UNIVERSITÉ DE MONTRÉAL

SINGULAR SYSTEMS WITH TIME-VARYING DELAYS

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MÉMOIRE PRÉSENTÉ EN VUE DE L'OBTENTION
DU DIPLÔME DE MAÎTRISE ÈS SCIENCES APPLIQUÉES
(GÉNIE MÉCANIQUE)
AVRIL 2008

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Your file Votre référence
ISBN: 978-0-494-41561-0
Our file Notre référence
ISBN: 978-0-494-41561-0

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Ce mémoire intitulé:

SINGULAR SYSTEMS WITH TIME-VARYING DELAYS

présenté par: HAIDAR Ahmad

en vue de l'obtention du diplôme de: Maîtrise ès sciences appliquées

a été dûment accepté par le jury d'examen constitué de:

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In the Name of Allah, Most Gracious, Most Merciful

Gratefully dedicated to the Prophet of Mercy, Peace and Love, Muhammad

and to my mother, father, brother and sisters

ACKNOWLEDGMENTS

First and foremost, I would like to express my special thanks to my thesis supervisor, Professor El-Kébir Boukas, for his guidance, his readiness to help, encouragement, support, and especially for providing the friendly atmosphere. I am deeply thankful to Professor Boukas and his family for the help and support they provided me outside the university beyond the duties of any supervisor. I would like to thank him for always pushing me to work hard, he always wanted the best for me. For all this, and more, I shall always be grateful.

I would like to thank Mathieu, Aliyu, Lixian and Jamila for their friendship. I will never forget the nice time we spent together.

I would like to thank my uncle Mehdi El-Mousawi, his great wife and all his family for their moral support, love and encouragement. I wish all the best for Rana and Nawar and their husbands in their new marital lives.

I would like to thank my roommates, Seyed Ebrahim and Mohamed Raef for their continuous support and encouragement. I wish all the best for the company they established with their friends. I am pretty sure that it will be one day as big as Google :).

I would like to thank Al-Amine restaurant staff for the delicious food they used to provide me almost everyday.

I would like to thank all those who inspired me, in particular, many thanks to Dr. Ziad Najem, Dr. Zafer Mahmud and Dr. Nawaf Al-Mutawa. I would like to thank also Dr. Mohamed Fahim and Dr. Mohamed Zribi for introducing me to the world of automatic control. My middle school math teacher, Kamal, must also be thanked for his invaluable guidance in my early stages.

Finally, but most important, I would like thank my mother and father who made me what I am today. Simply, without their love, support and prayers, this work would have not been completed.

RÉSUMÉ

Les systèmes singuliers avec retard ont été utilisés pour représenter plusieurs systèmes pratiques comme les avions, les procédés chimique, etc. Par conséquent, plus d'attention a été consacré à l'étude du comportement de la solution, analyse et la commande de cette classe de systèmes. Cependant, la pluspart des travaux reportés dans la littérature traitent le cas de retards constants. L'objectif de cette recherche est de s'attaquer à l'analyse et à la synthèse des systèmes continus singuliers avec retards variants dans le temps.

Pour le comportement de la solution, l'existence et l'unicité de la solution de cette classe de systèmes ont été discutés. Le problème de la compatibilité des conditions initiales est aussi discuté.

Pour le problème de stabilité, des conditions suffisantes dépendantes du retard sont developpées. Ensuite, des conditions assurants une stabilité exponentielle et faisant intervenir des bornes (inférieure et supérieure) du retard sont aussi développées. De plus, une estimation du taux de convergence des états est présentée. Ce résultat est aussi généralisé au cas de plusieurs retards variants dans le temps. Aussi, une expression des variables algébriques en fonction des conditions initiales et des variables rapides est donnée.

Pour la synthèse, un contrôleur par retour d'état est considéré. La commande $\mathscr{H}_\infty$ avec un contrôleur par retour d'état est aussi considérée. Ensuite, la commande par retour de sortie est considérée pour les systèmes singuliers avec saturation des contrôleurs et retard sur les entrées.

Les résultats proposés sur l'analyse et la synthèse sont basés sur l'aproche de Lyapunov-Krasoviskii, les inégalités matricielles linéaires et bilinéaires, et la théorie des graphes. L'outil SEDUMI est utilisé pour la résolution des problèmes d'optimisation convexe, tandis que l'outil PENBMI est utilisé pour la résolution des inégalités matricielles bilinéaires. L'outil YALMIP est utilisé dans les deux cas comme interface. Des exemples numériques sont présentés pour montrer l'efficacité des résultats proposés.

ABSTRACT

Time-delay singular systems have been used to represent many practical systems, including aircraft, chemical processes, lossless transmission lines, etc. Therefore, more attention has been paid to study the solution behavior, analysis and control of this class of systems. However, most of the results presented in the literature tackled the case of constant time delays. The objective of this research is to deal with the analysis and synthesis of the continuous-time singular systems with time-varying delays.

For the solution behavior, existence and uniqueness of solution for this class of systems are discussed. Impulsive behavior of its solutions is also discussed and conditions for smooth solutions are presented. The problem of compatible initial conditions is also presented.

For the stability problem, delay-dependent stability conditions for this class of systems are developed. Then, delay-range-dependent exponential stability conditions are also developed. Moreover, an estimate of the convergence rate of the state is presented. This result is generalized to the case of multiple time-varying delays. Also, an explicit expression of the algebraic variables in term of the initial condition and the fast variables is given.

For the synthesis problem, state feedback stabilizing controller is considered. $\mathscr{H}_\infty$ state feedback controller is also considered. Then, static output feedback controller is considered for the time-delay singular system with saturating controllers and input delays.

The proposed results on analysis and synthesis are based on Lyapunov-Krasovskii approach, linear and bilinear matrix inequality techniques and graph theory. The free solver SEDUMI is used to solve the convex optimization problems while PENBMI is used to solve the BMI problems. YALMIP is used as an interface in both cases. Some numerical examples are presented to show the effectiveness of the proposed results.

CONDENSÉ EN FRANÇAIS

Introduction

Au cours des dernières décennies, la théorie des systèmes utilisant le concept du variable d'état a été largement étudiée et elle est maintenant bien développée. Un système représenté par un model d'état est décrit par un ensemble d'équations différentielles ordinaires (ODEs). Toutefois, dans plusieurs systèmes physiques comme les procédés chimiques, les systèmes de circuits et les systèmes économiques, les variables d'état peuvent être liées aussi algébriquement, résultant en une classe plus générale de systèmes, appelés systèmes singuliers. Ainsi, un système singulier est décrit par un ensemble d'équations différentielles et d'équations algébriques, qui traduisent les contraintes dynamiques et statiques décrivant le système réel. Pour cette classe de systèmes, il existe dans la littérature plusieurs résultats sur la solution, l'analyse et le contrôle de ces type de systèmes.

Les retards peuvent être une des principales causes d'instabilité et sont rencontrés dans plusieurs systèmes physiques comme les procédés chimiques, les réacteurs nucléaires et le transport des lignes. Les systèmes avec retard représente une classe des systèmes de dimenstion infinie qui a été largement étudiée et plusieurs résultats ont été publiés dans la littérature.

Les systèmes singuliers avec retard ont été utilisés pour représenter plusieurs systèmes pratiques comme les avions, les procédés chimique, etc (Kumar and Daoutidis, 1999; Brayton, 1968; Halanay and Rasvan, 1997; Niculescu and Rasvan, 2000). Mathématiquement, les systèmes singuliers avec retards sont équations différentielles avec retard couplée avec équations aux différences. L'étude de ces systèmes est plus compliquée que celle de systèmes avec retard ou systèmes singuliers. L'existence et l'unicité de la solution d'un système singulier avec retard ne sont pas toujours garanties et le système peut aussi incorporer des effets impulsifs qui sont en général indésirables. .

L'objectif de cette recherche est de s'attaquer à l'analyse et à la synthèse des systèmes continus singuliers avec retards variants dans le temps.

Motivation

Les points suivants ont motivé les travaux dans ce mémoire.

- Les systèmes singuliers avec retard ont été utilisés pour représenter plusieurs systèmes pratiques. Ils peuvent être utilisés pour décrire les pertes des lignes de transmission (Brayton, 1968), le régime transitoire des turbines hydrauliques (Halanay and Rasvan, 1997), le régime transitoires de pression de vapeur dans la production combinée de chaleur et d'électricité (Halanay and Rasvan, 1997) et les systèmes en génie chimique (Kumar and Daoutidis, 1999).

- De plus en plus d'attention est accordée à l'importance théorique des systèmes singuliers avec retard. Par exemple, récemment, (Fridman and Shaked, 2002b) il a intoduit une approche qui transforme les systèmes avec retard et les systèmes neutres en systèmes singuliers. Cette approche a été utilisée pour développer plusieurs résultats pour ces deux types de systèmes, voir par exemple (Haurani, 2003).

- Dans la littérature, la plupart des résultats dépendants du retard traitent le cas des retards constants et peu de chercheurs comme (Yue et al., 2005; Yue and Lam, 2005; Yue and Han, 2005a) ont abordé le cas de retards variables.

- Un nouveau concept a été récemment étudié, où les retards sont supposés varier dans un intervalle et qui est plus approprié pour des systèmes pratiques (He et al., 2007a). À notre connaissance, des conditions assurant la stabilité et faisant intervenir des bornes (inférieure et supérieure) du retard n'ont pas encore été développées pour les systèmes singuliers avec retard (cas continu).

- Le problème de stabilisation pour les systèmes singuliers avec retard en présence de saturations du contrôleur n'a pas été encore totalement résolu.
- L'existence de très peu d'articles qui permettent de résoudre le problème de la commande par retour de sortie pour les systèmes singuliers avec retard.

Méthodologie

Les résultats proposés pour l'analyse et la synthèse sont basés sur l'approche de Lyapunov-Krasoviskii, les inégalités matricielles linéaires (LMI) et bilinéaires (BMI), et la théorie des graphes. L'outil SEDUMI est utilisé pour la résolution des problèmes d'optimisation convexe, tandis que l'outil PENBMI est utilisé pour la résolution des inégalités matricielles bilinéaires. L'outil YALMIP est utilisé dans les deux cas comme interface.

Préliminaires sur les systèmes singuliers avec retard variable

Un système singulier linéaire avec retards variables (cas continu), invariant dans le temps est régi par les équations suivantes:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t - d(t)) \\ x(t) &= \phi(t), \quad t \in [-\overline{d}, 0] \end{aligned}$$

où $x(t) \in \mathbb{R}^n$ désigne le vecteur d'état, la matrice $E \in \mathbb{R}^{n\times n}$ peut être singulier, et nous supposons que $\text{rank}(E) = r \leqslant n$, les matrices A et A_d sont réelles, constantes et de dimensions compatibles, $\phi(t) \in C^v_\tau$ est un vecteur de dimension compatible et a valeur fonction continue, $d(t)$ est le délai qui est supposé pour être une fonction continue limitée et $\overline{d}$ est un constant positif.

Les systèmes singuliers avec retard peuvent être considérée sous deux angles différents; un qu'ils sont des équations différentielles algébriques (DAEs) (c'est-à-dire les systèmes singulier) dont la formulation implique des termes retardés, l'autre qu'ils sont équations différentielles avec retard (DDEs) (c'est-à-dire systèmes de type retard) soumis à des contraintes. La discussion des systèmes singuliers avec retard hérite plusieurs des idées et de la terminologie des deux les systèmes singulier et systèmes de type retard. Toutefois, le comportement de solutions de cette classe de systèmes correspond en quelque sorte à ce que l'on peut attendre d'un systèmes singulier et dans un certain sens, à ce que l'on peut attendre d'un systèmes de type retard.

Voici quelques caractéristiques des systèmes singuliers avec retard:

- L'existence et l'unicité de la solution à un système singulier avec retard ne sont pas toujours garanties. Ainsi, la notion de la régularité doit être présentée. Donc, il est important de développer des conditions qui garantissent que le système donné est non seulement stable, mais aussi régulièr.

- La réponse d'un système singulier avec retard peut avoir des impulsions ainsi que les dérivés de ces impulsions. Ainsi, la notion de non-impulsif doit être présentée. Donc, il est important de développer des conditions qui garantissent que le système donné est régulier, non-impulsif et stable (c'est-à-dire admissible).

- Même si un système singulier avec retard est non-impulsif, il peut aussi avoir des discontinuities fini en raison de la condition initiale qui risque d'être incompatible. De plus, ces sauts peuvent se propager dans la solution en raison de l'existence de termes retardés. Donc, la notion de conditions initiales compatibles doit être présentée.

Stabilité des système singuliers avec retards variants

Le problème de stabilité est un des problèmes les plus importants de la théorie de contrôle. Comme mentionné auparavant, la plupart des résultats dans la littérature traitent le cas des retards constants et un nombre réduits d'article ont abordé le cas de retards variables. Entre autre le problème de stabilité pour ces systèmes singuliers n'a pas été complètement adressé. Dans cette mémoire, nous avons étudié la stabilité des systèmes singuliers avec retards variants simples et multiples.

Pour le problème de stabilité, des conditions suffisantes dépendantes du retard sont developpées. Ensuite, des conditions assurants une stabilité exponentielle et faisant intervenir des bornes (inférieure et supérieure) du retard sont aussi développées. De plus, une estimation du taux de convergence des états est présentée. Ce resultat est aussi généralisé au cas de plusieurs retards variants dans le temps. Aussi, une expression des variables algébriques en fonction des conditions initiales et des variables rapides est donnée.

D'abord, des conditions suffisantes dépendantes du retard pour les systèmes singuliers avec retard variant sont developpées tel que le système est régulier, non-impulsif et stable. Ce résultat peut utilisé pour généraliser beaucoup de résultats apparaissent dans la littérature pour les systèmes singuliers avec retards constants au cas de retards variants. Ensuite, des conditions assurants une stabilité exponentielle et faisant intervenir des bornes (inférieure et supérieure) du retard sont aussi développées. De plus, une estimation du taux de convergence des états est présentée. Ce resultat est aussi généralisé au cas de plusieurs retards variants dans le temps. Ces résultats sont basés surtout sur l'approche de Lyapunov-Krasovskii et quelques résultats algébriques. Aussi, la théorie des graphes est employé pour le cas de plusieurs retards variants dans le temps.

Commande par retour d'état pour système singuliers avec retards variants

Considérez le système singuliers linéaire avec retards suivant:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-d(t)) + Bu(t) + B_{\omega 1}\omega(t) \\ z(t) &= Cx(t) + B_{\omega 2}\omega(t) \\ x(t) &= \phi(t), \quad t \in [-\overline{d}, 0] \end{aligned}$$

où $u(t) \in \mathbb{R}^m$ désigne le vecteur de commande, $\omega(t) \in R^p$ représente la perturbation externe à énergie finie, $z(t) \in R^s$ désigne le vecteur de sortie contrôlé, les matrices C, B, $B_{\omega 1}$ et $B_{\omega 2}$ sont réelles, constantes et de dimensions compatibles et les autres variables comme défini auparavant.

Le contrôleur utilisé est donné par l'expression suivante:

$$u(t) = Kx(t), \;\; K \in R^{m \times n}$$

En appliquant ce contrôleur au système, nous obtenons le système en boucle fermé comme suit:

$$E\dot{x}(t) = (A+BK)x(t)+A_d x(t-d(t))+B_{\omega 1}\omega(t) = A_{cl}x(t)+A_d x(t-d(t))+B_{\omega 1}\omega(t)$$

Tout d'abord, $\omega(t)$ est supposé être nulle et des conditions en forme de LMI sont fournies pour le design du contrôleur par retour d'état. Maintenant, si $\omega(t)$ n'est pas égal à zéro, on s'attaque au problème de la commande $\mathscr{H}_\infty$.

L'idée de la commande $\mathscr{H}_\infty$ est de concevoir un controller qui stabilisent le système et rejett aussi les perturbations externes. Deux méthodes sont d'habitude utilisées pour s'attaquer au problème de la commande $\mathscr{H}_\infty$ pour les systèmes: la méthode de fonction de transfert (domaine de fréquence) et la méthode de domaine de temps.

Dans la méthode de fonction de transfert, un contrôleur stabilisant est conçu de manière à ce que le gain de la fonction de transfert en boucle fermée entre la perturbation w et la sortie contrôlée z est le moins possible, c'est-à-dire, l'effet de la perturbation sur la sortie contrôlée est minimisé. Pourtant, pour les systèmes singuliers avec retards, aucune fonction de transfert n'est définie pour les retards variants. Ainsi, la méthode de domaine de temps est utilisée pour s'attaquer au problème de la commande $\mathscr{H}_\infty$ et les conditions de LMI sont fournies afin de résoudre le problème de commande.

Commande par retour de sortie pour les systèmes singuliers avec retard variant

Considérez le système singuliers linéaire avec retard suivant:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-d(t)) + B\mathrm{sat}(u(t)) \\ y(t) &= Cx(t) \\ x(t) &= \phi(t), \quad t \in [-\overline{d}, 0] \end{aligned}$$

où $y(t) \in \mathbb{R}^q$ désigne le vecteur de sortie mesurée, $\mathrm{sat}(u(t)) = [\mathrm{sat}(u_1(t)), ..., \mathrm{sat}(u_m(t))]$ avec $\mathrm{sat}(u_i(t)) = \mathrm{sign}(u_i(t)) \min(|u_i(t)|, \overline{u}_i)$ et les autres variables comme défini auparavant. L'introduction de la saturation (nonlinéarities) est motivée par le fait que c'est impossible pour les actionneurs de conduire le signal avec une amplitude illimitée.

Le contrôleur employé des cette partie est donné par l'expression suivante:

$$u(t) = Ky(t), \ K \in R^{m \times q}$$

En appliquant ce contrôleur au système, nous obtenons le système en boucle fermée comme suit:

$$E\dot{x}(t) = Ax(t) + A_d x(t-d(t)) + B\mathrm{sat}(KCx(t))$$

En raison de l'existence du terme de saturation, généralement, le système ne peut pas être stabilisé globalement et notre problème est considéré comme un problème local de stabilisation. Donc, un ensemble de conditions initiales et augmentations de contrôleur devraient être donnés tel que le système en boucle fermée est régulier, non-impulsif et stable.

Il est commun de se rapprocher de la région d'attraction par les ensembles construits de Lyapunov fonctionnel. Dans cette mémoire, l'approche de Lyapunov utilisée pour l'analyse de stabilité est adopté pour se rapprocher de la région d'attraction.

Le système non linéaire est transformé à un système de polyhèdres linéaires en utilisant des inclusions différentielles (Molchanov and Pyatnitskiy, 1989; Tarbouriech and Gomes da Silva, 2000). Un algorithme LMI itératif est adopté pour résoudre les gains de contrôleur. L'algorithme proposé est employé pour élargir l'ensemble des conditions initiales le plus possible.

Le problème est étendu au cas où le retard apparaît dans le contrôle. Des conditions de BMI doivent être résolues afin d'obtenir les gains de contrôleur et l'ensemble des conditions initiales.

Travail futur

Les résultats de ce mèmoire peuvent être étendus pour les cas suivants:

- PID.

  La méthode la plus populaire utilisée dans l'industrie est aujourd'hui le contrôleur PID. De plus, beaucoup de processus chimiques peuvent être modelé comme des systèmes singuliers avec retard. Donc, le design de contrôleur de PID pour les systèmes singuliers avec retard est un problème intéressant de s'attaquer.

- Commandabilité et observabilité.

  La commandabilité et l'observabilité sont des propriétés importantes des la théorie moderne et elles jouent un rôle crucial dans beaucoup de problèmes de contrôle. À notre connaissance, aucun travail n'a été annoncé dans la littérature sur la commandabilité ou l'observabilité de systèmes singuliers avec retard.

- Systèmes singuliers nonlinéaires avec retard.

  Mathématiquement, un système singulier non linéaire avec retard est régi par les équations suivantes:

  $$\dot{x}(t) = F(t, x(t), x(t-d(t)), y(t), y(t-d(t)), u(t))$$

  $$0 = G(t, x(t), x(t-d(t), y(t), y(t-d(t)), u(t))$$

  La stabilité asymptotic de cette classe de systèmes est étudiée dans (Pepe, 2005; Pepe et al., 2006; Pepe and Verriest, 2003). Des extension sont à faire dans cette direction.

- Systèmes singuliers stochastiques avec retard.

  Une classe de systèmes stochastiques déterministes par morceaux a été utilisée pour modéliser beaucoup de systèmes pratiques, où des bris aléatoires et des réparations pourraient se produire. Il y a seulement peu de papiers des systèmes singuliers stochastiques avec retard (Boukas et al., 2005a; Boukas et al., 2005b). Des extension sont aussi à faire sur cette classe de systèmes. Les résultats dans (Boukas, 2008) peuvent être étendus au cas où le système à des retards variants dans le temps.

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LIST OF NOTATIONS AND SYMBOLS

$\mathbb{R}^n$	n dimensional Euclidean space
$\mathbb{R}^{n\times m}$	the set of all $n \times m$ real matrices
$^\mathsf{T}$	matrix transposition
$X \geq Y$	$X - Y$ is positive semi-definite, where X and Y are symmetric matrices
$X > Y$	$X - Y$ is positive definite, where X and Y are symmetric matrices
$\lambda_{max}(P)$	the maximal eigenvalue of matrix P
$\lambda_{min}(P)$	the minimal eigenvalue of matrix P
co $\{\cdot\}$	convex hull
C_τ	the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence
C_τ^v	$\{\phi \in C_\tau; \|\phi\|_c < v, v > 0\}$
$\|\cdot\|$	the Euclidean norm
$\|\phi\|_c$	the norm of a function $\phi \in C_\tau$ defined as $\|\phi\|_c = \sup_{-\tau\leq t\leq 0}\|\phi(t)\|$
$L_2[0,\infty)$	the space of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}^q$ which are Lebesque integrable in the square over the interval $[0, \infty)$, with the standard norm $\|.\|_2$
LMI	Linear matrix inequality
BMI	Bilinear matrix inequality
$\star$	represents the transpose of the corresponding element in a symmetric matrix

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INTRODUCTION

In the past decades, standard state-space theory has been extensively studied and the theory is now well developed. A standard state-space system is described by a set of ordinary differential equations (ODEs). However, in many physical systems such as chemical processes, circuit systems and economic systems, the state variables may be related algebraically, resulting in a more general class of systems, called singular systems (Lewis, 1986; Dai, 1989). Thus, a singular system model is described by a set of coupled differential and algebraic equations, which include information on the static as well as dynamic constraints of a real plant. For this class of systems, there are many results in the literature on the solution, analysis and control, and for more details, we refer the reader to (Xu and Lam, 2006; Dai, 1989; Wang et al., 2006; Lewis, 1986; Cobb, 1984) and the references therein.

Delays are one of the most important causes of instability and are encountered in many physical systems such as chemical processes, rolling mills, nuclear reactors, long transmission lines, and microwave oscillators (Boukas and Liu, 2002; Haurani, 2003; Niculescu, 2001; Gu et al., 2003). Time-delay systems (called also hereditary or with memory, deviating arguments, after-effects, post actions, dead-time, or time-lag) represent a class of infinite-dimensional systems that has been extensively studied and many results have been published in the literature. We refer the reader to (Boukas and Liu, 2002; Richard, 2003; Haurani, 2003; Niculescu, 2001; Gu et al., 2003) and references therein for an extensive treatment of this class of systems.

Time-delay singular systems, which have both delay and algebraic constraints, may in fact be systems of advanced type (Fridman, 2002; Baker et al., 2002). Time-delay singular systems often appear in various engineering systems, including aircraft stabilization, chemical engineering systems, lossless transmission lines, etc, (see (Brayton, 1968; Halanay and Rasvan, 1997; Niculescu and Rasvan, 2000; Niculescu, 2001; Pepe and Verriest, 2003; Kumar and Daoutidis, 1999) and the references therein). It is worth noting that this class of systems are also referred to in the literature as delay differential-algebraic equations, implicit systems with delay or descriptor systems with delay.

Mathematically, time-delay singular systems are delay differential equations coupled with difference equations. The study of such systems is much more complicated than that for standard state-space time-delay systems or singular systems. The existence and uniqueness of a solution to a given time-delay singular system is not always guaranteed and the system can also have undesired impulsive behavior.

It is worth mentioning that throughout this thesis, only continuous-time singular time-delay systems will be discussed.

0.1 Literature Review

The class of continuous-time time-delay singular systems represents an important class of systems that has attracted a lot of researchers from control and mathematics communities. Both delay-independent and delay-dependent stability conditions for singular time-delay systems have been derived using the time domain method, see (Xu et al., 2002; Fridman, 2002; Zhu et al., 2007; Feng et al., 2002; Yue et al., 2005) and references therein. However, most of the delay-dependent results in the literature tackle only the case of constant time delay where two approaches were used to prove the stability of the system. The first approach consists of decomposing the system into fast and slow subsystems and the stability of the fast subsystem is proved using some Lyapunov functional. Then, the fast variables is expressed explicitly by an iterative equation in terms of the slow variables (Xu et al., 2002). The stability of the fast variables can be guaranteed if the eigenvalues of some matrix are inside a unit circle. The second approach introduced by (Fridman, 2002) and it consists of constructing a Lyapunov-Krasovskii functional that corresponds directly to the descriptor form of the system. However, this approach is based on the assumption of the stability of a certain operator. This assumption is shown to be satisfied if the eigenvalues of some matrix expression are inside a unit circle. Indeed, in the case of a single delay, it can be shown easily that this condition is equivalent to the one used in (Xu et al., 2002) to prove the stability of the fast variables. The extension of these approaches to time-varying delays has not been addressed yet. In (Yue et al., 2005), where time-varying delays are considered, the response of the fast variables has been bounded by an exponential term using a different approach. Using this approach, it is not possible to give an estimate of the convergence rate of the states of the system. Also, the bounding approach of the derivative of the Lyapunov functional is conservative due to the ignorance of some useful terms (see (He et al., 2004) for a discussion on different bounding approaches).

Recently, a free-weighting matrices method is proposed in (He et al., 2004), (Wu et al., 2004) and (Xu and Lam, 2005) to study the delay-dependent stability for time-delay systems with constant and time-varying delays, in which the bounding techniques on some cross product terms are not involved. The new method has been shown to be more effective in reducing conservatism entailed in previous results, especially for uncertain systems. In 2007, Zhu et al. adopted this technique for time-delay singular systems (Zhu et al., 2007).

Formally speaking, these conditions provide only the asymptotic stability of time-delay singular systems. In (Sun, 2003), the global exponential stability for a class of singular systems with multiple constant time delays is investigated and an estimate of the convergence rate of such systems is presented. One may ask if there exists a possibility to use the LMI approach for deriving exponential estimates for solutions of time-delay singular systems. In (Yue et al., 2005), exponential stability conditions in terms of LMIs are given but no estimate of the convergence rate is presented.

The state feedback stabilization problem has also attracted a lot of researchers and many results appear in the literature (Boukas, 2007; Zhou and Lam, 2003; Xu et al., 2002; Feng et al., 2002; Gao et al., 2005; Li and Xing, 2005; Yang et al., 2007; Yue and Lam, 2005; Yue et al., 2003; Fridman and Shaked, 2002a; Feng et al., 2005; Yue and Han, 2005a). These results are based on the stability conditions and consequently, most of them treat the constant time delay case. In (Yue and Lam, 2005; Yue and Han, 2005a), the stabilization problem is tackled for systems with time-varying delay and the results are based on the approach of (Yue et al., 2005). Note that the results in (Feng et al., 2002; Gao et al., 2005; Yang et al., 2007; Yue and Lam, 2005) tackle the state feedback guaranteed cost stabilization problem, while the results in (Boukas, 2007; Yue et al., 2003; Fridman and Shaked, 2002a; Feng et al., 2005; Yue and Han, 2005a) tackle the $\mathscr{H}_\infty$ state feedback stabilization problem.

The problem of stabilizing linear systems with saturating controls has been widely studied because of its practical interest (Bernstein and Michel, 1995). Control saturation constraint comes from the impossibility of actuators to drive signal with unlimited amplitude or energy to the plants. However, only few works have dealt with stability analysis and the stabilization of singular linear systems in the presence of actuator saturation, see for example (Lan and Huang, 2003). It is established in (Lan and Huang, 2003) that a singular linear system with actuator saturation is semi-globally asymptotically stabilizable by linear state feedback if its reduced system under actuator saturation is semi-globally asymptotically stabilizable by linear feedback.

The static output feedback problem is probably the most important challenging question in control engineering. In contrast to the linear systems, there are only few papers solving the static output feedback problems for singular systems, see (Kuo and Fang, 2003; Castelan and Silva, 2002). In (Kuo and Fang, 2003), the authors introduce an equality constraint in order to get a linear matrix inequality (LMI) sufficient conditions for admissibility of closed-loop systems. However, this equality constraint introduces conservatism. This approach has been generalized by (Boukas, 2004) to time-delay singular systems. In (Castelan and Silva, 2002), singular systems is assumed to have some characteristics in advance: regularity and absence of direct action of control inputs on the algebraic variables, which is not always the case.

Regarding the numerical methods, some work has been done to solve a general delay differential-algebraic equation (DDAE), where the class of time-delay singular systems appears as a special case (Baker and Paul, 2006; Pepe, 2005; Baker et al., 2002; Shampine and Gahinet, 2006; Zhu and Petzold, 1997; Pepe and Verriest, 2003; Ascher and Petzold, 1995). However, there are numerous difficulties that can arise when solving a general DDAE. For example, it is well known that DDAEs may have one or more solution components that have jump discontinuities. These discontinuities can appear even if the initial function is continuous. It is also well known that these jump discontinuities propagate in the solution. This propagation is due to the existence of delayed solution terms. The problem of the propagation of discontinuities in DDAEs is still not adequately dealt with by existing DDAE strategies. Discontinuity tracking is generally complex and computationally expensive. It is worth noting that some approaches are based on transforming the problem of solving DDAEs to the one of solving delay differential equations (DDEs), neutral delay differential equations (NDDEs) or even singularly perturbed delay differential equations (SPDDEs). However, such a transformation is not always possible. We can say that numerical methods for DDAEs have received little attention in the literature and there still have much to learn about how to solve them numerically. For more information on numerical methods for DDAEs, we refer the reader to the references mentioned above and the ones therein.

0.2 Motivation

The following points motivate our work in this thesis.

- Time-delay singular systems find their way into the representation of many real applications. Time-delay singular systems can be used to describe lossless transmission lines (Brayton, 1968), transients of a hydraulic turbines under waterhammer conditions (Halanay and Rasvan, 1997), transients of extracted steam pressure in the combined generation of heat and electricity (Halanay and Rasvan, 1997), chemical engineering systems (Kumar and Daoutidis, 1999), etc.

- More and more attention is paid to the theoretical importance of time-delay singular systems. For example, recently, (Fridman and Shaked, 2002b) introduces a descriptor (singular) model transformation for retarded and neutral type systems. This transformation has been used to develop many results in the context of time-delay and neutral systems, see for example (Haurani, 2003).

- Most of the delay-dependent results in the literature tackle only the case of constant time delay and few researchers have tackled the time-varying case. There is still some work that has to be done for singular systems with time-varying delays.

- Delay-range-dependent concept was recently studied, where the delays are considered to vary in a range and thereby more applicable in practice (He et al., 2007a). To the best of the author knowledge, delay-range-dependent stability problem for time-delay singular systems has not been addressed yet.

- The stabilization problem for time-delay singular systems in the presence of actuator saturation has not been fully addressed yet.

- The existence of very few papers solving the static output feedback problem for time-delay singular systems.

0.3 Contribution of This Thesis and Methodology

The contributions of this thesis are summarized as follows:

- Delay-range-dependent exponential stability conditions for time-delay singular systems with single delay are established and an estimate of the convergence rate of the state is presented. Free weighting matrices are used in order to reduce the conservativeness of the conditions.

- Delay-range-dependent exponential stability conditions for singular systems with multiple time-varying delays are established and an estimate of the convergence rate of the state is presented. It has been shown also that this rate depends on the minimum bounds of the delays.

- An explicit expression of the algebraic variables is given, which can be seen as a generalization of the expression presented in (Xu et al., 2002). This implies that many of the existing results for singular systems with constant time delays can be extended easily to the systems with time-varying delays.

- An iterative linear matrix inequality (ILMI) algorithm is used to design a stabilizing static output feedback controller for time-delay singular systems in the presence of actuator saturation.

- Delay-dependent conditions are proposed to design a stabilizing static output feedback controller for time-delay singular systems in the presence of actuator saturation and input delay.

The work throughout the thesis takes place in time-domain and is based on Lyapunov-Krasovskii approach. In Chapter 2, some graph theory terminology has been used. The results are mainly presented in the LMI setting. Some of the results are presented in the bilinear matrix inequality (BMI) setting. The free solver SEDUMI is used to solve the convex optimization problems while PENBMI is used to solve the BMI problems. YALMIP is used as an interface in both cases. For a brief introduction on SEDUMI and PENBMI, we refer the reader to (Lofberg, 2004) and references therein.

0.4 Preview of Chapters

The summary of each chapter of this thesis is given below.

Chapter 1 gives some preliminary results on the class of singular time-delay systems. The solution behavior of singular systems, time-delay systems and time-delay singular systems is discussed. Conditions for existence and uniqueness of solution of time-delay singular systems are given. Sources of impulses in the solution of time-delay singular systems are investigated and conditions to avoid such impulses are also presented. A numerical example is given to illustrate the results presented in the chapter. Finally, some physical examples that can be described by the class of time-delay singular systems are presented.

Chapter 2 discusses the stability of time-delay singular systems. First, delay-dependent conditions that guarantee the asymptotic stability of singular systems with time-varying delays are given. Then, delay-range-dependent exponential stability conditions for time-delay singular systems with single delay are established and an estimate of the convergence rate of the state are presented. The latter result is extended to the case of multiple time-delays. Some numerical examples are given to show the effectiveness of the proposed results.

Chapter 3 deals with the state feedback stabilization of time-delay singular systems. Two results are presented in this chapter. First, delay-dependent conditions that can be used to design a stabilizing state feedback controller are presented. Then, the conditions are extended to allow the stabilizing controller to satisfy some $\mathscr{H}_\infty$ performance level as well. Some numerical examples are given to show the effectiveness of the proposed results.

Chapter 4 deals with the static output feedback controller design of time-delay singular systems with saturating actuators. First, an iterative LMI algorithm is proposed to solve for the controller gain. Then, the system is extended to include input delays and BMI conditions are given to solve for the controller gains. Some numerical examples are given to show the effectiveness of the proposed results.

CHAPTER 1

PRELIMINARIES ON TIME-DELAY SINGULAR SYSTEMS

In this chapter, some preliminary results and definitions for time-delay singular systems are presented. First, the mathematical model of time-delay singular systems is given. Then, the solution behavior of singular systems, standard time-delay systems and singular time-delay systems are discussed. The existence and uniqueness of solutions, impulsive behavior and compatible initial conditions for time-delay singular systems are also discussed. A numerical example to illustrate some of the features of time-delay singular systems is presented. Then, lossless propagation models and chemical processes are presented as classes of systems that can be described by time-delay singular systems. Finally, a conclusion is given.

1.1 System Description

The state space system description of time-delay singular systems is given by:

$$\begin{cases} E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) \\ x(t) = \phi(t), -\overline{d} \leq t \leq 0 \end{cases} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, the matrix $E \in \mathbb{R}^{n \times n}$ may be singular, and we assume that $\text{rank}(E) = r \leqslant n$, A and A_d are known real constant matrices, $\phi(t) \in C^v_\tau$ is a compatible vector valued continuous function (see Section 1.4.1) and $d(t)$ is the time delay and that is assumed to satisfy the following:

$$\begin{cases} 0 < d(t) \leq \overline{d} \\ \dot{d}(t) \leq \mu < 1 \end{cases}$$

with $\overline{d}$ and μ are given scalars.

Time-delay singular systems may be viewed from two different perspectives; one that they are differential algebraic equations (DAEs) (i.e. singular systems) whose formulation involves delayed solution terms, the other that they are delay differential equations (DDEs) (i.e. time-delay systems) subject to constraints. The discussion of time-delay singular systems inherits many of the ideas and much of the terminology of both singular systems and standard time-delay systems, which makes it necessary at this point to have an overview on the solution behavior of these major classes of systems.

1.2 Solution Behavior of Singular Systems

The unforced state space description of standard singular systems is given by:

$$E\dot{x}(t) = Ax(t) \tag{1.2}$$

System (1.1) becomes a standard singular system if $A_d = 0$. For this class of systems, there are many results in the literature on the solution, analysis and control, and for more details, we refer the reader to (Xu and Lam, 2006; Dai, 1989; Wang et al., 2006; Lewis, 1986; Cobb, 1984) and the references therein. The following features in singular systems are not usually found in state-space systems:

- The transfer function of a singular system may not be strictly proper (see Section 2.6 in (Dai, 1989)).
- For an arbitrary finite initial condition, the time response of a singular system may exhibit impulsive or non-causal behavior along with the derivatives of these impulses (see Section 1.4 in (Dai, 1989)).
- A singular system usually contains three kinds of modes: finite dynamic modes, infinite dynamic modes and nondynamic modes; the undesired im-

pulsive behavior in a singular system can be generated by infinite dynamic modes (see Section 3.2 in (Dai, 1989)).

- Even if a singular system is impulse-free, it can still have initial finite discontinuities due to inconsistent initial conditions.

The existence and uniqueness of a solution to a given singular system are not always guaranteed and the system can also have undesired impulsive behavior. Therefore, the definitions of regularity and non-impulsiveness (in the continuous case) or causality (in the discrete case) have to be introduced. For a good discussion on this subject, we refer the reader to (Dai, 1989).

Definition 1.2.1. *(Dai, 1989) The pair (E, A) is said to be regular if the characteristic polynomial,* $\det(sE - A)$ *is not identically zero.*

The regularity of the pair (E, A) ensures the existence and uniqueness of solution. The existence of impulsive solutions is usually studied in terms of the Weierstrass canonical form and the index of the system which are defined as follows: if the pair (E, A) is regular, then there exist two nonsingular matrices M_1 and N_1 such that (Dai, 1989):

$$M_1 E N_1 = \begin{bmatrix} \mathbb{I}_{n_1} & 0 \\ 0 & J \end{bmatrix}, \quad M_1 A N_1 = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & \mathbb{I}_{n_2} \end{bmatrix} \tag{1.3}$$

and the singular system corresponding to the new variable $y = \mathrm{col}\{y_1, y_2\} = {N_1}^{-1} x$ has the following canonical form:

$$\dot{y}_1(t) = \widetilde{A} y_1(t), \tag{1.4a}$$

$$J \dot{y}_2(t) = y_2(t) \tag{1.4b}$$

where $n_1 + n_2 = n$, $J \in \mathbb{R}^{n_2 \times n_2}$ and $\widetilde{A} \in \mathbb{R}^{n_1 \times n_1}$ are in Jordan form. The matrix J is nilpotent of index ν, i.e., $J^{\nu} = 0$, $J^{\nu-1} \neq 0$. The index of the singular system is the index of nilpotency ν of J. Solving (1.4) (see Section 1.4 in (Dai, 1989)):

$$y_1(t) = e^{\tilde{A}t} y_1(0),$$
$$y_2(t) = -\sum_{k=0}^{\nu-2} \delta^k(t) J^{k+1} y_2(0^-),$$

where $\delta(t)$ is the Dirac delta-function and superscript k denotes the k^{th} distributional derivative. If $\nu > 1$, the Dirac delta-function will appear in the solution and the system will admit impulsive solutions.

Remark 1.2.1. *The systems that we are considering here are unforced systems. It has been shown that these systems can exhibit impulses at $t = 0$ due to the appearance of Dirac delta-function. Moreover, for a forced system, if $\nu > 1$, smooth inputs can generate impulses at all instances of the solution (see Section 1.4 in (Dai, 1989) for more details). This is why for stability analysis we seek the pair (E, A) to be impulse-free, i.e., system of index one.*

From the above discussion, one can deduce the following lemma.

Lemma 1.2.1. *(Xu and Lam, 2006) Suppose that the pair (E, A) is regular, and two nonsingular matrices M_1 and N_1 are found such that (1.3) holds, then we have:*

(a) *The pair (E, A) is impulse-free if and only if $J = 0$.*

(b) *The pair (E, A) is stable if and only if* $\text{eig}(A) < 0$.

(c) *The pair (E, A) is admissible if and only if $J = 0$ and* $\text{eig}(A) < 0$.

When the regularity of the pair (E, A) is not known, it is always possible to choose two nonsingular matrices M_2 and N_2 such that

$$M_2 E N_2 = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 A N_2 = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{1.5}$$

The decomposition can be obtained via a singular value decomposition on E and followed by scaling of the bases. The singular system corresponding to the new variable $y = \mathrm{col}\{y_1, y_2\} = {N_2}^{-1}x$ has the following form:

$$\begin{aligned} \dot{y}_1(t) &= A_1 y_1(t) + A_2 y_2(t), \\ 0 &= A_3 y_1(t) + A_4 y_2(t) \end{aligned}$$

Then, we have the following lemma.

Lemma 1.2.2. *(Xu and Lam, 2006) The pair (E, A) is impulse-free if and only if A_4 is nonsingular.*

Remark 1.2.2. *Both Lemmas 1.2.1 and 1.2.2 present equivalent conditions on the impulse-freeness of the pair (E, A). Note that both conditions involve the decomposition of the matrices of the original singular system; that is, we have to find the two nonsingular matrices M_1 and N_1, and M_2 and N_2, which is sometimes numerically unreliable, especially in the case when the order of the system is relatively large. In view of that, all the conditions throughout the thesis will be in terms of LMIs and BMIs, which involve no decomposition of the system matrices.*

As mentioned before, even if a singular system is impulse-free, it can still have initial finite discontinuities due to inconsistent initial conditions. To illustrate this feature of this class of systems, consider the singular system with the following matrices:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -0.5 & 1 \\ -0.5 & -1 \end{bmatrix}$$

which is equivalent to

$$\begin{aligned} \dot{x}_1(t) &= -0.5x_1(t) + x_2(t) \\ 0 &= -0.5x_1(t) - x_2(t) \end{aligned}$$

The algebraic constraint at $t = 0$ is $x_2(0) = -0.5x_1(0)$. Thus, any initial condition that does not satisfy this constraint at $t = 0$ will result in an initial jump discontinuity. As it will be discussed later, this jump may propagate in the solution if there exist delayed terms in the system.

1.3 Solution Behavior of Standard Time-Delay Systems

The state space description of time-delay systems with single delay is given by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - d(t)) \\ x(t) &= \phi(t), -\overline{d} \leq t \leq 0 \end{aligned}$$

System (1.1) becomes a standard time-delay system if the matrix E is an invertible matrix. Time-delay systems represents a class of infinite-dimensional systems largely used to describe propagation and transport phenomena or population dynamics and many results has been published in the literature, see (Richard, 2003) for a good survey on time-delay systems.

It is well known that the solutions of time-delay systems may have discontinuous derivatives. One of the most significant features of this class of systems is the propagation of such discontinuities in the solution. The propagation of jumps in derivatives of the solution of time-delay systems is by now well-explored (Neves and Feldstein, 1976; Willé and Baker, 1992) and is summarized in (Bellen and Zennaro, 2003).

To illustrate this feature by a simple example, consider the standard time-delay system with the following matrices:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

with $d(t) = 1$, which is equivalent to

$$\dot{x}_1(t) = x_2(t-1) \tag{1.6a}$$

$$\dot{x}_2(t) = x_2(t) - x_2(t-1) \tag{1.6b}$$

Consider the following initial condition $\phi(t) = [3,\ 5]$, $t \in [-1,\ 0]$. Then at $t = 0^-$, $\phi(t)$ implies $\dot{x}_1(0^-) = \dot{x}_2(0^-) = 0$, but at $t = 0^+$, the relations in (1.6) implies $\dot{x}_1(0^+) = 5$ and $\dot{x}_2(0^+) = 2$, which means a jump in the first derivative at $t = 0$. Now, the second derivative can be represented by:

$$\ddot{x}_1(t) = \dot{x}_2(t-1)$$
$$\ddot{x}_2(t) = \dot{x}_2(t) - \dot{x}_2(t-1)$$

which means that the jump will propagate to the second derivative at $t = 1$ since

$$\ddot{x}_1(1) = \dot{x}_2(0)$$
$$\ddot{x}_2(1) = \dot{x}_2(1) - \dot{x}_2(0)$$

and so on with higher derivatives at coming time instances. Thus, the discontinuities are propagating between different times (see Figure 1.1). More precisely, for a general delay function, if there is a jump at t_k, this jump will propagate to $t_{k+1} = t_k + d(t_{k+1})$, i.e. when the argument of a delayed term crosses a previous jump. This feature for time-delay systems results in jump discontinuities in the algebraic variables in time-delay singular systems, as it will be seen in the next section.

1.4 Solution Behavior of Time-Delay Singular Systems

Time-delay singular systems are delay differential equations coupled with difference equations. The existence and uniqueness of a solution to a given time-delay singular

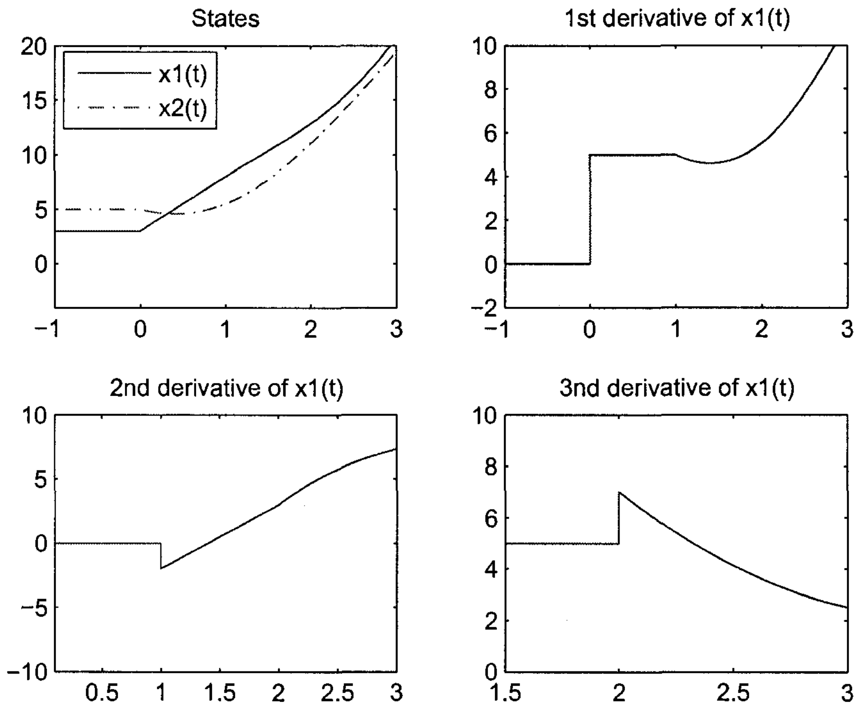

Figure 1.1 Solution behavior of time-delay system (1.6)

system are not always guaranteed and the system can also have undesired impulsive behavior. The behavior of solutions of this class of systems corresponds in some sense to what we might expect from a singular system and in some sense, to what we might expect from a time-delay system. The following are some features of time-delay singular systems:

- The existence and uniqueness of a solution to a given singular time-delay system are not always guaranteed.
- The time response of a time-delay singular system may exhibit undesired impulsive or non-causal behavior along with the derivatives of these impulses.
- Even if a time-delay singular system is impulse-free, it can still have finite discontinuities due to incompatible initial condition. Moreover, these jumps may propagate in the solution due to the existence of delayed terms.

For time-delay singular systems, the interaction of algebraic constraints with delayed solution terms gives rise to behavior that is not seen with either singular systems or time-delay systems. As has been pointed out in (Fridman, 2002; Baker et al., 2002), time-delay singular systems merit a separate investigation in their own right and can be considered as systems of advanced type.

Assume that the pair (E, A) in (1.1) is regular. Then, for M_1 and N_1 as in (1.3) and $y = N_1^{-1}x$, the time-delay singular system (1.1) has the following canonical form:

$$\begin{aligned} \dot{y}_1(t) &= \widetilde{A}y_1(t) + C_1y_1(t-d(t)) + C_2y_2(t-d(t)) \\ J\dot{y}_2(t) &= y_2(t) + C_3y_1(t-d(t)) + C_4y_2(t-d(t)) \end{aligned} \tag{1.7}$$

where

$$M_1A_dN_1 = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}.$$

The index of time-delay singular (1.1) is defined as the index of the corresponding singular system without delay (1.2). By continuously taking derivatives with respect to t on both sides of the second equation in (1.7), and left multiplying both sides by matrix J, we obtain the following equations:

$$\begin{aligned} J\dot{y}_2(t) &= y_2(t) + C_3y_1(t-d(t)) + C_4y_2(t-d(t)) \\ J^2y_2^{(2)}(t) &= J\dot{y}_2(t) + JC_3\dot{y}_1(t-d(t)) + JC_4\dot{y}_2(t-d(t)) \\ &\vdots \\ J^\nu y_2^{(\nu)}(t) &= J^{\nu-1}y_2^{(\nu-1)}(t) + J^{\nu-1}C_3y_1^{(\nu-1)}(t-d(t)) + J^{\nu-1}C_4y_2^{(\nu-1)}(t-d(t)) \end{aligned}$$

where $y_2^{(k)}$ stands for the k^{th} derivative of $y_2(t)$. From the addition of these equations and the fact that $J^\nu = 0$, we have the following expression for $y_2(t)$:

$$y_2(t) = -\sum_{k=0}^{\nu-2}\delta^k(t)J^{k+1}y_2(0^-) - \sum_{k=0}^{\nu-1}J^k\left[C_3y_1^{(k)}(t-d(t)) + C_4y_2^{(k)}(t-d(t))\right].$$

Therefore, as in the case without delay, system (1.7) admits impulsive solutions for $\nu > 1$:

$$y_1(t) = e^{\widetilde{A}t} y_1(0) + \int_0^t e^{\widetilde{A}(t-s)} \left[C_3 y_1(s - d_(s)) + C_4 y_2(s - d(s)) \right],$$
$$y_2(t) = -\sum_{k=0}^{\nu-2} \delta^k(t) J^{k+1} y_2(0^-) - \sum_{k=0}^{\nu-1} J^k \left[C_3 y_1^{(k)}(t - d(t)) + C_4 y_2^{(k)}(t - d(t)) \right].$$

Remark 1.4.1. *Noting the above equations, if $\nu > 1$, the appearance of the Dirac delta-function will admit impulsive solutions. Also, any jump in higher derivatives of the states will propagate to a jump in the states themselves. This is why for stability analysis we seek the pair (E, A) to be impulse-free, i.e., system of index one.*

Definition 1.4.1. *System (1.1) is said to be regular and impulse-free if the pair (E, A) is regular and impulse-free.*

If system (1.1) is regular and impulse-free, then it can be written in the form of (1.7) with $J = 0$ (see Lemma 1.2.1):

$$\begin{aligned} \dot{y}_1(t) &= \widetilde{A} y_1(t) + C_1 y_1(t - d(t)) + C_2 y_2(t - d(t)) \\ y_2(t) &= -C_3 y_1(t - d(t)) + C_4 y_2(t - d(t)). \end{aligned} \tag{1.8}$$

whose unique solution can be easily seen to exist since knowing the past, one can always obtain a unique $y_1(t)$ and $y_2(t)$. Moreover, J is equal to 0, i.e. no impulsive terms. However, as indicated previously, even if a time-delay singular system is impulse-free, it can still have finite discontinuities due to incompatible initial condition. If the initial condition does not satisfy the second equation of (1.8) at $t = 0$, the system will have jump discontinuities.

To illustrate this feature, let us consider a singular time-delay system with the following matrices:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & 0.1 \\ 1 & 0 \end{bmatrix}$$

with $d(t) = 0.6 + 0.1\sin(t)$, which is equivalent to

$$\begin{aligned} \dot{x}_1(t) &= -0.5x_1(t) - x_1(t-d(t)) + 0.1x_2(t-d(t)) \\ 0 &= x_2(t) + x_1(t-d(t)) \end{aligned} \tag{1.9}$$

From (1.9), $x_2(0) = -x_1(t-0.6)$ should be satisfied and any initial condition that does not satisfy it will result in an initial jump. Figure 1.2 shows the simulation results of the system with initial condition $\phi(t) = [-1, \ -2]$. The discontinuity due to this incompatible initial condition can be seen easily. It has been mentioned that

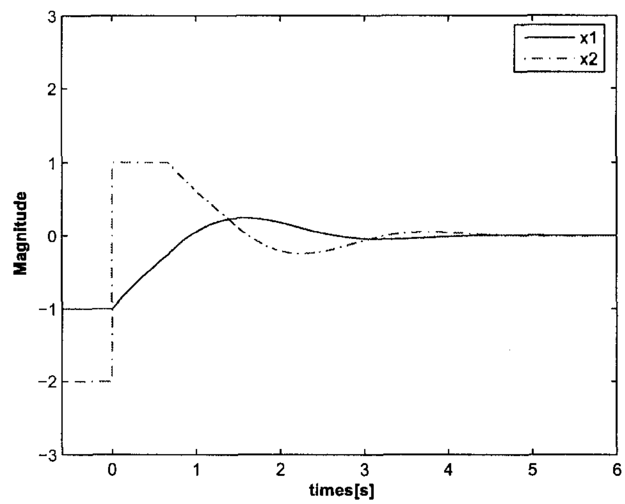

Figure 1.2 Inconsistent initial condition

unlike standard singular systems, discontinuities in time-delay singular systems can propagate between different times due to the existence of delayed solution terms. To illustrate this point, consider a time-delay singular system with the following matrices:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$

with $d(t) = 0.6$, which is equivalent to

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) - x_1(t-0.6) + 0.1x_2(t-0.6) \\ 0 &= x_2(t) + 0.5x_1(t-0.6) + 0.5x_2(t-0.6) \end{aligned} \tag{1.10}$$

Consider the following initial condition $\phi(t) = [1,\ -1]$. Using (1.10), $\dot{x}_1(0^+) = -2.1$ and $x_2(0^+) = 0$. Then at $t = 0$, $\dot{x}_1(0^-) \neq \dot{x}_1(0^+)$ and $x_2(0^-) \neq x_2(0^+)$, which means the appearance of jumps in $\dot{x}_1(t)$ and $x_2(t)$ at $t = 0$. Consequently, this jumps will propagate to the time instance $t = 0.6$ since

$$\begin{aligned} \dot{x}_1(0.6) &= -x_1(0.6) - x_1(0) + 0.1x_2(0) \\ 0 &= x_2(0.6) + 0.5x_1(0) + 0.5x_2(0) \end{aligned}$$

and so on with instances equal to multiples of 0.6. Thus, the jumps are propagating by the delayed terms whenever their arguments crosses a previous jump. Figure 1.3 shows the simulation results. The discontinuities throughout the state response can be seen easily, this feature is not found in neither singular systems nor time-delay systems. As discussed in (Shampine and Gahinet, 2006), the jumps can occur only in the algebraic variables or in the derivatives of the differential variables, i.e., the differential variables are always continuous.

1.4.1 Compatible Initial conditions

In this section, we present the compatible initial conditions such that the solution of the impulse-free time-delay singular systems is defined and continuous on $[-\overline{d}, \infty)$. It has been shown in the previous section that when the algebraic constraints

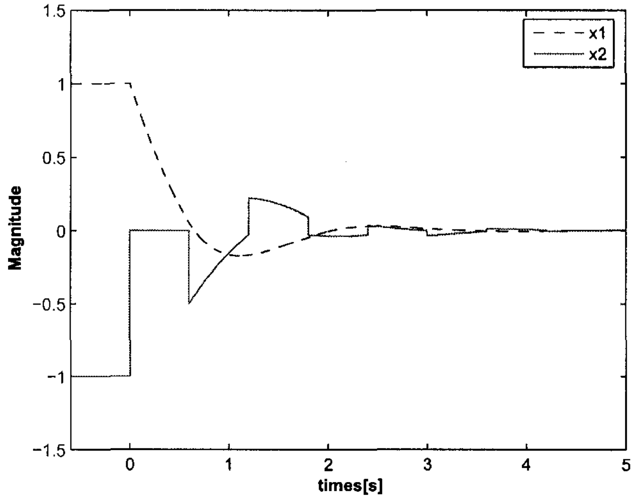

Figure 1.3 Inconsistent initial condition

are not satisfied at $t = 0$, the system will experience jump discontinuities in its solution. Thus, any initial condition that satisfies the algebraic constraints at $t = 0$ is considered as compatible initial condition. However, for the general form of (1.1), the algebraic constraints are implicit in the system model and, generally, they don't appear explicitly. Thus, the compatible initial conditions will be given explicitly to certain forms of the matrices E and A and a method will be presented to get the compatible initial conditions for the general form of E and A. In this section, the pair (E, A) is assumed to be regular and impulse-free.

Suppose that the matrices E and A are in the following form:

$$E = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \qquad A_d = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \tag{1.11}$$

Then, system (1.1) can be rewritten in the following form:

$$\begin{aligned} \dot{y}_1(t) &= \widetilde{A}y_1(t) + C_1 y_1(t - d(t)) + C_2 y_2(t - d(t)) \\ y_2(t) &= -C_3 y_1(t - d(t)) + C_4 y_2(t - d(t)). \end{aligned} \tag{1.12}$$

Note that any regular and impulse-free system can be written in this form (see Lemma 1.2.1). Consider the following initial conditions:

$$y_1(t) = \phi_1(t), \qquad y_2(t) = \phi_2(t), \qquad t \in [-\overline{d}, 0]. \tag{1.13}$$

Substituting initial condition $\phi(t) = \mathrm{col}(\phi_1(t), \phi_2(t))$ into the second functional equation of (1.12) we have

$$0 = \phi_2(0) + C_3\phi_1(-d(0)) + C_4\phi_2(-d(0)) \tag{1.14}$$

It has been discussed that the initial conditions should satisfy the algebraic constraints in order to prevent initial jumps, therefore, we have the following lemma.

Lemma 1.4.1. *For any continuous $\phi(t) = col(\phi_1(t), \phi_2(t))$ that satisfies (1.14), there exists a unique function $y(t)$ defined and continuous on $[-\overline{d}, \infty)$ that satisfies system (1.12) on $[0, \infty)$.*

Proof. The proof is similar to (Fridman, 2002) for the case of constant time-delay.
□

Suppose now a different form of the pair (E, A) where the matrices have the following form:

$$E = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \qquad A_d = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \tag{1.15}$$

where A_4 is nonsingular. This system can be rewritten as:

$$\begin{aligned} \dot{x}_1(t) &= A_1x_1(t) + A_2x_2(t) + C_1x_1(t-d(t)) + C_2x_2(t-d(t)) \\ 0 &= A_3x_1(t) + A_4x_2(t) + C_3x_1(t-d(t)) + C_4x_2(t-d(t)) \end{aligned} \tag{1.16}$$

Note that any regular and impulse-free system can be written in this form (see

Lemma 1.2.2). Consider the following initial conditions:

$$x_1(t) = \phi_1(t), \qquad x_2(t) = \phi_2(t), \qquad t \in [-\bar{d}, 0]. \tag{1.17}$$

Substituting initial condition $\phi(t) = \mathrm{col}(\phi_1(t), \phi_2(t))$ into the second functional equation of (1.16) we have

$$0 = A_3\phi_1(0) + A_4\phi_2(0) + C_3\phi_1(-d(0)) + C_4\phi_2(-d(0)) \tag{1.18}$$

And we have the following lemma for this form of time-delay singular systems.

Lemma 1.4.2. *For any continuous $\phi(t) = col(\phi_1(t), \phi_2(t))$ that satisfies (1.18), there exists a unique function $x(t)$ defined and continuous on $[-\bar{d}, \infty)$ that satisfies system (1.16) on $[0, \infty)$.*

Proof. The proof is similar to (Fridman, 2002) for the case of constant time-delay. □

For other forms of matrices E and A other than (1.11) and (1.15), the compatible initial conditions are the corresponding initial conditions to either (1.14) or (1.18) under the appropriate transformation. This will be demonstrated through the following example.

1.4.2 Numerical Example

Consider system (1.1) with the following matrices:

$$E = \begin{bmatrix} -1 & 4 & 3 \\ 2 & -1 & -6 \\ 3 & -8 & -9 \end{bmatrix}, \quad A = \begin{bmatrix} -30 & -13 & -22 \\ -123 & -30 & -23 \\ -2 & 7 & 34 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.5 & 0.2 & 0 \\ -1 & 0.1 & 0 \\ 0.5 & 0 & -1 \end{bmatrix}$$

with $d(t) = |0.3\sin(0.2t) + 0.1|$.

1. Regularity.

$$
\begin{aligned}
&\det(sE - A)\\
&= \det\left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -30 & -13 & -22 \\ -123 & -30 & -23 \\ -2 & 7 & 34 \end{bmatrix}\right)\\
&= \det\left(\begin{bmatrix} s+30 & 13 & 22 \\ 123 & s+30 & 23 \\ 2 & -7 & -34 \end{bmatrix}\right)\\
&= 2\begin{vmatrix} 13 & 22 \\ s+30 & 23 \end{vmatrix} + 7\begin{vmatrix} s+30 & 22 \\ 123 & 23 \end{vmatrix} - 34\begin{vmatrix} s+30 & 13 \\ 123 & s+30 \end{vmatrix}\\
&= -34s^2 + 2746s - 30797 \neq 0
\end{aligned}
$$

Therefore, by Definition 1.2.1, the system is regular.

2. Impulse-free.

   To check if the system is impulse-free, matrices M and N should be found such that the decomposition in (1.5) holds. As mentioned before, the decomposition can be obtained via a singular value decomposition on E and followed by scaling of the bases. Using Matlab, the function `[U,S,V]=svd(E)` is used to obtain the singular value decomposition of the matrix E. This function produces a diagonal matrix S and unitary matrices U and V such that $E = USV^\top$. The singular value decomposition is:

$$
\begin{aligned}
&[U, S, V] = \mathrm{svd}(E)\\
&U = \begin{bmatrix} -0.3395 & -0.4031 & 0.8498 \\ 0.3925 & -0.8818 & -0.2615 \\ 0.8548 & 0.2448 & 0.4576 \end{bmatrix}
\end{aligned}
$$

$$S = \begin{bmatrix} 14.4862 & 0 & 0 \\ 0 & 3.3393 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.2547 & -0.1875 & -0.9487 \\ -0.5929 & -0.8053 & 0 \\ -0.7640 & 0.5625 & -0.3162 \end{bmatrix}$$

Scale the first column of U by 14.4862 and the second column by 3.3393, then the matrices M and N are:

$$M = U^{-1} = \begin{bmatrix} -0.0234 & 0.0271 & 0.0590 \\ -0.1207 & -0.2641 & 0.0733 \\ 0.8498 & -0.2615 & 0.4576 \end{bmatrix}$$

$$N = V^{-\top} = \begin{bmatrix} 0.2547 & -0.1875 & -0.9487 \\ -0.5929 & -0.8053 & 0 \\ -0.7640 & 0.5625 & -0.3162 \end{bmatrix}$$

To verify the previous results,

$$MEN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Also,

$$MAN = \begin{bmatrix} -2.0938 & 1.6597 & 2.0062 \\ -5.3486 & -8.4857 & -37.6588 \\ -0.7325 & 0.5393 & -6.3671 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

and $A_4 = -6.3671$ is nonsingular. Therefore, from Lemma 1.2.2 the system is impulse-free.

3. Compatible initial conditions.

The compatible initial conditions are the corresponding initial conditions to (1.18) under the transformation M and N. Note that

$$MA_dN = \begin{bmatrix} 0.0439 & -0.0299 & 0.0275 \\ 0.1472 & -0.0456 & -0.2048 \\ 0.4974 & -0.5448 & -0.7235 \end{bmatrix}$$

Thus, the compatible initial conditions are the corresponding initial conditions to:

$$\begin{aligned} 0 &= [-0.7325 \quad 0.5393]\,\widetilde{\phi}_1(0) - 6.3671\widetilde{\phi}_2(0) \\ &\quad + [0.4974 \quad - 0.5448]\,\widetilde{\phi}_1(-0.1) - 0.7235\widetilde{\phi}_2(-0.1) \end{aligned} \tag{1.19}$$

where $\widetilde{\phi}(t) = N^{-1}\phi(t)$. Therefore,

$$\begin{aligned} \widetilde{\phi}(t) = \begin{bmatrix} \widetilde{\phi}_1(t) \\ \widetilde{\phi}_2(t) \end{bmatrix} &= \begin{bmatrix} \widetilde{\phi}_{11}(t) \\ \widetilde{\phi}_{12}(t) \\ \widetilde{\phi}_2(t) \end{bmatrix} \\ &= \begin{bmatrix} 0.2547 & -0.5929 & -0.7640 \\ -0.1875 & -0.8053 & 0.5625 \\ -0.9487 & 0 & -0.3162 \end{bmatrix} \begin{bmatrix} \phi_{11}(t) \\ \phi_{12}(t) \\ \phi_2(t) \end{bmatrix} \\ &= \begin{bmatrix} 0.2547\phi_{11}(t) - 0.5929\phi_{12}(t) - 0.7640\phi_2(t) \\ -0.1875\phi_{11}(t) - 0.8053\phi_{12}(t) + 0.5625\phi_2(t) \\ -0.9487\phi_{11}(t) - 0.3162\phi_2(t) \end{bmatrix} \end{aligned}$$

Substituting this into (1.19), after some algebraic manipulations, we get the following condition for compatible initial conditions:

$$\begin{aligned} 0 &= [5.7528 \quad 0]\,\phi_1(0) + 2.87626\phi_2(0) \\ &\quad + [0.9152 \quad 0.1438]\,\phi_1(-0.1) - 0.4577\phi_2(-0.1) \end{aligned} \tag{1.20}$$

The response of this system with the following compatible initial condition:

$$\phi(t) = \begin{bmatrix} 0.3867 + 0.0255\cos(40t) - 0.0562\sin(30t) \\ -1.9911 - 0.0593\cos(40t) - 0.2416\sin(30t) \\ -0.9438 - 0.0764\cos(40t) + 0.1687\sin(30t) \end{bmatrix} \tag{1.21}$$

and $d(t) = |0.3\sin(0.2t) + 0.1|$ is shown in Figure 1.4.

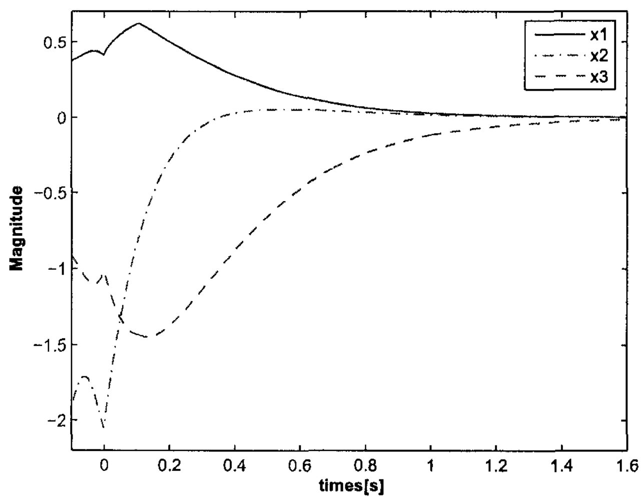

Figure 1.4 Solution behavior of x_1, x_2 and x_3 with $d(t) = |0.3\sin(0.2t) + 0.1|$ and $\phi(t)$ defined in (1.21).

4. Incompatible initial conditions.

The response of the system with the following incompatible initial condition:

$$\phi(t) = \begin{bmatrix} -1.2677 \\ 1.0176 \\ -2.5214 \end{bmatrix} \tag{1.22}$$

and $d(t) = |0.3\sin(0.2t) + 0.1|$ is shown in Figure 1.5. Note the jump discontinuities due to incompatible initial condition.

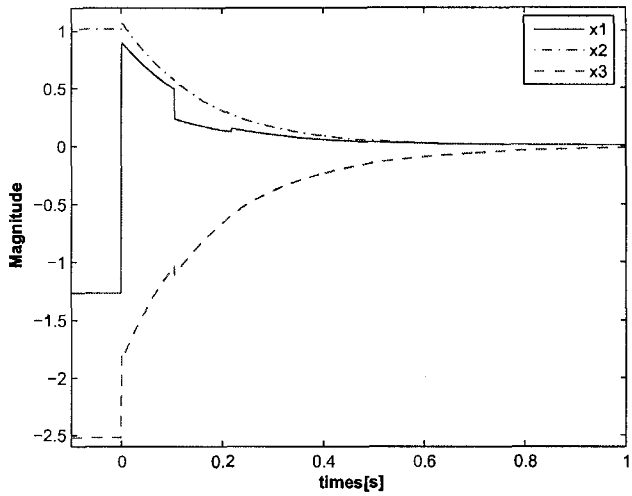

Figure 1.5 Solution behavior of x_1, x_2 and x_3 with $d(t) = |0.3\sin(0.2t) + 0.1|$ and $\phi(t)$ defined in (1.22).

1.5 Neutral Systems Viewed as Time-Delay Singular Systems

Consider the following classical form of linear functional differential equation of neutral type:

$$\frac{d}{dt}\left[x_2(t) - Dx_2(t)(t - d(t))\right] = Ax_2(t) + Bx_2(t - d(t))$$

then denoting: $x_1(t) = x_2(t) - Dx_2(t)(t - d(t))$, we obtain the system (see Section 3.3.4 in (Niculescu, 2001)):

$$\begin{aligned} \dot{x}_1(t) &= Ax_1(t) + (B + AD)x_2(t - d(t)) \\ x_2(t) &= x_1(t) + Dx_2(t - d(t)) \end{aligned}$$

which is a time-delay singular system. This means that the results appear in this thesis can be applied to neutral systems as well. Note that the converse is not true, i.e., the general time-delay singular system cannot be always transformed to neutral system. However, such a transformation exists under certain conditions.

1.6 Examples of Time-Delay Singular Systems

Time-delay singular systems find their way into the representation of many real applications. In what follows, two systems that can be described by time-delay singular systems will be presented.

1.6.1 Lossless Propagation Models

Lossless propagation is associated to transmission lines without losses or to lossless steam, water or gas pipes (Niculescu and Rasvan, 2000). Some dynamics with respect to this topics are the transients of a hydraulic turbines under waterhammer conditions (Halanay and Rasvan, 1997), the transients of extracted steam pressure in the combined generation of heat and electricity (Halanay and Rasvan, 1997), the dynamics of electrical and computer networks containing lossless transmission line (Brayton, 1968), etc.

The mathematical model describing the lossless propagation consists of hyperbolic partial differential equations. By using a well-known result about the wave equation, the partial differential equations are replaced by difference equations (Brayton, 1968). These difference equations are combined with the ordinary differential equations describing the remainder of the system yielding a system of difference-differential equations.

It has been shown in (Halanay and Rasvan, 1997) that the transients of extracted steam pressure in the combined generation of heat and electricity can be described by the following system:

$$
\begin{aligned}
\dot{x}_1 &= Ax_1(t) + B_1\eta_1(t-h) + B_2\eta_2(t-d(t)) \\
\eta_1(t) &= C_1x_1(t) + D_1\eta_2(t-d(t)) \\
\eta_2(t) &= C_2x_1(t) + D_2\eta_1(t-d(t))
\end{aligned}
$$

where $x_1(t) \in \mathbb{R}^{n_1}$, $\eta_1(t) \in \mathbb{R}^{n_2}$ and $\eta_2(t) \in \mathbb{R}^{n_3}$ are the state variables, A, B_1, B_2, C_1, C_2, D_1 and D_2 are real constant matrices and $d(t)$ is a positive continuous bounded function.

This system can be rewritten in the following form:

$$\widetilde{E}\dot{x}(t) = \widetilde{A}x(t) + \widetilde{A}_d x(t - d(t))$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ \eta_1(t) \\ \eta_2(t) \end{bmatrix}, \qquad \widetilde{E} = \begin{bmatrix} \mathbb{I}_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{A} = \begin{bmatrix} A & 0 & 0 \\ C_1 & -\mathbb{I}_{n_2} & 0 \\ C_2 & 0 & -\mathbb{I}_{n_3} \end{bmatrix}, \qquad \widetilde{A}_d = \begin{bmatrix} 0 & B_1 & B_2 \\ 0 & 0 & D_1 \\ 0 & D_2 & 0 \end{bmatrix}.$$

For more details on lossless propagation models, we refer the reader to (Niculescu and Rasvan, 2000; Halanay and Rasvan, 1997) and the references therein.

1.6.2 Chemical Processes

Chemical processes are typically modeled by coupled differential and algebraic equations (DAEs). The differential equations arise from dynamic conservation equations, while the algebraic equations commonly arise from thermodynamic equilibrium relations, empirical correlations, pseudo-steady-state assumptions, closure conditions, and so on (Kumar and Daoutidis, 1999).

From the other side, delays often appear in chemical processes such as transportation delays caused by finiteness of time of heat transport along different elements of the circulation contours, warming up time of a reactor and so on (Haurani, 2003).

When both delays and algebraic constraints arise in a chemical process, it can be modeled as time-delay singular system.

A broad class of chemical processes modeled by time-delay singular systems consists of multiphase systems where the individual phases are in thermodynamic equilibrium. In what follows we will consider a vapor-liquid reaction system in a continuous stirred-tank reactor (CSTR), with the two phases in physical equilibrium (Kumar and Daoutidis, 1995).

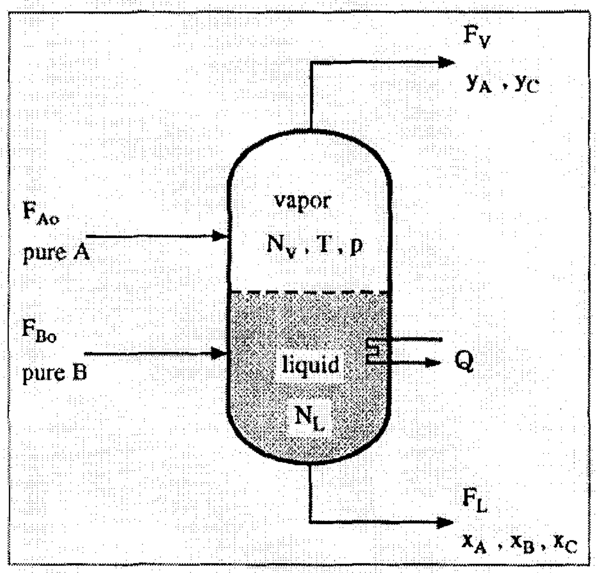

Figure 1.6 Two-phase reactor

Consider the two-phase (liquid- and vapor-phase) reactor shown in Figure 1.6. Reactants A and B are fed to the CSTR as pure vapor and liquid streams, respectively, at molar flow rates F_{AO}, and F_{BO}, while the two outlet streams from the liquid and vapor phases have molar flow rates F_L and F_V, respectively. It is assumed that the individual phases are well-mixed and they are in physical equilibrium at pressure p and temperature T, that is, the chemical reaction is slow compared to the mass transfer across the interface. The molar specific heat capacity c_p, density ρ, and latent heat of vaporization ΔH^u are also assumed to be constant and equal for all the species. Reactant A diffuses into the liquid phase, where a reaction of the form:

$$A + B \rightarrow C$$

takes place. Product C then diffuses out into the vapor phase (product phase). Reactant B is assumed to be nonvolatile, that is, only the reactant A and the product C are present in the vapor phase while all the three species are present in the liquid phase.

The dynamic conservation equations for this process consist of the total mole balances in the liquid and vapor phases, the mole balance for the species A in the vapor phase, the mole balances for species A and B in the liquid phase, and the total enthalpy balance.

In addition to these differential equations, the model consists of algebraic relations which include phase-equilibrium relations for the species A and C present in both phases, and the ideal gas law for the vapor phase. Moreover, transportation delay along different elements of the circulation contour appears in the system as well. Also, assume that a disturbance is introduced when the reactor is fed by the reactant A. Under some assumptions (see (Kumar and Daoutidis, 1995)), the dynamics of the system is governed by the following equations:

Pure differential equations

$$\begin{aligned}
\frac{dN_v}{dt} &= F_{A0} + \omega(t) - N_A + N_C - F_V \\
\frac{dy_A}{dt} &= \frac{(F_{A0} + \omega(t))(1 - y_A)}{N_v} - \frac{1 - y_A}{N_V} N_A - \frac{y_A}{N_V} N_C \\
\frac{dN_L}{dt} &= F_{B0} - F_L - R_C + N_A - N_C \\
\frac{dx_A}{dt} &= \frac{F_{B0} x_A + R_C(1 - x_A)}{N_L} + \frac{1 - x_A}{N_L} N_A + \frac{x_A}{N_L} N_C \\
\frac{dx_B}{dt} &= \frac{F_{B0}(1 - x_B) - R_C(1 - x_B)}{N_L} - \frac{x_B}{N_L} N_A + \frac{x_B}{N_L} N_C
\end{aligned}$$

Delay differential equations

$$\frac{dT}{dt} = \frac{F_{Ao}+\omega(t)}{N_L+N_v}(T_{Ao}-T) + \frac{F_{Bo}}{N_L+N_v}(T_{Bo}-T) + \frac{R_c}{N_L+N_v}(T - \frac{\Delta H_R}{c_p})$$
$$+ \frac{\Delta H^v}{(N_L+N_v)c_p} + \frac{1}{(N_L+N_v)c_p}Q(t-d(t))$$

Algebraic constraints

$$0 = -x_A P_A^s + p y_A$$
$$0 = -(1 - x_A - x_B)P_c^s + p(1 - y_A)$$
$$0 = -N_v RT + p\frac{V_{T\rho - N_L}}{\rho}$$

In the above equations, N_v is the vapor-phase molar holdup, y_A is the mole fraction of species A in vapor phase, N_L is the liquid-phase molar holdup, x_A and x_B are the mole fractions of the reactants A and B in the liquid phase, N_A is the molar rate of transfer of reactant A from the vapor of the liquid phase, N_c is the molar rate of transfer of product C from the liquid to the vapor phase, Q is the heat input to the reactor, and $d(t)$ is the transportation delay. A detailed description of the process parameters and variables is given in (Kumar and Daoutidis, 1995) along with their nominal steady-state values.

It is desired to control the composition of the vapor phase y_A and temperature T, using the vapor stream outlet flow rate F_V and the heat input Q as the manipulated inputs. Define now the differential variables x_i, $i = 1, \ldots 6$, the algebraic variables x_i, $i = 7, 8, 9$, the controlled outputs z_i, $i = 1, 2$ and the manipulated inputs u_i, $i = 1, 2$ as given below:

$$x_1 = N_v; \quad x_2 = y_A; \quad x_3 = N_L; \quad x_4 = x_A;$$
$$x_5 = x_B; \quad x_6 = T; \quad x_7 = N_A; \quad x_8 = N_c;$$
$$x_9 = p; \quad z_1 = y_A; \quad z_2 = T; \quad u_1 = F_V;$$
$$u_2 = Q.$$

Linearizing the delay differential, the differential and the algebraic equations around the nominal steady-state values, the two-phase reactor system can be modeled by the following system:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 u(t - d(t)) + B_{\omega 1}\omega(t) \\ z(t) &= C_z x(t) + B_{\omega 2}\omega(t) \end{aligned}$$

where $x(t) \in \mathbb{R}^9$ is the state, $u(t) \in \mathbb{R}^2$ is the control input, $z(t) \in \mathbb{R}^3$ is the controlled output, $E = \begin{bmatrix} \mathbb{I}_6 & 0 \\ 0 & 0 \end{bmatrix}$, A, B_1, B_2, $B_{\omega 1}$, $B_{\omega 2}$ and C_z are known real constant matrices, and $d(t)$ is the time-delay that is assumed to be a positive continuous bounded function.

The models of the presented examples (lossless propagation models and the two phase reactor) belong to the following general form of time-delay singular systems:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t - d_1(t)) + B_1 u(t) + B_2 u(t - d_2(t)) + B_{\omega 1}\omega(t) \\ y(t) &= C_y x(t) + D_y u(t) \\ z(t) &= C_z x(t) + B_{\omega 2}\omega(t) \\ x(t) &= \phi(t), \quad t \in [-\overline{d}, 0] \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^q$ is the measurement, $z(t) \in \mathbb{R}^s$ is the controlled output, the matrix $E \in \mathbb{R}^{n \times n}$ may be singular, and we assume that $\text{rank}(E) = r \leq n$, A, A_d, B_1, B_2, $B_{\omega 1}$, $B_{\omega 2}$, C_y, D_y and C_z are known real constant matrices, $\phi(t) \in C_{\tau}^{v}$ is a compatible vector valued continuous function and $d_1(t)$ and $d_2(t)$ are the time-delays that are assumed to be positive continuous functions that are bounded by $\overline{d}$.

In the following chapters, we will deal with the analysis and synthesis of this class of systems.

1.7 Conclusion

In this chapter, some preliminary results and definitions for time-delay singular systems have been presented. The solution behavior of singular systems and the impulsive behavior of its solution are studied. It has been shown that the jump discontinuities propagate via the algebraic constraints, and consequently, they propagate between different states. Also, the solution behavior of time-delay systems and the propagation of jump discontinuities in the higher derivatives of the solution vector are studied. Due to the existence of delayed terms, these jump discontinuities propagate between different time instances as well. For time-delay singular systems, where both constraints and delayed terms exists, jumps propagate between different time instances and different state components. This behavior is inherited from both singular systems and time-delay systems. Some definitions are presented to insure the existence and uniqueness of solutions and the absence of any impulsive behavior. Incompatible initial conditions can give rise to impulses as well and the notion of compatible initial conditions has been discussed. Some physical examples that can be modeled by the class of time-delay singular systems are presented to motivate the work in this thesis.

CHAPTER 2

STABILITY OF TIME-DELAY SINGULAR SYSTEMS

Stability problem is one of the most important problems in control theory. As mentioned in the introduction, most of the results in the literature deal with singular systems with constant delays, and the stability problem for singular systems with time varying delays has not been fully addressed. This chapter is devoted to study the stability of singular systems with time-varying delays.

In this chapter, first, delay-dependent stability conditions for singular systems with time-varying delay are derived. Then, delay-range-dependent exponential stability conditions for singular systems with time-varying delay are derived. The latter results are also extended for the case of multiple time-varying delays. The results of this chapter are mainly based on the Lyapunov-Krasovskii approach and some algebraic results. Also, a graph theoretic analysis is employed for the case of multiple time-varying delays. It should be noted that the free-weighting matrices method will be used in order to have less conservative results. All the results will be in the LMI setting.

2.1 Problem Statement and Preliminaries

Consider the linear time-delay singular system:

$$\begin{cases} E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) \\ x(t) = \phi(t), -\overline{d} \leq t \leq 0 \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, the matrix $E \in \mathbb{R}^{n \times n}$ may be singular, and we assume that $\operatorname{rank}(E) = r \leqslant n$, A and A_d are known real constant matrices, $\phi(t) \in C_\tau^v$ is a compatible vector valued continuous function and $d(t)$ is the time delay and that is assumed to satisfy:

$$\begin{cases} 0 < d(t) \leq \overline{d} \\ \dot{d}(t) \leq \mu < 1 \end{cases} \tag{2.2}$$

with $\overline{d}$ and μ are given scalars.

The following definitions and lemmas will be used in the rest of this chapter:

Definition 2.1.1. *i. System (2.1) is said to be regular and impulse-free if the pair (E, A) is regular and impulse-free.*

ii. The singular delay system is said to be stable if, for any $\epsilon > 0$ there exists a scalar $\delta(\epsilon) > 0$ such that, for any compatible initial conditions $\phi(t)$ satisfying $\|\phi\|_c < \delta(\epsilon)$, the solution $x(t)$ to the time-delay singular system satisfies $\|x(t)\| < \epsilon$ and a scalar $\delta > 0$ can be chosen such that $\|\phi\|_c < \delta$, implies $x \to 0$ as $t \to \infty$.

iii. System (2.1) is said to be admissible if it is regular, impulse-free and stable.

iv. System (2.1) is said to be exponentially stable if there exist $\sigma > 0$ and $\gamma > 0$ such that, for any compatible initial conditions $\phi(t)$, the solution $x(t)$ to the time-delay singular system satisfies

$$\|x(t)\| \leq \gamma e^{-\sigma t} \|\phi\|_c$$

v. System (2.1) is said to be exponentially admissible if it is regular, impulse-free and exponentially stable.

Lemma 2.1.1. *(Xu and Lam, 2006) Consider the function $\varphi : \mathbb{R}^+ \to \mathbb{R}$. If $\dot{\varphi}$ is bounded on $[0, \infty)$; that is, there exists a scalar $\alpha > 0$ such that $\|\dot{\varphi}(t)\| \leq \alpha$ for all $t \in [0, \infty)$, then φ is uniformly continuous on $[0, \infty)$.*

Lemma 2.1.2 (Barbalat's Lemma). *Consider the function* $\varphi : \mathbb{R}^+ \to \mathbb{R}$. *If* φ *is uniformly continuous and* $\int_0^t \varphi(s)ds < \infty$, *then*

$$\lim_{t\to\infty} \varphi(t) = 0.$$

2.2 Delay-Dependent Stability Conditions for Singular Systems with Time-Varying Delay

In this section, delay-dependent stability conditions for system (2.1) are developed and the following theorem gives such result.

Theorem 2.2.1. *Let* $\bar{d} > 0$ *and* $\mu < 1$ *be given scalars. System (2.1) is admissible if there exist matrices* P, T_1, T_2, $Q > 0, Z > 0$ *and free matrices* $N_i, i = 1, 2, 3$ *such that:*

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \bar{d}N_1 \\ \star & \Pi_{22} & \Pi_{23} & \bar{d}N_2 \\ \star & \star & \Pi_{33} & \bar{d}N_3 \\ \star & \star & \star & -\bar{d}Z \end{bmatrix} < 0 \tag{2.3}$$

$$E^\top P = P^\top E \geq 0 \tag{2.4}$$

$$E^\top T_1^\top = T_1 E \geq 0 \tag{2.5}$$

where:

$\Pi_{11} = Q + T_1 A + A^\top T_1^\top + N_1 E + (N_1 E)^\top$

$\Pi_{12} = T_1 A_d + (N_2 E)^\top - N_1 E$

$\Pi_{13} = P^\top - T_1 + (N_3 E)^\top + A^\top T_2^\top$

$\Pi_{22} = -(1-\mu)Q - N_2 E - (N_2 E)^\top$

$\Pi_{23} = -(N_3 E)^\top + A_d^\top T_2^\top$

$\Pi_{33} = \bar{d}Z - T_2 - T_2^\top$

Proof. First, we will show that the system is regular and impulse-free. For this purpose, choose two nonsingular matrices R and L such that

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{A} = RAL = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{2.6}$$

Now, let

$$\bar{A}_d = RA_dL = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}, \qquad \bar{T}_1 = R^{-\top}T_1^\top L = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{2.7}$$

$$\bar{N}_i = L^\top N_i R^{-1} = \begin{bmatrix} N_{i11} & N_{i12} \\ N_{i21} & N_{i22} \end{bmatrix}, \quad \bar{Q} = L^\top QL = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}. \tag{2.8}$$

From (2.5)-(2.7), we conclude that $T_{12} = 0$ and $T_{11} > 0$.
Also, from (2.3), we get $\Pi_{11} < 0$ which gives $T_1A + A^\top T_1^\top + N_1E + (N_1E)^\top < 0$. Based on (2.6)-(2.8), pre- and post-multiply this inequality by $L^\top$ and L, respectively, we have

$$\bar{T}_1^\top \bar{A} + \bar{A}^\top \bar{T}_1 + \bar{N}_1\bar{E} + (\bar{N}_1\bar{E})^\top < 0. \tag{2.9}$$

Noting that

$$\bar{N}_1\bar{E} = \begin{bmatrix} N_{111} & 0 \\ N_{121} & 0 \end{bmatrix},$$

Eq. (2.9) gives

$$\begin{bmatrix} \star & \star \\ \star & A_{22}^\top T_{22} + T_{22}^\top A_{22} \end{bmatrix} < 0 \quad \text{that implies in turn that} \quad A_{22}^\top T_{22} + T_{22}^\top A_{22} < 0.$$

Therefore A_{22} is nonsingular, which implies in turn that system (2.1) is regular and impulse-free (Lemma 1.2.2). Next, we show the stability of system (2.1). Since system (2.1) is regular and impulse-free, there exist two other matrices R and L such that (Lemma 1.2.1)

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{A} = RAL = \begin{bmatrix} A_1 & 0 \\ 0 & \mathbb{I}_{n-r} \end{bmatrix} \tag{2.10}$$

Define $\bar{A}_d$, $\bar{T}_1$, $\bar{N}_i$, $\bar{Q}$ in a similar manner as (2.7)-(2.8), $\bar{T}_2 = R^{-\top} T_2^{\top} L$ and $\bar{Z} = R^{-\top} Z R^{-1}$. Using (2.3) and Schur complement, we get

$$\begin{bmatrix} \Pi_{11} & \Pi_{21} \\ \star & \Pi_{22} \end{bmatrix} < 0$$

Substitute (2.10) into the previous inequality, pre- and post-multiply by $\mathrm{diag}\{L^{\top}, L^{\top}\}$ and $\mathrm{diag}\{L, L\}$, respectively, and using Schur complement, we have

$$\begin{bmatrix} T_{22}^{\top} + T_{22} + Q_{22} & T_{22}^{\top} A_{d22} \\ A_{d22}^{\top} T_{22} & -(1-\mu) Q_{22} \end{bmatrix} < 0$$

Pre- and post-multiplying this by $\begin{bmatrix} -A_{d22}^{\top} & \mathbb{I} \end{bmatrix}$ and its transpose, and noting that $Q > 0$ and $\mu \geq 0$ (since if $\mu < 0$, the first condition in (2.2) will be violated), we get

$$A_{d22}^{\top} Q_{22} A_{d22} - Q_{22} < 0 \qquad \text{which implies} \qquad \rho(A_{d22}) < 1. \tag{2.11}$$

Let $\zeta(t) = L^{-1} x(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}$, where $\zeta_1(t) \in \mathbb{R}^r$ and $\zeta_2(t) \in \mathbb{R}^{n-r}$. Then, system (2.1) becomes equivalent to the following one:

$$\dot{\zeta}_1(t) = A_1 \zeta_1(t) + A_{d11} \zeta_1(t - d(t)) + A_{d12} \zeta_2(t - d(t)), \tag{2.12}$$

$$0 = \zeta_2(t) + A_{d21} \zeta_1(t - d(t)) + A_{d22} \zeta_2(t - d(t)). \tag{2.13}$$

Now, choose the Lyapunov functional as follows:

$$V(t) = \zeta^\top(t)\bar{E}^\top\bar{P}\zeta(t) + \int_{t-d(t)}^{t} \zeta^\top(s)\bar{Q}\zeta(s)ds + \int_{-\bar{d}}^{0}\int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top\bar{Z}\bar{E}\dot{\zeta}(s)dsd\theta$$

Then, the time-derivative of $V(t)$ along the solution of (2.12) and (2.13) is given by

$$\begin{aligned}\dot{V}(t) &= 2\zeta^\top(t)\bar{P}^\top\bar{E}\dot{\zeta}(t) + \zeta^\top(t)\bar{Q}\zeta(t) - (1-\dot{d}(t))\zeta^\top(t-d(t))\bar{Q}_3\zeta(t-d(t)) \\ &+ \bar{d}(\bar{E}\dot{\zeta}(t))^\top\bar{Z}\bar{E}\dot{\zeta}(t) - \int_{t-\bar{d}}^{t}(\bar{E}\dot{\zeta}(s))^\top\bar{Z}\bar{E}\dot{\zeta}(s)ds \end{aligned} \tag{2.14}$$

Note that

$$2\left[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2 + \dot{\zeta}^\top(t)\bar{N}_3\right] \times \left[\bar{E}\zeta(t) - \bar{E}\zeta(t-d(t)) - \int_{t-d(t)}^{t}\bar{E}\dot{\zeta}(s)ds\right] = 0 \tag{2.15}$$

$$2\left[\zeta^\top(t)\bar{T}_1^\top + \dot{\zeta}^\top(t)\bar{T}_2^\top\right]\cdot\left[\bar{E}\dot{\zeta}(t) - \bar{A}\zeta(t) - \bar{A}_d\zeta(t-d(t))\right] = 0 \tag{2.16}$$

and for a semipositive-definite matrix $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ \star & X_{22} & X_{23} \\ \star & \star & X_{33} \end{bmatrix}$, the following holds:

$$\bar{d}\eta^\top(t)X\eta(t) - \int_{t-d(t)}^{t}\eta^\top(t)X\eta(t)ds \geq 0 \tag{2.17}$$

where $\eta(t) = \begin{bmatrix}\zeta^\top(t) & \zeta^\top(t-d(t)) & \dot{\zeta}^\top(t)\end{bmatrix}^\top$. Then, adding the terms on the left of (2.15) and (2.16) to $\dot{V}(t)$ allows us to express it as

$$\dot{V}(t) \leq \eta^\top(t)\Xi\eta(t) - \int_{t-d(t)}^{t}\xi^\top(t,s)\Psi\xi(t,s)ds$$

where

$$
\begin{aligned}
\xi(t,s) &= \begin{bmatrix}\eta^\top(t) \ \dot{\zeta}^\top(s)\end{bmatrix} \\
\Xi &= \begin{bmatrix} \bar{\Pi}_{11} + \bar{d}X_{11} & \bar{\Pi}_{12} + \bar{d}X_{12} & \bar{\Pi}_{13} + \bar{d}X_{13} \\ \star & \bar{\Pi}_{22} + \bar{d}X_{22} & \bar{\Pi}_{23} + \bar{d}X_{23} \\ \star & \star & \bar{\Pi}_{33} + \bar{d}X_{33} \end{bmatrix} \\
\Psi &= \begin{bmatrix} X_{11} & X_{12} & X_{13} & \bar{N}_1 \\ \star & X_{22} & X_{23} & \bar{N}_2 \\ \star & \star & X_{33} & \bar{N}_3 \\ \star & \star & \star & \bar{Z} \end{bmatrix} \\
\Pi_{11} &= \bar{Q} + \bar{T}_1\bar{A} + \bar{A}^\top\bar{T}_1^\top + \bar{N}_1\bar{E} + (\bar{N}_1\bar{E})^\top \\
\Pi_{12} &= \bar{T}_1\bar{A}_d + (\bar{N}_2\bar{E})^\top - \bar{N}_1\bar{E} \\
\Pi_{13} &= \bar{P}^\top - \bar{T}_1 + (\bar{N}_3\bar{E})^\top + \bar{A}^\top\bar{T}_2^\top \\
\Pi_{22} &= -(1-\mu)\bar{Q} - \bar{N}_2\bar{E} - (\bar{N}_2\bar{E})^\top \\
\Pi_{23} &= -(\bar{N}_3\bar{E})^\top + \bar{A}_d^\top\bar{T}_2^\top \\
\Pi_{33} &= \bar{d}\bar{Z} - \bar{T}_2 - \bar{T}_2^\top
\end{aligned}
$$

If we select $X = \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_3 \end{bmatrix} \bar{Z}^{-1} \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_3 \end{bmatrix}^\top$, this ensures that $X \geq 0$ and $\Psi \geq 0$. In this case if $\Xi < 0$, which is equivalent to (2.3) by Schur complement and basic algebraic manipulations, then $\dot{V}(t) < 0$. Thus, (2.3) implies

$$
\begin{aligned}
\lambda_1\|\zeta_1(t)\|^2 - V(0) &\leq \zeta_1^\top(t)\bar{E}\bar{P}\zeta_1(t) - V(0) \\
&\leq V(t) - V(0) \leq \int_0^t \dot{V}(s)ds \\
&\leq -\lambda_2 \int_0^t \|\zeta(s)\|^2 ds \leq -\lambda_2 \int_0^t \|\zeta_1(s)\|^2 ds < 0
\end{aligned}
$$

where $\lambda_1 = \lambda_{min}(\bar{E}\bar{P}) > 0$ and $\lambda_2 = -\lambda_{max}(\Xi) > 0$. Taking this into account, we can deduce

$$\lambda_1 \|\zeta_1(t)\|^2 + \lambda_2 \int_0^t \|\zeta_1(t)\|^2 ds \leq V(0).$$

Therefore,

$$\|\zeta_1(t)\|^2 \leq \frac{1}{\lambda_1} V(0) \qquad \text{and} \qquad \int_0^t \|\zeta_1(s)\|^2 ds \leq \frac{1}{\lambda_2} V(0) \tag{2.18}$$

Thus $\|\zeta_1(t)\|$ is bounded. In order to prove the exponential stability of the fast subsystem, the relation in (2.13) should be used. For constant time delay, an explicit equation of $\zeta_2(t)$ is found by an iterative method (Xu et al., 2002). It can be seen that $\zeta_2(t)$ depends on $\zeta_2(t-\tau)$, where τ is the constant delay, and $\zeta_2(t-\tau)$ depends on $\zeta_2(t-2\tau)$, and so on. In the case of time-varying delay, such a direct relation cannot be found. Thus, some new variables will be defined in order to model the dependency of $\zeta_2(t)$ on past instances. Now, define

$$\begin{aligned} t_i &= t_{i-1} - d(t_{i-1}), \; i = 1, 2, \ldots \\ t_0 &= t \end{aligned}$$

It can be seen that the value of $\zeta(t)$ at $t = t_i$ depends on the value of $\zeta(t)$ at $t = t_{i-1}$. From (2.13), we get

$$\begin{aligned} \zeta_2(t) &= -A_{d21}\zeta_1(t - d(t)) - A_{d22}\zeta_2(t - d(t)) \\ &= -A_{d21}\zeta_1(t_1) - A_{d22}\zeta_2(t_1) \end{aligned} \tag{2.19}$$

Now, $\zeta_2(t_1)$ can be computed from (2.13) as follows

$$\begin{aligned} \zeta_2(t_1) &= \zeta_2(t - d(t)) \\ &= -A_{d21}\zeta_1(t - d(t) - d(t - d(t))) - A_{d22}\zeta_2(t - d(t) - d(t - d(t))) \\ &= -A_{d21}\zeta_1(t_1 - d(t_1)) - A_{d22}\zeta_2(t_1 - d(t_1)) \\ &= -A_{d21}\zeta_1(t_2) - A_{d22}\zeta_2(t_2) \end{aligned}$$

Substituting this in (2.19), we get

$$\begin{aligned}\zeta_2(t) &= -A_{d21}\zeta_1(t_1) - A_{d22}[-A_{d21}\zeta_1(t_2) - A_{d22}\zeta_2(t_2)] \\ &= -A_{d21}\zeta_1(t_1) - A_{d22}[-A_{d21}\zeta_1(t_2) - A_{d22}[-A_{d21}\zeta_1(t_3) - A_{d22}\zeta_2(t_3)]]\end{aligned}$$

and so on.

Note that $t_i < t_{i-1}$, therefore, there exists a positive finite integer $k(t)$ such that (see Figure 2.1)

$$\zeta_2(t) = (-A_{d22})^{k(t)}\zeta_2(t_{k(t)}) - \sum_{i=0}^{k(t)-1}(-A_{d22})^i A_{d21}\zeta_1(t_{i+1}) \tag{2.20}$$

and $t_{k(t)} \in [-\bar{d}, 0]$. Considering this and the fact that $\|\zeta_1(t)\|$ is bounded, then, $\|\zeta_2(t)\|$ is bounded too. Hence, it follows from (2.12) that $\|\dot{\zeta}(t)\|$ is bounded, therefore, $\frac{d}{dt}\|\zeta(t)\|^2$ is also bounded. By Lemma 2.1.1 we have that $\|\zeta(t)\|^2$ is uniformly continuous. Therefore, noting (2.18) and using Lemma 2.1.2, we obtain

$$\lim_{t\to\infty}\|\zeta_1(t)\| = 0. \tag{2.21}$$

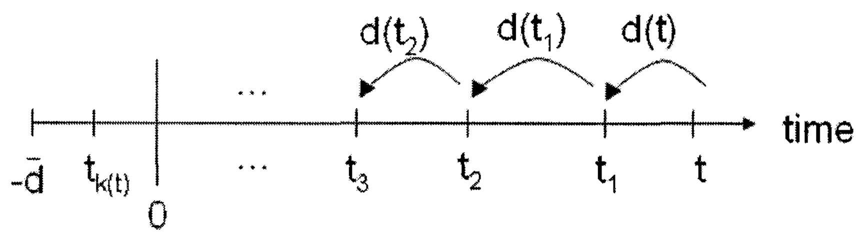

Figure 2.1 The relation between different t_i, $i = 1, 2, \ldots$.

This, together with (2.11) and (2.20), implies

$$\lim_{t\to\infty}\|\zeta_2(t)\| = 0.$$

Thus, the time-delay singular system is asymptotically stable. Finally, as that this system is also regular and impulse-free, by Definition (2.1.1), we then have that the system is admissible. This completes the proof. □

Remark 2.2.1. *Eq. (2.20) can be seen as a generalization of the iterative equation in (Xu et al., 2002) for systems with constant time delay. Also, based on (2.11), which is equivalent to (20) in (Xu et al., 2002), the stability of the fast subsystem has been shown for the case of time-varying delay. Thus, the results in (Xu et al., 2002), and in (Zhu et al., 2007) and (Feng et al., 2002) as well, can be extended easily to the case of time-varying delay.*

2.3 Delay-Range-Dependent Exponential Stability for Singular Systems with Time-Varying Delay

The conditions in Theorem (2.2.1) ensures only asymptotic stability. In this section, exponential stability conditions is derived. Moreover, an estimate of the convergence rate of the states of system (2.1) is presented. Also, the delay will be assumed to vary in a range and thereby more applicable in practice (He et al., 2007a). The time delay is assumed to satisfy:

$$\begin{cases} 0 < \underline{d} \leq d(t) \leq \overline{d} \\ \dot{d}(t) \leq \mu < 1 \end{cases} \tag{2.22}$$

with $\overline{d}$, $\underline{d}$ and μ are given scalars.

Before proceeding to the main results, the following preliminary is needed.

Lemma 2.3.1. *(Kharitonov et al., 2005) Given a matrix D, let a positive-definite matrix S and a positive scalar $\eta \in (0, 1)$ exist such that*

$$D^{\mathsf{T}} S D - \eta^2 S < 0$$

then, the matrix D satisfies the bound

$$\|D^i\| \leq \chi e^{-\lambda i} \text{ with } \chi = \sqrt{\frac{\lambda_{max}(S)}{\lambda_{min}(S)}} \text{ and } \lambda = -ln(\eta)$$

where i is a positive integer.

Now, we have the following result.

Theorem 2.3.1. *Let $0 < \underline{d} < \overline{d}$, $\mu < 1$ and $\alpha > 0$ be given scalars. System (2.1) is exponentially admissible with $\sigma = \alpha$ if there exist a nonsingular matrix P, symmetric and positive-definite matrices Q_1, Q_2, Q_3, Z_1 and Z_2, and matrices M_i, N_i and S_i, $i = 1, 2$ such that the following LMI holds:*

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & e^{\alpha\underline{d}}M_1E & -e^{\alpha\overline{d}}S_1E & \frac{e^{2\alpha\overline{d}}-1}{2\alpha}N_1 & cS_1 & cM_1 & \Pi_{18} \\ \star & \Pi_{22} & e^{\alpha\underline{d}}M_2E & -e^{\alpha\overline{d}}S_2E & \frac{e^{2\alpha\overline{d}}-1}{2\alpha}N_2 & cS_2 & cM_2 & A_d^\top U \\ \star & \star & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -Q_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\frac{e^{2\alpha\overline{d}}-1}{2\alpha}Z_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -c(Z_1+Z_2) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -cZ_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -U \end{bmatrix} < 0 \quad (2.23)$$

with the following constraint:

$$E^\top P = P^\top E \geq 0 \quad (2.24)$$

where

$$\Pi_{11} = P^\top A + A^\top P + \sum_{i=1}^{3} Q_i + N_1E + (N_1E)^\top + 2\alpha E^\top P$$

$$\Pi_{12} = P^\top A_d + (N_2E)^\top - N_1E + S_1E - M_1E$$

$$\Pi_{22} = -(1-\mu)e^{-2\alpha\overline{d}}Q_3 + S_2E + (S_2E)^\top - N_2E - (N_2E)^\top - M_2E - (M_2E)^\top$$

$$\underline{\overline{d}} = \overline{d} - \underline{d}, \quad U = \overline{d}Z_1 + \underline{\overline{d}}Z_2, \quad \Pi_{18} = A^\top U, \quad c = \frac{e^{2\alpha\overline{d}} - e^{2\alpha\underline{d}}}{2\alpha}.$$

Proof. The system can be shown to be regular and impulse-free. Therefore, there exist two matrices R and L such that (Lemma 1.2.1)

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{A} = RAL = \begin{bmatrix} A_1 & 0 \\ 0 & \mathbb{I}_{n-r} \end{bmatrix}. \tag{2.25}$$

Define $\bar{A}_d$, $\bar{N}_i$ in a similar manner as (2.7)-(2.8), $\bar{M}_i$, $\bar{S}_i$ similar to $\bar{N}_i$, $\bar{Z}_i = R^{-\top} Z_i R^{-1}$ and

$$\bar{P} = R^{-\top} PL = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \qquad \bar{Q}_i = L^{\top} Q_i L = \begin{bmatrix} Q_{i11} & Q_{i12} \\ Q_{i21} & Q_{i22} \end{bmatrix}. \tag{2.26}$$

Using (2.23) and Schur complement, we get

$$\begin{bmatrix} \Pi_{11} & \Pi_{21} \\ \star & \Pi_{22} \end{bmatrix} < 0$$

Substitute (2.25) into the previous inequality, pre- and post multiply by diag$\{L^{\top}, L^{\top}\}$, diag$\{L, L\}$ and using Schur complement, we have

$$\begin{bmatrix} P_{22}^{\top} + P_{22} + \sum_{j=1}^{3} Q_{i22} & P_{22}^{\top} A_{d22} \\ A_{d22}^{\top} P_{22} & -(1-\mu) e^{-2\alpha \bar{d}} Q_{322} \end{bmatrix} < 0$$

Pre- and post-multiplying by $\begin{bmatrix} -A_{d22}^{\top} & \mathbb{I} \end{bmatrix}$ and its transpose, and noting that $Q_i > 0$ and $\mu \geq 0$ (since if $\mu < 0$, the first condition in (2.22) will be violated), we get

$$A_{d22}^{\top} Q_{322} A_{d22} - e^{-2\alpha \bar{d}} Q_{322} < 0 \qquad \text{which implies} \qquad \rho(e^{\alpha \bar{d}} A_{d22}) < 1 \tag{2.27}$$

So there exist constants $\beta > 1$ and $\gamma \in (0,1)$ such that

$$\| e^{i\alpha d_2} A_{d22}^{i} \| \leq \beta \gamma^{i}, \ i = 1, 2, \cdots . \tag{2.28}$$

Let $\zeta(t) = L^{-1}x(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}$, where $\zeta_1(t) \in \mathbb{R}^r$ and $\zeta_2(t) \in \mathbb{R}^{n-r}$. Then, system (2.1) becomes equivalent to the following one

$$\dot{\zeta}_1(t) = A_1\zeta_1(t) + A_{d11}\zeta_1(t-d(t)) + A_{d12}\zeta_2(t-d(t)), \tag{2.29}$$

$$0 = \zeta_2(t) + A_{d21}\zeta_1(t-d(t)) + A_{d22}\zeta_2(t-d(t)). \tag{2.30}$$

Now, choose the Lyapunov functional as follows:

$$\begin{aligned} V(t) = {} & \zeta^\top(t)\bar{E}^\top\bar{P}\zeta(t) + \int_{t-\underline{d}}^{t}\zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_1\zeta(s)ds + \int_{t-\bar{d}}^{t}\zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_2\zeta(s)ds \\ & + \int_{t-d(t)}^{t}\zeta(s)^\top e^{2\alpha(s-t)}\bar{Q}_3\zeta(s)ds + \int_{-\bar{d}}^{0}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_1\bar{E}\dot{\zeta}(s)dsd\theta \\ & + \int_{-\bar{d}}^{-\underline{d}}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_2\bar{E}\dot{\zeta}(s)dsd\theta \end{aligned} \tag{2.31}$$

Then, the time-derivative of $V(t)$ along the solution of (2.29) and (2.30) is given by

$$\begin{aligned} \dot{V}(t) = {} & 2\zeta^\top(t)\bar{P}^\top\bar{E}\dot{\zeta}(t) + \zeta^\top(t)\bar{Q}_1\zeta(t) - \zeta^\top(t-\underline{d})e^{-2\alpha\underline{d}}\bar{Q}_1\zeta(t-\underline{d}) \\ & + \zeta^\top(t)\bar{Q}_2\zeta(t) - \zeta^\top(t-\bar{d})e^{-2\alpha\bar{d}}\bar{Q}_2\zeta(t-\bar{d}) \\ & + \zeta^\top(t)\bar{Q}_3\zeta(t) - (1-\dot{d}(t))\zeta^\top(t-d(t))e^{-2\alpha d(t)}\bar{Q}_3\zeta(t-d(t)) \\ & + \bar{d}(\bar{E}\dot{\zeta}(t))^\top\bar{Z}_1\bar{E}\dot{\zeta}(t) - \int_{t-\bar{d}}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_1\bar{E}\dot{\zeta}(s)ds \\ & + (\bar{d}-\underline{d})(\bar{E}\dot{\zeta}(t))^\top\bar{Z}_2\bar{E}\dot{\zeta}(t) - \int_{t-\bar{d}}^{t-\underline{d}}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_2\bar{E}\dot{\zeta}(s)ds \\ & - 2\alpha\int_{t-\underline{d}}^{t}\zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_1\zeta(s)ds - 2\alpha\int_{t-\bar{d}}^{t}\zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_2\zeta(s)ds \\ & - 2\alpha\int_{t-d(t)}^{t}\zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_3\zeta(s)ds \\ & - 2\alpha\int_{-\bar{d}}^{0}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_1\bar{E}\dot{\zeta}(s)dsd\theta \\ & - 2\alpha\int_{-\bar{d}}^{-\underline{d}}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_2\bar{E}\dot{\zeta}(s)dsd\theta \end{aligned} \tag{2.32}$$

Adding now these terms to (2.32)

$$2\left[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\right] \times \left[\bar{E}\zeta(t) - \bar{E}\zeta(t-d(t)) - \int_{t-d(t)}^{t} \bar{E}\dot{\zeta}(s)ds\right] = 0$$

$$2\left[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\right] \times \left[\bar{E}\zeta(t-d(t)) - \bar{E}\zeta(t-\bar{d}) - \int_{t-\bar{d}}^{t-d(t)} \bar{E}\dot{\zeta}(s)ds\right] = 0$$

$$2\left[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\right] \times \left[\bar{E}\zeta(t-\underline{d}) - \bar{E}\zeta(t-d(t)) - \int_{t-d(t)}^{t-\underline{d}} \bar{E}\dot{\zeta}(s)ds\right] = 0$$

gives

$$\begin{aligned}
\dot{V}(\zeta_t) \leq & \sum_{i=1}^{9} \Psi_i + (\bar{E}\dot{\zeta}(t))^\top \left[\bar{d}\bar{Z}_1 + \underline{\bar{d}}_1\bar{Z}_2\right](\bar{E}\dot{\zeta}(t)) \\
& - 2\left[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\right] \int_{t-d(t)}^{t} \bar{E}\dot{\zeta}(s)ds \\
& - \int_{t-d(t)}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_1\bar{E}\dot{\zeta}(s)ds \\
& - 2\left[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\right] \int_{t-\bar{d}}^{t-d(t)} \bar{E}\dot{\zeta}(s)ds \\
& - \int_{t-\bar{d}}^{t-d(t)} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\left(\bar{Z}_1 + \bar{Z}_2\right)\bar{E}\dot{\zeta}(s)ds \\
& - 2\left[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\right] \int_{t-d(t)}^{t-\underline{d}} \bar{E}\dot{\zeta}(s)ds \\
& - \int_{t-d(t)}^{t-\underline{d}} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_2\bar{E}\dot{\zeta}(s)ds \\
& - 2\alpha\int_{t-\underline{d}}^{t} \zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_1\zeta(s)ds - 2\alpha\int_{t-\bar{d}}^{t} \zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_2\zeta(s)ds \\
& - 2\alpha\int_{t-d(t)}^{t} \zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_3\zeta(s)ds \\
& - 2\alpha\int_{-\bar{d}}^{0}\int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_1\bar{E}\dot{\zeta}(s)dsd\theta \\
& - 2\alpha\int_{-\bar{d}}^{-\underline{d}}\int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_2\bar{E}\dot{\zeta}(s)dsd\theta
\end{aligned}$$

where

$$\Psi_1 = \zeta^\top(t)\left[\bar{P}^\top\bar{A} + \bar{A}^\top\bar{P} + \sum_{i=1}^{3}\bar{Q}_i + \bar{N}_1\bar{E} + (\bar{N}_1\bar{E})^\top\right]\zeta(t)$$

$$\Psi_2 = 2\zeta^\top(t)\left[\bar{P}^\top \bar{A}_d + (\bar{N}_2\bar{E})^\top - \bar{N}_1\bar{E} + \bar{S}_1\bar{E} - \bar{M}_1\bar{E}\right]\zeta(t-d(t))$$

$$\Psi_3 = \zeta^\top(t-d(t))\big[-(1-\mu)e^{-2\alpha\bar{d}}\bar{Q}_3 + \bar{S}_2\bar{E} + (\bar{S}_2\bar{E})^\top - \bar{N}_2\bar{E} - (\bar{N}_2\bar{E})^\top - \bar{M}_2\bar{E} - (\bar{M}_2\bar{E})^\top\big]\zeta(t-d(t))$$

$$\Psi_4 = 2\zeta^\top(t)\bar{M}_1\bar{E}\zeta(t-\underline{d}) \qquad \Psi_5 = -2\zeta^\top(t)\bar{S}_1\bar{E}\zeta(t-\bar{d})$$

$$\Psi_6 = 2\zeta^\top(t-d(t))\bar{M}_2\bar{E}\zeta(t-\underline{d}) \qquad \Psi_7 = -2\zeta^\top(t-d(t))\bar{S}_2\bar{E}\zeta(t-\bar{d})$$

$$\Psi_8 = -\zeta^\top(t-\underline{d})e^{-2\alpha\underline{d}}\bar{Q}_1\zeta(t-\underline{d}) \quad \Psi_9 = -\zeta^\top(t-\bar{d})e^{-2\alpha\bar{d}}\bar{Q}_2\zeta(t-\bar{d})$$

Noting that $\bar{Z}_1 > 0$ and $\bar{Z}_2 > 0$, adding and subtracting these terms:

$$+\int_{t-\bar{d}}^{t}\left[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\right]\bar{Z}_1^{-1}e^{-2\alpha(s-t)}\left[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\right]^\top ds$$
$$-\int_{t-d(t)}^{t}\left[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\right]\bar{Z}_1^{-1}e^{-2\alpha(s-t)}\left[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\right]^\top ds$$
$$+\int_{t-\bar{d}}^{t-\underline{d}}\left[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\right]\left(\bar{Z}_1 + \bar{Z}_2\right)^{-1}\times e^{-2\alpha(s-t)}\left[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\right]^\top ds$$
$$-\int_{t-\bar{d}}^{t-d(t)}\left[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\right]\left(\bar{Z}_1 + \bar{Z}_2\right)^{-1}\times e^{-2\alpha(s-t)}\left[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\right]^\top ds$$
$$+\int_{t-\bar{d}}^{t-\underline{d}}\left[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\right]\bar{Z}_2^{-1}e^{-2\alpha(s-t)}\left[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\right]^\top ds$$
$$-\int_{t-d(t)}^{t-\underline{d}}\left[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\right]\bar{Z}_2^{-1}e^{-2\alpha(s-t)}\left[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\right]^\top ds$$

gives

$$\dot{V}(\zeta_t) + 2\alpha V(\zeta_t) \leq$$
$$\eta^\top(t)[\Pi + \widetilde{A}^\top\left(\bar{d}\bar{Z}_1 + \bar{d}_1\bar{Z}_2\right)\widetilde{A} + \frac{e^{2\alpha\bar{d}}-1}{2\alpha}\widetilde{N}\bar{Z}_1^{-1}\widetilde{N}^\top$$
$$+\frac{e^{2\alpha\bar{d}} - e^{2\alpha\underline{d}}}{2\alpha}\widetilde{S}\left(\bar{Z}_1 + \bar{Z}_2\right)^{-1}\widetilde{S}^\top + \frac{e^{2\alpha\bar{d}} - e^{2\alpha\underline{d}}}{2\alpha}\widetilde{M}\bar{Z}_2^{-1}\widetilde{M}^\top]\eta(t)$$
$$-\int_{t-d(t)}^{t}\left[\eta^\top(t)\widetilde{N} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_1\right]e^{-2\alpha(s-t)}\bar{Z}_1^{-1}\left[\eta^\top(t)\widetilde{N} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_1\right]^\top ds$$

$$
\begin{aligned}
&-\int_{t-\bar{d}}^{t-d(t)} \left[\eta^\top(t)\widetilde{S} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\left(\bar{Z}_1+\bar{Z}_2\right)\right] e^{-2\alpha(s-t)}\left(\bar{Z}_1+\bar{Z}_2\right)^{-1} \times \\
&\qquad\qquad\qquad \left[\eta^\top(t)\widetilde{S} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\left(\bar{Z}_1+\bar{Z}_2\right)\right]^\top ds \\
&-\int_{t-d(t)}^{t-\underline{d}} \left[\eta^\top(t)\widetilde{M} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_2\right] e^{-2\alpha(s-t)}\bar{Z}_2^{-1}\left[\eta^\top(t)\widetilde{M} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_2\right]^\top ds \\
&\le \eta^\top(t)[\Pi + \widetilde{A}^\top\left(\bar{d}\bar{Z}_1 + \underline{d}_1\bar{Z}_2\right)\widetilde{A} + \frac{e^{2\alpha\bar{d}}-1}{2\alpha}\widetilde{N}\bar{Z}_1{}^{-1}\widetilde{N}^\top \\
&\quad + \frac{e^{2\alpha\bar{d}}-e^{2\alpha\underline{d}}}{2\alpha}\widetilde{S}\left(\bar{Z}_1+\bar{Z}_2\right)^{-1}\widetilde{S}^\top + \frac{e^{2\alpha\bar{d}}-e^{2\alpha\underline{d}}}{2\alpha}\widetilde{M}\bar{Z}_2{}^{-1}\widetilde{M}^\top]\eta(t)
\end{aligned}
$$

where

$$
\eta(t) = \begin{bmatrix} \zeta(t) \\ \zeta(t-d(t)) \\ \zeta(t-\underline{d}) \\ \zeta(t-\bar{d}) \end{bmatrix} \quad
\Pi = \begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} & \bar{M}_1\bar{E} & -\bar{S}_1\bar{E} \\ \star & \bar{\Pi}_{22} & \bar{M}_2\bar{E} & -\bar{S}_2\bar{E} \\ \star & \star & -e^{-2\alpha\underline{d}}\bar{Q}_1 & 0 \\ \star & \star & 0 & -e^{-2\alpha\bar{d}}\bar{Q}_2 \end{bmatrix}
$$

$$
\widetilde{N} = \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ 0 \\ 0 \end{bmatrix} \quad
\widetilde{M} = \begin{bmatrix} \bar{M}_1 \\ \bar{M}_2 \\ 0 \\ 0 \end{bmatrix} \quad
\widetilde{S} = \begin{bmatrix} \bar{S}_1 \\ \bar{S}_2 \\ 0 \\ 0 \end{bmatrix} \quad
\widetilde{A} = \begin{bmatrix} \bar{A}^\top \\ \bar{A}_d{}^\top \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{aligned}
\bar{\Pi}_{11} &= \bar{P}^\top\bar{A} + \bar{A}^\top\bar{P} + \sum_{i=1}^{3}\bar{Q}_i + \bar{N}_1\bar{E} + (\bar{N}_1\bar{E})^\top + 2\alpha\bar{E}^\top\bar{P} \\
\bar{\Pi}_{12} &= \bar{P}^\top\bar{A}_d + (\bar{N}_2\bar{E})^\top - \bar{N}_1\bar{E} + \bar{S}_1\bar{E} - \bar{M}_1\bar{E} \\
\bar{\Pi}_{22} &= -(1-\mu)e^{-2\alpha\bar{d}}\bar{Q}_3 + \bar{S}_2\bar{E} + (\bar{S}_2\bar{E})^\top - \bar{N}_2\bar{E} - (\bar{N}_2\bar{E})^\top - \bar{M}_2\bar{E} - (\bar{M}_2\bar{E})^\top
\end{aligned}
$$

Pre- and post-multiply (2.23) by diag$\left\{L^\top, L^\top, e^{-\alpha\underline{d}}L^\top, e^{-\alpha\bar{d}}L^\top, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}\right\}$ and its transpose, respectively, gives

$$\begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} & \bar{M}_1\bar{E} & -\bar{S}_1\bar{E} & \frac{e^{2\alpha\overline{d}}-1}{2\alpha}L^\top N_1 & cL^\top S_1 & cL^\top M_1 & \bar{\Pi}_{18} \\ \star & \bar{\Pi}_{22} & \bar{M}_2\bar{E} & -\bar{S}_2\bar{E} & \frac{e^{2\alpha\overline{d}}-1}{2\alpha}L^\top N_2 & cL^\top S_2 & cL^\top M_2 & L^\top A_d^\top U \\ \star & \star & -e^{-2\alpha\underline{d}}\bar{Q}_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -e^{-\alpha\overline{d}}\bar{Q}_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\frac{e^{2\alpha\overline{d}}-1}{2\alpha}Z_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -c(Z_1+Z_2) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -cZ_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\bar{U} \end{bmatrix} < 0$$

where

$$\begin{aligned} \bar{\Pi}_{11} &= \bar{P}^\top \bar{A} + \bar{A}^\top \bar{P} + \sum_{i=1}^{3} \bar{Q}_i + \bar{N}_1\bar{E} + (\bar{N}_1\bar{E})^\top + 2\alpha\bar{E}^\top\bar{P} \\ \bar{\Pi}_{12} &= \bar{P}^\top \bar{A}_d + (\bar{N}_2\bar{E})^\top - \bar{N}_1\bar{E} + \bar{S}_1\bar{E} - \bar{M}_1\bar{E} \\ \bar{\Pi}_{22} &= -(1-\mu)e^{-2\alpha\overline{d}}\bar{Q}_3 + \bar{S}_2\bar{E} + (\bar{S}_2\bar{E})^\top - \bar{N}_2\bar{E} - (\bar{N}_2\bar{E})^\top - \bar{M}_2\bar{E} - (\bar{M}_2\bar{E})^\top \\ \underline{\overline{d}} &= \overline{d} - \underline{d}, \quad U = \overline{d}Z_1 + \underline{\overline{d}}Z_2, \quad \bar{\Pi}_{18} = L^\top A^\top U, \quad c = \frac{e^{2\alpha\overline{d}} - e^{2\alpha\underline{d}}}{2\alpha}. \end{aligned}$$

Then, using Schur complement implies

$$\begin{aligned} \Pi + \widetilde{A}^\top \left(\overline{d}\bar{Z}_1 + \overline{\underline{d}}_1\bar{Z}_2\right)\widetilde{A} + \frac{e^{2\alpha\overline{d}}-1}{2\alpha}\widetilde{N}\bar{Z}_1{}^{-1}\widetilde{N}^\top \\ + \frac{e^{2\alpha\overline{d}} - e^{2\alpha\underline{d}}}{2\alpha}\widetilde{S}\left(\bar{Z}_1+\bar{Z}_2\right)^{-1}\widetilde{S}^\top + \frac{e^{2\alpha\overline{d}} - e^{2\alpha\underline{d}}}{2\alpha}\widetilde{M}\bar{Z}_2{}^{-1}\widetilde{M}^\top < 0 \end{aligned}$$

which implies in turn

$$\dot{V}(\zeta_t) + 2\alpha V(\zeta_t) \leq 0 \qquad \text{which leads to} \qquad V(\zeta_t) \leq e^{-2\alpha t}V(\phi(t))$$

Then, the following estimation is obtained

$$\lambda_1\|\zeta_1(t)\|^2 \leq V(\zeta_t) \leq e^{-2\alpha t}V(\phi(t)) \leq \lambda_2 e^{-2\alpha t}\|\phi\|_c^2$$

where $\lambda_1 = \lambda_{min}(\bar{P}_{11}) > 0$ and $\lambda_2 > 0$ is sufficiently large and can be found since $V(\phi(t))$ is a bounded quadratic functional of $\phi(t)$. This leads to

$$\|\zeta_1(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|_c e^{-\alpha t} \tag{2.33}$$

Now, define

$$\begin{aligned} t_i &= t_{i-1} - d(t_{i-1}),\ i = 1, 2, \ldots \\ t_0 &= t \end{aligned}$$

Then, similar to what we did in the proof of Theorem 2.2.1, there exists a positive integer $k(t)$ such that

$$\zeta_2(t) = (-A_{d22})^{k(t)}\zeta_2(t_{k(t)}) - \sum_{i=0}^{k(t)-1}(-A_{d22})^i A_{d21}\zeta_1(t_{i+1})$$

and $t_{k(t)} \in [-\bar{d}, 0]$. Therefore, from (2.28), (2.27), (2.33), Lemma 2.3.1 and noting that

$$k(t)\bar{d} \geq t, \qquad t_i = t - \sum_{j=0}^{i-1} d(t_j) \geq t - i\bar{d}$$

we get,

$$\begin{aligned} \|\zeta_2(t)\| &\leq \|A_{d22}{}^{k(t)}\|\, \|\phi\|_c + \|A_{d21}\| \sum_{i=0}^{k(t)-1} \left\|A_{d22}{}^i\right\| \|\zeta_1(t_{(i+1)})\| \\ &\leq \chi e^{-\alpha\bar{d}k(t)}\, \|\phi\|_c + \|A_{d21}\| \sqrt{\frac{\lambda_2}{\lambda_1}}\, \|\phi\|_c \sum_{i=0}^{k(t)-1} \left\|A_{d22}{}^i\right\| e^{-\alpha(t-(i+1)\bar{d})} \\ &\leq \left[\chi\, \|\phi\|_c + \|A_{d21}\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha\bar{d}}\, \|\phi\|_c \sum_{i=0}^{k(t)-1} \|A_{d22}\|^i e^{i\alpha\bar{d}}\right] e^{-\alpha t} \\ &\leq \left[\chi + \|A_{d21}\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha\bar{d}} M\right] \|\phi\|_c\, e^{-\alpha t} \end{aligned}$$

where

$$M = \frac{\beta}{1-\gamma}, \qquad \chi = \sqrt{\frac{\lambda_{max}(Q_{322})}{\lambda_{min}(Q_{322})}}$$

Thus, the time-delay singular system is exponentially stable with a minimum decaying rate $= \alpha$. Finally, as this system is also regular and impulse-free, by Definition 2.1.1, we then have that the system is exponentially admissible. This completes the proof. $\square$

Remark 2.3.1. *(Xu and Lam, 2006) It is noted that the condition in (2.24) is non-strict LMI, which contains equality constraints; this may result in numerical problems when checking such non-strict LMI conditions since equality constraints are fragile and usually not satisfied perfectly. Therefore, strict LMI conditions are more desirable than non-strict ones from the numerical point of view. Considering this, Eqs. (2.23) and (2.24) can be combined into a single strict LMI. Let* $P > 0$ *and* $S \in R^{n\times(n-r)}$ *be any matrix with full column rank and satisfies* $E^\top S = 0$. *Changing* P *to* $PE + SQ$ *in (2.23) yields the strict LMI. This remark applies to Theorem 2.4.1 as well.*

Remark 2.3.2. *Taking the limits of the elements of (2.23) as* $\alpha \to 0$, *Theorem 2.3.1 yields an admissibility conditions for time-delay singular systems. Moreover, when* $E = \mathbb{I}$, *the singular delay system in (2.1) reduces to a state-space delay system and the result of Theorem 2.2.1 as* $\alpha \to 0$ *coincides exactly with the result in (He et al., 2007a).*

Remark 2.3.3. *If* $\underline{d}_1 = 0$, $Q_1 = \epsilon_1 I$, $Q_2 = \epsilon_2 I$, $Z_2 = \epsilon_3 I$, *with* $\epsilon_i > 0$, $i = 1, 2, 3$, *being sufficiently small scalars,* $M_1 = M_2 = S_1 = S_2 = 0$, *the result of Theorem 2.3.1 as* $\alpha \to 0^+$ *is equivalent to Theorem 1 in (Zhu et al., 2007). Therefore, Theorem 1 in (Zhu et al., 2007) can be applied to the case of time-varying delays.*

Remark 2.3.4. *If* $\alpha < 0$, *the proof of Theorem 2.3.1 will be the same except the fact that* $-e^{-2\alpha d(t)}$ *will be bounded by* $-e^{-2\alpha d_1}$ *instead of* $-e^{-2\alpha d_2}$. *Therefore, if* Π_{22} *is rewritten as* $\Pi_{22} = -(1-\mu)e^{-2\alpha d(\alpha)}Q_3 + S_2E + (S_2E)^\top - N_2E - (N_2E)^\top - M_2E -$

$(M_2E)^\top$, *where*

$$d(\alpha) = \begin{cases} d_1 & \textit{if } \alpha < 0 \\ d_2 & \textit{if } \alpha > 0 \end{cases}$$

then Theorem 2.3.1 will be generalized to include negative decaying rates, i.e. $\|x(t)\| \leq \gamma e^{-\alpha t}\|\phi\|_c$, $\alpha < 0$. *This fact will be discussed again in Remark 4.2.6.*

Example: Consider the time-delay singular system studied in (Zhu et al., 2007) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.5 & 0 \\ -1 & -1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & 0 \\ 0 & \beta \end{bmatrix}.$$

Let $\beta = 0$, we know from (Zhu et al., 2007) that this system is asymptotically stable for constant delay $\tau < \tau^*$ and unstable for constant delay $\tau > \tau^*$, where $\tau^* = 1.2092$. Now, allowing time-varying delay, the exponential stability of this system will be investigated using Theorem 2.3.1. For various $\overline{d}$, the maximum allowable decay rates α, which guarantee the exponential stability for given lower bound $\underline{d}$ and derivative bound μ, are listed in Table 2.1. As it is clear from the table, if we increase $\overline{d}$, then we obtain smaller decay rates α. Figure 2.2 gives the solution behavior of x_1 and x_2 as compared to $e^{-0.3t}$ when $d(t) = |0.05\cos(3t) + 0.4|$ and the initial condition is $\phi(t) = [1+0.1\sin(40t),\ -(1+0.1\sin(20t))]^\top, t \in [-0.405, 0]$. From Figure 2.2, we can see that the states x_1 and x_2 exponentially converge to zero with a decay rate more than 0.3.

Now, let $\beta = 0.5$ (i.e. the delay appear also in the algebraic constraint). For $\underline{d} = 0.2$, $\overline{d} = 0.5$ and $\mu = 0.5$, the maximum allowable decay rate is $\alpha = 0.32$. □

Table 2.1 Maximum allowable decay rates α for different $\overline{d}$ with $\underline{d} = 0.2$ and $\mu = 0.5$

$\overline{d}$	0.5	0.6	0.7	0.8	0.9	1	1.1
α	0.3239	0.3014	0.2816	0.2642	0.2411	0.1323	0.0290

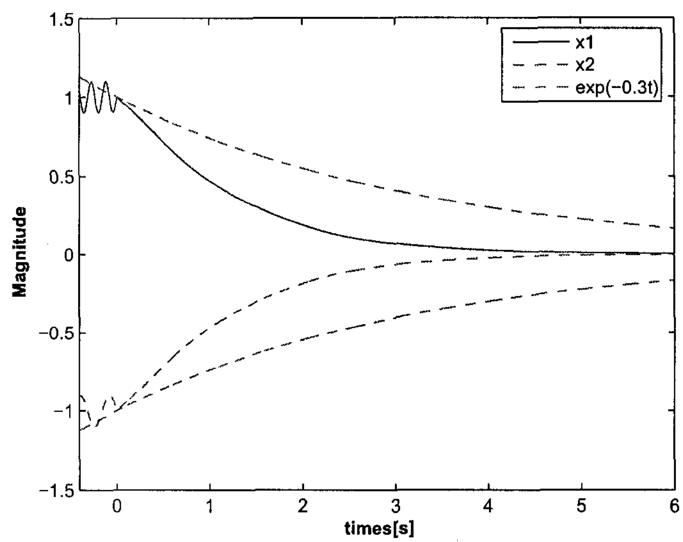

Figure 2.2 Solution behavior of x_1 and x_2 as compared to $e^{-0.3t}$

2.4 Delay-Range-Dependent Exponential Stability for Singular Systems with Multiple Time-Varying Delays

As a natural extension of the results of the previous section, it is of theoretical and practical importance to consider the case with multiple time delays. Systems with many feedback paths are most commonly to encounter multiple delays with different lower and upper bounds. The delays are assumed to be time-varying and an estimate of the convergence rate of the states of the system (2.1) is presented. It is shown also that this rate depends on the minimum bounds of the delays.

Consider the linear time-delay singular system:

$$\begin{cases} E\dot{x}(t) = Ax(t) + \sum_{k=1}^{p} A_k x(t - d_k(t)) \\ x(t) = \phi(t), -\overline{d} \leq t \leq 0 \end{cases} \tag{2.34}$$

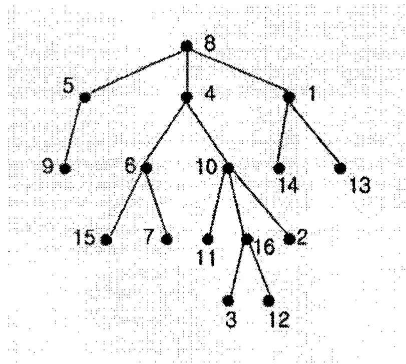

Figure 2.3 An example of a tree

where the time delays $d_k(t)$, $k = 1, \ldots, p$, are assumed to satisfy:

$$\begin{cases} 0 < \underline{d}_k \leq d_k(t) \leq \overline{d}_k \\ \dot{d}_k(t) \leq \mu < 1 \end{cases} \tag{2.35}$$

with $\underline{d}_k$, $\overline{d}_k$ and μ are given scalars. Also, $\overline{d}$ and $\underline{d}$ are positive scalars with $\overline{d} = \max\{\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_p\}$ and $\underline{d} = \min\{\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_p\}$.

In the rest of this section, the following terminology borrowed from graph theory will be used (Cormen et al., 2001).

- A tree structure is a way of representing the hierarchical nature of a structure in a graphical form (see Figure (2.3)).
- The topmost node in a tree is called the root node.
- A node is a parent of another node (child) if it is one step higher in the hierarchy and closer to the root node.
- Nodes at the bottommost level of the tree are called leaf nodes.

Now, we have the following results.

Theorem 2.4.1. *Let $\underline{d}_k$ and $\overline{d}_k$, with $0 < \underline{d}_k < \overline{d}_k$, $k = 1, \ldots, p$, $\mu < 1$ and $\alpha > 0$ be given scalars. System (2.34) is exponentially admissible with $\sigma = \alpha$ if there exist a nonsingular matrix P, symmetric and positive-definite matrices Q_{k1}, Q_{k2}, Q_{k3}, Z_{k1} and Z_{k2}, $k = 1, \ldots, p$, and matrices M_{ki}, N_{ki} and S_{ki}, $i = 1, 2$, $k = 1, \ldots, p$, such that the following LMI hold*

$$\begin{bmatrix} \Pi & \Upsilon & \tilde{A}U \\ \star & T & 0 \\ \star & \star & -U \end{bmatrix} < 0 \tag{2.36}$$

with the following constraint

$$E^\top P = P^\top E \geq 0 \tag{2.37}$$

where

$$\begin{aligned}
T &= \operatorname{diag}\left\{ -\frac{2\alpha}{e^{2\alpha\overline{d}_k} - 1} Z_{k1}, -\frac{2\alpha}{e^{2\alpha\overline{d}_k} - e^{2\alpha\underline{d}_k}} (Z_{k1} + Z_{k2}), \right. \\
&\quad \left. -\frac{2\alpha}{e^{2\alpha\overline{d}_k} - e^{2\alpha\underline{d}_k}} Z_{k2} \right\}, \quad k = 1, \ldots, p \\
\widetilde{A}^\top &= \begin{bmatrix} A & \tilde{A}_1 & \ldots & \tilde{A}_p \end{bmatrix}, \ \textit{with } \tilde{A}_k = \begin{bmatrix} A_k & 0 & 0 \end{bmatrix} \\
U &= \sum_{k=1}^{p} \left(\overline{d}_k Z_{k1} + \overline{\underline{d}}_k Z_{k2} \right) \ \textit{with } \overline{\underline{d}}_k = \overline{d}_k - \underline{d}_k \\
\Pi &= \begin{bmatrix} \Pi_1 & F \\ \star & G \end{bmatrix}, \ \Upsilon = \begin{bmatrix} \widetilde{N}_1 & \widetilde{S}_1 & \widetilde{M}_1 & \ldots & \widetilde{N}_p & \widetilde{S}_p & \widetilde{M}_p \end{bmatrix} \\
\widetilde{N}_k^\top &= \begin{bmatrix} N_{k1}^\top & \mathbf{0}_{n\times 3n(k-1)} & N_{k2}^\top & 0 & 0 & \mathbf{0}_{n\times 3n(p-k)} \end{bmatrix}, \quad k = 1, \ldots, p \\
\widetilde{M}_k^\top &= \begin{bmatrix} M_{k1}^\top & \mathbf{0}_{n\times 3n(k-1)} & M_{k2}^\top & 0 & 0 & \mathbf{0}_{n\times 3n(p-k)} \end{bmatrix}, \quad k = 1, \ldots, p \\
\widetilde{S}_k^\top &= \begin{bmatrix} S_{k1}^\top & \mathbf{0}_{n\times 3n(k-1)} & S_{k2}^\top & 0 & 0 & \mathbf{0}_{n\times 3n(p-k)} \end{bmatrix}, \quad k = 1, \ldots, p \\
\Pi_1 &= P^\top A + A^\top P + \sum_{k=1}^{p} \left\{ \sum_{i=1}^{3} Q_{ki} + N_{k1}E + (N_{k1}E)^\top \right\} + 2\alpha E^\top P \\
F &= \begin{bmatrix} \Pi_{12} & e^{\alpha\underline{d}_1} M_{11}E & -e^{\alpha\overline{d}_1} S_{11}E & \ldots & \Pi_{p2} & e^{\alpha\underline{d}_p} M_{p1}E & -e^{\alpha\overline{d}_p} S_{p1}E \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
G &= \operatorname{diag}\{J_1, \ldots, J_p\} \\
J_k &= \begin{bmatrix} \Pi_{k3} & e^{\alpha \underline{d}_k} M_{k2} E & -e^{\alpha \bar{d}_k} S_{k2} E \\ \star & -Q_{k1} & \mathbf{0} \\ \star & \star & -Q_{k2} \end{bmatrix}, \quad k = 1, \ldots, p \\
\Pi_{k2} &= P^\top A_k + (N_{k2} E)^\top - N_{k1} E + S_{k1} E - M_{k1} E, \quad k = 1, \ldots, p \\
\Pi_{k3} &= -(1-\mu) e^{-2\alpha \bar{d}_k} Q_{k3} + S_{k2} E + (S_{k2} E)^\top - N_{k2} E \\
&\quad - (N_{k2} E)^\top - M_{k2} E - (M_{k2} E)^\top, \quad k = 1, \ldots, p.
\end{aligned}
$$

Proof. The system can be shown to be regular and impulse-free. Therefore, there exist two matrices R and L such that (Lemma 1.2.1)

$$
\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{A} = RAL = \begin{bmatrix} \widehat{A} & 0 \\ 0 & \mathbb{I}_{n-r} \end{bmatrix}. \tag{2.38}
$$

Now, define $\bar{P}$ similar to (2.26), $\bar{Z}_{ki} = R^{-\top} Z_{ki} R^{-1}$ and

$$
\bar{A}_{kd} = RA_k L = \begin{bmatrix} A_{kd11} & A_{kd12} \\ A_{kd21} & A_{kd22} \end{bmatrix}, \qquad \bar{Q}_{ki} = L^\top Q_{ki} L = \begin{bmatrix} Q_{ki11} & Q_{ki12} \\ Q_{ki21} & Q_{ki22} \end{bmatrix}.
$$

Using (2.36) and Schur complement, we get

$$
\begin{bmatrix} \Pi_1 & \Xi \\ \star & \Omega \end{bmatrix} < 0 \text{ where } \quad \begin{aligned} \Xi &= [\Pi_{12}, \ldots \Pi_{p2}] \\ \Omega &= \operatorname{diag}\{\Pi_{13}, \ldots, \Pi_{p3}\} \end{aligned}
$$

Substitute (2.38) into this inequality, pre- and post-multiply by $\operatorname{diag}\{L^\top, \ldots, L^\top\}$, $\operatorname{diag}\{L, \ldots, L\}$ and using Schur complement, we have

$$
\begin{bmatrix} P_{22}^\top + P_{22} + \sum_{k=1}^{p} \sum_{i=1}^{3} Q_{ki22} & X \\ \star & H \end{bmatrix} < 0 \tag{2.39}
$$

where $X = \begin{bmatrix} P_{22}^\top A_{1d22} & \ldots & P_{22}^\top A_{pd22} \end{bmatrix}$

$$H = \operatorname{diag}\left\{-(1-\mu)e^{-2\alpha\bar{d}_i}Q_{i322}\right\},\ i = 1,\dots,p$$

Pre- and post-multiplying (2.39) by $\operatorname{diag}\left\{\mathbb{I},\ e^{\alpha\bar{d}_1}\mathbb{I},\ \cdots,\ e^{\alpha\bar{d}_p}\mathbb{I}\right\}$ and its transpose, respectively, and noting that $\mu \geq 0$ (since if $\mu < 0$, the first condition in (2.35) will be violated), Lemma 2 in (Fridman, 2002) implies

$$\rho(\sum_{k=1}^{p} e^{\alpha\bar{d}_k}A_{kd22}) < 1 \tag{2.40}$$

From (2.40), there exist constants $\beta > 1$ and $\gamma \in (0,1)$ such that

$$\|\sum_{k=1}^{p} e^{\alpha\bar{d}_k}A_{kd22}\| \leq \beta\gamma^i,\ i = 1,2,\cdots. \tag{2.41}$$

Let $\zeta(t) = L^{-1}x(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}$, where $\zeta_1(t) \in \mathbb{R}^r$ and $\zeta_2(t) \in \mathbb{R}^{n-r}$. System (2.34) is equivalent to the following one

$$\dot{\zeta}_1(t) = \widehat{A}\zeta_1(t) + \sum_{k=1}^{p}\{A_{kd11}\zeta_1(t-d_k(t)) + A_{kd12}\zeta_2(t-d_k(t))\} \tag{2.42}$$

$$0 = \zeta_2(t) + \sum_{k=1}^{p}\{A_{kd21}\zeta_1(t-d_k(t)) + A_{kd22}\zeta_2(t-d_k(t))\}. \tag{2.43}$$

Now, choose the Lyapunov functional as follows:

$$\begin{aligned} V(t) = {} & \zeta^\top(t)\bar{E}^\top\bar{P}\zeta(t) + \sum_{k=1}^{p}\left\{\int_{t-\underline{d}_k}^{t}\zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_{k1}\zeta(s)ds\right. \\ & + \int_{t-\bar{d}_k}^{t}\zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_{k2}\zeta(s)ds \\ & + \int_{t-d_k(t)}^{t}\zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_{k3}\zeta(s)ds \\ & + \int_{-\bar{d}_k}^{0}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_{k1}\bar{E}\dot{\zeta}(s)dsd\theta \\ & \left. + \int_{-\bar{d}_k}^{-\underline{d}_k}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_{k2}\bar{E}\dot{\zeta}(s)dsd\theta\right\} \end{aligned} \tag{2.44}$$

Then, the following estimation can be obtained

$$\|\zeta_1(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|_c e^{-\alpha t} \tag{2.45}$$

where λ_1 and λ_2 are positive integers. In order to prove the exponential stability of the fast subsystem, the relation in (2.43) should be used. In the case of multiple delays, a relation similar to (2.20) cannot be found. Thus, a tree structure will be adopted to model the dependency of $\zeta_2(t)$ on past instances. Now, define

$$t_{00} = t \tag{2.46}$$

$$t_{ij} = t_{(i-1)\nu_j} - d_{(\text{mod}(j,p)+1)}(t_{(i-1)\nu_j}) \tag{2.47}$$

$$\Theta = \{t_{ij} \mid t_{ij} \in (-\overline{d}, 0] \ \text{ and } \ t_{(i-1)\nu_j} \notin (-\overline{d}, 0]\} \tag{2.48}$$

$$\hat{A}_{00} = \mathbb{I} \tag{2.49}$$

$$\hat{A}_{ij} = \left(\hat{A}_{(i-1)\nu_j}\right) \times \left(-A_{(\text{mod}(j,p)+1)d22}\right) \tag{2.50}$$

where

$$\begin{aligned} \nu_j = & \text{ the greatest integer less than or equal to } \frac{j}{p}, \\ \text{mod}(j,p) = & \text{ the remainder of the integer division } \frac{j}{p}, \text{ and} \end{aligned}$$

t_{ij} and $\hat{A}_{ij}$ are undefined if $t_{(i-1)\nu_j} \in (-\overline{d}, 0]$.

If we let the parents of t_{ij} and $\hat{A}_{ij}$ to be $t_{(i-1)\nu_j}$ and $\hat{A}_{(i-1)\nu_j}$, respectively, t_{ij}'s and $\hat{A}_{ij}$'s will represent two trees with the same structure (see Figure 2.4), with roots t and $\mathbb{I}$, respectively.

Remark 2.4.1. *Figure 2.4 represents an example with $p = 2$. Take note that $\zeta_2(t_{ij})$ depends on the value of ζ at all times indicated by the children of t_{ij} in the tree. Note also that the values of the leaf nodes of the t_{ij}'s tree belongs to Θ.*

Noting that $\text{mod}(j,p) = j$ if $j < p$, then from (2.43), and using the definitions

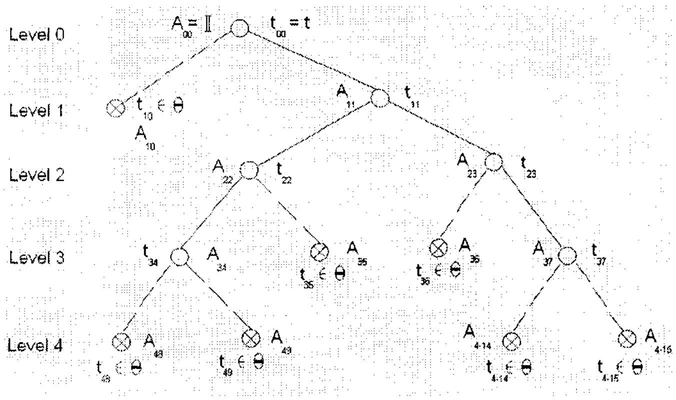

Figure 2.4 An example with $p = 2$.

(2.46)-(2.50), we get

$$
\begin{aligned}
\zeta_2(t) &= -\sum_{k=1}^{p} \{A_{kd21}\zeta_1(t-d_k(t)) + A_{kd22}\zeta_2(t-d_k(t))\} \\
\zeta_2(t) &= \sum_{j=0}^{p-1} \left\{-A_{(j+1)d21}\zeta_1(t_{1j}) + \hat{A}_{1j}\zeta_2(t_{1j})\right\} \\
&= \sum_{t_{1j}\in\Theta} \left\{\hat{A}_{1j}\zeta_2(t_{1j})\right\} + \sum_{j=0}^{p-1} \{-A_{(j+1)d21}\zeta_1(t_{1j})\} + \sum_{\substack{j=0 \\ t_{1j}\notin\Theta}}^{p-1} \left\{\hat{A}_{1j}\zeta_2(t_{1j})\right\} \qquad (2.51)
\end{aligned}
$$

if $t_{1j} \notin \Theta$, from (2.43) and (2.46)-(2.50), we get

$$
\begin{aligned}
\hat{A}_{1j}\zeta_2(t_{1j}) &= \hat{A}_{1j} \sum_{r=jp}^{(j+1)p-1} \{-A_{(\mathrm{mod}(r,p)+1)d21}\zeta_1(t_{2r}) - A_{(\mathrm{mod}(r,p)+1)d22}\zeta_2(t_{2r})\} \\
&= \sum_{r=jp}^{(j+1)p-1} \left\{-\hat{A}_{1j}A_{(\mathrm{mod}(r,p)+1)d21}\zeta_1(t_{2r})\hat{A}_{1j}A_{(\mathrm{mod}(r,p)+1)d22}\zeta_2(t_{2r})\right\} \\
&= \sum_{r=jp}^{(j+1)p-1} \left\{-\hat{A}_{1j}A_{(\mathrm{mod}(r,p)+1)d21}\zeta_1(t_{2r}) + \hat{A}_{2r}\zeta_2(t_{2r})\right\}
\end{aligned}
$$

thus, $\zeta_2(t)$ in (2.51) can be computed from

$$
\zeta_2(t) = \sum_{t_{1j}\in\Theta} \left\{\hat{A}_{1j}\zeta_2(t_{1j})\right\} + \sum_{j=0}^{p-1} \{-A_{(j+1)d21}\zeta_1(t_{1j})\}
$$

$$+\sum_{\substack{j=0\\t_{1j}\notin\Theta}}^{p-1}\sum_{r=jp}^{(j+1)p-1}\left\{-\hat{A}_{1j}A_{(\mathrm{mod}(r,p)+1)d21}\zeta_1(t_{2r})+\hat{A}_{2r}\zeta_2(t_{2r})\right\}$$

$$=\sum_{t_{1j}\in\Theta}\left\{\hat{A}_{1j}\zeta_2(t_{1j})\right\}+\sum_{j=0}^{p-1}\left\{-A_{(j+1)d21}\zeta_1(t_{1j})\right\}$$

$$+\sum_{\substack{j=0\\t_{1\nu_j}\notin\Theta}}^{p^2-1}\left\{-\hat{A}_{1j}A_{(\mathrm{mod}(j,p)+1)d21}\zeta_1(t_{2j})+\hat{A}_{2j}\zeta_2(t_{2j})\right\}$$

$$=\sum_{t_{1j}\in\Theta}\left\{\hat{A}_{1j}\zeta_2(t_{1j})\right\}-\sum_{i=0}^{1}\sum_{\substack{j=0\\t_{i\nu_j}\notin\Theta}}^{p^{i+1}-1}\left\{\hat{A}_{ij}A_{(\mathrm{mod}(j,p)+1)d21}\zeta_1(t_{(i+1)j})\right\}$$

$$+\sum_{\substack{j=0\\t_{1\nu_j}\notin\Theta}}^{p^2-1}\left\{\hat{A}_{2j}\zeta_2(t_{2j})\right\}$$

$$=\sum_{i=1}^{2}\sum_{t_{ij}\in\Theta}\left\{\hat{A}_{ij}\zeta_2(t_{ij})\right\}-\sum_{i=0}^{1}\sum_{\substack{j=0\\t_{i\nu_j}\notin\Theta}}^{p^{i+1}-1}\left\{\hat{A}_{ij}A_{(\mathrm{mod}(j,p)+1)d21}\zeta_1(t_{(i+1)j})\right\}$$

$$+\sum_{\substack{j=0\\t_{2j}\notin\Theta}}^{p^2-1}\left\{\hat{A}_{2j}\zeta_2(t_{2j})\right\}$$

Continuing in the same manner, if $t_{2j}\notin\Theta$,

$$\begin{aligned}\hat{A}_{2j}\zeta_2(t_{2j})&=\sum_{r=jp}^{(j+1)p-1}\left\{-\hat{A}_{2j}A_{(\mathrm{mod}(r,p)+1)d21}\zeta_1(t_{3r})-\hat{A}_{2j}A_{(\mathrm{mod}(r,p)+1)d22}\zeta_2(t_{3r})\right\}\\&=\sum_{r=jp}^{(j+1)p-1}\left\{-\hat{A}_{2j}A_{(\mathrm{mod}(r,p)+1)d21}\zeta_1(t_{3r})+\hat{A}_{3r}\zeta_2(t_{3r})\right\}\end{aligned}$$

we get,

$$\zeta_2(t)=\sum_{i=1}^{3}\sum_{t_{ij}\in\Theta}\left\{\hat{A}_{ij}\zeta_2(t_{ij})\right\}-\sum_{i=0}^{2}\sum_{\substack{j=0\\t_{i\nu_j}\notin\Theta}}^{p^{i+1}-1}\left\{\hat{A}_{ij}A_{(\mathrm{mod}(j,p)+1)d21}\zeta_1(t_{(i+1)j})\right\}$$

$$+\sum_{\substack{j=0\\t_{3j}\notin\Theta}}^{p^3-1}\left\{\hat{A}_{3j}\zeta_2(t_{3j})\right\}$$

Note that

$$\begin{aligned} t_{ij} &= t_{(i-1)\nu_j} - d_{(\mathrm{mod}(j,p)+1)}(t_{(i-1)\nu_j}) \\ &\leq t_{(i-1)\nu_j} - \underline{d}_{(\mathrm{mod}(j,p)+1)} < t_{(i-1)\nu_j} \end{aligned}$$

which means that the time of a child is always less than the time of its parent. Therefore, there exists a positive finite integer $k(t)$ such that

$$\zeta_2(t) = \sum_{i=1}^{k(t)} \sum_{t_{ij}\in\Theta} \left\{ \hat{A}_{ij}\zeta_2(t_{ij}) \right\} - \sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j}\notin\Theta}}^{p^{i+1}-1} \left\{ \hat{A}_{ij} A_{(\mathrm{mod}(j,p)+1)d21}\zeta_1(t_{(i+1)j}) \right\}$$

and $t_{ij} \in [-\overline{d}, 0]$. Thus, we get

$$\begin{aligned} \|\zeta_2(t)\| \leq & \sum_{i=1}^{k(t)} \sum_{t_{ij}\in\Theta} \left\{ \|\hat{A}_{ij}\| \right\} \|\phi\|_c \\ & + \sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j}\notin\Theta}}^{p^{i+1}-1} \left\{ \|\hat{A}_{ij}\| \|A_{(\mathrm{mod}(j,p)+1)d21}\| \|\zeta_1(t_{(i+1)j})\| \right\} \end{aligned} \tag{2.52}$$

Now, in order to estimate $\|\zeta_2(t)\|$, the two terms in (2.52) have to be estimated. For the first term, from (2.49)-(2.50), $\hat{A}_{ij}$ can be written as

$$\hat{A}_{ij} = \left(\hat{A}_{(i-1)\nu_j}\right) \times \left(-A_{(\mathrm{mod}(j,p)+1)d22}\right) = \left(\hat{A}_{(i-1)\nu_j}\right) \times \left(-A_{k_1 d22}\right)$$

Iterating on $\hat{A}_{(i-1)\nu_j}$ gives after $(i-1)$ iterations $\hat{A}_{ij} = A_{k_i d22} \dots A_{k_1 d22}$, where $k_1, \dots, k_i$ are integers between 1 and p. Then, we have

$$\hat{A}_{ij} = A_{k_i d22} e^{\alpha \overline{d}_{k_i}} \dots A_{k_1 d22} e^{\alpha \overline{d}_{k_1}} e^{-\alpha \hat{d}_{ij}} \tag{2.53}$$

with $\hat{d}_{ij} = \sum_{e=1}^{i} \overline{d}_{k_e}$. Note also that since $t_{ij} \in \Theta$, $\hat{d}_{ij}$ is greater than or equal to t.

Therefore, using (2.53), the first term in (2.52) can be bounded by

$$\sum_{i=1}^{k(t)} \sum_{t_{ij}\in\Theta} \left\{ \|\hat{A}_{ij}\| \right\} \|\phi\|_c \leq \sum_{i=1}^{k(t)} \sum_{t_{ij}\in\Theta} \left\{ \|\hat{A}_{ij} e^{\alpha \hat{d}_{ij}}\| \right\} \|\phi\|_c e^{-\alpha t} \tag{2.54}$$

Now, the summation $\sum_{t_{ij}\in\Theta}$ sums over the leaves in level i (see Figure 2.4). This summation is bounded by the summation over all the nodes in level i, which has the worst case sum when all the nodes exist in the level (i.e. p^i nodes). Therefore, $\sum_{t_{ij}\in\Theta} \left\{ \|\hat{A}_{ij} e^{\alpha \hat{d}_{ij}}\| \right\}$ can be bounded by

$$\sum_{j=0}^{p^i-1} \left\{ \|\hat{A}_{ij} e^{\alpha \hat{d}_{ij}}\| \right\} = \sum_{j=0}^{p-1} \|\hat{A}_{ij} e^{\alpha \hat{d}_{ij}}\| + \cdots + \sum_{j=p^{i-1}}^{p^i-1} \|\hat{A}_{ij} e^{\alpha \hat{d}_{ij}}\|$$

Note that $\hat{A}_{ij}$'s in each summation have the same parent. The parent of $\hat{A}_{ij}$'s in the first summation is $\hat{A}_{(i-1)\nu_0}$, in the second summation is $\hat{A}_{(i-1)\nu_1}$ and so on. Therefore, using (2.46)-(2.50), $\sum_{j=0}^{p^i-1} \left\{ \|\hat{A}_{ij} e^{\alpha \hat{d}_{ij}}\| \right\}$ equals to

$$\begin{aligned}
&\sum_{j=0}^{p-1} \|\hat{A}_{(i-1)\nu_0} e^{\alpha \hat{d}_{(i-1)\nu_0}} A_{(\mathrm{mod}(j,p)+1)d22} e^{\alpha \bar{d}_{(\mathrm{mod}(j,p)+1)}}\| + \cdots \\
&\quad + \sum_{j=p^{i-1}}^{p^i-1} \|\hat{A}_{(i-1)\nu_{(p^{i-1}-1)}} e^{\alpha \hat{d}_{(i-1)\nu_{(p^{i-1}-1)}}} A_{(\mathrm{mod}(j,p)+1)d22} e^{\alpha \bar{d}_{(\mathrm{mod}(j,p)+1)}}\| \\
&\leq \|\hat{A}_{(i-1)\nu_0} e^{\alpha \hat{d}_{(i-1)\nu_0}}\| \sum_{j=0}^{p-1} A_{(\mathrm{mod}(j,p)+1)d22} e^{\alpha \bar{d}_{(\mathrm{mod}(j,p)+1)}} + \cdots \\
&\quad + \|\hat{A}_{(i-1)\nu_{(p^{i-1}-1)}} e^{\alpha \hat{d}_{(i-1)\nu_{(p^{i-1}-1)}}}\| \sum_{j=p^{i-1}}^{p^i-1} A_{(\mathrm{mod}(j,p)+1)d22} e^{\alpha \bar{d}_{(\mathrm{mod}(j,p)+1)}}
\end{aligned} \tag{2.55}$$

From the definition of $\mathrm{mod}(j,p)$, all the summations in (2.55) are equal to $\sum_{k=1}^{p} A_{kd22} e^{\alpha \bar{d}_k}$. Therefore,

$$\sum_{j=0}^{p^i-1} \|\hat{A}_{ij} e^{\alpha \hat{d}_{ij}}\| \leq \sum_{j=0}^{p^{i-1}-1} \|\hat{A}_{(i-1)j} e^{\alpha \hat{d}_{(i-1)j}}\| \sum_{k=1}^{p} A_{kd22} e^{\alpha \bar{d}_k}$$

Note that $\sum_{j=0}^{p^{i-1}-1} \|\hat{A}_{(i-1)j} e^{\alpha \hat{d}_{(i-1)j}}\|$ is the summation over all the nodes in the previous level (i.e. level $i-1$). Following the same procedure with level $(i-1)$ and iterating until level 0 gives

$$\sum_{j=0}^{p^i-1} \|\hat{A}_{ij} e^{\alpha \hat{d}_{ij}}\| \leq \left[\sum_{k=1}^{p} A_{kd22} e^{\alpha \overline{d}_k} \right]^i \tag{2.56}$$

Therefore, using (2.41) and (2.54), the first term in (2.52) can be bounded by

$$\|\phi\|_c \sum_{i=1}^{k(t)} \left[\sum_{k=1}^{p} \left\{ A_{kd22} e^{\alpha \overline{d}_k} \right\} \right]^i e^{-\alpha t} \leq \frac{\beta}{1-\gamma} \|\phi\|_c e^{-\alpha t}. \tag{2.57}$$

Now, in order to estimate the second term in (2.52), define

$$\|\check{A}_1\| = \max\{\|A_{1d21}\|, \ldots, \|A_{pd21}\|\}$$

Then, from (2.46)-(2.50), we get

$$\begin{aligned} \|\hat{A}_{ij}\| e^{-\alpha(t_{(i+1)j})} &\leq \|\hat{A}_{(i-1)\nu_j} A_{(\mathrm{mod}(j,p)+1)d22}\| e^{-\alpha(t_{i\nu_j})} e^{\alpha d_{(\mathrm{mod}(j,p)+1)}(t_{i\nu_j})} \\ &\leq \|\hat{A}_{(i-1)\nu_j} e^{-\alpha(t_{i\nu_j})} A_{(\mathrm{mod}(j,p)+1)d22} e^{\alpha \overline{d}_{(\mathrm{mod}(j,p)+1)}}\| \\ &\leq \|\hat{A}_{(i-1)\nu_j} e^{-\alpha(t_{i\nu_j})} A_{k_1 d22} e^{\alpha \overline{d}_{k_1}}\| \end{aligned} \tag{2.58}$$

Iterating on $\hat{A}_{(i-1)\nu_j}$ and $t_{i\nu_j}$ gives after $(i-1)$ iterations,

$$\begin{aligned} \|\hat{A}_{ij}\| e^{-\alpha(t_{(i+1)j})} &\leq \|\mathbb{I} e^{-\alpha t_{1k_{i+1}}} A_{k_i d22} e^{\alpha \overline{d}_{k_i}} \ldots A_{k_1 d22} e^{\alpha \overline{d}_{k_1}}\| \\ &\leq \|A_{k_i d22} e^{\alpha \overline{d}_{k_i}} \ldots A_{k_1 d22} e^{\alpha \overline{d}_{k_1}}\| e^{-\alpha t} e^{\alpha \overline{d}} \\ &\leq \|\hat{A}_{ij} e^{e^{\alpha \hat{d}_{ij}}}\| e^{-\alpha t} e^{\alpha \overline{d}} \end{aligned} \tag{2.59}$$

Now, using (2.45) and noting that $\|A_{(\mathrm{mod}(j,p)+1)d21}\| \leq \|\check{A}_1\|$ for any integer j, the second term in (2.52) can be estimated as

$$\sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \|\hat{A}_{ij}\| \|A_{(\mathrm{mod}(j,p)+1)d21}\| \|\zeta_1(t_{(i+1)j})\|$$
$$\leq \|\check{A}_1\| \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\|_c \sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \|\hat{A}_{ij}\| e^{-\alpha(t_{(i+1)j})}$$

Using (2.59), this term is bounded by

$$\|\check{A}_1\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha\bar{d}} \sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \left\{ \|\hat{A}_{ij} e^{e^{\alpha\hat{d}_{ij}}}\| \right\} \|\phi\|_c e^{-\alpha t} \tag{2.60}$$

Note that for any i, $\sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \|\check{A}_e\|^i = m\|\check{A}_e\|^i$, where m equals to the number of nodes in level $i+1$ (see Figure 2.4). It can be seen easily that the worst case is when all the nodes exist in the level (i.e. p^{i+1} nodes), and we get

$$\sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \left\{ \|\hat{A}_{ij} e^{e^{\alpha\hat{d}_{ij}}}\| \right\} \leq \sum_{i=0}^{k(t)-1} \sum_{j=0}^{p^{i+1}-1} \left\{ \|\hat{A}_{ij} e^{e^{\alpha\hat{d}_{ij}}}\| \right\}$$

Using now (2.56) and (2.41)

$$\sum_{i=0}^{k(t)-1} \sum_{j=0}^{p^{i+1}-1} \left\{ \|\hat{A}_{ij} e^{e^{\alpha\hat{d}_{ij}}}\| \right\} \leq \frac{\beta}{1-\gamma} \tag{2.61}$$

Therefore, using (2.60) and (2.61), the second term in (2.52) can be estimated as

$$\sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \|\hat{A}_{ij}\| \|A_{(\mathrm{mod}(j,p)+1)d21}\| \|\zeta_1(t_{(i+1)j})\|$$
$$\leq \|\check{A}_1\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha\bar{d}} \frac{\beta}{1-\gamma} \|\phi\|_c e^{-\alpha t} \tag{2.62}$$

Now, from (2.57) and (2.62), $\|\zeta_2(t)\|$ in (2.52) is estimated by

$$\|\zeta_2(t)\| \leq \left[\frac{\beta}{1-\gamma} + \|\breve{A}_1\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha\bar{d}} \frac{\beta}{1-\gamma}\right] \|\phi\|_c e^{-\alpha t}$$

Thus, system in (2.42) and (2.43) is exponentially stable with a minimum decaying rate equals to α. Finally, as this system is also regular and impulse-free, by Definition (2.1.1), we conclude that the system (2.34) is exponentially admissible. This completes the proof. $\square$

Example: Consider the following singular system with multiple time delays:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 2 & 0 \\ 0 & -5 & 1 \\ 1 & 0 & 2 \end{bmatrix},$$

$$A_{1d} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.1 \\ -0.1 & 0 & -0.1 \end{bmatrix}, \qquad A_{2d} = \begin{bmatrix} 0.05 & 0.1 & 0 \\ 0 & -0.4 & 0.05 \\ 0.1 & 0.1 & -0.1 \end{bmatrix}.$$

Let $\underline{d}_1 = 0.5$, $\bar{d}_1 = 0.6$, $\underline{d}_2 = 0.6$, $\bar{d}_2 = 0.8$ and $\mu = 0.5$. Using Theorem 2.4.1, the guaranteed convergence rate of this system is given by $\sigma = 2.4$. Figure 2.5 gives the solution behavior of x_1, x_2 and x_3 when $d_1(t) = 0.55 + 0.04\sin(5t)$, $d_2(t) = 0.7 + 0.05\sin(5t)$ and the initial condition is $\phi(t) = [\ 2 \quad -1 \quad -1.0556\ ]^\top, t \in [-0.7, 0]$. As it is expected, $\|x(t)\|$ is bounded by $\gamma e^{-2.4t}$. $\square$

Example: Consider the following time-delay singular system:

$$E = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}, \qquad A = \begin{bmatrix} -4.7 & 0.4 \\ -4.9 & 0.8 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.7 & -0.95 \\ 1.1 & -1.75 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & -0.8 \\ 1.4 & -1.3 \end{bmatrix}$$

Simulation results show that this systems is unstable for large delays. Now, let

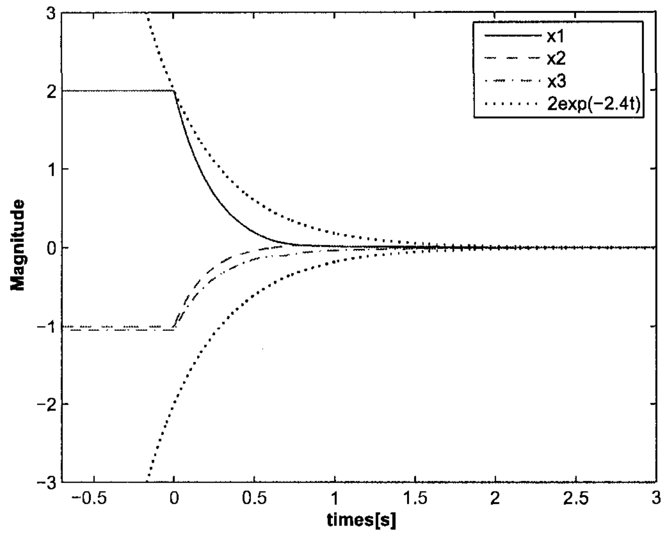

Figure 2.5 Solution behavior of x_1, x_2 and x_3

$\underline{d}_1 = 0.1$, $\overline{d}_1 = 0.5$, $\underline{d}_2 = 0.2$, and $\mu = 0.3$. For various α, the maximum allowable $\overline{d}_2$, for which the system is exponentially stable are listed in Table 2.2. Note that as α increases, the maximum allowable $\overline{d}_2$ decreases. Figure 2.6 gives the simulation results of x_1 and x_2 when $d_1(t) = |0.3 + 0.1\sin(2t)|$, $d_2(t) = |2 + 0.25\sin(t)|$ and the initial function is $\phi(t) = [\ -0.059 \quad 0.47\ ]^\top, t \in [-2, 0]$. As it is expected from Table 2.2, $\|x(t)\|$ is bounded by $\gamma e^{-0.2t}$. □

Table 2.2 Maximum allowable $\overline{d}_2$ for different α

α	0.01	0.1	0.15	0.2	0.25	0.3	0.35
$\overline{d}_2$	5.73	3.30	2.79	2.43	1.84	1.01	0.32

2.5 Conclusion

This chapter dealt with the stability of the class of singular systems with single and multiple time-varying delays. Delay-dependent and delay-range-dependent conditions have been stated. Also, an estimate of the convergence rate of such stable systems has been presented. The results are expressed in terms of LMIs. The results have been illustrated with examples. It is worth mentioning that the stabil-

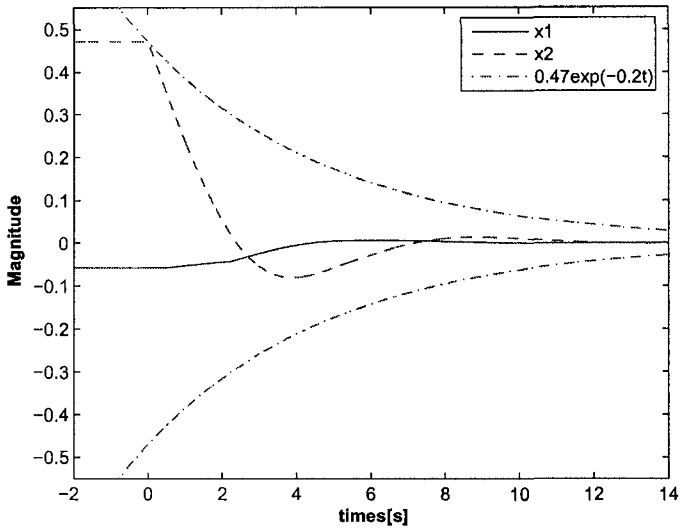

Figure 2.6 Solution behavior of x_1 and x_2.

ity results developed in this chapter will play important roles in dealing with the stabilization problem as can be seen in the following chapters.

CHAPTER 3

STATE FEEDBACK CONTROLLER FOR TIME-DELAY SINGULAR SYSTEMS

In this chapter, the state feedback stabilization problem for singular time-delay systems will be tackled. First, LMI conditions will be presented that help in designing a state feedback controller such that the closed-loop time-delay singular system is regular, impulse-free and stable. Then, these LMI conditions are adopted to find the controller gains such that the closed-loop time-delay singular system is admissible and also satisfies $\mathscr{H}_\infty$ performance level. The results are based on Theorem 2.2.1 of Chapter 2. We assume in this chapter that the whole state vector is accessible.

3.1 State Feedback Stabilization for Singular Systems with Time-Varying Delay

In this section, the state feedback stabilization problem for time-delay singular systems will be tackled. Consider the linear time-delay singular system:

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + Bu(t) \tag{3.1a}$$

$$x(t) = \phi(t), \quad t \in [-\overline{d}, 0] \tag{3.1b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, the matrix $E \in \mathbb{R}^{n \times n}$ may be singular, and we assume that $\text{rank}(E) = r \leqslant n$, A, A_d and B are known real constant matrices, $\phi(t) \in C_\tau^v$ is a compatible vector valued continuous function and $d(t)$ is a time-varying continuous function that satisfies:

$$0 < d(t) \leq \overline{d} \qquad \text{and} \qquad \dot{d}(t) \leq \mu < 1 \tag{3.2}$$

Consider the following state feedback controller:

$$u(t) = Kx(t), \;\; K \in R^{m \times n} \tag{3.3}$$

Applying this controller to system in (3.1), we obtain the closed-loop system as follows:

$$E\dot{x}(t) = (A + BK)x(t) + A_d x(t - d(t)) = A_{cl}x(t) + A_d x(t - d(t)) \tag{3.4}$$

Then, we have the following result.

Theorem 3.1.1. *Let $\bar{d} > 0$ and $\mu < 1$ be given scalars. System (3.1) is stabilizable if there exist matrices P, Y, $T > 0$, $Q > 0$, $Z > 0$, an invertible matrix G and free matrices N_i, $i = 1, 2, 3$ such that:*

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \bar{d}N_1 \\ \star & \Pi_{22} & \Pi_{23} & \bar{d}N_2 \\ \star & \star & \Pi_{33} & \bar{d}N_3 \\ \star & \star & \star & -\bar{d}Z \end{bmatrix} < 0 \tag{3.5}$$

$$E^\top P \;=\; P^\top E \geq 0 \tag{3.6}$$

$$TB \;=\; BG \tag{3.7}$$

$$E^\top T^\top \;=\; TE \geq 0 \tag{3.8}$$

where:

$\Pi_{11} = Q + TA + BY + A^\top T^\top + Y^\top B^\top + N_1 E + (N_1 E)^\top$

$\Pi_{12} = TA_d + (N_2 E)^\top - N_1 E$

$\Pi_{13} = P^\top - T + (N_3 E)^\top + A^\top T^\top + Y^\top B^\top$

$\Pi_{22} = -(1 - \mu)Q - N_2 E - (N_2 E)^\top$

$\Pi_{23} = -(N_3 E)^\top + A_d^\top T^\top$

$\Pi_{33} = \bar{d}Z - T - T^\top$

Moreover, a stabilizing state feedback control law is given by:

$$K = G^{-1}Y$$

Proof. Let T be a positive-definite matrix, then TA_{cl} can be written as

$$TA_{cl} = T(A + BK) = TA + TBK$$

Now, assume that there exists an invertible matrix G such that $TB = BG$. Thus,

$$TA + TBK = TA + BGK$$

Let $Y = GK$, gives

$$TA + BGK = TA + BY$$

Applying Theorem 2.2.1 to the closed-loop system in (3.4), and letting $T = T_1 = T_3$, the results of Theorem 3.1.1 follows immediately. □

Remark 3.1.1. *It is noted that some conservativeness is introduced due to the constraint in (3.7). Moreover, the matrix B should be of full column rank. However, if we impose the free matrices N_i, $i = 1, 2, 3$, to be equal to zero (which will also introduce some conservativeness), LMI conditions could be obtained without introducing the constraint in (3.7). Therefore, both approaches (introducing the constraints or letting the free matrices equal to zero) will give LMI conditions. Based on some numerical examples, the advantage of either of the approaches over the other turns out to depend on the example itself.*

Example: Consider the time-delay singular system described by:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.5 & 0 & 1 \\ 0 & -1 & -0.2 \\ 0.05 & 1 & 0.1 \end{bmatrix},$$

$$A_d = \begin{bmatrix} -1 & 0.1 & 0 \\ 0 & 0.5 & 0.1 \\ 0.1 & 0 & 0.2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}.$$

with $\overline{d} = 1$ and $\mu = 0.5$. The open-loop response of this system with $\phi(t) = [2,\ -1,\ 2.333]^\top$ and $d(t) = |0.05\sin(5t) + 0.8|$ is shown in Figure 3.1. As it can be seen, the open-loop system is unstable. In order to stabilize this system, solving the LMIs of Theorem 3.1.1, gives

$$K = \begin{bmatrix} -0.7907 & -0.2976 & -0.9838 \end{bmatrix},$$

Figure 3.2 and Figure 3.3 show the closed-loop response of the system. □

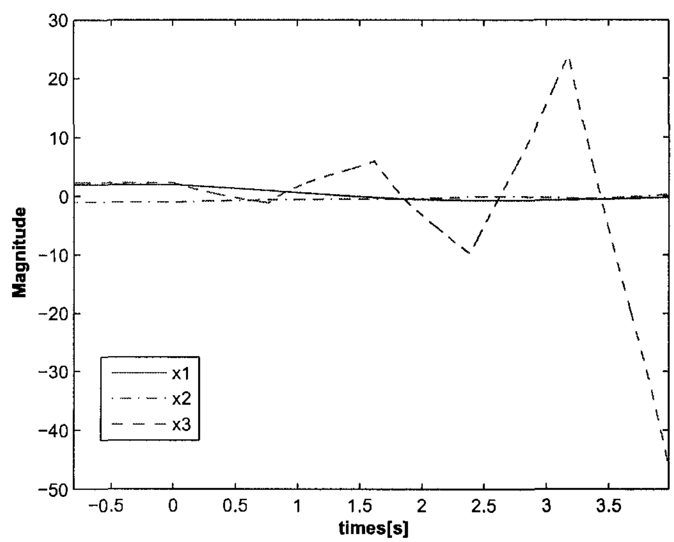

Figure 3.1 The open-loop solution behavior of x_1 and x_2 with $d(t) = |0.05\sin(5t) + 0.8|$ and $\phi(t) = [2,\ -1,\ 2.333]^\top$

Example: Consider the time-delay singular system described by:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix},$$

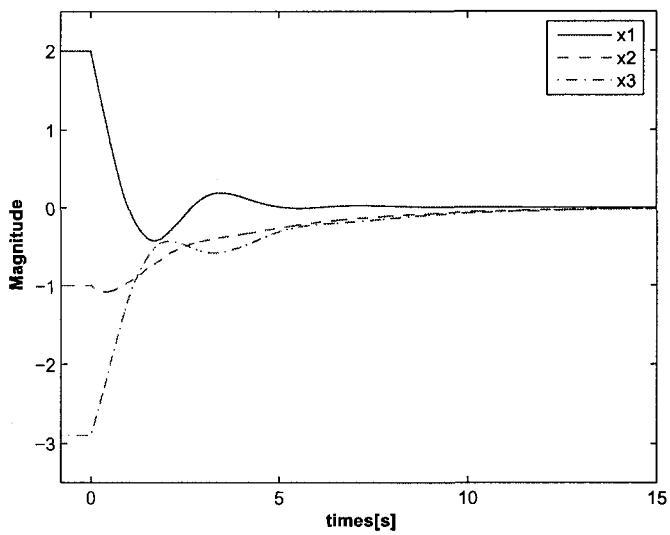

Figure 3.2 Behavior of x_1 and x_2 with $d(t) = |0.05\sin(5t) + 0.8|$ and $\phi(t) = [2,\ -1,\ -2.9011]^\top$ for the closed-loop dynamics.

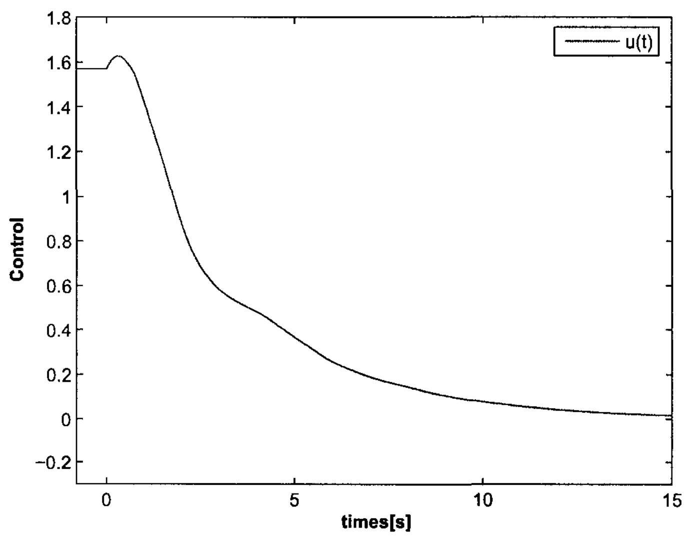

Figure 3.3 Behavior of the controller

$$A_d = \begin{bmatrix} -3 & 1 \\ 0 & 0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

with $\overline{d} = 1.2$ and $\mu = 0.55$. The open-loop response of this system with $\phi(t) = [1 + 0.1\sin(40t),\ 2]^\top$ and $d(t) = |0.1\cos(2t) + 1|$ (see Figure 3.5 for the variation of the delay) is shown in Figure 3.4. As it can be seen, the open-loop system is unstable. In order to stabilize this system, solving the LMIs of Theorem 3.1.1, gives

$$K = \begin{bmatrix} -17.2949 & 2.3204 \end{bmatrix},$$

Figure 3.6 and Figure 3.7 show the closed-loop response of the system. □

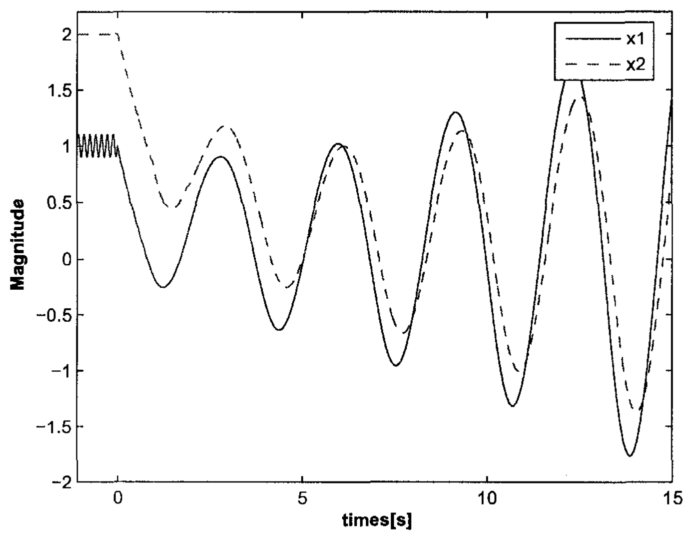

Figure 3.4 The open-loop solution behavior of x_1 and x_2 with $d(t) = |0.1\cos(2t) + 1|$ and $\phi(t) = [1 + 0.1\sin(40t),\ 2]^\top$

3.2 $\mathscr{H}_\infty$ Stabilization for Singular Systems with Time-Varying Delay

The problem of $\mathscr{H}_\infty$ control for standard state space systems, delay systems and singular systems has attracted a lot of researchers and a huge amount of results

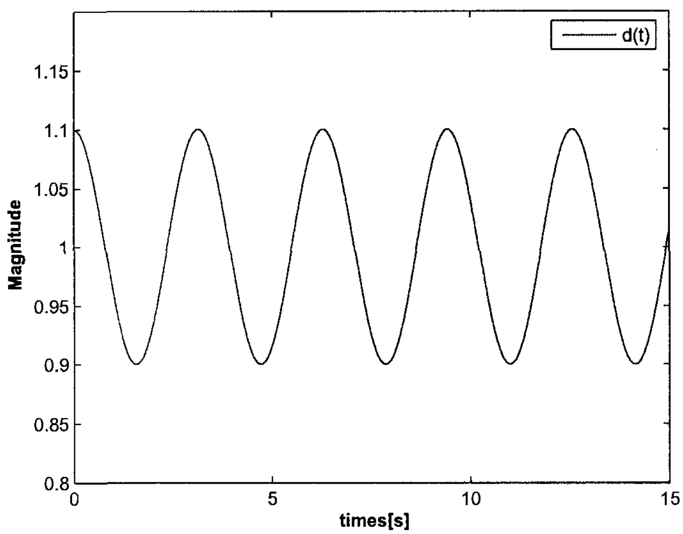

Figure 3.5 The time-varying delay $d(t) = |0.1\cos(2t) + 1|$

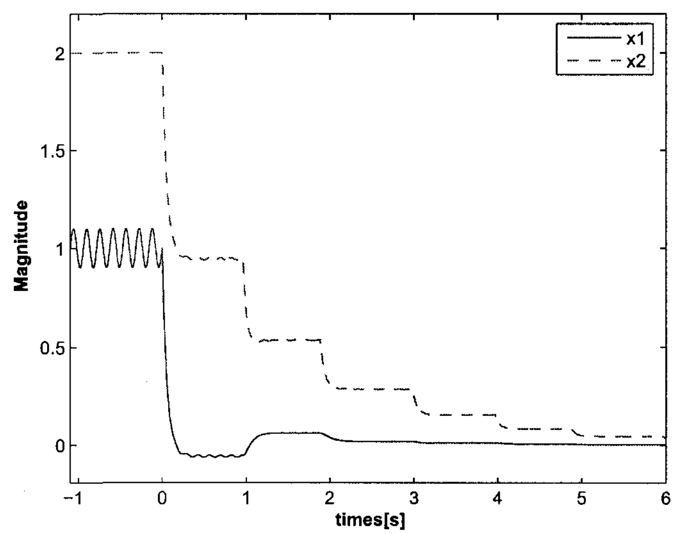

Figure 3.6 Behavior of x_1 and x_2 with $d(t) = |0.1\cos(2t) + 1|$ and $\phi(t) = [1 + 0.1\sin(40t),\ 2]^\top$ for the closed-loop dynamics.

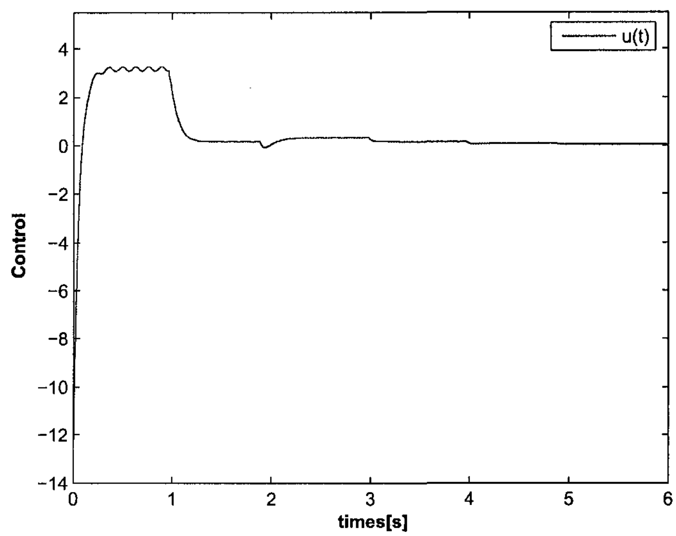

Figure 3.7 Behavior of the controller

can be found in the literature. The idea of $\mathscr{H}_\infty$ control is to design a stabilizing controller that not only stabilize the system but also reject external disturbances. Two methods are usually used to tackle the $\mathscr{H}_\infty$ control problem for time-delay singular systems: the transfer function method (frequency domain) and the functional method (time domain).

In the transfer function method, a stabilizing controller is designed such that the resulting closed-loop transfer function from the disturbance w to the controlled output z, say, $T_{zw}(s)$, is as small as possible, i.e., the effect of the disturbance on the controlled output is minimized. The transfer function $T_{zw}(s)$ is a function of frequency and it is hard to tell if it is big or small. Therefore, the $\mathscr{H}_\infty$-norm can be used as a measure of the size of the transfer function. However, for time-delay singular systems, no transfer function is defined for time-varying delays. Thus, the functional method (time domain) has to be used to tackle the $\mathscr{H}_\infty$ control problem.

Consider the linear time-delay singular system:

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + Bu(t) + B_{\omega 1}\omega(t) \tag{3.9a}$$

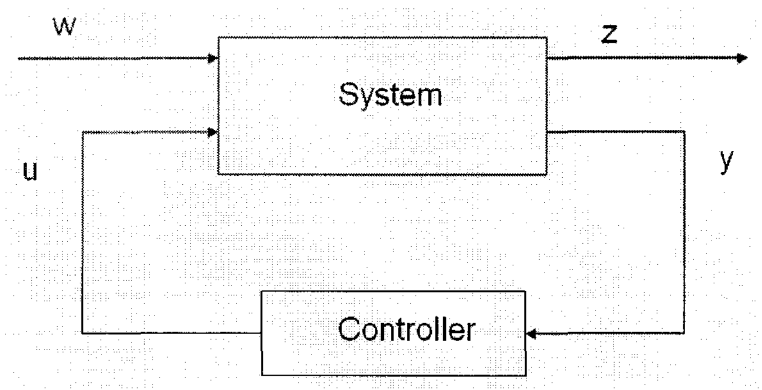

Figure 3.8 The standard $\mathscr{H}_\infty$ control problem

$$z(t) = Cx(t) + B_{\omega 2}\omega(t) \tag{3.9b}$$

$$x(t) = \phi(t), \quad t \in [-\overline{d}, 0] \tag{3.9c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $\omega(t) \in R^p$ is the disturbance input that belongs to $L_2[0, \infty)$, $z(t) \in R^s$ is the controlled output, the matrix $E \in \mathbb{R}^{n \times n}$ may be singular, and we assume that $\text{rank}(E) = r \leqslant n$, A, A_d and B are known real constant matrices, $\phi(t) \in C_\tau^v$ is a compatible vector valued continuous function and $d(t)$ is a time-varying continuous function that satisfies:

$$0 < d(t) \leq \overline{d} \qquad \text{and} \qquad \dot{d}(t) \leq \mu < 1 \tag{3.10}$$

The following definitions will be used in the sequel.

Definition 3.2.1. *The time-delay singular system (3.9) is said to be admissible, if the system with $u(t) = 0$ and $\omega(t) = 0$ is admissible in the sense of Definition 2.1.1.*

Definition 3.2.2. *For a given scalar $\gamma > 0$, system (3.9) with $u(t) = 0$ is said to be admissible with $\mathscr{H}_\infty$ performance γ if it is admissible in the sense of Definition 3.2.1 and under zero initial condition, $\|z(t)\|_2 < \gamma \|\omega(t)\|_2$ for any non-zero $\omega(t) \in L_2[0, \infty)$.*

Now, consider the following state feedback controller:

$$u(t) = Kx(t), \;\; K \in R^{m\times n} \tag{3.11}$$

The problem under consideration is to design a state feedback controller such that the closed-loop system is admissible and satisfies a prescribed $\mathscr{H}_\infty$ performance level. Then, we have the following results.

Theorem 3.2.1. *Let $\overline{d} > 0$ and $\mu < 1$ be given scalars. System (3.9) is stabilizable with $\mathscr{H}_\infty$ performance γ if there exist matrices $P,\ Y,\ T > 0,\ Q > 0,\ Z > 0,$ an invertible matrix G and free matrices $N_i, i = 1, 2, 3$ such that:*

$$\Omega = \begin{bmatrix} \Pi_{11} + C^\top C & \Pi_{12} & \Pi_{13} & d_2 N_1 & TB_{\omega 1} \\ \star & \Pi_{22} & \Pi_{23} & d_2 N_2 & 0 \\ \star & \star & \Pi_{33} & d_2 N_3 & TB_{\omega 1} \\ \star & \star & \star & -d_2 Z & 0 \\ \star & \star & \star & \star & -\gamma^2 \mathbb{I} + B_{\omega 2}^\top B_{\omega 2} \end{bmatrix} < 0 \tag{3.12}$$

$$E^\top P = P^\top E \geq 0 \tag{3.13}$$

$$TB = BG \tag{3.14}$$

$$E^\top T^\top = TE \geq 0 \tag{3.15}$$

where:

$\Pi_{11} = Q + TA + BY + A^\top T^\top + Y^\top B^\top + N_1 E + (N_1 E)^\top$

$\Pi_{12} = TA_d + (N_2 E)^\top - N_1 E$

$\Pi_{13} = P^\top - T + (N_3 E)^\top + A^\top T^\top + Y^\top B^\top$

$\Pi_{22} = -(1 - \mu)Q - N_2 E - (N_2 E)^\top$

$\Pi_{23} = -(N_3 E)^\top + TA_d$

$\Pi_{33} = d_2 Z - T - T^\top$

Moreover, a stabilizing state feedback control law is given by:

$$K = G^{-1}Y$$

Proof. Following the steps in the proof of Theorem 2.2.1, the system can be shown to be regular, impulse-free and stable. In order to show that the closed-loop system satisfies the $\mathscr{H}_\infty$ performance level γ, let us define the following performance function:

$$J_\infty = \int_0^\infty [z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t)]dt$$

According to $V(x_t)|_{t=0} = 0$ under zero initial condition, and following the steps in the proof of Theorem 2.2.1, we obtain

$$\begin{aligned} J_\infty &\leq \int_0^\infty [z^\top(t)z(t) - \gamma^2\omega^\top(t)\omega(t) + \dot{V}(x_t)]dt \\ &\leq \int_0^\infty \eta^\top(t)\Omega\eta(t)dt \end{aligned}$$

where $\eta(t) = [x^\top(t) \;\; x^\top(t-d(t)) \;\; (E\dot{x}(t))^\top \;\; \omega^\top(t)]^\top$. From (3.12), it is easy to see that $J_\infty < 0$, that is $\|z(t)\|_2 < \gamma\|\omega(t)\|_2$ for any non-zero $\omega(t) \in L_2[0,\infty)$. $\square$

In order to optimize the performance, one should solve the following optimization problem.

$$\begin{aligned} &\text{OP1.} \quad \min_{P,G,Y,T>0,Q>0,Z>0,N_i,i=1,2,3} \gamma \\ &s.t.\ \gamma > 0 \\ &\qquad (3.12) - (3.15). \end{aligned}$$

Example: Consider the time-delay singular system described by:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -0.5 & 0 & 1 \\ 0 & -1 & 0.2 \\ 0.1 & 0.1 & -0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -3 & 1 & 0 \\ 0 & 0.5 & 0.2 \\ 0 & 0 & 0.3 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}, \qquad B_{w1} = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad B_{w2} = 0.$$

with $\overline{d} = 0.6$ and $\mu = 0.2$. The open-loop response of this system with $\phi(t) = [1 \ \ -2 \ \ -0.5]^{\top}$ and $d(t) = |0.03\sin(5t) + 0.5|$ is shown in Figure 3.9, which is unstable. In order to stabilize this system and reject external disturbance, solving the optimization problem OP1, gives

$$K = \begin{bmatrix} -3.4826 & -1.3746 & -1.2942 \end{bmatrix},$$
$$\gamma = 0.65$$

Figure 3.10 and Figure 3.11 show the closed-loop response of the system with $d(t) = |0.03\sin(5t) + 0.5|$ and $\phi(t) = [1 \ \ -2 \ \ -0.7617]^{\top}$.

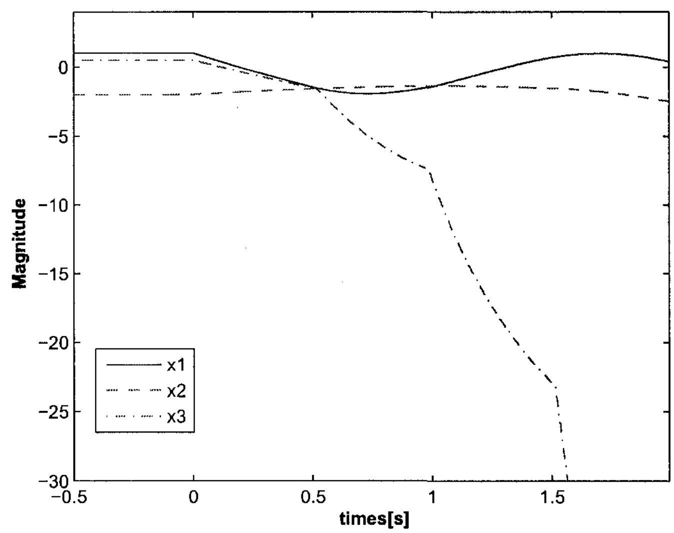

Figure 3.9 The open-loop response of x_1 and x_2 with $d(t) = |0.03\cos(5t) + 0.5|$ and $\phi(t) = [1 \ \ -2 \ 0.5]^{\top}$

Moreover, to illustrate the disturbance rejection, the closed-loop system with zero initial condition will be simulated under a finite energy disturbance. The dis-

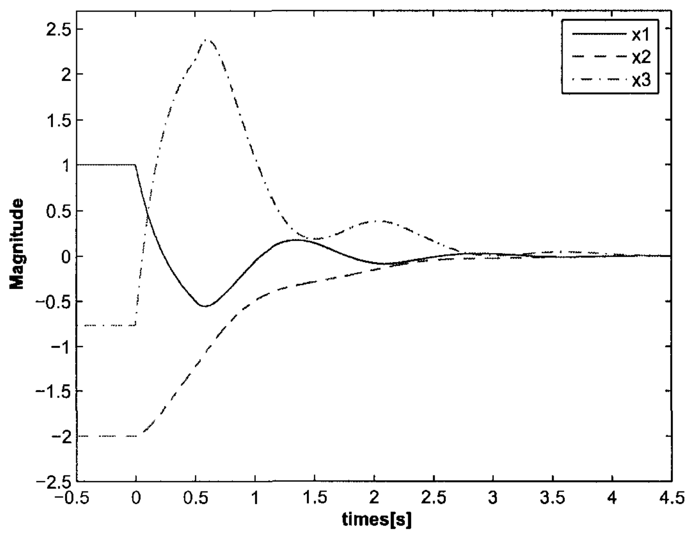

Figure 3.10 Behavior of x_1 and x_2 with $d(t) = |0.03\cos(5t) + 0.5|$ and $\phi(t) = [1\ -2\ -0.7617]^\top$ under the state feedback controller.

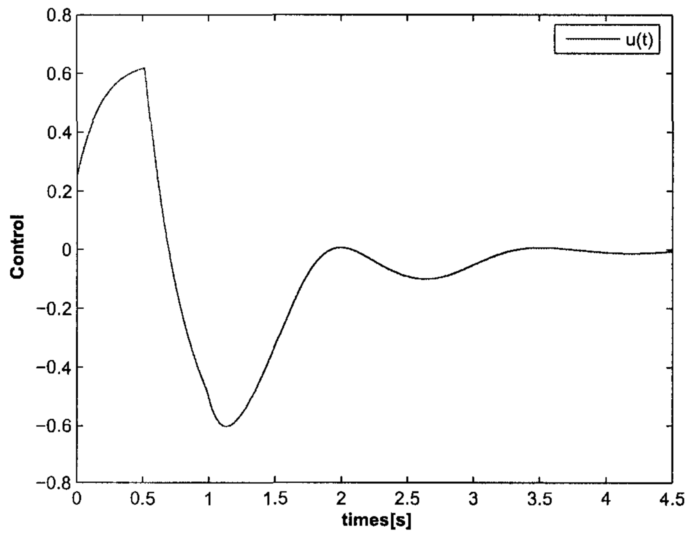

Figure 3.11 Behavior of the controller

turbance is shown in Figure 3.12 and its norm is $\|w(t)\|_2 = 54.1363$. The disturbed output is shown in Figure 3.13 and its norm is $\|z(t)\|_2 = 0.4782$. Note that $\frac{\|z(t)\|_2}{\|w(t)\|_2} = 0.009 < \gamma = 0.65$. In this simulation, the delay function is chosen to be $d(t) = 0.5$. $\square$

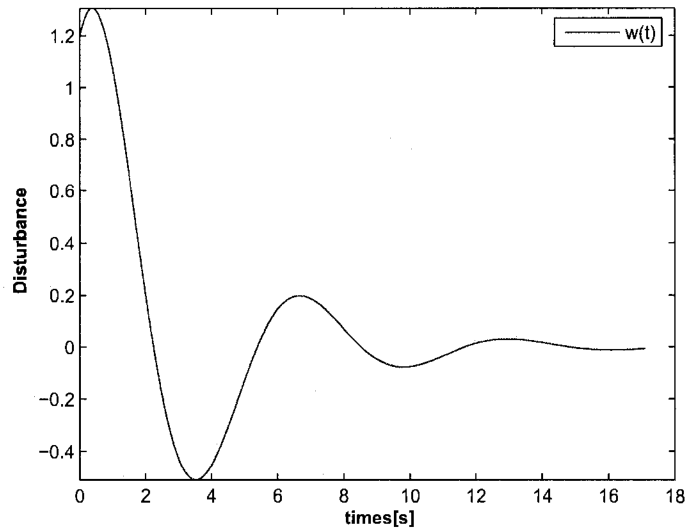

Figure 3.12 Disturbance $w(t) = 2\sin(t + 0.9)e^{-0.3(t+0.9)}$.

Example: Consider the time-delay singular system described by:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -3 & 1 \\ 0 & 0.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \qquad B_{w1} = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad B_{w2} = 0.$$

with $\overline{d} = 1.2$ and $\mu = 0.2$. The open-loop response of this system with $\phi(t) = [1\ 0]^{\top}$ and $d(t) = |0.03\sin(5t) + 1|$ is shown in Figure 3.14, which is unstable. In order to stabilize this system and reject external disturbance, solving the optimization

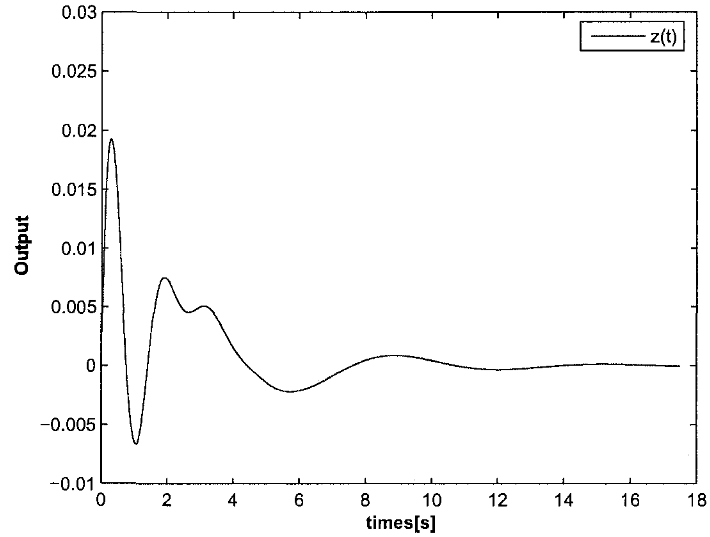

Figure 3.13 Solution behavior of the disturbed output $\|z(t)\|_2 = 0.4782$

problem OP1, gives

$$K = \begin{bmatrix} -9.8918 & -0.6905 \end{bmatrix},$$
$$\gamma = 2.1$$

Figure 3.15 and Figure 3.16 show the closed-loop response of the system with $d(t) = |0.03\sin(5t) + 1|$.

Moreover, to illustrate the disturbance rejection, the closed-loop system with zero initial condition will be simulated under a finite energy disturbance. The disturbance is shown in Figure 3.17 and its norm is $\|w(t)\|_2 = 54.1363$. The disturbed output is shown in Figure 3.18 and its norm is $\|z(t)\|_2 = 1.9286$. Note that $\frac{\|z(t)\|_2}{\|w(t)\|_2} = 0.0356 < \gamma = 2.1$. In this simulation, the delay function is chosen to be $d(t) = 0.9$. □

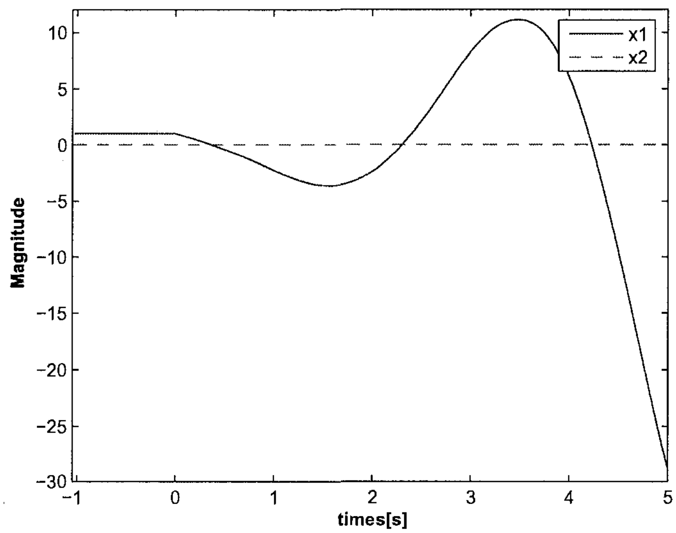

Figure 3.14 The open-loop response of x_1 and x_2 with $d(t) = |0.03\cos(5t) + 1|$ and $\phi(t) = [1,\ 0]^\top$

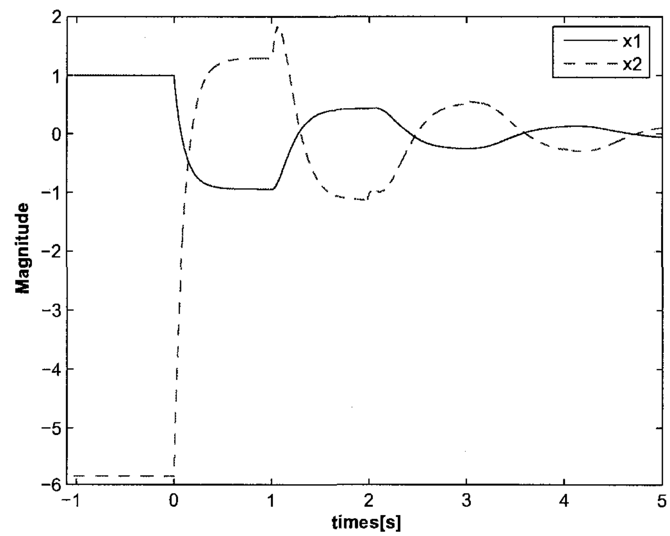

Figure 3.15 Behavior of x_1 and x_2 with $d(t) = |0.03\cos(5t) + 1|$ and $\phi(t) = [1,\ -5.8514]^\top$ under the state feedback controller.

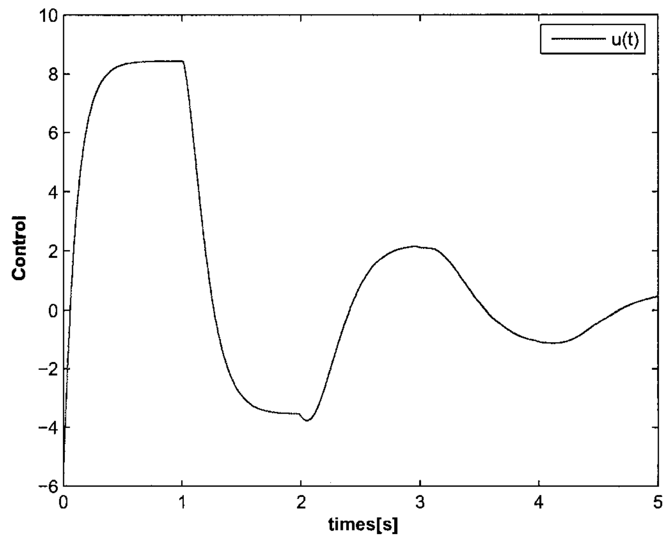

Figure 3.16 Behavior of the controller

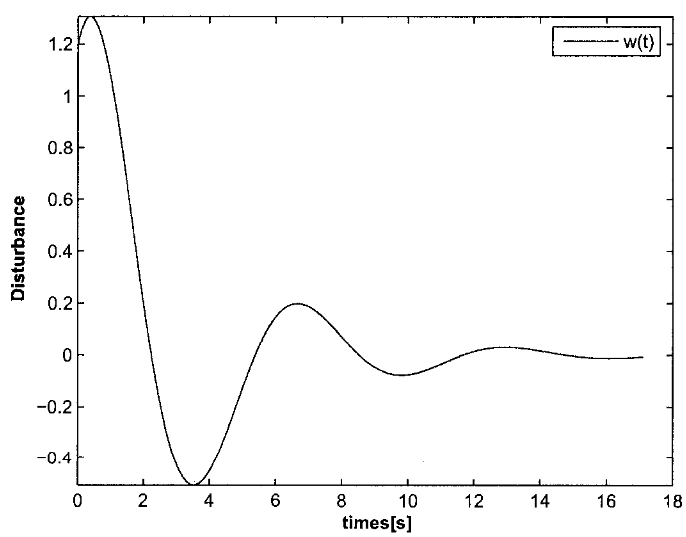

Figure 3.17 Disturbance $w(t) = 2\sin(t + 0.9)e^{-0.3(t+0.9)}$.

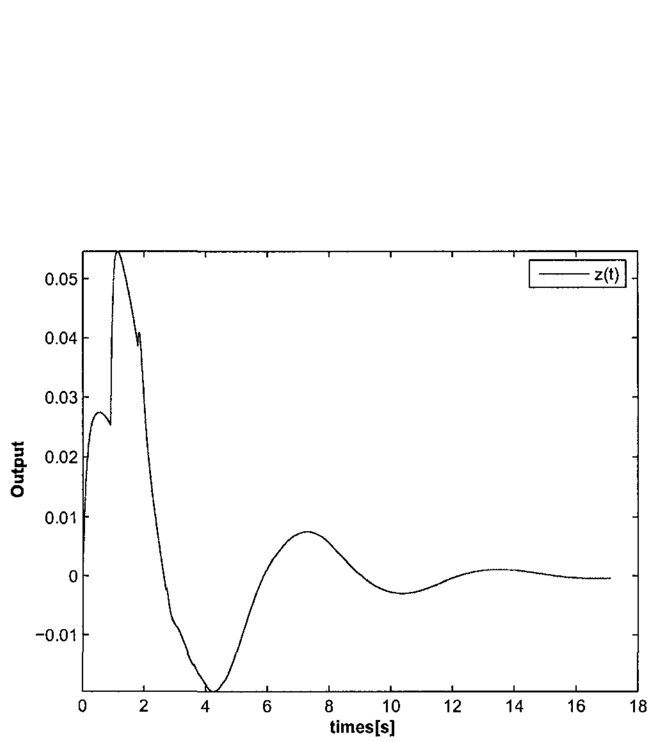

Figure 3.18 Solution behavior of the disturbed output $\|z(t)\|_2 = 1.9286$

3.3 Conclusion

This chapter dealt with the state feedback stabilization problem of time-delay singular systems. First, LMI conditions are provided in order to design a stabilizing state feedback controller. Then, LMI conditions are provided in order to solve the state feedback $\mathscr{H}_\infty$ stabilization problem. Some numerical examples have been given to illustrate the effectiveness of the results.

CHAPTER 4

STATIC OUTPUT FEEDBACK CONTROLLER FOR TIME-DELAY SINGULAR SYSTEMS WITH SATURATING ACTUATORS

In this chapter, the static output feedback stabilization problem for singular time-delay systems in the presence of saturating actuators will be tackled. The problem will be considered in the absence and presence of input delay. In the absence of input delay, an iterative LMI (ILMI) algorithm is proposed to design a stabilizing static output feedback controller. In the presence of input delay, PENBMI is used to solve a set of bilinear matrix inequalities (BMI) in order to design a stabilizing static output feedback controller. The objective of the control designs is twofold. It consists in determining both a static output feedback control law to guarantee that the system is regular, impulse-free and exponentially stable with a predefined decaying rate for the closed-loop system, and a set of safe initial conditions for which the exponential stability of the closed-loop system is guaranteed and at the same time the control remains between some given bounds. The results presented here are delay-range-dependent. Some numerical examples will be given to demonstrate the effectiveness of the results.

4.1 System Description

Consider the following time-delay singular system:

$$E\dot{x}(t) = Ax(t) + A_d x(t-d(t)) + B\text{sat}(u(t)) \tag{4.1a}$$

$$y(t) = Cx(t) \tag{4.1b}$$

$$x(t) = \phi(t), \quad t \in [-\bar{d}, 0] \tag{4.1c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the saturating control input, $y(t) \in \mathbb{R}^q$ is the measurement, the matrix $E \in \mathbb{R}^{n\times n}$ may be singular, and we assume that $\text{rank}(E) = r \leq n$, A, A_d, B and C are known real constant matrices, $\text{sat}(u(t)) = [\text{sat}(u_1(t)), ..., \text{sat}(u_m(t))]$, where $\text{sat}(u_i(t)) = \text{sign}(u_i(t)) \min(|u_i(t)|, \overline{u}_i)$, $\phi(t) \in C_\tau^v$ is a compatible vector valued continuous function, and $d(t)$ is the time-delay that is assumed to satisfy:

$$0 < \underline{d} \leq d(t) \leq \overline{d} \qquad \text{and} \qquad \dot{d}(t) \leq \mu < 1$$

where $\underline{d}$, $\overline{d}$ and μ are given scalars.

Now, consider the following static output feedback controller:

$$u(t) = Ky(t), \;\; K \in R^{m\times q} \tag{4.2}$$

Applying this controller to system (4.1), we obtain the closed-loop system as follows:

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + B\text{sat}(KCx(t)) \tag{4.3}$$

The following definition will be used in the sequel.

Definition 4.1.1. *(Khalil, 1992) A set Ω is called a region of attraction if for all $\phi(t) \in \Omega$,*

$$\lim_{t\to\infty} x(t) = 0$$

Due to the existence of the saturation term, generally, the system cannot be stabilized globally (see Figure 4.1). Therefore, our problem is considered as a local stabilization problem. Generally, for a given stabilizing static output feedback K, it is not possible to determine exactly the region of attraction of the origin with respect to system (4.3). Hence, a domain of initial conditions, for which the exponential stability of system (4.3) is ensured, has to be determined.

It is of common practice to approximate the region of attraction by sets constructed

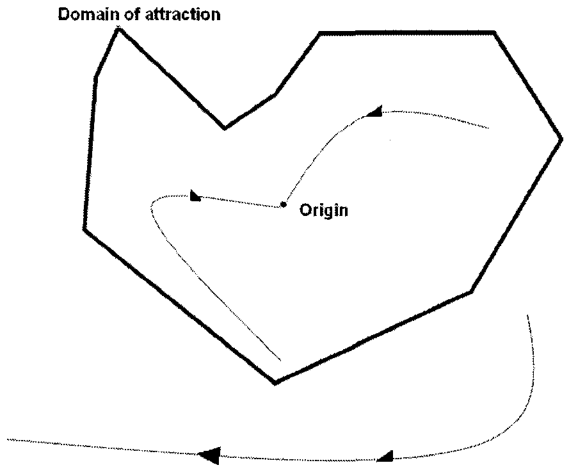

Figure 4.1 Region of attraction.

from the Lyapunov functional. It is shown in (Khalil, 1992) using LaSalle's theorem that the set

$$\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\}$$

is a subset of the region of attraction and it can be used to approximate it (see Figure 4.2).

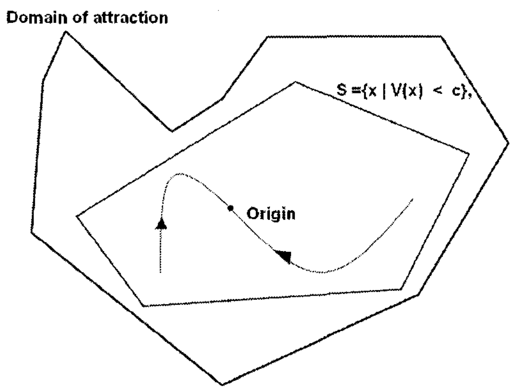

Figure 4.2 A subset of the region of attraction constructed by Lyapunov functional.

In this chapter, the Lyapunov functional used in Chapter 2 will be adopted; namely (2.31) and (2.44). It can be seen that even the set Ω_c cannot be determined exactly. However, this set can be approximated by an ellipsoid as it will be seen later (see

Figure 4.3). For more details on this subject, we refer the reader to (Khalil, 1992;

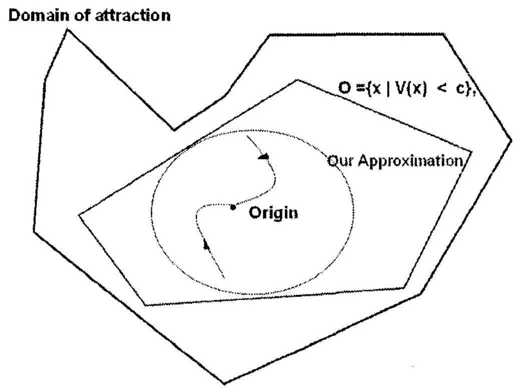

Figure 4.3 Approximation of the region of attraction.

Tarbouriech and Gomes da Silva, 2000; Molchanov and Pyatnitskiy, 1989; Cao et al., 2002; Hu et al., 2002; Bernstein and Michel, 1995) and references therein.

4.2 Static Output Feedback Controller of Singular Systems with Time-Varying Delay and Saturating Actuators

The problem to be tackled in this section can be summarized as follows: Find a static output feedback law of the form (4.2) and a set of initial conditions such that the closed-loop system (4.3) is exponentially admissible with a predefined minimum decaying rate. The technique introduced in (Tarbouriech and Gomes da Silva, 2000); namely, the differential inclusion, will be adopted in order to write the saturated nonlinear system (4.3) in a linear polytopic form. Let us write the saturation term as

$$\mathrm{sat}(KCx(t)) = D(\rho(x))KCx(t), \quad D(\rho(x)) \in R^{m\times m} \tag{4.4}$$

where $D(\rho(x))$ is a diagonal matrix for which the diagonal elements $\rho_i(x)$ are defined for $i = 1, ..., m$ as

$$\rho_i(x) = \begin{cases} -\frac{\overline{u}_i}{(KC)_i x} & \text{if } (KC)_i x \leq -\overline{u}_i \\ 1 & \text{if } -\overline{u}_i < (KC)_i x < \overline{u}_i \\ \frac{\overline{u}_i}{(KC)_i x} & \text{if } (KC)_i x \geq \overline{u}_i \end{cases}$$

where $(KC)_i$ is the i^{th} row of (KC). Using this, system (4.3) can be written as follows:

$$E\dot{x}(t) = (A + BD(\rho(x))KC)x(t) + A_d x(t - d(t)) \tag{4.5}$$

The coefficient $\rho_i(x)$ can be viewed as an indicator of the degree of saturation of the i^{th} entry of the control vector. In fact, the smaller $\rho_i(x)$, the farther is the state vector from the region of linearity defined by:

$$S(K, 1) = \{x \in R^n \mid -\overline{u}_i \leq (KC)_i x \leq \overline{u}_i\}$$

Since we address the problem of local stabilization, a limit is imposed on x yielding a lower bound for $\rho_i(x)$. Define the following region of interest in the state space:

$$S(K, \overline{u}^\rho) = \{x \in R^n \mid -\overline{u}^\rho \leq KCx \leq \overline{u}^\rho\}$$

where every component of the vector $\overline{u}^\rho$ is defined by $\overline{u}_i/\underline{\rho}_i$ with $0 < \underline{\rho}_i \leq 1$. This implies

$$-\frac{\overline{u}_i}{\underline{\rho}_i} \leq (KC)_i x \leq \frac{\overline{u}_i}{\underline{\rho}_i}$$

which can be rewritten as:

$$-\frac{1}{\underline{\rho}_i} \leq \frac{(KC)_i x}{\overline{u}_i} \leq \frac{1}{\underline{\rho}_i}$$

Since $(KC)_i x$ can be positive or negative, we have

$$\begin{array}{lllll} \text{if } (KC)_i x > 0 & \Rightarrow & \dfrac{\overline{u}_i}{(KC)_i x} \geq \underline{\rho}_i & \Rightarrow & \underline{\rho}_i \leq \rho_i(x) \\ \text{if } (KC)_i x < 0 & \Rightarrow & \dfrac{\overline{u}_i}{(KC)_i x} \leq -\underline{\rho}_i & \Rightarrow & \underline{\rho}_i \leq \rho_i(x) \end{array}$$

which means that for any $x \in S(K, \overline{u}^\rho)$, one gets

$$\underline{\rho}_i \leq \rho_i(x) \leq 1, \ \forall i = 1, \dots, m$$

which allow us to define the vector $\underline{\rho} = [\underline{\rho}_1, ..., \underline{\rho}_m]$ and the following matrices

$$A_j = A + BD(\gamma_j)KC, \ j = 1, ..., 2^m$$

where $D(\gamma_j)$ is a diagonal matrix of positive scalars $\gamma_{j(i)}$ for $i = 1, ..., m$, which arbitrarily take the value one or $\underline{\rho}_i$. Note that we have 2^m matrices since the matrices $D(\gamma_j)$ have m diagonal elements, each have two possibilities.

Now, if $x \in S(K, \overline{u}^\rho)$, we have

$$\begin{aligned} D(\rho(x)) = & \begin{bmatrix} \rho_1(x) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \rho_m(x) \end{bmatrix} \\ & \in \left\{ \begin{bmatrix} \underline{\rho}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \underline{\rho}_m \end{bmatrix}, \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \underline{\rho}_m \end{bmatrix}, \dots, \begin{bmatrix} \underline{\rho}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 \end{bmatrix} \right\} \\ & \in \{D(\gamma_1), \dots, D(\gamma_{2^m})\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & D(\rho(x)) \in \{D(\gamma_1), \dots, D(\gamma_{2^m})\} \\ \Rightarrow & (A + BD(\rho(x))KC) \in co\{A_1, ..., A_{2^m}\} \end{aligned}$$

$$\Rightarrow (A + BD(\rho(x))KC) = \sum_{j=1}^{2^m} \lambda_{j,t} A_j$$

with $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$ and $\lambda_{j,t} \geq 0$, which implies that $E\dot{x}(t)$ can be determined from the following polytopic model:

$$E\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_{j,t} A_j x(t) + A_d x(t - d(t)) \tag{4.6}$$

Remark 4.2.1. *(Haurani et al., 2004) Notice that the trajectories of the polytopic system (4.6) includes all trajectories of the saturated system (4.3), but the converse is not necessarily true. This means that the stability of system (4.6) is only a sufficient condition to the stability of system (4.3). Thus, some unavoidable conservativeness is introduced.*

Remark 4.2.2. *Different control saturation models are proposed in the literature, i.e. regions of saturation, differential inclusion and sector modeling. In (Gomes da Silva et al., 2002), a comparative analysis of these models is presented, and concluded that the differential inclusion model leads to the least conservative design. Based on that, the differential inclusion model for the actuator saturation is used here.*

Remark 4.2.3. *Using this saturation model, the problem of controlling the nonlinear system (4.5) is transformed to the problem of controlling the linear time-variant system (4.6). However, the time-variant matrix $A(t)$ evolves with time inside a convex polyhedron of matrices (see Figure 4.4). Now, the question is as follows: if we proof the stabilizability of the 2^m linear time-invariant systems that uses the vertices of that convex polyhedron as its A's matrices, does this imply the stabilizability of the linear time-variant system? The answer is yes, and this will be the result of the next theorem.*

Theorem 4.2.1. *Let $0 < \underline{d} < \overline{d}$, $\alpha > 0$, a vector $\underline{\rho}$ and $\mu < 1$ be given. If there exist symmetric and positive-definite matrix P, a matrix Q, symmetric and*

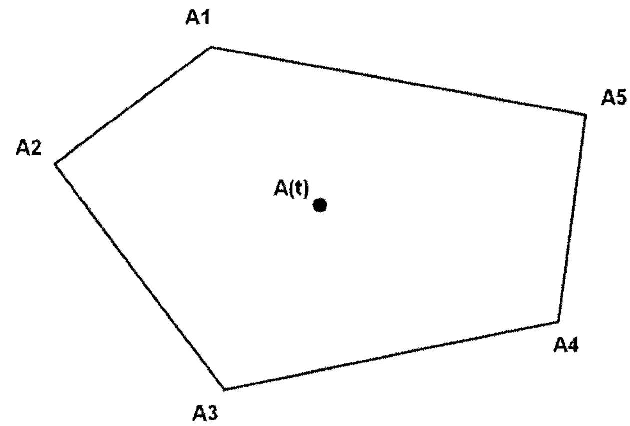

Figure 4.4 $A(t)$ evolves with time inside a convex polyhedron of matrices.

positive-definite matrices Q_1, Q_2, Q_3, Z_1 *and* Z_2, *matrices* M_i, N_i *and* S_i, $i = 1, 2$, *a matrix* K *and a positive scalar* κ *such that*

$$\begin{bmatrix} \Pi_{j11} & \Pi_{12} & e^{\alpha \underline{d}} M_1 E & -e^{\alpha \bar{d}} S_1 E & \frac{e^{2\alpha \bar{d}}-1}{2\alpha} N_1 & cS_1 & cM_1 & \Pi_{j18} \\ \star & \Pi_{22} & e^{\alpha \underline{d}} M_2 E & -e^{\alpha \bar{d}} S_2 E & \frac{e^{2\alpha \bar{d}}-1}{2\alpha} N_2 & cS_2 & cM_2 & A_d^\top U \\ \star & \star & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -Q_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\frac{e^{2\alpha \bar{d}}-1}{2\alpha} Z_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -c(Z_1 + Z_2) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -cZ_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -U \end{bmatrix} < 0 \quad (4.7)$$

$$j = 1, ..., 2^m$$

$$\begin{bmatrix} E^\top (PE + SQ) & \underline{\rho}_i (KC)_i^\top \\ \underline{\rho}_i (KC)_i & \kappa \bar{u}_i^2 \end{bmatrix} \geq 0, \quad i = 1, ..., m \quad (4.8)$$

where

$$
\begin{aligned}
\Pi_{j11} &= (PE+SQ)^\top A + A^\top (PE+SQ) + \sum_{i=1}^{3} Q_i + N_1 E + (N_1 E)^\top \\
&\quad + (PE+SQ)^\top BD(\gamma_j)KC + \left((PE+SQ)^\top BD(\gamma_j)KC\right)^\top \\
&\quad + 2\alpha E^\top (PE+SQ) \\
\Pi_{12} &= (PE+SQ)^\top A_d + (N_2 E)^\top - N_1 E + S_1 E - M_1 E \\
\Pi_{22} &= -(1-\mu)e^{-2\alpha\overline{d}} Q_3 + S_2 E + (S_2 E)^\top - N_2 E - (N_2 E)^\top - M_2 E - (M_2 E)^\top \\
\underline{\overline{d}} &= \overline{d} - \underline{d},\ U = \overline{d} Z_1 + \underline{\overline{d}} Z_2,\ \Pi_{j18} = A^\top U + (BD(\gamma_j)KC)^\top U,\ c = \frac{e^{2\alpha\overline{d}} - e^{2\alpha\underline{d}}}{2\alpha}
\end{aligned}
$$

where $S \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^\top S = 0$, then there exists a static output feedback controller (4.2) such that the closed-loop system (4.3) is locally exponentially admissible with $\sigma = \alpha$ for any compatible initial condition belonging to:

$$
\Omega(\nu_1, \nu_2) = \{\phi \in C_{\overline{d}}^{v} \quad : \quad \frac{\|\phi\|_c^2}{\nu_1} + \frac{\|\dot{\phi}\|_c^2}{\nu_2} \leq 1\}. \tag{4.9}
$$

where

$$
\begin{aligned}
&\nu_1 = \frac{\kappa^{-1}}{\chi_1}, \qquad\qquad \nu_2 = \frac{\kappa^{-1}}{\chi_2} \\
&\chi_1 = \lambda_{max}(E^\top PE) + \lambda_{max}(Q_1)\frac{1-e^{-2\alpha\underline{d}}}{2\alpha} + \lambda_{max}(Q_2)\frac{1-e^{-2\alpha\overline{d}}}{2\alpha} \\
&\quad + \lambda_{max}(Q_3)\frac{1-e^{-2\alpha\overline{d}}}{2\alpha} \\
&\chi_2 = \lambda_{max}(Z_1)\lambda_{max}(E^\top E)\frac{2\alpha\overline{d} - 1 + e^{-2\alpha\overline{d}}}{4\alpha^2} \\
&\qquad\qquad + \lambda_{max}(Z_2)\lambda_{max}(E^\top E)\frac{2\alpha\underline{\overline{d}} - e^{-2\alpha\underline{d}} + e^{-2\alpha\overline{d}}}{4\alpha^2}
\end{aligned}
$$

Proof. Assume that $x(t) \in S(K, \overline{u}^{\rho})$, $\forall t > 0$ (will be proved later). Therefore, $E\dot{x}(t)$ can be determined from the polytopic system (4.6). Applying Remark 2.3.1 to (2.23)-(2.24) in Theorem 2.3.1 yields a single matrix inequality. Then, if we apply this matrix inequality 2^m times to the parameters A_j with $j = 1, \ldots, 2^m$, A_d,

E, $\underline{d}$, $\overline{d}$ and μ, we will have (4.7). Now, we will show that the system is regular and impulse-free. For this purpose, choose two nonsingular matrices R, L such that

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix}$$

and let

$$\bar{A}_j = RA_jL = \begin{bmatrix} A_{j11} & A_{j12} \\ A_{j21} & A_{j22} \end{bmatrix}, \qquad \bar{P} = R^{-\top}PL = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Now, proceeding in a similar way as for the proof of regularity and impulse-free in Theorem 2.2.1, yields

$$A_{j22}^{\top}P_{22} + P_{22}^{\top}A_{j22} < 0, \ \ j = 1, \ldots 2^m$$

Using the fact that $\lambda_{j,t} \geq 0$ and $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$,

$$\lambda_{j,t}A_{j22}^{\top}P_{22} + P_{22}^{\top}\lambda_{j,t}A_{j22} \leq 0, \ \ j = 1, \ldots 2^m, \ \ \forall t \in (0, \infty)$$

which gives

$$\left[\sum_{j=1}^{2^m} \lambda_{j,t}A_{j22}\right]^{\top} P_{22} + P_{22}^{\top}\sum_{j=1}^{2^m} \lambda_{j,t}A_{j22} < 0$$

which implies

$$\sum_{j=1}^{2^m} \lambda_{j,t}A_{j22} \text{ is nonsingular} \quad \forall t \in [0, \infty).$$

Therefore, system (4.6) is regular and impulse-free. Now, choose a Lyapunov functional as in Theorem 2.3.1, and proceeding in a similar manner as in the proof of

Theorem 2.3.1, then

$$\dot{V}(t) + 2\alpha V(t) \leq$$

$$\eta^\top(t)[\Pi + \widetilde{A}^\top \left(\bar{d}\bar{Z}_1 + \underline{d}\bar{Z}_2\right) \widetilde{A} + \frac{e^{2\alpha\bar{d}} - 1}{2\alpha} \widetilde{N} \bar{Z}_1{}^{-1} \widetilde{N}^\top$$
$$+ \frac{e^{2\alpha\bar{d}} - e^{2\alpha\underline{d}}}{2\alpha} \widetilde{S} \left(\bar{Z}_1 + \bar{Z}_2\right)^{-1} \widetilde{S}^\top + \frac{e^{2\alpha\bar{d}} - e^{2\alpha\underline{d}}}{2\alpha} \widetilde{M} \bar{Z}_2{}^{-1} \widetilde{M}^\top] \eta(t)$$

with all the variables as defined in Theorem 2.3.1 and A is replaced by $\sum_{j=1}^{2^m} \lambda_{j,t} A_j$.

Then, by convexity and noting that $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$ with $\lambda_{j,t} \geq 0$, condition (4.7) yields:

$$\dot{V}(t) + 2\alpha V(t) \leq 0$$

The rest of the proof is similar to the proof of Theorem 2.3.1, and the details is omitted.

Now, we will prove that $x(t) \in S(K, \overline{u}^\rho)$, $\forall t > 0$. By virtue of condition (4.8), one has

$$E^\top(PE + SQ)E - \underline{\rho}_i(KC)_i^\top \kappa^{-1} \overline{u}_i^{-2} \underline{\rho}_i(KC)_i \geq 0$$
$$\Rightarrow \kappa E^\top(PE + SQ)E \geq \underline{\rho}_i(KC)_i^\top \overline{u}_i^{-2} \underline{\rho}_i(KC)_i.$$

Therefore,

$$\kappa E^\top(PE + SQ)E \leq 1 \quad \text{implies} \quad \underline{\rho}_i(KC)_i^\top \overline{u}_i^{-2} \underline{\rho}_i(KC)_i \leq 1$$

which means that the ellipsoid defined by $\Gamma = \{x \in \mathbb{R}^n : x^\top E^\top(PE+SQ)x \leq \kappa^{-1}\}$ is included in the set $S(K, \overline{u}^\rho)$. Suppose now that the initial condition $\phi(t)$ satisfies (4.9). The Lyapunov functional in (2.31) at $t = 0$ can be rewritten as follows:

$$V(0) = \phi^\top(0)E^\top(PE + SQ)\phi(0) + \int_{-\underline{d}}^{0} \phi^\top(s) e^{2\alpha s} Q_1 \phi(s) ds$$

$$+\int_{-\bar{d}}^{0}\phi^{\top}(s)e^{2\alpha s}Q_2\phi(s)ds+\int_{-d(0)}^{0}\phi(s)^{\top}e^{2\alpha s}Q_3\phi(s)ds$$
$$+\int_{-\bar{d}}^{0}\int_{\theta}^{0}(E\dot{\phi}(s))^{\top}e^{2\alpha s}Z_1E\dot{\phi}(s)dsd\theta$$
$$+\int_{-\bar{d}}^{-\underline{d}}\int_{\theta}^{0}(E\dot{\phi}(s))^{\top}e^{2\alpha s}Z_2E\dot{\phi}(s)dsd\theta$$

where P, Q_1, Q_2, Q_3, Z_1 and Z_2 are symmetric positive-definite matrices. Now, the terms of the Lyapunov functional can be bounded as follows:

$$\phi^{\top}(0)E^{\top}(PE+SQ)\phi(0)=\phi^{\top}(0)E^{\top}PE\phi(0)\leq\lambda_{max}(E^{\top}PE)\|\phi\|_c^2$$
$$\int_{-\underline{d}}^{0}\phi^{\top}(s)e^{2\alpha s}Q_1\phi(s)ds\leq\lambda_{max}(Q_1)\int_{-\underline{d}}^{0}e^{2\alpha s}ds\|\phi\|_c^2\leq\lambda_{max}(Q_1)\frac{1-e^{-2\alpha\underline{d}}}{2\alpha}\|\phi\|_c^2$$
$$\int_{-\bar{d}}^{0}\phi^{\top}(s)e^{2\alpha s}Q_2\phi(s)ds\leq\lambda_{max}(Q_2)\frac{1-e^{-2\alpha\bar{d}}}{2\alpha}\|\phi\|_c^2$$
$$\int_{-d(0)}^{0}\phi(s)^{\top}e^{2\alpha s}Q_3\phi(s)ds\leq\lambda_{max}(Q_3)\frac{1-e^{-2\alpha\bar{d}}}{2\alpha}\|\phi\|_c^2$$
$$\int_{-\bar{d}}^{0}\int_{\theta}^{0}(E\dot{\phi}(s))^{\top}e^{2\alpha s}Z_1E\dot{\phi}(s)dsd\theta\leq\lambda_{max}(Z_1)\lambda_{max}(E^{\top}E)\int_{-\bar{d}}^{0}\int_{\theta}^{0}e^{2\alpha s}ds\|\dot{\phi}\|_c^2$$
$$\leq\lambda_{max}(Z_1)\lambda_{max}(E^{\top}E)\frac{2\alpha\bar{d}-1+e^{-2\alpha\bar{d}}}{4\alpha^2}\|\dot{\phi}\|_c^2$$
$$\int_{-\bar{d}}^{-\underline{d}}\int_{\theta}^{0}(E\dot{\phi}(s))^{\top}e^{2\alpha s}Z_2E\dot{\phi}(s)dsd\theta\leq\lambda_{max}(Z_2)\lambda_{max}(E^{\top}E)\frac{2\alpha\bar{\underline{d}}-e^{-2\alpha\underline{d}}+e^{-2\alpha\bar{d}}}{4\alpha^2}\|\dot{\phi}\|_c^2$$

Therefore, from the definition of $V(0)$, it follows that $x^{\top}(0)E^{\top}(PE+SQ)x(0)\leq V(0)\leq\chi_1\|\phi\|_c^2+\chi_2\|\dot{\phi}\|_c^2\leq\kappa^{-1}$ and, in this case, one has $x(0)\in\Gamma\subset S$. Now, since $\dot{V}(0)<0$, we conclude that $x^{\top}(t)E^{\top}(PE+SQ)x(t)\leq V(t)\leq V(0)\leq\chi_1\|\phi\|_c^2+\chi_2\|\dot{\phi}\|_c^2\leq\kappa^{-1}$, which means that $x(t)\in S$, $\forall t>0$ (See Figure 4.5 for a graphical representation of different sets). Therefore, $E\dot{x}(t)$ can be determined from the polytopic system (4.6). This completes the proof. □

Remark 4.2.4. *Being inside the set $\Omega(\nu_1,\nu_2)$, the compatibility of the initial condition is also very important especially when saturation is present. As discussed in Chapter 1, incompatible initial conditions results in jump discontinuities due to the singular structure. Such jump discontinuities may take the system outside the set*

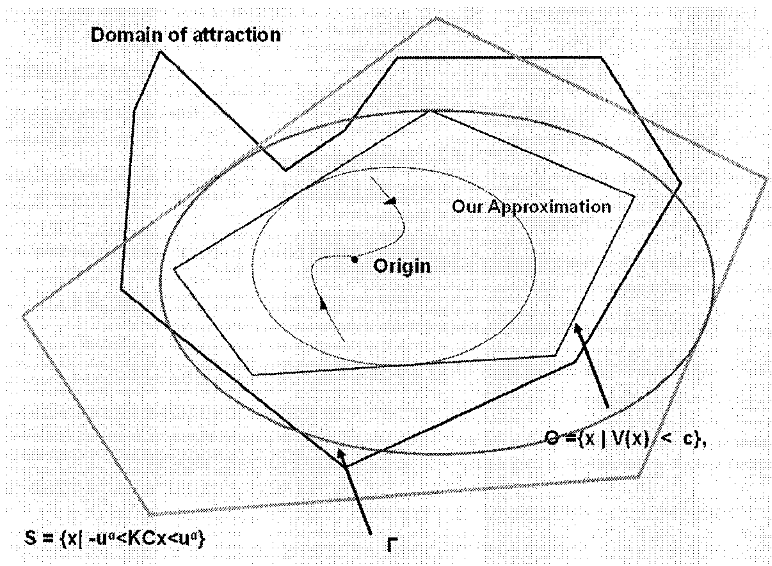

Figure 4.5 Different sets in the proof of Theorem 4.2.1

$\Omega(\nu_1, \nu_2)$, *where the controller may be unable to stabilize the system.*

It is obvious that (4.7) is a BMI, and consequently its solution is very difficult. Thus, an ILMI approach similar to (Cao et al., 1998) and (Zheng et al., 2002) will be proposed. The derivation of the algorithm is similar to (Cao et al., 1998) and (Zheng et al., 2002) and the details will be omitted. This algorithm has the same disadvantages as those in (Cao et al., 1998) and (Zheng et al., 2002), i.e. based on a sufficient conditions. The following is the proposed algorithm and the explanation is given later.

ILMI Algorithm.

- Step 1. OP1.

$$\min_{P_0>0,Q,Q_1>0,Q_2>0,Q_3>0,Z_{p0}>0,M_p,N_p,S_p,p=1,2,\kappa} \beta_0$$
$$s.t.(4.12)-(4.13)$$
$$K=0 \text{ and } X_0=E.$$

Set $i = 1$, $X_1 = E$, $Z_{11} = Z_{10}$ and $Z_{21} = Z_{20}$.

- Step 2. OP2.

$$\begin{aligned} &\min_{P_i>0,Q,Q_1>0,Q_2>0,Q_3>0,M_p,N_p,S_p,p=1,2,K,\kappa} \beta_i \\ &s.t.(4.12) - (4.13) \end{aligned}$$

Let ${\beta_i}^*$ and K^* be the solution of OP2. If ${\beta_i}^* \le -\alpha$, where α is a prescribed decay rate, then K^* is a stabilizing static output feedback gain, go to step 5, otherwise, go to step 3.

- Step 3. OP3.

$$\begin{aligned} &\min_{P_i>0,Q,Q_1>0,Q_2>0,Q_3>0,Z_{pi}>0,M_p,N_p,S_p,p=1,2,\kappa} \mathrm{tr}(E^\top T_i) \\ &s.t.(4.12) - (4.13) \\ &\quad \beta_i = \beta_i^* \text{ and } K = K^*. \end{aligned}$$

If $\|X_i B - T_i^* B\| < \epsilon$, go to step 4, else set $i = i + 1$, $X_i = T_{i-1}^*$, $Z_{1i} = Z_{1(i-1)}^*$ and $Z_{2i} = Z_{2(i-1)}^*$, then go to step 2.

- Step 4. The system may not be stabilizable via static output feedback. Stop.
- Step 5. OP4.

$$\begin{aligned} &\min_{P>0,Q,Q_1>0,Q_2>0,Q_3>0,Z_p>0,M_p,N_p,S_p,p=1,2,K,\kappa} r \\ &s.t.(4.12) - (4.13) \quad \beta_i = \alpha \\ &\delta_1 \mathbb{I} \ge E^\top P E \qquad \delta_2 \mathbb{I} \ge Q_1 \qquad \delta_3 \mathbb{I} \ge Q_2 \qquad (4.10) \\ &\delta_4 \mathbb{I} \ge Q_3 \qquad \delta_5 \mathbb{I} \ge Z_1 \qquad \delta_6 \mathbb{I} \ge Z_2 \qquad (4.11) \end{aligned}$$

where

$$r = w_1 \left(\delta_1 + \frac{1 - e^{-2\alpha \underline{d}}}{2\alpha} \delta_2 + \frac{1 - e^{-2\alpha \bar{d}}}{2\alpha} \delta_3 + \frac{1 - e^{-2\alpha \bar{d}}}{2\alpha} \delta_4 \right)$$

$$+ w_2 \left(\lambda_{max}(E^\top E) \frac{2\alpha\overline{d} - 1 + e^{-2\alpha\overline{d}}}{4\alpha^2} \delta_5 + \lambda_{max}(E^\top E) \frac{2\alpha\overline{d} - e^{-2\alpha\underline{d}} + e^{-2\alpha\overline{d}}}{4\alpha^2} \delta_6 \right)$$

$+ w_3\kappa$, and w_1, w_2 and w_3 are weighting factors.

We solve this problem iteratively in two steps as follows:

a) Fix K, and solve for $P > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_p > 0$, M_p, N_p, S_p, $p = 1, 2$, and κ.

b) Fix Z_1 and Z_2, and solve for $P > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, M_p, N_p, S_p, $p = 1, 2$, K and κ. Set $X = T$.

The set (4.9) is calculated from the matrices that solve OP4.

$$\left[\begin{array}{ccccc}
\Pi_{11} & \begin{bmatrix} (B^\top T_i \\ +D(\gamma_j)KC)^\top \end{bmatrix} & \Pi_{12} & e^{-\beta_i \underline{d}} M_1 E & -e^{-\beta_i \overline{d}} S_1 E \\
\begin{bmatrix} (B^\top T_i \\ +D(\gamma_j)KC) \end{bmatrix} & -\mathbb{I} & 0 & 0 & 0 \\
\star & 0 & \Pi_{22} & e^{-\beta_i \underline{d}} M_2 E & -e^{-\beta_i \overline{d}} S_2 E \\
\star & 0 & \star & -Q_1 & 0 \\
\star & 0 & \star & \star & -Q_2 \\
\star & 0 & \star & \star & \star \\
\star & 0 & \star & \star & \star \\
\star & 0 & \star & \star & \star \\
\star & 0 & \star & \star & \star
\end{array} \right.$$

$$
\left.\begin{matrix}
\frac{e^{-2\beta_i\overline{d}}-1}{-2\beta_i}N_1 & \frac{e^{-2\beta_i\overline{d}}-e^{-2\beta_i\underline{d}}}{-2\beta_i}S_1 & \frac{e^{-2\beta_i\overline{d}}-e^{-2\beta_i\underline{d}}}{-2\beta_i}M_1 & \Pi_{j18} \\
0 & 0 & 0 & 0 \\
\frac{e^{-2\beta_i\overline{d}}-1}{-2\beta_i}N_2 & \frac{e^{-2\beta_i\overline{d}}-e^{-2\beta_i\underline{d}}}{-2\beta_i}S_2 & \frac{e^{-2\beta_i\overline{d}}-e^{-2\beta_i\underline{d}}}{-2\beta_i}M_2 & {A_d}^{\top}U \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{e^{-2\beta_i\overline{d}}-1}{-2\beta_i}Z_{1i} & 0 & 0 & 0 \\
\star & -\frac{e^{-2\beta_i\overline{d}}-e^{-2\beta_i\underline{d}}}{-2\beta_i}(Z_{1i}+Z_{2i}) & 0 & 0 \\
\star & \star & -\frac{e^{-2\beta_i\overline{d}}-e^{-2\beta_i\underline{d}}}{-2\beta_i}Z_{2i} & 0 \\
\star & \star & \star & -U
\end{matrix}\right] < 0
$$

$$j = 1, ..., 2^m \quad (4.12)$$

$$
\begin{bmatrix} E^{\top}T_i & \underline{\rho}_r {(KC)_r}^{\top} \\ \underline{\rho}_r (KC)_r & \kappa {\overline{u}_r}^2 \end{bmatrix} \geq 0, \quad r = 1, ..., m \qquad (4.13)
$$

where

$$
\begin{aligned}
\Pi_{11} &= T_i^{\top}A + A^{\top}T_i + \sum_{i=1}^{3} Q_i + N_1E + (N_1E)^{\top} \\
&\quad - X_iBB^{\top}T_i - (X_iBB^{\top}T_i)^{\top} + X_iBB^{\top}X_i - 2\beta_iE^{\top}T_i \\
\Pi_{j18} &= A^{\top}U + (BD(\gamma_j)KC)^{\top}U \\
\Pi_{22} &= -(1-\mu)e^{2\beta d(\beta)}Q_3 + S_2E + (S_2E)^{\top} - N_2E - (N_2E)^{\top} - M_2E - (M_2E)^{\top} \\
T_i &= (P_iE + SQ) \qquad d(\beta) = \begin{cases} \underline{d} & \text{if } \beta > 0 \\ \overline{d} & \text{if } \beta < 0 \end{cases}
\end{aligned}
$$

and the other variables as defined previously.

Remark 4.2.5. *The core of this algorithm is OP2 and OP3. As shown in (Cao et al., 1998), OP2 guarantees the progressive reduction of β_i while OP3 guarantees the convergence of the algorithm. Yet, in (Cao et al., 1998), only X needs to be fixed in order to get LMIs, while in our case, we have also to fix either Z_1 and Z_2 or K to get LMIs. Thus, we will fix Z_1 and Z_2 in OP2, and K in OP3. This way* *of solving this problem will not affect the convergence of the algorithm. It is worth noting that although this ILMI algorithm is convergent, we may not achieve the solution because β may not always converge to its minimum. For more details on the numerical properties of the algorithm, we refer the reader to (Cao et al., 1998).*

Remark 4.2.6. *Noting Remark 2.3.4, as β decreases in the algorithm, this is nothing but an increasing decaying rate, and as β becomes negative, the system becomes exponentially stable. This fact resembles the facts in (Cao et al., 1998) and (Zheng et al., 2002) that all eigenvalues of $(A - BKC)$ are shifted progressively toward the left-half-plane through the reduction of β.*

Remark 4.2.7. *In order to start the algorithm, OP2 should have a solution for $i = 1$. Yet, the solution depends on the initial matrix X. In (Zheng et al., 2002), some Riccati equation is proposed in order to select an initial X for the descriptor version of this algorithm. In (Lin et al., 2004), it has been proved that this Riccati equation may not have a solution and an initial value of $X = \mathbb{I}$ is proposed instead. Actually, the identity matrix may not do the job for even some simple systems, an example of such systems is*

$$(A, B, C) = (\mathbb{I}, \mathbb{I}, \mathbb{I}), \qquad E = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{bmatrix}$$

Our numerical experience indicates that an initial choice of $X_0 = E$ often leads to a convergent result. With this X_0, OP1 is used here to get initial values for Z_1 and Z_2.

Remark 4.2.8. *The minimization of β in OP1 and OP2 should be done using the bisection method. The lower bound of the bisection method can be any value less than $-\alpha$ since we are not interested in minimizing β less than $-\alpha$. The upper bound of the bisection method can be any sufficiently large number. These upper and lower bounds should be chosen only once and can be fixed throughout the algorithm.*

Remark 4.2.9. *OP4 is used in order to enlarge the set of initial conditions (4.9). The satisfaction of (4.10)-(4.11) means that $\chi_1 \leq \delta_1 + \frac{1-e^{-2\alpha d}}{2\alpha}\delta_2 + \frac{1-e^{-2\alpha d}}{2\alpha}\delta_3 +$*

$\frac{1-e^{-2\alpha\underline{d}}}{2\alpha}\delta_4$ and $\chi_2 \leq \lambda_{max}(E^\top E)\frac{2\alpha\bar{d}-1+e^{-2\alpha\bar{d}}}{4\alpha^2}\delta_5 + \lambda_{max}(E^\top E)\frac{2\alpha\underline{d}-e^{-2\alpha\underline{d}}+e^{-2\alpha\bar{d}}}{4\alpha^2}\delta_6$. Therefore, because $\nu_i = \frac{\kappa^{-1}}{\chi_i}$, if we minimize the criterion as defined in OP4, then greater the bounds on $\|\phi\|_c^2$ and $\|\dot{\phi}\|_c^2$ tend to be. In other words, by using OP4, we orient the solutions of LMIs (4.7)-(4.8) in a sense to obtain the set $\Omega(\nu_1, \nu_2)$ as large as possible. For more discussion on this topic, we refer the reader to (Tarbouriech and Gomes da Silva, 2000).

Example: Consider the time-delay singular system described by:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0 & 0 & 0.3 \\ 0 & 0.4 & 0 \\ 0.2 & 0.3 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & -2 \\ 0.1 & 0.3 \\ 0.1 & -0.3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This system is originally unstable for all values of delay. Now, allowing time-varying delay, the exponential stabilizability of this system will be investigated using Theorem 4.2.1 and the iterative algorithm. Letting $\underline{d} = 0.2$, $\bar{d} = 0.6$, $\mu = 0.5$, $\bar{u} = 7$ and $\alpha = 0.3$, the ILMI algorithm gives after 14 iterations

$$K = \begin{bmatrix} -1.4186 & -1.2682 \\ 1.3943 & 0.8652 \end{bmatrix}, \qquad \nu_1 = 14.8960, \qquad \nu_2 = 82.6586.$$

Figures 4.6 and 4.7 give the simulation results for the closed-loop system when $d(t) = |0.4 + 0.15\sin(3t)|$ and the initial condition is $\phi(t) = [5,\ 12,\ 9.6]^\top, t \in [-0.6, 0]$. Changing the control amplitude saturation level, Figure 4.8 presents the functional dependence of ν_1 and ν_2 on the level of control saturation $\bar{u}$.

For various α, the values ν_1 and ν_2 for which we guarantee the exponential admis-

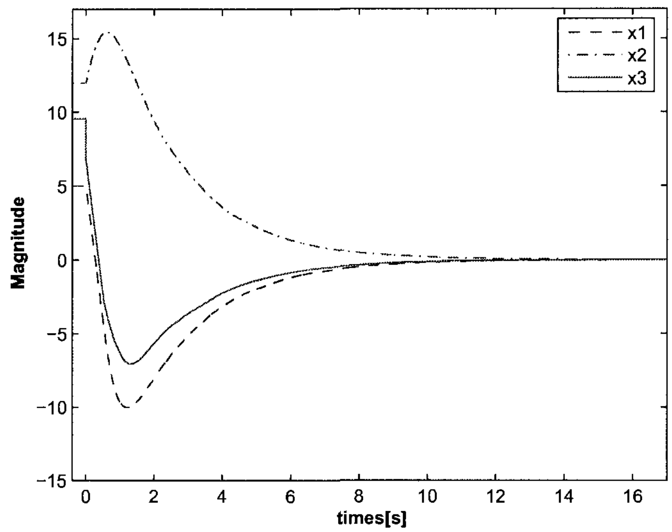

Figure 4.6 Behavior of x_1, x_2 and x_3 versus time t.

sibility of the saturated system are listed in Table 4.1. The number of iterations are also listed in Table 4.1. It is shown that as the predefined decaying rate α increases, the size of the set of the initial conditions $\Omega(\nu_1, \nu_2)$ decreases.

Table 4.1 Computation results of example 2 with $\overline{u} = 15$

α	0.001	0.2	0.4	0.6	0.8	1	1.2
ν_1	192.1172	97.0467	48.7601	25.8165	14.0812	7.9295	5.5883
ν_2	967.1209	509.6311	268.5460	165.2845	90.6967	37.1311	28.2688
Iterations	11	13	14	15	16	17	18

4.3 Extension to the Case of Input Delay

Considering the transfer delays of sensor-to-controller and controller-to-actuator that appear in many control systems. The stabilization problem of system (4.1) in the presence of input delay will be tackled. Therefore, system (4.1) is rewritten in the following form:

$$E\dot{x}(t) = Ax(t) + A_d x(t - d_1(t)) + B_1 \text{sat}(u(t)) + B_2 \text{sat}(u(t - d_2(t))) \tag{4.14}$$

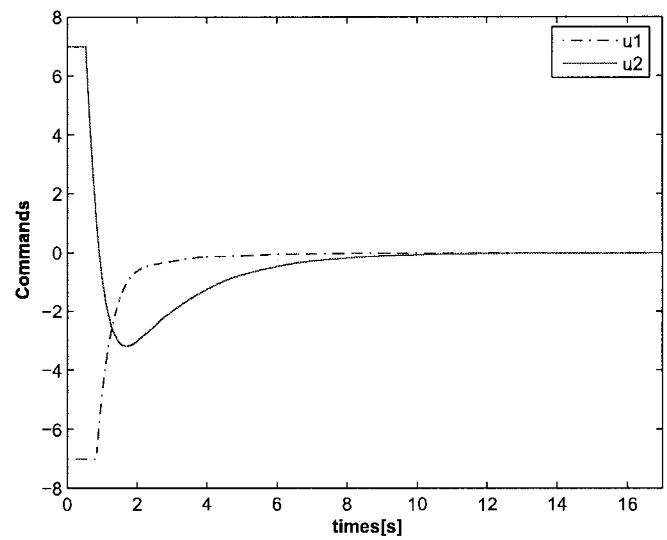

Figure 4.7 Behavior of the controllers

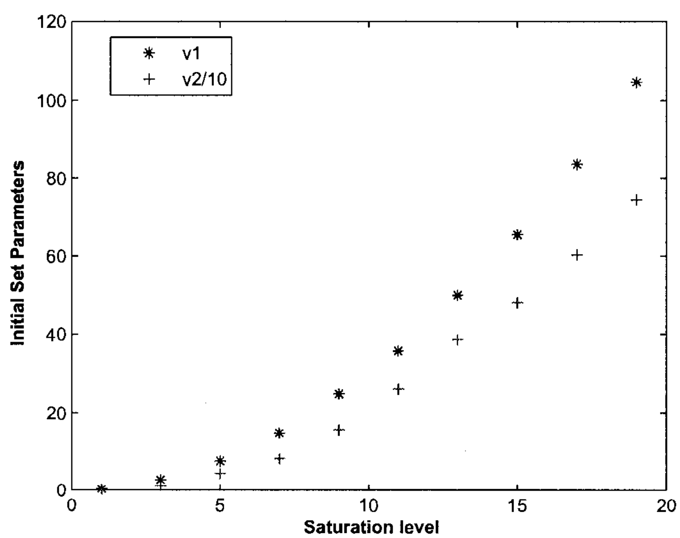

Figure 4.8 Values of ν_1 and ν_2 for which the exponential admissibility is guaranteed as a function of the control amplitude saturation level

$$y(t) = Cx(t) \tag{4.14b}$$

$$x(t) = \phi(t), \quad t \in [-\overline{d}, 0] \tag{4.14c}$$

with

$$0 < \underline{d}_k \leq d_k(t) \leq \overline{d}_k \qquad \text{and} \qquad \dot{d}_k(t) \leq \mu < 1 \qquad k{=}1{,}2$$

where $\underline{d}_k$ and $\overline{d}_k$ are given positive scalars. Also, $\overline{d}$ and $\underline{d}$ are positive scalars with $\overline{d} = \max\{\overline{d}_1, \overline{d}_2\}$ and $\underline{d} = \min\{\underline{d}_1, \underline{d}_2\}$.

The closed-loop system with controller (4.2) is rewritten as follows:

$$E\dot{x}(t) = Ax(t) + A_d x(t - d_1(t)) + B_1 \text{sat}(KCx(t)) + B_2 \text{sat}(KCx(t - d_2(t))) \tag{4.15}$$

The problem to be tackled in this section can be summarized as follows: Find a static output feedback law of the form (4.2) and a set of initial conditions such that the closed-loop system (4.15) is exponentially admissible with a predefined minimum decaying rate.

Express the saturation term as in (4.4), then system (4.15) can then be written in the following equivalent form:

$$\begin{aligned} E\dot{x}(t) = {} & (A + B_1 D(\rho(x(t)))KC)x(t) + A_d x(t - d_1(t)) \\ & + B_2 D(\rho(x(t - d_2(t))))KCx(t - d_2(t)) \end{aligned} \tag{4.16}$$

Similar to the previous section, let $0 \leq \underline{\rho}_i \leq 1$ be a lower bound to $\rho_i(x)$, and define a vector $\underline{\rho} = [\underline{\rho}_1, ..., \underline{\rho}_m]$. The vector $\underline{\rho}$ is associated to the following region in the state space:

$$S(K, \overline{u}^\rho) = \{x \in R^n \mid -\overline{u}^\rho \leq KCx \leq \overline{u}^\rho\}$$

where every component of the vector $\overline{u}^\rho$ is defined by $\overline{u}_i / \underline{\rho}_i$.

Define now matrices A_j, $j = 1, ..., 2^m$ and B_c, $c = 1, ..., 2^m$ as follows:

$$A_j = A + B_1 D(\gamma_j) KC$$
$$B_c = B_2 D(\gamma_c) KC$$

where $D(\gamma_j)$ and $D(\gamma_c)$ are diagonal matrices of positive scalars $\gamma_{j(i)}$ and $\gamma_{c(i)}$ for $i = 1, ..., m$, which arbitrarily take the value one or $\underline{\rho}_i$. Note that the matrices A_j and B_c are the vertices of two convex polytopes of matrices. If $x(t) \in S(K, \overline{u}^\rho)$ and $x(t - d_2(t)) \in S(K, \overline{u}^\rho)$, it follows that $(A + BD(\rho(x(t)))KC) \in co\{A_1, ..., A_{2^m}\}$ and $B_2 D(\rho(x(t - d_2(t))))KC \in co\{B_1, ..., B_{2^m}\}$. We conclude that if $x(t)$ and $x(t - d_2(t)) \in S(K, \overline{u}^\rho)$, then $E\dot{x}(t)$ can be determined from the following polytopic model:

$$E\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_{j,t} A_j x(t) + A_d x(t - d(t)) + \sum_{c=1}^{2^m} \tau_{c,t} B_c x(t - d_2(t)) \tag{4.17}$$

with $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$, $\lambda_{j,t} \geq 0$, $\sum_{c=1}^{2^m} \tau_{c,t} = 1$ and $\tau_{c,t} \geq 0$.

Theorem 4.3.1. *Let $0 < \underline{d}_k < \overline{d}_k$, $k = 1, 2$, $\mu < 1$ and α be given scalars. If there exist a non-singular matrix P, symmetric and positive-definite matrices Q_{k1}, Q_{k2}, Q_{k3}, Z_{k1} and Z_{k2}, $k = 1, 2$, and matrices M_{ki}, N_{ki} and S_{ki}, $i = 1, 2$, $k = 1, 2$, and K and a positive scalar κ such that*

$$\begin{bmatrix} \Pi_{jc} & \Upsilon & \tilde{A}_{jc} U \\ \star & T & 0 \\ \star & \star & -U \end{bmatrix} < 0 \quad j = 1, \ldots, 2^m, c = 1, \ldots, 2^m \tag{4.18}$$

$$\begin{bmatrix} E^\top P & \underline{\rho}_i (KC)_i^\top \\ \underline{\rho}_i (KC)_i & \kappa \overline{u}_i^2 \end{bmatrix} \geq 0, \quad i = 1, ..., m \tag{4.19}$$

with the following constraint

$$E^\top P = P^\top E \geq 0 \tag{4.20}$$

where

$$T = \mathrm{diag}\left\{ -\frac{e^{2\alpha\overline{d}_k}-1}{2\alpha} Z_{k1}, -\frac{e^{2\alpha\overline{d}_k}-e^{2\alpha\underline{d}_k}}{2\alpha}(Z_{k1}+Z_{k2}), -\frac{e^{2\alpha\overline{d}_k}-e^{2\alpha\underline{d}_k}}{2\alpha} Z_{k2} \right\}$$

$$k = 1, 2$$

$$\widetilde{A}_{jc}^\top = \begin{bmatrix} A + B_1 D(\gamma_j)KC & A_d & 0 & 0 & B_2 D(\gamma_c)KC & 0 & 0 \end{bmatrix}$$

$$U = \sum_{k=1}^{2} \left\{ \left(\overline{d}_k Z_{k1} + \overline{\underline{d}}_k Z_{k2} \right) \right\} \ \mathit{with}\ \overline{\underline{d}}_k = \overline{d}_k - \underline{d}_k$$

$$\Pi_{jc} = \begin{bmatrix} \Pi_{1j} & F_c \\ \star & G \end{bmatrix} \ \mathit{and}\ \Upsilon = \begin{bmatrix} \tilde{N}_1 & \tilde{S}_1 & \tilde{M}_1 & \tilde{N}_2 & \tilde{S}_2 & \tilde{M}_2 \end{bmatrix}$$

$$\widetilde{N}_k^\top = \begin{bmatrix} N_{k1}^\top & \mathbf{0}_{n\times 3n(k-1)} & N_{k2}^\top & 0 & 0 & \mathbf{0}_{n\times 3n(2-k)} \end{bmatrix}$$

$$\widetilde{M}_k^\top = \begin{bmatrix} M_{k1}^\top & \mathbf{0}_{n\times 3n(k-1)} & M_{k2}^\top & 0 & 0 & \mathbf{0}_{n\times 3n(2-k)} \end{bmatrix}$$

$$\widetilde{S}_k^\top = \begin{bmatrix} S_{k1}^\top & \mathbf{0}_{n\times 3n(k-1)} & S_{k2}^\top & 0 & 0 & \mathbf{0}_{n\times 3n(2-k)} \end{bmatrix}$$

$$\begin{aligned} \Pi_{1j} &= P^\top (A + B_1 D(\gamma_j)KC) + (A + B_1 D(\gamma_j)KC)^\top P \\ &\quad + \sum_{k=1}^{2} \left\{ \sum_{i=1}^{3} Q_{ki} + N_{k1}E + (N_{k1}E)^\top \right\} + 2\alpha E^\top P \end{aligned}$$

$$F_c = \begin{bmatrix} \Pi_{12} & e^{\alpha\underline{d}_1} M_{11}E & -e^{\alpha\overline{d}_1} S_{11}E & \Pi_{22c} & e^{\alpha\underline{d}_2} M_{21}E & -e^{\alpha\overline{d}_2} S_{21}E \end{bmatrix} \ \mathit{and}$$

$$G = \mathit{diag}\{J_k\} \quad k = 1, 2$$

$$J_k = \begin{bmatrix} \Pi_{k3} & e^{\alpha\underline{d}_k} M_{k2}E & -e^{\alpha\overline{d}_k} S_{k2}E \\ \star & -Q_{k1} & \mathbf{0} \\ \star & \star & -Q_{k2} \end{bmatrix}$$

$$\Pi_{12} = P^\top A_d + (N_{12}E)^\top - N_{11}E + S_{11}E - M_{11}E$$

$$\Pi_{22c} = P^\top B_2 D(\gamma_c)KC + (N_{22}E)^\top - N_{21}E + S_{21}E - M_{21}E$$

$$\begin{aligned} \Pi_{k3} &= -(1-\mu)e^{-2\alpha\overline{d}_k} Q_{k3} + S_{k2}E + (S_{k2}E)^\top - N_{k2}E - (N_{k2}E)^\top - M_{k2}E \\ &\quad - (M_{k2}E)^\top. \end{aligned}$$

then, there exists a static output feedback controller (4.2) such that the closed-loop system (4.15) is locally exponentially admissible with $\sigma = \alpha - \frac{1}{\underline{d}}$ for any compatible initial condition belonging to:

$$\Omega(\nu_1, \nu_2) = \{\phi \in C_{d_2}^{v} \quad : \quad \frac{\|\phi\|_c^2}{\nu_1} + \frac{\|\dot{\phi}\|_c^2}{\nu_2} \leq 1\} \tag{4.21}$$

where

$$\begin{aligned}
\nu_1 &= \frac{\kappa^{-1}}{\chi_1}, \qquad\qquad \nu_2 = \frac{\kappa^{-1}}{\chi_2} \\
\chi_1 &= \lambda_{max}(E^\top PE) + \sum_{k=1}^{2}\left\{\lambda_{max}(Q_{k1})\frac{1-e^{-2\alpha\underline{d}_k}}{2\alpha} + \lambda_{max}(Q_{k2})\frac{1-e^{-2\alpha\overline{d}_k}}{2\alpha}\right. \\
&\quad \left. + \lambda_{max}(Q_{k3})\frac{1-e^{-2\alpha\overline{d}_k}}{2\alpha}\right\} \\
\chi_2 &= \sum_{k=1}^{2}\left\{\lambda_{max}(Z_{k1})\lambda_{max}(E^\top E)\frac{2\alpha\overline{d}_k - 1 + e^{-2\alpha\overline{d}_k}}{4\alpha^2}\right. \\
&\quad \left. + \lambda_{max}(Z_{k2})\lambda_{max}(E^\top E)\frac{2\alpha\overline{\underline{d}_k} - e^{-2\alpha\underline{d}_k} + e^{-2\alpha\overline{d}_k}}{4\alpha^2}\right\}
\end{aligned}$$

Proof. Assume that $x(t) \in S(K, \overline{w}^\rho)$, $\forall t > 0$. Therefore, $E\dot{x}(t)$ can be determined from the polytopic system (4.17). Note that system (4.17) is equivalent to the following system:

$$E\dot{x}(t) = \sum_{j=1}^{2^m}\sum_{c=1}^{2^m} \lambda_{j,t}\tau_{c,t}\left\{A_j x(t) + A_d x(t-d(t)) + B_c x(t-d_2(t))\right\} \tag{4.22}$$

with $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$, $\lambda_{j,t} \geq 0$, $\sum_{c=1}^{2^m} \tau_{c,t} = 1$ and $\tau_{c,t} \geq 0$.

Then, if we apply Theorem (2.4.1) 4^m times to the parameters A_j with $j = 1, \ldots, 2^m$, B_c with $c = 1, \ldots, 2^m$, A_d, E, d_1, d_2 and μ, we will have (4.18) and (4.20). First, we will show that the system is regular and impulse-free. For this purpose, choose two nonsingular matrices R, L such that

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix}$$

and let

$$\bar{A}_j = RA_jL = \begin{bmatrix} A_{j11} & A_{j12} \\ A_{j21} & A_{j22} \end{bmatrix} \qquad \bar{P} = R^{-\top}PL = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Now, proceeding in a similar way as for the proof of regularity and impulse-free in Theorem 2.2.1, yields

$$A_{j22}^{\top}P_{22} + P_{22}^{\top}A_{j22} < 0, \; j = 1, \dots 2^m$$

Using the fact that $\lambda_{j,t} \geq 0$,

$$\lambda_{j,t}A_{j22}^{\top}P_{22} + P_{22}^{\top}\lambda_{j,t}A_{j22} < 0, \; j = 1, \dots 2^m, \; \forall t \in (0, \infty)$$

adding these inequalities together, gives

$$\left[\sum_{j=1}^{2^m} \lambda_{j,t}A_{j22}\right]^{\top} P_{22} + P_{22}^{\top}\sum_{j=1}^{2^m} \lambda_{j,t}A_{j22} < 0$$

which implies

$$\sum_{j=1}^{2^m} \lambda_{j,t}A_{j22} \text{ is nonsingular} \quad \forall t \in (0, \infty).$$

which implies that system (4.17) is regular and impulse-free. Now, choose a Lyapunov functional as in Theorem (2.4.1), and proceeding in a similar manner as in the proof of Theorem (2.4.1), then

$$\dot{V}(\zeta_t) + 2\alpha V(\zeta_t) \leq$$

$$\leq \eta^{\top}(t)\left[\Pi + \sum_{k=1}^{2}\left\{\widetilde{A}^{\top}\left(\overline{d}_k\bar{Z}_{k1} + \underline{d}_k\bar{Z}_{k2}\right)\widetilde{A} + \frac{e^{2\alpha\overline{d}_k}-1}{2\alpha}\widetilde{N}_k\bar{Z}_{k1}^{-1}\widetilde{N}_k^{\top}\right.\right.$$
$$\left.\left. + \frac{e^{2\alpha\overline{d}_k}-e^{2\alpha\underline{d}_k}}{2\alpha}\widetilde{S}_k\left(\bar{Z}_{k1}+\bar{Z}_{k2}\right)^{-1}\widetilde{S}_k^{\top} + \frac{e^{2\alpha\overline{d}_k}-e^{2\alpha\underline{d}_k}}{2\alpha}\widetilde{M}_k\bar{Z}_{k2}^{-1}\widetilde{M}_k^{\top}\right\}\right]\eta(t)$$

with all the variables as defined in Theorem (2.4.1), A is replaced by $\sum_{j=1}^{2^m}\lambda_{j,t}A_j$ and A_{d2} is replaced by $\sum_{c=1}^{2^m}\tau_{c,t}B_c$. Then, by convexity, condition (4.18), (4.20) and noting that $\sum_{j=1}^{2^m}\lambda_{j,t}=1$, $\lambda_{j,t}\geq 0$, $\sum_{j=1}^{2^m}\tau_{j,t}=1$ and $\tau_{j,t}\geq 0$,

$$\dot{V}(\zeta_t) + 2\alpha V(\zeta_t) \leq 0$$

Completing the proof in a similar manner as in Theorem (2.4.1), yields the exponential stability result.

Now, by virtue of condition (4.19), $x(t)\in S(K,\overline{w}^{\rho})$, $\forall t>0$. Therefore, $E\dot{x}(t)$ can be determined from the polytopic system (4.17). This completes the proof. □

In order to maximize the ball of initial conditions, following the idea in (Tarbouriech and Gomes da Silva, 2000), we suggest the following optimization problem:
OP: Minimize $w_1\sum_{k=1}^{2}\left(\delta_{k1} + \frac{1-e^{-2\alpha\underline{d}_k}}{2\alpha}\delta_{k2} + \frac{1-e^{-2\alpha\overline{d}_k}}{2\alpha}\delta_{k3} + \frac{1-e^{-2\alpha\overline{d}_k}}{2\alpha}\delta_{k4}\right) +$
$w_2\sum_{k=1}^{2}\left(\lambda_{max}(E^{\top}E)\frac{2\alpha\overline{d}_k-1+e^{-2\alpha\overline{d}_k}}{4\alpha^2}\delta_{k5} + \lambda_{max}(E^{\top}E)\frac{2\alpha\underline{d}_k-e^{-2\alpha\underline{d}_k}+e^{-2\alpha\overline{d}_k}}{4\alpha^2}\delta_{k6}\right)$
$+w_3\kappa$ subject to the previous BMIs and the following LMIs:

$$\delta_1\mathbb{I}\geq E^{\top}PE \qquad \delta_{k2}\mathbb{I}\geq Q_{k1} \qquad \delta_{k3}\mathbb{I}\geq Q_{k2}$$
$$\delta_{k4}\mathbb{I}\geq Q_{k3} \qquad \delta_{k5}\mathbb{I}\geq Z_{k1} \qquad \delta_{k6}\mathbb{I}\geq Z_{k2}$$

where $k=1,2$ and w_1, w_2 and w_3 are weighting factors.

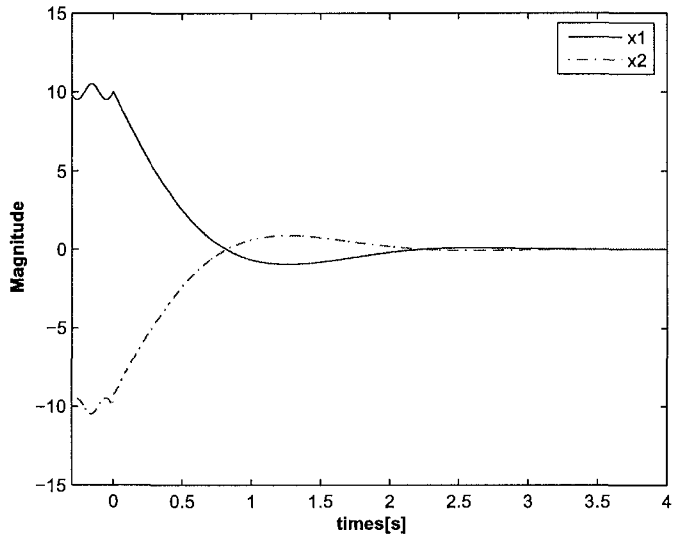

Figure 4.9 Behavior of x_1 and x_2 versus time t.

Example: Consider the time-delay singular system described by:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & -0.5 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Letting $\underline{d}_1 = 0.2$, $\overline{d}_1 = 0.4$, $\underline{d}_2 = 0.1$, $\overline{d}_2 = 0.3$, $\mu = 0.4$, $\overline{u} = 20$ and $\alpha = 0.1$, using PENBMI, gives

$$K = \begin{bmatrix} -1.4452 \\ 0.0308 \end{bmatrix}, \qquad \nu_1 = 146.69 \qquad \nu_2 = 213.22$$

Figures 4.9 and 4.10 gives the simulation results for the closed-loop system when $d_1(t) = |0.3 + 0.3\sin(t)|$, $d_2(t) = |0.2 + 0.3\sin(t)|$ and the initial condition is $\phi(t) = [10 + 0.5\sin(30t),\ -(10 + 0.5\sin(30t))]^\top, t \in [-0.4, 0]$.

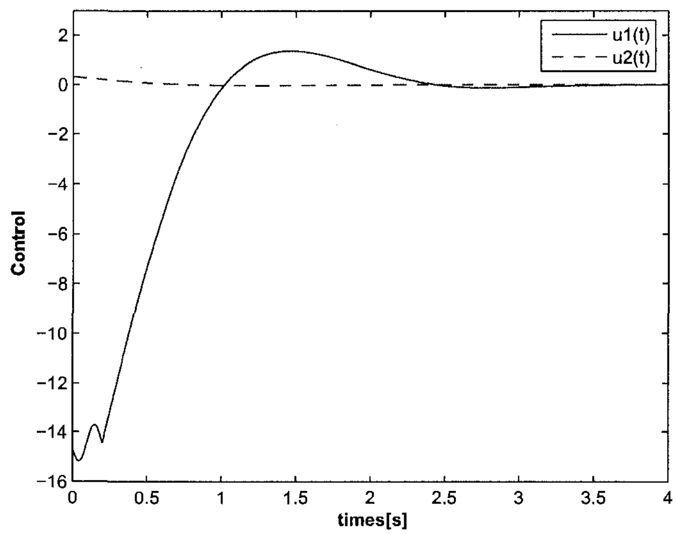

Figure 4.10 Behavior of the controllers

4.4 Conclusion

This chapter has dealt with the static output feedback stabilization of the class of time-delay singular systems in the presence of actuator saturation. A delay-range-dependent static output feedback controller has been designed and an ILMI algorithm has been proposed to compute the controller gains. Then, system dynamics is generalized to include input delay and a set of BMIs has to be solved to give the controller gains. The effectiveness of the results has been illustrated through examples.

CONCLUSION

In this thesis, the class of singular systems with time-varying delays has been treated. The results are based on time domain methods and the conditions are mainly presented in the LMI setting. There are efficient numerical methods to determine whether an LMI is feasible or to solve a convex optimization problem with LMI constraints (Boyd et al., 1994); namely, interior point methods. The free solver SEDUMI is used to solve the convex optimization problems while PENBMI is used to solve the BMI problems. YALMIP is used as an interface in both cases. For a brief introduction on SEDUMI and PENBMI, we refer the reader to (Lofberg, 2004).

In chapter 1, some preliminary results and definitions for time-delay singular systems have been presented. The solution behavior of singular systems and the impulsive behavior of the solutions are studied. It has been shown that the jump discontinuities propagate via the algebraic constraints, and consequently, they propagate between different states. Also, the solution behavior of time-delay systems and the propagation of jump discontinuities in the higher derivatives of the solution vector are studied. Due to the existence of delayed terms, these jump discontinuities also propagate between different time instances. For time-delay singular systems, where both constraints and delayed terms exists, jumps propagate between different time instances and different state components. This behavior is inherited from both singular systems and time-delay systems. Some definitions are presented to insure the existence and uniqueness of solutions and the absence of any impulsive behavior. Incompatible initial conditions can also give rise to impulses and the notion of compatible initial conditions has been discussed. Physical example that can be modeled by the class of time-delay singular systems are presented to motivate the work in this thesis.

Chapter 2 has dealt with the stability of the class of singular systems with single and multiple time-varying delays. The free-weighting matrices approach is adopted to reduce the conservatism of the results. First, delay-dependent conditions have been stated such that the singular time-delay system is regular, impulse-free and stable. This result can be used to generalize many results appear in the literature for singular systems with constant time-delays to the case of time-varying delays. Then, delay-range-dependent conditions have been stated such that the singular time-delay system is regular, impulse-free and exponentially stable. Moreover, an estimate of the convergence rate of such stable systems has been presented. Then, this result has been extended to the case of multiple time-varying delays. Some graph theory terminologies and concepts have been adopted in order to prove the stability of the algebraic subsystem. The results in this chapter are expressed in terms of LMIs.

Chapter 3 has dealt with the state feedback stabilization problem of this class of systems. First, LMI conditions are provided in order to design a stabilizing state feedback controller. Then, LMI conditions are provided in order to solve the state feedback $\mathscr{H}_\infty$ stabilization problem. Note that an equality constraint was imposed in order to change the problem from BMI to LMI. This equality constraint introduces some inevitable conservatism.

In Chapter 4, some nonlinearities are introduced in the model of the system. These nonlinearities are introduced in the control as saturation nonlinearities. This was motivated by the fact that every physical actuator is subject to saturation. The static output feedback stabilization problem is then tackled. A local stabilization problem is considered where the controller gains and a set of initial conditions should be given such that the system is guaranteed to be regular impulse-free and stable. Also, the results allow us to predefine a decaying rate for the states that the closed-loop system has to satisfy. The nonlinear system is transformed to a linear polytopic system. The conditions for solving the problem turned to be BMIs, whose efficient softwares to solve are not available up to date. Solving BMIs is an N-P hard problem. Therefore, an iterative LMI algorithm was proposed in order to solve for the controller gains. The algorithm was proposed to enlarge the initial conditions set as large as possible. The problem is extended to the case where input delay appears. BMI conditions have to be solved in order to get the controller gains and the set of initial conditions. It is not easy to get an iterative algorithm for these BMIs and PENBMI is used to solve for the controller gains. Note that PENBMI gives only a local solution.

As a future work, we would propose the following.

- PID.

  The most popular method used in the industry today is the PID controller. As it was discussed in Chapter 1, many chemical processes can modeled as time-delay singular systems. Therefore, PID controller design for singular time-delay systems is an interesting problem to tackle. A work concerning PID controller design for singular systems and its application to chemical processes can be found in (Rao et al., 2003).

- Controllability and observability.

  Controllability is an important property of a control system, and the controllability property plays a crucial role in many control problems, such as stabilization of unstable systems by feedback. To the best of the author's knowledge, no work has been reported in the literature regarding the controllability or observability of time-delay singular systems.

- Nonlinear time-delay singular systems.

  It would be of considerable interest to study the extensions to the nonlinear case. The general mathematical description of nonlinear time-delay singular systems is given by the following form:

$$\dot{x}(t) = F(t, x(t), x(t-d(t)), y(t), y(t-d(t)), u(t))$$

$$0 = G(t, x(t), x(t-d(t), y(t), y(t-d(t)), u(t))$$

The asymptotic stability of this class of systems is studied in (Pepe, 2005; Pepe et al., 2006; Pepe and Verriest, 2003). There is a lot of work that has to be done in the context of analysis and control of this class of nonlinear systems.

- Markovian jump singular systems with delays.

  A class of stochastic systems driven by continuous-time Markov chains has been used to model many practical systems, where random failures and repairs and sudden environment changes may occur. This class of systems is referred to in the literature as Markovian jump systems. Applications of Markovian jump systems include failures and repairs of machine in manufacturing systems, modifications of the operating point of a linearized model of a nonlinear system, power systems and economics systems. Also, networked control systems are modeled as jump linear systems (Chan and Ozguner, 1995; Krtolica et al., 1994). For more details on what has been done on this class of systems, we refer the reader to the recent books by Boukas (Boukas and Liu, 2002; Boukas, 2005). For the class of Markovian jump singular systems, we refer the reader also to the other recent book by Boukas (Boukas, 2008). There are only few papers on Markovian jump singular systems with delays (Boukas et al., 2005a; Boukas et al., 2005b) and there is a lot of work that has to be done in this context.

- In (Michiels et al., 2004), two recent proposed simple modifications/generalizations of static output feedback are investigated; namely, introducing time-delay in the control law and making the gain time-varying. Both approaches have been shown to be complementary and existing results are brought together in a unifying framework. Motivated by this work, the generalization of the static output feedback controller presented here could be the subject of a forthcoming work.

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