{59}$$

Pure competition in a large market amounts to assuming the stronger form of the law of supply and demand for a large market:[26]

$$\text{sign}(H) = \text{sign}(D - S), \tag{60}$$

where the sign function is defined as

$$\text{sign}(x) = \begin{cases} +1, x > 0, \\ \quad 0, x = 0, \\ -1, x < 1. \end{cases}$$

The following proposition is a direct implication of the large-market assumption:

Proposition 3 *(Unique Large-Market Equilibrium). A large market's value center coincides with the market-clearing price, which is unique.*

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[25] We are implicitly assuming bounded supports; otherwise, if $c_{\max} = \infty$ or $v_{\max} = \infty$, then the left and right limits are excluded, and we assume then $\mathbb{E}(c) < \infty$ and $\mathbb{E}(v) < \infty$.

[26] That price change and excess demand have the same sign, and the short-side ratioing constraint (57) are significant elements in Adam Smith's verbal description of the dynamics of market price formation. (See Inoua & Smith, 2020c, Subsection 2.6.)

Proof. A large market's excess demand $Z = D - S$ is a strictly decreasing function: hence there is a unique $p^* = \arg\min V(x) = \{x \in \mathbb{R} : Z(x) = 0\}$, by the strict convexity of V. ■

The following result emphasizes weaker conditions and mechanisms under which competitive equilibrium emerges in a large market (illustrated in Figure 7).

Theorem 2*. Consider a large market. The strict law of supply and demand,* $\mathrm{sign}(H) = \mathrm{sign}(D - S)$, *is sufficient for global convergence to the competitive equilibrium price. Under short-side rationing, any price adjustment rule that maximizes the number of transactions* $Q(p) = \min\{S(p), D(p)\}$ *also leads to competitive equilibrium; maximum surplus is reached under this trading institution even* if $\mathrm{cov}(v, q_1 \mid v \geq p) = \mathrm{cov}(c, q_2 \mid c \leq p) = 0,$ in which case the surplus can be written as $\Lambda(p) = Q(p)\,[s_1(p) + s_2(p)]$.

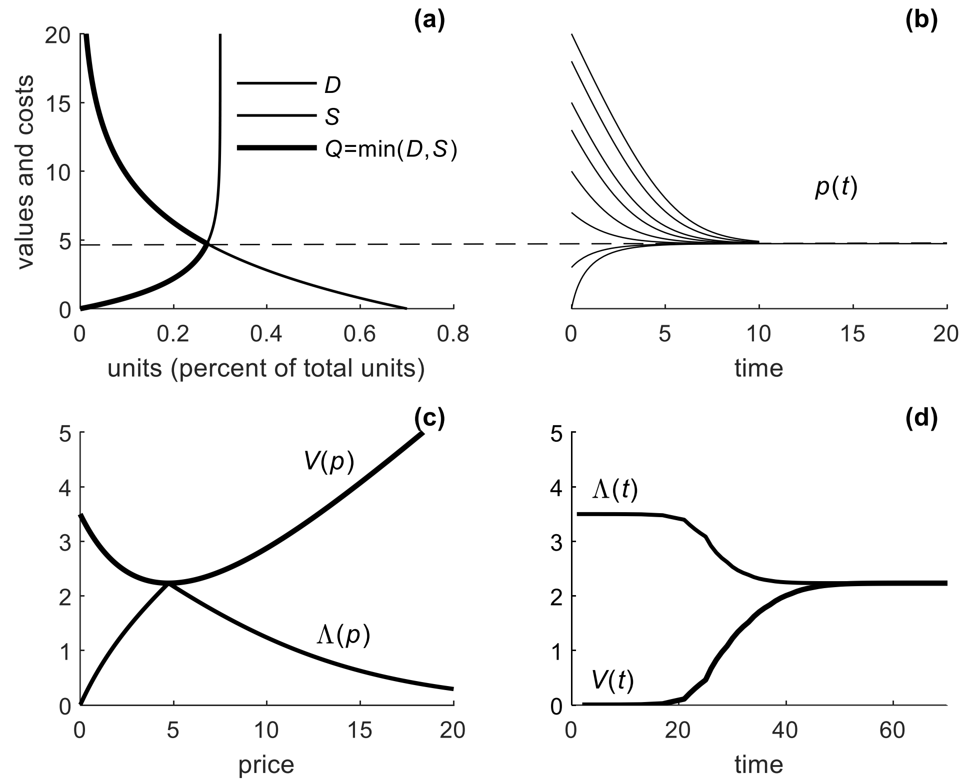

Figure 7. A Large market's competitive dynamics. (a) Supply and demand (exponential value and cost distributions [mean(value)=5, mean(cost)=3, H(Z)=10Z] and number of transactions under short-side rationing. (b) Price trajectories are shown for various initial conditions. (c) Potential versus actual surplus (the latter calculated under the no-covariance assumption.). (d) Dynamics of potential versus actual surplus.

Proof. Since $\dot{V}(p) = -Z(p)H(p)$, and $\mathrm{sign}(H) = \mathrm{sign}(Z)$, we have $E = \{x : \dot{V}(x) = 0\} = \{x : Z(x) = H(p^*) = 0\} = \{p^*\}$, which is an invariant set, since $H(p^*) = 0$. It follows from the invariance principle that $p(t) \to p^*$, for any initial p_0, since $V(x) \to \infty$ as $|x| \to \infty$. Under short-side rule, $Q = \min(S, D)$. If $p < p^*$, $Q'(p) = S'(p) > 0$, and if $p > p^*$, $Q'(p) = Q'(p) < 0$, therefore $Q(p)$ is maximum at $p = p^*$. A large market's mean surplus is $\Lambda = \mathbb{E}(vq_1) - \mathbb{E}(cq_1)$. Since $q_1 =$

0 if $v < p$, we have $\mathbb{E}(vq_1) = \mathbb{E}(vq_1 \mid v \geq p)\mathbb{P}(v \geq p)$ and $\mathbb{E}(q_1) = \mathbb{E}(q_1 \mid v \geq p)\mathbb{P}(v \geq p)$. By $\text{cov}(v, q_1 \mid v \geq p) = 0$ is meant

$$\mathbb{E}(vq_1 \mid v \geq p) = \mathbb{E}(v \mid v \geq p)\mathbb{E}(q_1 \mid v \geq p).$$

Then

$$\mathbb{E}(vq_1) = \mathbb{E}(v \mid v \geq p)\ [\mathbb{E}(q_1 \mid v \geq p)\mathbb{P}(v \geq p)] = \mathbb{E}(v \mid v \geq p)\ Q(p) = [s_1(p) + p]Q(p).$$

Similarly, $\mathbb{E}(cq_2) = [p - s_2(p)]q_2 = [p - s_2(p)]Q(p)$, and, therefore,

$$\Lambda(p) = [s_1(p) + s_2(p)]Q(p).$$

Thus $\Lambda' = Q'(s_1 + s_2) + Q(s_1' + s_2')$, for $p \neq p^*$ (where the prime denotes differentiation with respect to price). It follows from (53)-(54) that

$$s_1' = -\ (S'/\ S)s_1 - 1 \text{ and } s_2' = -(D'/\ D)s_2 + 1,$$

hence,

$$\Lambda' = Q(Q'/\ Q)\{s_1 + s_2\} + Q\{-(S'/S)s_1 - (D'/\ D)s_2\}.$$

Thus, after rearranging terms, we get

$$\Lambda' = Q\{(Q'/\ Q - S'/\ S)s_1 + (Q'/\ Q - D'/\ D)s_2\}.$$

If $p < p^*$, $Q = \min(S, D) = S$, and $\Lambda' = S(S'/\ S - D'/\ D)s_2 > 0$. If $p > p^*$, $Q = D$, and $\Lambda' = D(D'/\ D - S'/\ S)s_2 < 0$. Thus $\Lambda(p)$ is maximum when $p = p^*$. ■

4 Application: Specific Market Institutions as Examples Embedded in the Theory

This section reviews a few well-known observed market institutions that are contained in the theory. The goal is to illustrate the breadth of the old view on competitive market price formation, and to emphasize more specifically how, behind the mechanics of a few well-known markets, lies at work the PMI. That is, we illustrate the theory as it applies to a few standard cases, which, while commonly viewed intuitively as driven by competition, yet are not usually analyzed in terms of the basic supply and demand diagram, due to dominant belief in neoclassical requirements for competitive equilibrium: large number of traders, price-taking behavior, market clearance, complete information. Paradoxically, increased public information on value, or sophisticated individual rationality more generally, may hinder or limit collective market rationality and efficiency, where strategic behavior may be transactions costly. Experiments have been reported in which complete public, compared with incomplete, or private, information on values and costs slows convergence to the center of attraction. (V. Smith, 1980, p. 357-360) Essentially, more information is worse because it invites strategies that are not sustainable, which delays reaching the Center. We emphasize the generality of the value center C in explaining the competitive outcome of known market cases, illustrating in the process the calculation of C, although its graphical determination is easy enough. Needless to say, that this section is not intended to be a systematic comparative analysis of market institutions and conditions; the goal is simply to bring back to the realm of standard competitive price theory a number of market cases and institutions that a century-and-half of marginalist tradition in economics has excluded, because the theoretical requirements of neoclassical price theory are often taken to be requirements of the empirical price mechanism itself.

From these illustrations will emerge the proposition that the relevant taxonomy of markets is not one in terms of the number of traders, but one in terms of the forms of competition operating in each market, and the constraints imposed by the trading institution (the trading rules organizing trade in a market). As long as a market is driven by at least one form of competition, competitive equilibrium naturally emerges within the boundaries set by the trading institution. A trading rule usually emerges from trial-and-error practice and usually acts as a catalyzer of competition, by fostering competitive price discovery (for example, the bid-ask reduction rule); but in the process it may also impose constraints on purely competitive dynamics by acting as friction.

Unless stated otherwise, competitive dynamics is simulated in all illustrations using a log-linear approximation of the law of supply and demand:

$$p_{t+1} = p_t \exp\{\lambda[D(p_t) - S(p_t)]\}, \tag{61}$$

where the positive parameter λ is chosen to yield $V(p_{t+1}) \leq V(p_t)$ approximately. (The discontinuity of supply and demand typically creates oscillations in the dynamics.)[27]

4.1 Isolated Exchange: The Smallest Market

Consider the simplest market, a casual, isolated, bargain between a buyer (with valuation v_0) and a seller (with valuation c_0) on a unit of a good $(m = n = 1)$. (Let $c_0 \leq v_0$, for a transaction to occur.) This market is a special case of a market configuration studied experimentally by Smith

[27] It goes without saying that $t = 1, \ldots, T$ denotes *real transaction time*, unlike in the tatonnement story, which, by having imposed that all traders must take price as exogenous parameter beyond their control, is led to ascribe the fundamental task of price adjustment to a fictional auctioneer in "virtual time". That is, the difficulty in the tatonnement story is not the (linear version of) formula (61) per se, which is a reasonable approximation of the law of supply and demand, but the derivation of supply and demand from price-taking behavior and the law of one price.

and Williams (the so-called box design). Its value center is $C=[c_0,v_0]$. Hence the bargain would be concluded at some price in $[c_0,v_0]$, as is known.

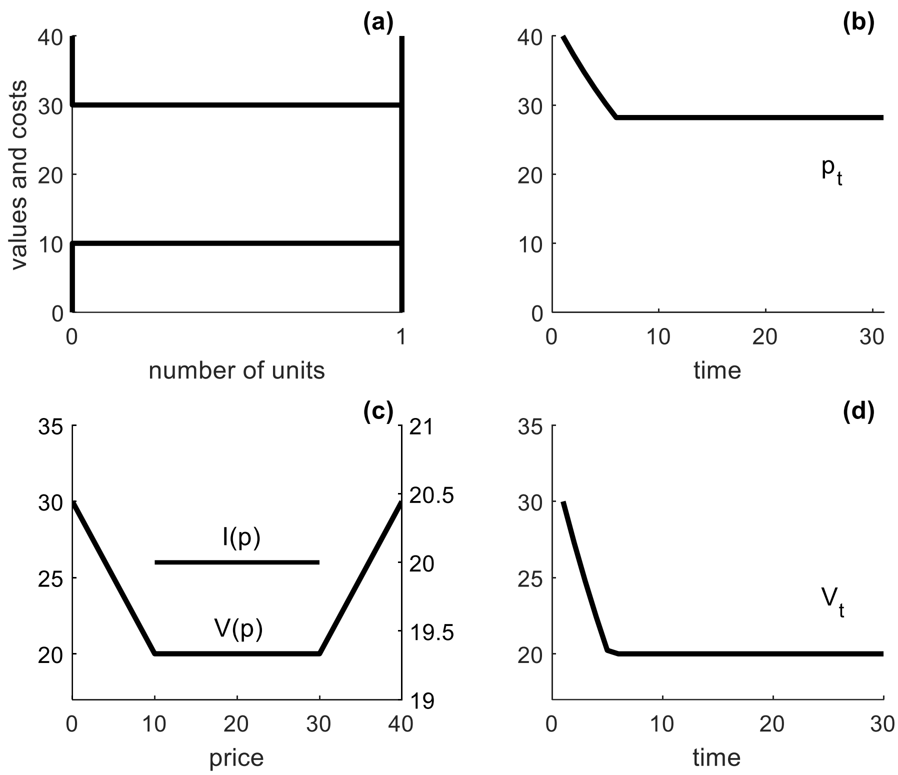

Figure 8. Isolated Buyer-Seller Haggling Simulated: (a) Supply and Demand (b) Price dynamics: $p_0>v_{max}$ means we are modeling the standing ask price (see footnote 10). (c) Potential surplus function V (left scale) and information function I (in bits, right scale). (d) Dynamics of V.

Chamberlin's (1948) experiments can be viewed as a collection of isolated exchanges, each of which is to be treated in all rigor as an isolated market in itself. Thus, by an appropriate definition of the market size, competitive equilibrium may still apply: each transaction price in these distance-separated, or "isolated," trades, is a competitive equilibrium, and their multiplicity simply reflect the multiplicity of relevant markets.[28] V. L. Smith's (1962) experiments are in this

[28] John List's replication of Chamberlin's (1948) bargaining markets over time demonstrate that "publicity of…bids and asks are not necessary for markets to equilibrate… List (2004, p. 1154)

sense the first experiments of an organized market, where the oral double-auction trading rule allows contiguous buyers and sellers to participate in a distinct, collective higgling and bargaining, which can then be treated as a unique market.

4.2 Swastika and Box Supply-Demand Configurations

Consider the following market, well-stud$[c_0, v_0]$, ied in the lab (V. L. Smith, 1965; V. L. Smith & Williams, 1990), involving $\mathbf{r} = [v_0, ..., v_0; c_0, ..., c_0]$, with $v_0 \geq c_0$ so that trade is possible. We have $V(p) = n(v_0 - p)I(v_0 \geq p) + m(p - c_0)I(c_0 \leq p)$. By Lemma 2, $C = \arg\min_\Theta V(p)$, $\Theta = [c_0, v_0]$. For p in Θ, $V(p) = n(v_0 - p) + m(p - c_0) = (nv_0 - mc_0) + (m - n)p$. So three cases should be contrasted. If $m > n$, V increases (linearly) on Θ, so $C = \{c_0\}$. If $m < n$, V decreases on Θ, hence $C = \{v_0\}$. (These two cases are known as the 'swastika design'.) If $m = n$, V is constant on Θ, hence $C = \Theta = [c_0, v_0]$. This is the 'box design' originally suggest by Edgeworth as an example of equilibrium indeterminacy (multiple equilibria).[29] The above theory says that if price evolves purely competitively, then $p_t \to c_0$ if $\alpha > 1$, $p_t \to v_0$; if. $\alpha < 1$, .and $\alpha = 1$, $p_t \to [c_0, v_0]$, that is, p_t can be anywhere in for t big enough. This prediction, $p_t \to C$, matches the experimental findings within a few cents. Of course, the lab DA competition cannot be perfectly pure due to *transaction costs* and other *frictions*, as emphasized more formally below (Subsection 4.6). First, trade of an abstract commodity representation (lab) differs from trade of a commodity bought for its use-value (field). For a commodity you think is worth v_0 dollars use-value, you would be

[29] "Suppose a market, consisting of an equal number of masters and servants, offering respectively wages and services; subject to the condition that no man can serve two masters, no master employ more than one man" Edgeworth, 1881 p. 59)

willing to pay up to (and including) that value; and the seller, who values the commodity's production cost to be c_0 dollars would be willing to accept that amount, *provided that the cost is classically defined*, namely as reservation prices including a minimum profit requirement. In the lab, however, a subject-trader that buys or sells at the limit prices would make no monetary benefit through the trade (since the object of trade is an abstract commodity representation). Thus, it was common to pay the lab subject-traders a small commission to induce trade at the limit prices, notably near CE. The "swastika" is a good design for measuring this transaction cost, the minimum amount required to induce trade at the limit price, since by construction all trades must occur at a limit price (V. L. Smith & Williams, 1990).[30]

[30] In the experiments, a buyer (seller) earns zero at price equal to their value (cost). The observations in the swastika experiments yield a value minus price distance measure of the minimum profit a buyer requires, and similarly for a seller. Uncertainty as to this profit implies that the attractor price can only be specified subject to this same uncertainty.

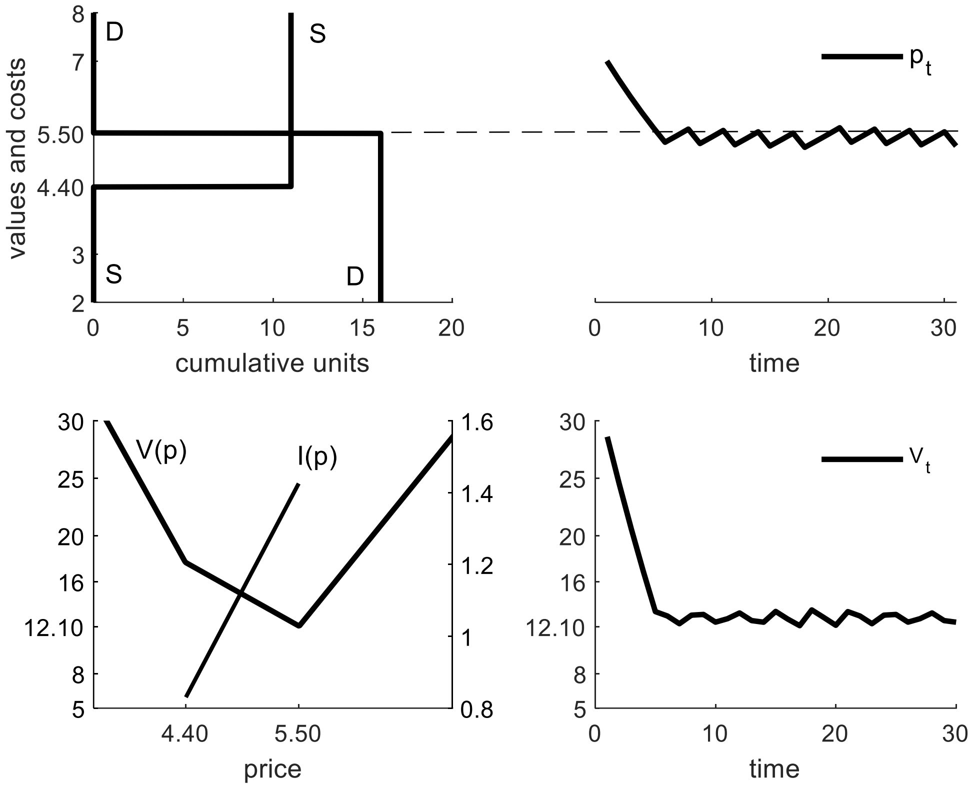

Figure 9. Simulation of a non-clearing market studied in the lab (V. L. Smith & Williams, 1990, Figure 10, Condition A). (a) "Swastika" supply and demand configuration. (b) Price dynamics. (c) Potential surplus function, V, and value and cost information amount, I, revealed by price (in bits). (d) Dynamics of V.

4.3 Auctions: English versus Other Institutions

The English auction of a single unit (say an artwork) is the most popular auction form: the auctioneer announced an initial minimum acceptable bid, and the bidders sequentially raise each standing bid price, until no bidder is willing to raise it further, at which point the standing bidder wins the unit by paying the standing bid price. Classically, there is nothing special about the English auction as a competitive market (the absence of competition on the supply side is irrelevant, since competition is on the buyers' side). The English auction is indeed the simplest example of a market that is entirely driven by buyer-buyer competition (a single passive seller, with buyers outbidding one another).[31] At an elementary practical level, the English auction illustrates the defining features of classical price formation theory in which traders bring their WTP (WTA) values to market; aggregation and price discovery are joint functional consequences of the market process. The market end-state is maximum value revelation—the allocation is to the buyer who most highly values the item, at a price profitable to the seller.

Because it is driven by ordinary competition, the English auction is no exception to standard competitive price theory in terms of supply and demand, and its outcome is a regular case of competitive equilibrium, provided only that supply, demand, and C.E. are as define in the foregoing theory (namely as value and cost distributions and value center, respectively).

The dominant approach to auctions today is of course the game-theoretical approach, starting from the seminal work of W. S. Vickrey (1961), and expanded into a vast literature (recently awarded a new recognition).[32] This approach produced many detailed important insights about the theory and design of auctions. Rather than reviewing these specific game-theoretic models, which is not possible here, we simply want to briefly contrast this usually parametric class of models with the qualitative (nonparametric) direct predictions of foregoing classical price theory and to establish as a direct consequence of the foregoing theory the superiority, ceteris paribus,

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[31] The "English auction" has a colorful history. Its origin is not English; rather, it was introduced into England by the Roman occupation. Roman auctioneers followed the soldiers who were paid in the spoils-of-war property they seized and needed to convert into gold or currency. It was an ascending bid procedure, as suggested by the Latin auctiō, from the Latin verb augere, "to increase". Further emphasizing the connection with warfare, auctions were conducted *sub hasta*, "under the spear." Thus, for example, in Spanish, "subhasta" refers to an auction or auction house (Cassady, 1967, pp 26-30)

[32] The literature on the game-theoretic approach to auctions is vast: for a review, see, e.g., the background to the recent Nobel Prize in Economics (2020), awarded to P. R. Milgrom and R.B. Wilson. It is notable that although auction theory was founded on distributions functions of value, meaning Max-WTP, this modelling perspective was not part of neoclassical/modern supply and demand theory. Implicitly, however, it was part of Adam Smith's verbal model of price formation in a market. (See Inoua & Smith, 2020c, Subsection 2.6.)

of the simple, ordinary competitive rationality of the English auction in terms of aggerate welfare and information efficiency.

Formally, the English auction of a single item is a market with $\mathbf{r} = [v_1, ..., v_n; c_0]$, where the seller's reserve price is c_0, and the buyers' valuations are as usual arranged in descending order, $v_1 \geq v_2 \geq ... \geq v_{n-1} \geq v_n$. The value center is easily determined graphically to be $C = [v_2, v_1]$, assuming $v_2 \geq c_0$. Formally, the endpoints of the value center are by Lemma 2 respectively the right and left endpoints of the following intervals:

$$\{x : D(x) > S(x)\} =]-\infty, v_2] \text{ and } \{x : D(x) < S(x)\} = [v_1, \infty[.$$

Thus, by the competition of the buyers, $p_t \to [v_2, v_1]$, since the rivalry will ultimately be between the two highest-value buyers, '1' and '2'. Clearly $p_t \to v_2$ is a possibility: if perchance '1' bids v_2, then this bid would conclude the auction ('2', whose limit price is reached, cannot raise the bid anymore). Yet v_2 is not a market-clearing price, since $D(v_2) - S(v_2) = 2 - 1 = 1$. More generally, the value center of an English auction is not hard to determine graphically (by considering the two cases $c \leq v_0$ and $v_1 \geq c_0 > v_2$). Formally, it is given by Lemma 2 as:

$$C = [\max(c_0, v_2), v_1]. \tag{62}$$

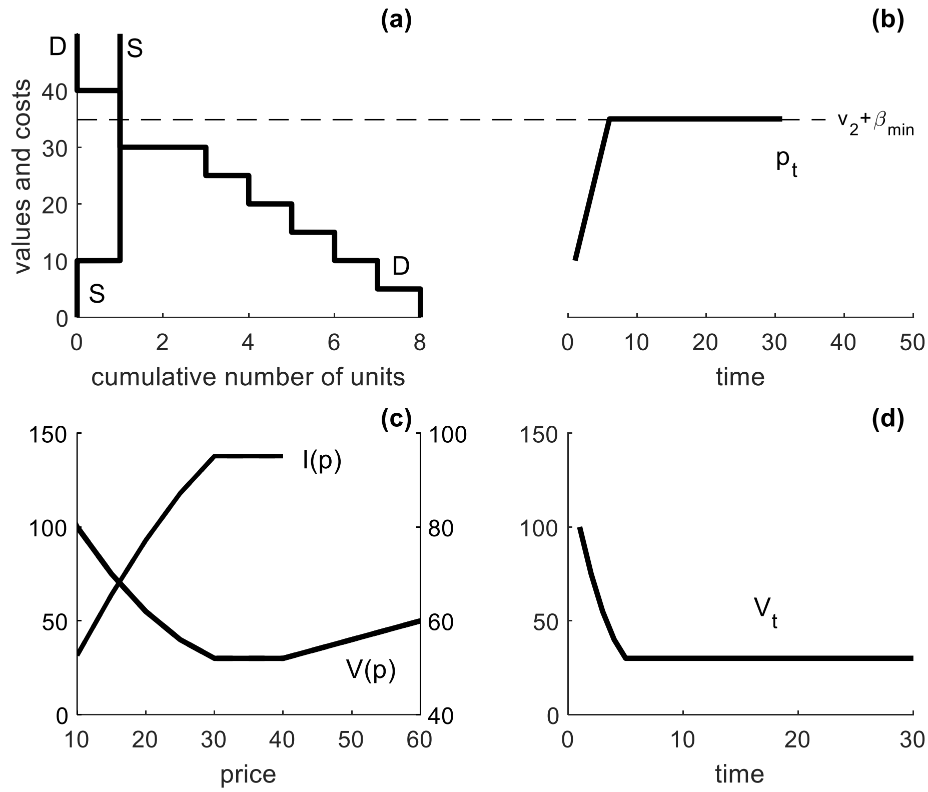

Figure 10. English Auction Simulated. (a) Cumulative distribution of values (5 to 40 by increment of 5) and cost (or seller reserve price, 10). (b) Price dynamics: minimum bid increment $\beta_{\min}$ = 5. (c) Potential surplus function, V, and Information, I, revealed by price (in bits). (d) Dynamics of the potential surplus, V.

More explicitly, the standing price in the English is $p_t = H^t(c_0)$, since by construction the initial price asked is $p_0 = c_0$ (the seller's reservation price). The other institutional constraint is the minimum price increment allowed, say $\beta_{\min}$. So the English auction price dynamics can be written in all generality as:[33]

$$p_0 = c_0, p_{t+1} = p_t + \beta_t I[D(p_t) > S(p_t)], \beta_t \geq \beta_{\min}. \quad (63)$$

[33] Reminder: $I(A)$ denotes the indicator of A, namely $I(A) = 1$ if A is true, $I(A) = 0$ if A is false.

That is, the Auctioneer announces the opening bid at $p_0 = c_0$ and the standing bid price increases by increments $\{\beta_t : \beta_t \geq \beta_{\min}\}$ so long as there is more than one bidder standing to announce their bid. In the reasonable case $v_2 \geq c_0$, and assuming for simplicity of discussion $\beta_t = \beta_{\min}$, we have the dynamics (assumed in the simulations in Figure 10 and Figure 11):

$$p_0 = c_0, p_{t+1} = p_t + \beta_{\min} I[\#\{i : v_i \geq p_t\} > 1]. \tag{64}$$

This simple exercise illustrates more specifically the practical interpretation of the general theory's vocabulary. The qualitative, nonparametric, prediction $p_t \to C$, predicts all possible outcome for an English auction's final price: the final transaction price can be anywhere in C. That general outcome is brought about by the pure competitive bidding process; the specific outcome depends on the institutional parameter $\beta_{\min}$ through the constraint $\beta_t \geq \beta_{\min}$, $t = 1, ..., T$.

The English auction has been extended to multiple units (McCabe, Rassenti, & Smith, 1990). Theoretically, the single item case extends easily to the generalized auction of m units: the value center in the general case is $C = [\max(c_0, v_{m+1}), v_m]$, where v_m are the mth and $(m+1)$th highest values (Figure 10).

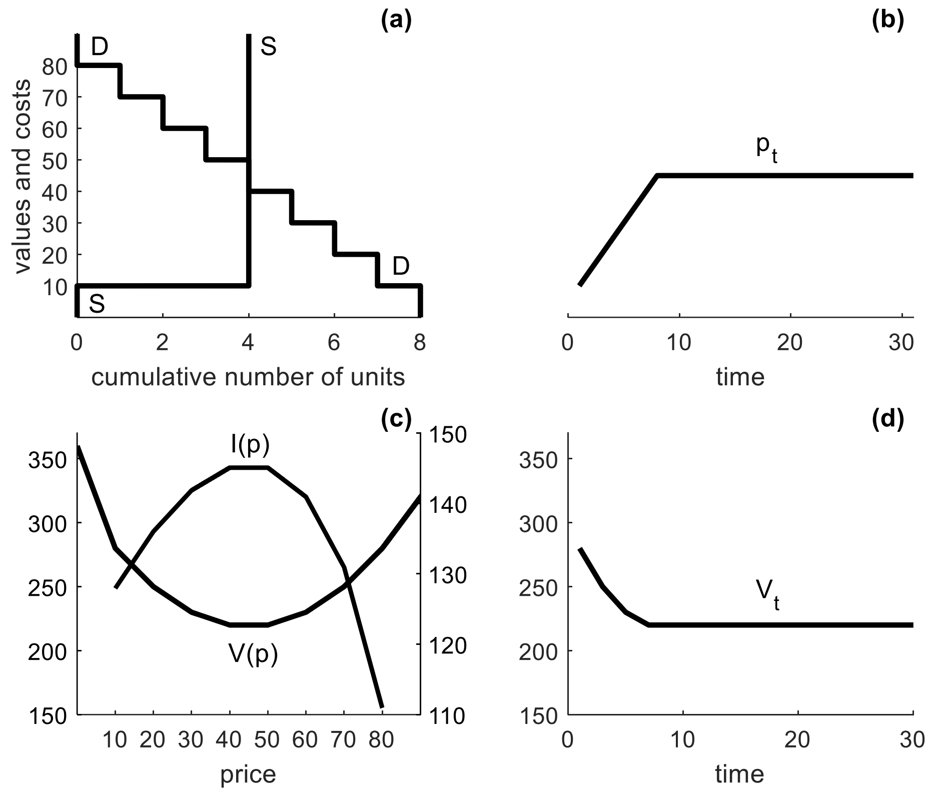

Figure 11. Multiple-Unit English Auction Simulated (Four Units). (a) Values and cost (seller reserve price). (b) Price dynamics: minimum bid increment $\beta_{\min}$ =5. (c) Potential surplus function, V, and Information, I, revealed by price (in bits). (d) Dynamics of the potential surplus, V.

Having emphasized the value center as the English auction's competitive outcome, the foregoing theory provides a qualitative (nonparametric) comparative prediction in terms of comparative efficiency of the English auction vis-à-vis any other auction form, notably the Dutch and sealed-bid auctions.

Proposition 4 (Superior Efficiency of English Auction). *For a given seller's reserve price and bidders' reservation value distribution, the English auction is at least as efficient as any other possible auction form, both informationally (price discovery or value revelation) and in terms of welfare* *(surplus). (Also, ceteris paribus, the English auction price or revenue is at least as high as in the second-price sealed-bid auction.)*

Proof. For simplicity, assume single-object auctions. Since $p_t \to C$ for the English auction, the first part merely specializes Theorem 1. The second part is essentially true by definition of reservation value. The English seller's revenue is $R(p_T) = (p_T - c_0)I(p_T \geq c_0) \geq R(v_2)$, where p_T is the transaction price (or final standing price if no trade happens). Considering a sealed-bid auction, let $b_1 \geq b_2 \geq ... \geq b_{n-1} \geq b_n$ be the buyers' bids, to be submitted to the auctioneer. Whatever the bidding strategy adopted by the bibbers in the sealed-bid second price, the seller's revenue is $R(b_2) = (b_2 - c_0)I(b_2 \geq c_0) \leq R(v_2)$, since by definition of reservation price $b_2 \leq v_2$.

■

We put the second part of Proposition 4 in parenthesis because it has not the same importance as the first part, and not merely mathematically: in traditional economic analysis, white matters is efficiency at the aggregate market level, rather than efficiency from the viewpoint of a particular agent (the seller). Both the welfare and informational optimality of the English auction holds even if we include the commission the seller pays to the auctioneer (after the sale).

The Dutch and sealed-bid auction processes reveal no public information as in the English auction and this introduces strategic considerations into rational action. [34] Yet sophisticated strategic (game-theoretical) rationality cannot outperform ordinary higgling and bargaining rationality in terms of price discovery and social welfare generated. The knowledge requirement in game

[34] For a comparative study of multiple-unit Dutch versus English auctions see McCabe et al. (1990).

theory, be it reminded, is usually strong (in fact infinite) knowledge requirement on the part of each bidder (the common knowledge assumption). A strategically rational bid requires a sense of how others will bid and a solution, as in game theory, postulates bidder knowledge of the distribution of values among the participants—an imperfect alternative but kindred to Jevons' perfect information hypothesis. Moreover, this knowledge must be common to all if a symmetric equilibrium is to exist. In the English auction, such prior knowledge is of worthless value, as others will publicly reveal that information during the auction. Concerning these different auction protocols, classical theory makes a qualitative, comparative, prediction, easily tested in the lab: for the same value distribution (and controlling for transaction costs), the more sophisticated forms of auction will not outperform the English (in terms of value-revelation and surplus extraction). More knowledge and more sophisticated rationality do not necessarily improve market performance.

English auctions, traditionally requiring bidders to be present, are thereby transactions costly, and were most suited to unique art items of high value. Sealed bid auctions therefore are more suitable for items of relatively low value. Since the highest bid wins in the English auction, it might have appeared natural, if incorrect, for practitioners seeking to broaden the use of auctions to apply the first price rule when the bids are submitted sealed and hidden from other bidders.

The second-price rule emerged naturally in free-entry markets long before auction theorists proved its equivalence to the English auction. It was used by stamp collectors in the 19th century. (Lucking-Reiley, 2000) The rule is also used in the London Stamp auction (and for art objects sold at English auction) where off-floor book bids can be submitted. Auctioneers apply the second price rule as a principle in this mixed-bid environment: If the floor bids stop at a price below the highest book bid, but above the second highest book bid, the award goes to the high book bid at the highest oral outcry price; if the floor bidding stops at a price below the second highest book bid, the auctioneer offers to sell it at that price, and if no one bids higher, the highest book bidder wins at a price equal to the second highest book bid.

4.4 Sealed-bid Auctions or Call Markets

Financial markets traditionally closed in late afternoon and reopened the next morning. In double auction trading they close at a final bid-ask spread, but the relevance of that spread is often invalidated by the overnight accumulation of information. This led to the demand for a means of aggregating that information before the restart of continuous double auction trading the next day. One solution was a "call" market procedure, adopted at the opening of some financial markets, which consisted of calling for buy and sell orders at sealed bid or ask prices, with an auctioneer fixing a transaction price that clears the market, or more generally maximizes trade (the number of transactions).[35]

Informationally, a call market (CM) session from $t=1$ to $t=T$ is formally a set of bid and ask messages (price-quantity) $\{(b_j, n_j) : i=1,\ldots,n\}$ and $\{(a_j, m_j) : j=1,\ldots,m\}$ collected during the period $t=1$ to $t=T$ by the auctioneer (market maker), who determines a price p that clears this market; or, if no clearing price exists, a price that maximize the trade volume; or, if many such prices exist, at the mid-price of the trade-maximizing price range.[36] In practice, many specific

[35] This aggregation procedure also is invoked anytime there is a major news event affecting a stock that triggers a halt in trading. A call reopens trading.

[36] Notice that here we have not followed the notational convention of treating individually each value and cost unit; but rather we wrote the multiplicities or frequencies explicitly: this is merely to come close to ordinary practice here, for we will not carry for long the heavier notation including the frequencies.

rules are designed to account for all these possible cases. In the language of the foregoing theory, we can summarize these rules in one line: the transaction price is the middle of the value center corresponding to the revealed supply and demand (the distribution of the submitted bids and asks). Formally, one can show that call-market-price setting rules comes down to applying the price algorithm ,

$$p^{CM} = \frac{r^+ + r^-}{2}, \tag{65}$$

where, as in Lemma 2,

$$r^+ = \sup\{p : d(p) > s(p)\}, \tag{66}$$

$$r^- = \inf\{p : d(p) < s(p)\}, \tag{67}$$

$$d(p) = \sum_{i=1}^{n} n_i I(b_i \geq p), \tag{68}$$

$$s(p) = \sum_{j=1}^{m} m_j I(a_j \leq p). \tag{69}$$

A call market is therefore a static proxy price fixing rule applied to a price emerging naturally from competition for a competitive market, notably a double-auction market (Subsection 4.6) where the main obvious difference is: the call-auction price reveals maximally the bid-ask submitted messages, but need not reveal maximally information about the true, underlying reservation values (because the traders need not reveal their true underlying valuations, and full willingness to trade, through a one-shot submission of bids and asks). The interesting question, therefore, is the extent to which, or the conditions under which, a call-market price approximates the underlying value center; or, better, the extent to which the true underlying value center can be inferred from the submitted bids and asks. Here is a sketch of how this could be answered.

By definition of reservation price, $v_i \geq b_i$ and $c_j \leq a_j$ for all i, j. Let D and S be the true (underlying) willingness to buy and sell (based on values and costs), which are only partly revealed by d and s (based on bids and asks). Let $\varepsilon_i = v_i - b_i$ and $\nu_j = a_j - c_j$, $\varepsilon = [\varepsilon_i]$ and $\nu = [\nu_j]$. It is not hard to derive the following (using an expectation notation to denote averaging with respect to the subscripted variate):[37]

$$D(p) = \mathbb{E}_{\varepsilon}[d(p - \varepsilon)], \tag{70}$$

$$S(p) = \mathbb{E}_{\nu}[s(p + \nu)]. \tag{71}$$

That is, the true underlying demand and supply curves can be in principle estimated from the submitted bids and asks (the buying and selling orders submitted to the auctioneer): the revealed demand supply curves tend to be shifted versions of the underlying true counterparts. Thus, if one could have an idea (through surveys or other empirical work) about the statistics of value-bid and cost-ask discrepancies in a given market, then one could guess the extent to which the revealed call-market price tends to differ from the true value center. For example, assume just for simplicity a sufficiently large market to have smooth demand and supply curves and a unique equilibrium price. (Proposition 3) Then the following first-order approximation is immediate:

$$D(p) \approx d(p) - d'(p)\mathbb{E}(\varepsilon). \tag{72}$$

$$S(p) \approx s(p) + s'(p)\mathbb{E}(\nu). \tag{73}$$

[37] Hint: for demand, e.g., one applies iterated expectations to $I(b_i \geq p) = I(v_i \geq p + \varepsilon)$.

Let p^* be the true (unobserved) clearing price, meaning $D(p^*) = S(p^*)$, which at a first-order approximation is given by (72) and (73) by the equation:

$$s(p^*) - d(p^*) + s'(p^*)\mathbb{E}(\nu) + d'(p^*)\mathbb{E}(\varepsilon) = 0. \tag{74}$$

Thus from mere knowledge of the average discrepancy between bids (asks) and the underlying reservation values (costs) in a market, one can infer approximately the true (unobserved) market-clearing price from the bids and asks submitted using (74).

4.5 Posted-Price Market

Consider the familiar posted-price institution as it appears in retail markets (in which the market clearing has a concrete meaning). This is the institution we are most familiar with as buyers and is most supportive of the concept of utility maximization subject to the constraint of given prices.

The posted-price institution, as it has been implanted in experiments, differs from the double-auction one (Subsection 4.6 below) by its less intense competition since it involves no buyer-buyer outbidding, and involves at most the two other forms (seller-seller undercutting and buyer-seller confrontation), and even buyer-seller confrontation is not a direct face-to-face haggling: a seller posts a price for a whole period, then buyers refrain from buying if the price is too high, leading the seller to cut the next period's price.[38] This less intense competition, combined with the fact that price is revised only with delay makes for a slower convergence to competitive

[38] In the US economy, with so-called "big ticket" goods like automobiles, appliances, and antiques, the posted price is considered an opening offer to be negotiated downward depending on dealer inventories and the buyer's cash available. Prices were routinely negotiated in the general merchandise store, which was replaced by the mass retailing innovations by R. H Macy, F. W. Woolworth, and Sears Roebuck, beginning in the 1870s. These innovations separated ownership and operation and introduced more centralized pricing in retail stores. (Ketcham, Smith, & Williams, 1984)

equilibrium compared to the more intense competition of the continuous double auction.[39] (Ketcham et al., 1984) V. L. Smith reports several experiments in which sellers (buyers) are free to actively change posted prices in real time, which they do not hesitate to do. (V. Smith, 1964)

Still, the number of sellers per se is not an intrinsically important parameter for convergence to equilibrium: as long as there is type 1 (buyer-seller competition), standard convergence to competitive equilibrium emerges even with one seller. (Classically, one seller in a market is not sufficient to define monopoly.[40]) This is intuitively clear; yet given neoclassical received doctrine (the requirement of "large number of sellers"), it might be useful to emphasize it more explicitly.

Formally, consider a seller who can produce m units of a commodity at the cost of $[c_1, \ ..., c_m]$. Assume the seller posts a price p_t for period (day or week) t. The seller can consider revising it in period $t+1$, for example upward when buyers' show strong fondness for the article $[D(p_t) > S(p_t)]$, or downward, to get rid of an excess supply. Classically this is a normal competitive price adjustment (that is qualitatively the same as any other competition involving more sellers): the competition in this posted-price case, to insist, is a confrontation between the lone seller and the buyers (who will refrain from buying if the price is too high, or if they can get a price concession). In appearance the seller is merely revising the price to clear the market

[39] These early experimental posted offer markets implement price with less flexibility than commonly associated with inventory management in large retail establishments. Thus, when slow moving items allow the inventory of perishables like milk to rise, the items are offered at a temporary discount from the accustomed posted price. Similarly, the rapid inventory stock-out of an item may lead to its being replenished at higher posted label prices. Classical higgling is thereby expressed in the form of rules for real time price adjustment.

[40] For an overview of the much misunderstood but deep and rigorous classical vocabulary see Inoua and Smith (2020a).

through the rule $Z(p_t)\Delta p_t \leq 0$, but more fundamentally, the competitive price revision process is a collectively value and cost revelation from the viewpoint of the foregoing theory, in the sense that $I(p_{t+1}) \geq I(p_t)$, at least approximately (for the reason emphasized in Proposition 1: the resulting law of supply and demand is only directional). If moreover, the seller revises the price based on accumulated inventories up to the revision date, then the value and cost revelation is more exact, in the sense that the price revision rule approximates better $V(p_{t+1}) \leq V(p_t)$. The crucial point to keep in mind is that for the competition principle to hold, it is irrelevant that there is no competition on the supply side: all that matters for the competition principle to hold is there is some buyer-seller confrontation, as is clear, again, from a multinomial expansion of

$$V^k(p) = \left[\sum_{j=1}^{m} | c_j - p | a_j(p) + \sum_{i=1}^{n} | v_i - p | a_i(p)\right]^k .$$

Classically, a lone seller is not treated as a monopolist unless the seller blocks free entry in the market and manages to sell the items in a noncompetitive way (which used to be possible with state privileges).[41]

More generally, a multiple-seller-posted-price market session from $t = 0$ to $t = T$, is informationally a series of price offer message $\{p_t^s : t = 0,...,T\}$ by each seller, s, followed by a sequence of quantity message $\{D^b : t = 0,...,T\}$ by each buyer, b, specifying quantity demanded at a seller's posted price. The competition and information principle applies with additional vivacity due to seller-seller underselling.

[41] On the subtle old view of monopoly, see, e.g., Inoua and Smith (2020a).

4.6 Double Auction

The double auction (DA) is perhaps the best institution in terms of intensity and frequency of competitive interactions, since all three forms of competition are in operation (buyer-buyer, seller-seller, and buyer-seller); moreover, the DA institution is also the richest institution informationally (by the higher frequency of quote and trade messages accumulated). Here we present synthetically how the forgoing theory specializes for DA competition. [42]

Informationally speaking, a DA market session from $t = 0$ to $t = T$ is a series of bid, ask, and trade messages $\{(a(t), b(t), \text{accept}(t) : 0 \leq t \leq T\}$, where each trade message is of the form $\{\text{accept}\; a(t)\}$ or $\{\text{accept}\; b(t)\}$. Thus the state of a DA market requires two state variables, the standing ask and standing bid, $a(t)$ and $b(t)$. Merely for simplicity of exposition, we will assume a smooth DA market, in particular a continuous-time dynamics, which seems reasonable for a DA.

DA competition is naturally summarized by the price-value distance (or potential surplus) function, which, for the DA, takes the form (valid up to normalization for continuous distributions):

$$V(a,b) = \sum\nolimits_{c \leq a} |c - a| + \sum\nolimits_{v \geq b} |v - b|. \tag{75}$$

That is, more simply:

$$V(a,b) = \sum\nolimits_{c \leq a} (a - c) + \sum\nolimits_{v \geq b} (v - b). \tag{76}$$

Clearly, this function decreases through both seller-seller underselling (whereby the standing ask decreases) and through buyer-buyer outbidding (whereby the standing bid rises); but it also

[42] Earlier models of the specific DA institution have been offered in the experimental literature (Anufriev, Arifovic, Ledyard, & Panchenko, 2013; Asparouhova, Bossaerts, & Ledyard, 2020; Cason & Friedman, 1996; Friedman, 1991; Gjerstad & Dickhaut, 1998; Gode & Sunder, 1993). For a review, see Friedman and Rust (1993 [2018]). A few DA models of order-book-driven financial price dynamics are also offered in the quantitative finance literature (e.g., Luckock, 2003; E. Smith, Farmer, Gillemot, & Krishnamurthy, 2003).

decreases through buyer-seller haggling in a less obvious way: that all forms of competition decreases $V(a,b)$ follows more formally through the multinomial formula (in essentially the same way emphasized in the Motivations: Section 2).

Alternatively, we can derive this characterization of DA competition from the law of supply and demand and the bid-ask reduction process, both implied by the three competition forms:

$$[D(b) - S(a)]\frac{db}{dt} \geq 0, \tag{77}$$

$$[D(b) - S(a)]\frac{da}{dt} \geq 0, \tag{78}$$

$$\frac{d}{dt}[a-b] \leq 0. \tag{79}$$

It is easily shown (as in Lemma 1) that DA price-value distance function can be written as:

$$V(a,b) = \int_0^a S(x)dx + \int_b^\infty D(x)dx. \tag{80}$$

By the chain rule,

$$\frac{d}{dt}V(a,b) = \frac{\partial V}{\partial a}\frac{da}{dt} + \frac{\partial V}{\partial b}\frac{db}{dt} = S(a)\frac{da}{dt} - D(b)\frac{db}{dt}.$$

Writing $a = b + (a - b)$ for the first term of the second equality, we get after basic manipulation:

$$\frac{d}{dt}V(a,b) = [S(a) - D(b)]\frac{db}{dt} + S(a)\frac{d}{dt}[a-b]. \tag{81}$$

Writing $b = a - (a - b)$ for the last term, we get similarly:

$$\frac{d}{dt}V(a,b) = [S(a) - D(b)]\frac{da}{dt} + D(b)\frac{d}{dt}[a-b]. \tag{82}$$

Or, adding the two equations, we get:

$$\frac{d}{dt}V(a,b) = [S(a) - D(b)]\frac{d}{dt}\left[\frac{a+b}{2}\right] + \frac{S(a)+D(b)}{2}\frac{d}{dt}[a-b]. \tag{83}$$

DA competition then reads simply, in terms of either one of the three formulas (81)-(83):

$$\frac{d}{dt}V(a,b) \leq 0. \tag{84}$$

Equivalently, DA competition can be characterized in terms of the information function. A bid message b reveals existence of a value $v \geq b,$ and an ask message reveals a cost $c \leq a.$ Thus, the information content that the standing bid and standing ask messages reveal about the value and cost distribution is measured by the function:

$$I(a,b) = -\int_{c_{\min}}^{a} \log \frac{S(x)}{m+n} dx - \int_{b}^{v_{\max}} \log \frac{D(x)}{m+n} dx. \tag{85}$$

As bids and asks are sent and revised competitively (hence the quote revisions obey the law of supply and demand and bid-ask reduction), it follows from a similar manipulation that yielded (81)-(83), that the information these messages reveal about the underlying value and cost distribution increases:

$$\frac{d}{dt}I(a,b) \geq 0. \tag{86}$$

So far, we are merely specializing the competition principle for the DA. Now we complement the discussion by further relating the vocabulary and implication of the foregoing theory to the DA. Since the bid and ask are clearly related (because they tend to come closer and closer), DA dynamics is perhaps better characterized in terms of, say, the standing ask $a,$ and the bid-ask spread $s = a - b,$ so that state of the DA is captured by the couple $(a,s),$ whose dynamics we can write as:

$$\frac{da}{dt} = F(a,s), \tag{87}$$

$$\frac{ds}{dt} = G(a,s). \tag{88}$$

We assume $G(a,0)=0,$ namely $s=0$ is an equilibrium bid-ask spread state.

The dynamics of the price-value distance function, as given by (82), reads:

$$\frac{d}{dt}V(a,s)=[S(a)-D(a-s)]F(a,s)+D(a-s)G(a,s). \tag{89}$$

Notice that the *zero-spread market-clearing states* $E=\{(a^{*},0):S(a^{*})=D(a^{*})\}$ are competitive-equilibrium (C.E.) states of the DA (where C.E., recall, means more generally stationary potential surplus or, equivalently, stationary information revealed). Any such state is by definition a *purely competitive equilibrium state* since it achieved the lowest possible potential surplus. Lab market dynamics, as we documented throughout, are well-approximated as convergence throughout many repetitions toward a purely competitive equilibrium state, because the *frictions* in the lab DA market are small or inexistent, measured by the many contracts within pennies of the WTP value, WTA cost. The lab DA market therefore is the closest realization we know of a purely competitive trade institution. In all generality, however, we should allow for frictions reflected in constraining $(F,G),$ we should theoretically allow for other equilibrium states, which in the language of the forgoing theory we would call constrained competitive DA equilibria or a competitive equilibrium under trade friction: a positive equilibrium spread, $s^{*}>0,$ being a notable example of a friction, namely a *transaction cost*. Transaction costs and other potential frictions in field DA markets are more diverse: not only the bid-ask spread paid to the market maker (as profit) in specialist markets, but also: commissions paid to brokers; the trading fees; minimum price tick and minimum trade volume size constraint; regulatory price controls.

5 Conclusion

What is the most fundamental law of value theory? The law of supply and demand does not qualify, since it derives from the competition principle, postulated as the main axiom of price theory in this paper, and which was well-recognized early in classical economics as the core law of the discipline: "only through the principle of competition has political economy any pretension to the character of a science" (J.S. Mill, 1848 [1965], vol. II, bk II, ch. IV, § 1, p. 239). But competition is not the sole regulator of price (as J.S. Mill goes on to emphasize, insisting on custom as the other price regulator). Section 3 above developed a theory of classical competitive price formation, in which the main result (Theorem 1) applies to the model of purely competitive dynamics. Convergence to the value center is a theoretical norm, not necessarily the case in all generality: in practice this means no limitation on price dynamics imposed by trading institution and other, external, institutional regulators of price, except traders' competition. Throughout the derivations in the pure-competition model, we assumed away all frictions. In practice, price dynamics depends more generally on a mix of: (1) competition in the market; (2) trading institution; and (3) exogenous constraints on price (such as State regulations). Thus, empirically, market equilibrium should reflect this mix of competition, institution, and regulation. It is in this sense that the information principle is the more general and more fundamental principle than the competition principle (and the law of supply and demand). In all generality, the PMI should be phrased as follows: equilibrium price reflects all the relevant data (or information) involved in its formation: the traders' valuations, the trading institution, and the exogenous constrained regulating trade in the market. (By the price-quantity duality, the trading institution and other legal constraint have dual implication on the allocation vector, which depends on whom is allowed to compete or trade with whom, and so on.)

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Mathematical Appendix: The Invariance Principle[43]

Definitions (Lyapunov function, invariant set, convergence to a set, radially unbounded function). Let $p_{t+1} = H(p_t), t \geq 0,$ be a "dynamical system", where the map $H : \Omega \subset \mathbb{R}^K \to \mathbb{R}^K$. A *continuous* scalar function V such that $V(H(x)) \leq V(x)$ for all $x \in \Omega_0 \subset \Omega$, is called a *Lyapunov function* of the system on Ω_0. [It is nonincreasing along any trajectory in $\Omega_0 : V(p_{t+1}) \leq V(p_t)$.] A function V is *radially unbounded* if $V(x) \to \infty$ as $||x|| \to \infty$. A set I is i*nvariant* with respect to the map H if $H(I) \subseteq I$. (That is, if $x \in A$ then $H(x) \in A$. Hence if the system ever reaches the set I, then it stays there forever: if $p_0 \in I$ then $H^t(p_0) \in I$ for all $t \geq 0$.) Let $E = \{x \in \Theta : V(H(x)) = V(x)\}$, the set over which V is stationary; and let A be the largest invariant set contained in the set E (that is, the union of all invariant subsets of E). The distance between a point p and a set A is defined as $\text{dist}(p, A) = \inf_{a \in A} ||p - a||$, using, say, the Euclidian norm. We say that p_t *converges to the set* A, and write $p_t \to A$ as $t \to \infty$, if $\text{dist}(p_t, A) \to 0$ as $p_t \to A$.

***Theorem** (Invariance Principle). Let $p_{t+1} = H(p_t)$, where $H : \Omega \subset \mathbb{R}^K \to \mathbb{R}^K$ is a continuous function. If the system has a radially unbounded Lyapunov function V on $\Omega_0 \subset \Omega$, then $p_t \to M$ for any $p_0 \in \Omega_0$, where M is the largest invariant set in $E = \{x \in \Omega_0 : V(H(x)) = V(x)\}$.*

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[43] La Salle (1976) is a standard exposition of the invariance theory and the proofs (discrete and continuous times). The theorem is covered in stability theory textbooks: for a succinct treatment see Sedaghat (2003, ch. 3); for a more comprehensive treatment (discrete and continuous times) see Haddad and Chellaboina (2008); see also (continuous time) Khalil (2002).

Variants of the invariance principle. The radial unboundedness of the Lyapunov function V guarantees that any trajectory $p_t = H^t(p_0)$ is bounded. Without the radial unboundedness property, the invariance theorem holds with the following proviso: (1) assuming bounded trajectory; (2) the attractor is $A = M \cap \{x \in \Omega_0 : V(x) = \mu\}$, where $\mu = \lim_{t\to\infty} V(H^t(p_0)) = \min_{t\geq 0}\{V(H^t(p_0))\}$ and M is as defined in the theorem.

The continuous-time version $dp/dt = H(p)$ is similarly stated, in terms of $\dot{V}(p) = dV(p)/dt = \text{grad}(V(p)) \cdot H(p) \leq 0$, $E = \{x \in \Omega_0 : \dot{V}(x) = 0\}$, and a set I is invariant if: $p_0 \in A$ implies $p(t) \in A$ for all $t \geq 0$).

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