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Improved linearized model collision operator for the highly collisional regime

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The linearized model collision operator for multiple species plasmas given by H. Sugama, T.-H. Watanabe, and M. Nunami [Phys. Plasmas **16**, 112503 (2009)] is improved to be properly applicable up to the highly collisional regime. The improved linearized model operator retains conservation laws of particles, momentum, and energy as well as it reproduces the same friction-flow relations as derived by the linearized Landau operator so that this model can be used to correctly evaluate neoclassical transport fluxes in all collisionality regimes. The adjointness relations and Boltzmann's H-theorem are exactly satisfied by the improved operator except in the case of collisions between unlike particle species with unequal temperatures where these relations and H-theorem still holds approximately because there is a large difference between the masses of the two species with significantly different temperatures. Even in the unequal-temperature case, the improved operator can also be modified so as to exactly satisfy the adjointness relations while it causes the values of the friction coefficients to deviate from those given by the Landau operator. In addition, for application to gyrokinetic simulations of turbulent transport, the improved operator is transformed into the gyrophase-averaged form with keeping the finite gyroradius effect.

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I. INTRODUCTION

Coulomb collisions are the main mechanism which causes classical and neoclassical transport in magnetically confined plasmas.[1–5] Even though plasma confinement is generally dominated by turbulent transport rather than by collisional transport, collisions still have impacts on structures of phase-space distribution functions of particles, growth rates of instabilities, and micro/macroscopic profiles of plasma flows so that they indirectly influence turbulent transport processes as well.[6–11] Also, transport processes of heavy impurities with high charge numbers which penetrate from the edge into the core region are greatly affected by Coulomb collisions.[12–15] So far, there have been numerous works on model collision operators[16–30] for application to theoretical and numerical studies of plasma transport.

A well-established Coulomb collision term is given by the Landau operator[31] which is nonlinear for like-species collisions or bilinear for unlike-species collisions. The linearized Landau operator[32–34] obtained by perturbatively expanding the distribution functions about the local Maxwellian is more tractable than the full Landau operator[35–37] and the former is preferred to be used for transport studies when the deviation from the Maxwellian is sufficiently small. Since the field particle part of the linearized Landau operator is not as easy to evaluate as its test particle part, several linearized model collision operators have been proposed, in which simplified versions of the field particle part are used.[17–23] As an example, Sugama *et al.*[23] presented a linearized model collision operator for multiple ion species plasmas which conserves particles, momentum, and energy, and satisfies adjointness relations and Boltzmann's H-theorem even for collisions between unlike particle species with unequal temperatures. This model called the Sugama operator has been successfully applied to studies of neoclassical and turbulent transport in relatively low collisional regimes.[38–45]

The difference between the field particle part of the Sugama operator and that of the exact linearized Landau operator is anticipated to increase in a highly collisional regime. Even in very-high-temperature fusion plasmas like the ITER plasma,[14,15,46] minority impurity ions such as tungsten are considered to remain in the Pfirsch-Schlüter regime even though bulk ions and electrons are in the banana regime. For such a case, it is necessary to use a collision model which is accurate in all collisionality regimes. In this work, the Sugama operator is improved to present the new linearized model collision operator, which is properly applicable to all cases from low to high collisionality. The improved model is constructed so as to give exactly the same friction-flow relations as those derived from the linearized Landau operator. Therefore, it can be used in drift kinetic simulations to accurately evaluate neoclassical transport fluxes in all collisionality regimes. Then, it is noted that the exact friction-flow relations no longer rigorously keep the symmetry property in the case of collisions between unlike particle species with unequal temperatures, where neither the improved model operator nor the linearized Landau operator is completely self-adjoint. Since the self-adjointness is practically useful for analytical or numerical derivation of the Onsager symmetric neoclassical transport coefficients,[1–5,47–50] further modification of the improved model for the unequal-temperature case is considered in the present paper to restore the adjointness relations by relaxing the accuracy of the friction-flow relations. In addition, the improved collision operator in the form suitable for application to gyrokinetic simulations of turbulent transport is derived by taking the gyrophase average with the finite gyroradius effect taken into account.

It is instructive to note here that Hirshman and Sigmar[16] presented a linearized model collision operator which is similar to ours in that spherical harmonic functions and Laguerre polynomials are used to expand distribution functions as well as key properties of the original linearized Landau collision operator are retained. In their work,[16] an elegant and skillful method of constructing novel basis functions is presented to approximate both test and field particle operators including spherical harmonic functions of all degree numbers (l's) although an explicit expression of their model collision operator is given in their paper only for the case where spherical harmonic functions of degrees $l > 2$ are dropped. To satisfy conservation laws of momentum and energy, the field particle part of the Sugama collision operator contains the $l = 0$ and 1 parts which are expressed using the test particle part and take similar forms to those of the Hirshman-Sigmar operator. In the present paper, the improved Sugama collision operator is given by adding the correction terms into only the $l = 1$ spherical harmonic component of the original Sugama operator in order to correctly reproduce the friction-flow relations which determine collisional transport and influence turbulent transport through controlling micro/macroscopic plasma flow profiles. However, the procedures shown in the present work can be extended to give correction terms to all other spherical harmonic components. It is also shown by Abel *et al.*[22] that, when the Hirshman-Sigmar model operator is transformed to its gyrophase-averaged form for application to the gyrokinetic equation, the problematic gyroradius dependence appears in the energy diffusion term in the test particle operator. Therefore, for the gyrokinetic case, our model operator is more favorable than the Hirshman-Sigmar model operator.

The rest of this paper is organized ad follows. In Sec. II, we briefly explain the Landau collision operator and its linearization, from which the associated matrix elements are defined to obtain the friction coefficients entering the friction-flow relations. Then, after reviewing the definition and properties of the original Sugama operator in Sec. III, its improved version is presented in Sec. IV, where we write down the correction term to reproduce the same matrix elements and friction coefficients as given by the linearized Landau operator. In Sec. V, the improved operator is expressed in the form suitable for gyrokinetic equations. Finally, conclusions are given in Sec. VI. In Appendix A, a collisional energy transfer rate between unlike species with unequal temperatures is estimated depending on the ratio between the masses of the two species. In Appendix B, effects of unequal temperatures of colliding particle species on the adjointness relations and matrix elements associated with the linearized Landau operator are discussed. The detailed expressions of the matrix elements are shown in Appendix C. In addition, Appendix D presents a modified version of the improved operator which exactly satisfies the adjointness relations even for collisions between unlike particle species with unequal temperatures although it consequently makes the values of the friction coefficients deviate from those given by the Landau operator.

II. LANDAU COLLISION OPERATOR AND FRICTION-FLOW RELATIONS

The Landau operator for collisions between particle species a and b is written as[31]

$$C_{ab}(f_a, f_b) \equiv -\frac{2\pi e_a^2 e_b^2 \ln\Lambda}{m_a}\frac{\partial}{\partial \mathbf{v}}\cdot\left[\int d^3v'\, \mathbf{U}(\mathbf{v}-\mathbf{v}') \cdot\left\{\frac{f_a(\mathbf{v})}{m_b}\frac{\partial f_b(\mathbf{v}')}{\partial \mathbf{v}'} - \frac{f_b(\mathbf{v}')}{m_a}\frac{\partial f_a(\mathbf{v})}{\partial \mathbf{v}}\right\}\right], \quad (1)$$

where

$$\mathbf{U}(\mathbf{v}-\mathbf{v}') \equiv \frac{|\mathbf{v}-\mathbf{v}'|^2\,\mathbf{I} - (\mathbf{v}-\mathbf{v}')(\mathbf{v}-\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|^3}, \quad (2)$$

and $\ln\Lambda$ is the Coulomb logarithm. The particle mass and charge are denoted by m_s and e_s, respectively, where the particle species is denoted by the subscript $s(= a, b)$. The distribution function $f_s(\mathbf{v})$ generally depends not only on the velocity $\mathbf{v}$ but also on the position and time variables $(\mathbf{x}, t)$ although the dependence on $(\mathbf{x}, t)$ are not explicitly shown here. Writing the distribution function by the sum of the equilibrium part and the small perturbation part as $f_s = f_{s0} + \delta f_s$, we obtain

$$C_{ab}(f_a, f_b) = C_{ab}(f_{a0}, f_{b0}) + C_{ab}(\delta f_a, f_{b0}) + C_{ab}(f_{a0}, \delta f_b) + C_{ab}(\delta f_a, \delta f_b), \quad (3)$$

where the last term $C_{ab}(\delta f_a, \delta f_b)$ is neglected hereafter.

We now assume the equilibrium distribution functions to take the Maxwellian form $f_{s0} = f_{sM} \equiv (n_s/\pi^{3/2}v_{Ts}^3)\exp(-v^2/v_{Ts}^2)$ where n_s is the density, $v_{Ts} \equiv (2T_s/m_s)^{1/2}$ is the thermal velocity, and T_s is the temperature. Then, the first term on the right-hand side of Eq. (3) is written as

$$C_{ab}(f_{aM}, f_{bM}) = -3\sqrt{\pi}\left(\frac{T_a}{T_b} - 1\right)\frac{f_{aM}}{\tau_{ab}}x_a \times\left[G(\alpha_{ab}x_a) - \frac{\alpha_{ab}}{2x_a}\Phi'(\alpha_{ab}x_a)\right], \quad (4)$$

where $x_a \equiv v/v_{Ta}$, $\alpha_{ab} \equiv v_{Ta}/v_{Tb}$, $G(x) \equiv [\Phi(x) - x\Phi'(x)]/(2x^2)$, $\Phi(x) \equiv 2\pi^{-1/2}\int_0^x e^{-t^2}dt$, and $\Phi'(x) \equiv 2\pi^{-1/2}e^{-x^2}$. The collision time τ_{ab} is defined by $(3\sqrt{\pi}/4)\tau_{ab}^{-1} \equiv 4\pi n_b e_a^2 e_b^2 \ln\Lambda/(m_a^2 v_{Ta}^3)$. It is easily seen that $C_{ab}(f_{aM}, f_{bM})$ vanishes for $T_a = T_b$. We hereafter assume that $T_a/T_b = \mathcal{O}(1)$. When $m_a/m_b = \mathcal{O}(1)$, we have $\alpha_{ab} = \mathcal{O}(1)$ and $C_{ab}(f_{aM}, f_{bM}) \sim -(T_a/T_b - 1) f_{aM}/\tau_{ab}$. In this case, as explained in Appendix A, we may consider that collisions cause species a and b to have the equal temperature $T_a = T_b$ after a time scale longer than τ_{ab}.

The second and third terms on the right-hand side of Eq. (3) are called the test and field particle parts, respectively, and the sum of them gives the linearized collision operator,

$$\begin{aligned} C^L_{ab}(\delta f_a, \delta f_b) &\equiv C_{ab}(\delta f_a, f_{bM}) + C_{ab}(f_{aM}, \delta f_b) \\ &\equiv C^T_{ab}(\delta f_a) + C^F_{ab}(\delta f_b). \end{aligned} \quad (5)$$

We now expand the perturbed distribution functions δf_s ($s = a, b$) as

$$\begin{aligned} \delta f_s(\mathbf{v}) &= \sum_{l=0}^{\infty} \delta f_s^{(l)}(\mathbf{v}), \\ \delta f_s^{(l)}(\mathbf{v}) &= \sum_{m=-l}^{l} (\delta f_s)^m_l(v) Y^m_l(\theta, \varphi), \end{aligned} \quad (6)$$

where $Y^m_l(\theta, \varphi)$ represent spherical harmonic functions and (v, θ, φ) are spherical coordinates in the velocity space. The $l = 1$ component $\delta f_s^{(l=1)}$ of the distribution function δf_s is further expanded in terms of the Laguerre polynomials $L_j^{(3/2)}(x_s^2)$ $[L_0^{(3/2)}(x_s^2) = 1, L_1^{(3/2)}(x_s^2) = \frac{5}{2} - x_s^2, \cdots]$ as

$$\begin{aligned} \delta f_s^{(l=1)} &= f_{sM} \frac{m_s}{T_s} \mathbf{v} \cdot \left[\mathbf{u}_s + \frac{2}{5} \frac{\mathbf{q}_s}{p_s} \left(x_s^2 - \frac{5}{2} \right) + \cdots \right] \\ &= f_{sM} \frac{m_s}{T_s} \mathbf{v} \cdot \sum_{j=0}^{\infty} \mathbf{u}_{sj} L_j^{(3/2)}(x_s^2), \end{aligned} \quad (7)$$

where $x_s \equiv v/v_{Ts}$. The flow vectors $\mathbf{u}_{sj}$ $(j = 0, 1, 2, \cdots)$ are defined by

$$\begin{aligned} \mathbf{u}_{sj} &\equiv \frac{c_j}{n_s} \int d^3v \, \delta f_s L_j^{(3/2)}(x_s^2) \mathbf{v}, \\ c_j &\equiv \frac{3 \cdot 2^j \cdot j!}{(2j+3)!!}. \end{aligned} \quad (8)$$

For $j = 0$ and $j = 1$, we can write $\mathbf{u}_{s0} = \mathbf{u}_s$ and $\mathbf{u}_{s1} = -(2/5)(\mathbf{q}_s/p_s)$, where $\mathbf{u}_s \equiv n_s^{-1} \int d^3v \ \delta f_s \mathbf{v}$ and $\mathbf{q}_s \equiv T_s \int d^3v \ \delta f_s \mathbf{v}(x_s^2 - \frac{5}{2})$ represent the fluid velocity and the heat flow, respectively.

We next use the Laguerre polynomials to expand the $l = 1$ spherical harmonic component of the collision operator before deriving the friction-flow relations in Eq. (12). The resultant expansion [given below in Eq. (9)] contains the coefficients (denoted by $\mathbf{C}_{abj}$) as functionals of distribution functions, into which the expression in Eq. (7) is substituted to define the matrix elements M^{jk}_{ab} and N^{jk}_{ab} for representing the the friction coefficients l^{ab}_{ij} later. The $l = 1$ component of the collision term in Eq. (5) is written as

$$\begin{aligned} C^L_{ab}(\delta f_a^{(l=1)}, \delta f_b^{(l=1)}) &\equiv C^T_{ab}(\delta f_a^{(l=1)}) + C^F_{ab}(\delta f_b^{(l=1)}) \\ &= f_{aM} \frac{m_a}{T_a} \mathbf{v} \cdot \sum_{j=0}^{\infty} \mathbf{C}_{abj} L_j^{(3/2)}(x_a^2). \end{aligned} \quad (9)$$

Here, $\mathbf{C}_{abj}$ $(j = 0, 1, 2, \cdots)$ are defined by

$$\begin{aligned} \mathbf{C}_{abj} &\equiv \frac{c_j}{n_a} \int d^3v \, \mathbf{v} L_j^{(3/2)}(x_a^2) C^L_{ab}(\delta f_a, \delta f_b) \\ &= \frac{c_j}{\tau_{ab}} \sum_{k=0}^{\infty} \left(M^{jk}_{ab} \mathbf{u}_{ak} + N^{jk}_{ab} \mathbf{u}_{bk} \right), \end{aligned} \quad (10)$$

where the matrix elements M^{jk}_{ab} and N^{jk}_{ab} $(j, k = 0, 1, 2, \cdots)$ are given from the test and field particle operators, respectively, as[3]

$$\begin{aligned} \frac{n_a}{\tau_{ab}} M^{jk}_{ab} &\equiv \int d^3v \ v_\parallel L_j^{(3/2)}(x_a^2) C^T_{ab} \left(f_{aM} L_k^{(3/2)}(x_a^2) \frac{m_a v_\parallel}{T_a} \right), \\ \frac{n_a}{\tau_{ab}} N^{jk}_{ab} &\equiv \int d^3v \ v_\parallel L_j^{(3/2)}(x_a^2) C^F_{ab} \left(f_{bM} L_k^{(3/2)}(x_b^2) \frac{m_b v_\parallel}{T_b} \right). \end{aligned} \quad (11)$$

In Eq. (11), $v_\parallel$ denotes the velocity component parallel to the background magnetic field although it can be replaced with the velocity component in any other direction because of the spherical symmetry of the collision operator.

Using the linear collision operator, the friction forces $\mathbf{F}_{ai}$ $(i = 1, 2, \cdots)$ are given by[3]

$$\begin{aligned} \mathbf{F}_{ai} &\equiv (-1)^{i-1} \int d^3v \ m_a \mathbf{v} L_{i-1}^{(3/2)}(x_a^2) \sum_b C^L_{ab}(\delta f_a, \delta f_b) \\ &= (-1)^{i-1} \frac{n_a m_a}{c_{i-1}} \sum_b \mathbf{C}_{ab,i-1} \\ &= (-1)^{i-1} \sum_b \sum_{j=1}^{\infty} l^{ab}_{ij} \mathbf{u}_{b,j-1} \quad (i = 1, 2, \cdots). \end{aligned} \quad (12)$$

Here, the first two-order friction forces are written as $\mathbf{F}_{a1} = n_a m_a \sum_b \mathbf{C}_{ab0} = \int d^3v \ \ m_a \mathbf{v} \sum_b$ and $\mathbf{F}_{a2} = -\frac{5}{2} n_a m_a \sum_b \mathbf{C}_{ab1} = \int d^3v \ \ m_a \mathbf{v} \left(x_a^2 - \frac{5}{2} \right) \sum_b$. The friction coefficients l^{ab}_{ij} $(i, j = 1, 2, \cdots)$ are defined by[3]

$$l^{ab}_{ij} \equiv n_a m_a \left[\left(\sum_c \frac{M^{i-1,j-1}_{ac}}{\tau_{ac}} \right) \delta_{ab} + \frac{N^{i-1,j-1}_{ab}}{\tau_{ab}} \right], \quad (13)$$

where δ_{ab} denotes the Kronecker delta ($\delta_{ab} = 1$ for $a = b$ and $\delta_{ab} = 0$ for $a \neq b$).

From the momentum conservation in collisions [see Eq. (B8) in Appendix B], we obtain

$$M^{0j}_{ab} + \frac{T_a v_{Ta}}{T_b v_{Tb}} N^{0j}_{ba} = 0 \quad (j = 0, 1, 2, \cdots), \quad (14)$$

and

$$\sum_a l^{ab}_{1j} = 0 \quad (j = 1, 2, \cdots). \quad (15)$$

The adjointness relations for the linearized Landau collision operator is written as

$$\begin{aligned} \int d^3v \ \frac{\delta f_a}{f_{aM}} C^T_{ab}(\delta g_a) &= \int d^3v \ \frac{\delta g_a}{f_{aM}} C^T_{ab}(\delta f_a), \\ T_a \int d^3v \ \frac{\delta f_a}{f_{aM}} C^F_{ab}(\delta f_b) &= T_b \int d^3v \ \frac{\delta f_b}{f_{bM}} C^F_{ba}(\delta f_a). \end{aligned} \quad (16)$$

Strictly speaking, the linearized Landau operator satisfies the adjointness relations in Eq. (16) rigorously only for the case of $T_a = T_b$. In this case, the symmetry properties of M^{ij}_{ab}, N^{ij}_{ab}, and l^{ab}_{ij} are derived from Eq. (16) as

$$M^{ij}_{ab} = M^{ji}_{ab}, \quad \frac{N^{ij}_{ab}}{T_a v_{Ta}} = \frac{N^{ji}_{ba}}{T_b v_{Tb}} \quad (i,j = 0,1,2,\cdots),$$
$$l^{ab}_{ij} = l^{ba}_{ji} \quad (i,j = 1,2,\cdots). \tag{17}$$

As explained in Appendix A, T_a and T_b are significantly different from each other only when $m_a/m_b \ll 1$ or $m_a/m_b \gg 1$. It is explained in Appendix B that, even for this case of unequal temperatures, the adjointness relations given in Eq. (16), the symmetry properties in Eq. (17), and Boltzmann's H-theorem in the form shown later in Eq. (30) are regarded as approximately valid because of the large difference between m_a and m_b. We also should note that the Onsager symmetry for collisional transport coefficients is derived from the adjointness relations which are also used to give useful methods for solving drift kinetic equations and evaluating neoclassical transport fluxes.[1–5,47–50]

III. SUGAMA OPERATOR

The linearized model collision operator for collisions between species a and b given by Sugama *et al.*[23] is written here as

$$C^{LS}_{ab}(\delta f_a, \delta f_b) = C^{TS}_{ab}(\delta f_a) + C^{FS}_{ab}(\delta f_b). \tag{18}$$

The test-particle part $C^{TS}_{ab}(\delta f_a)$ of the Sugama operator is defined by

$$C^{TS}_{ab}(\delta f_a) = \mathcal{Q}_{ab}\, C^{T0}_{ab}\, \mathcal{Q}_{ab}\, \delta f_a, \tag{19}$$

where C^{T0}_{ab} is defined by Eq. (B2) in Appendix B and the operator $\mathcal{Q}_{ab}$ is given for an arbitrary distribution function g by

$$\mathcal{Q}_{ab}\, g \equiv g + (\theta_{ab} - 1)(\mathcal{P}_{1a}\, g + \mathcal{P}_{2a}\, g), \tag{20}$$

with the dimensionless parameter θ_{ab},

$$\theta_{ab} \equiv \left[\frac{T_a\left(\frac{1}{m_a} + \frac{1}{m_b}\right)}{\left(\frac{T_a}{m_a} + \frac{T_b}{m_b}\right)} \right]^{1/2} = \left(\frac{T_a/T_b + \alpha^2_{ab}}{1 + \alpha^2_{ab}} \right)^{1/2}. \tag{21}$$

The projection operators $\mathcal{P}_{1a}$ and $\mathcal{P}_{2a}$ is defined by

$$\mathcal{P}_{1a}\, g \equiv f_{aM} \frac{m_a}{T_a} \mathbf{u}_a[g] \cdot \mathbf{v},$$
$$\mathcal{P}_{2a}\, g \equiv f_{aM} \frac{\delta T_a[g]}{T_a} \left(x_a^2 - \frac{3}{2} \right), \tag{22}$$

where $\mathbf{u}_a[g] \equiv n_a^{-1} \int d^3v\, g \mathbf{v}$ and $\delta T_a[g]/T_a \equiv n_a^{-1} \int d^3v\, g(m_a v^2/3T_a - 1)$. The definition of θ_{ab} is given so as to satisfy $\int d^3v\, m_a \mathbf{v}\, C^{TS}_{ab}(f_{aM} m_a \mathbf{v}/T_a) = \int d^3v\, m_a \mathbf{v}\, C^{T}_{ab}(f_{aM} m_a \mathbf{v}/T_a)$ where C^T_{ab} represents the test particle part of the linearized Landau collision operator given in Sec. II. We here note that C^{TS}_{ab} is defined such that the self-adjointness condition,

$$\int d^3v \frac{\delta f_a}{f_{aM}} C^{TS}_{ab}(\delta g_a) = \int d^3v \frac{\delta g_a}{f_{aM}} C^{TS}_{ab}(\delta f_a) \tag{23}$$

holds exactly even if $T_a \neq T_b$.

The field particle part $C^{FS}_{ab}(\delta f_b)$ of the Sugama operator is given by

$$C^{FS}_{ab}(\delta f_b) = -\mathbf{V}_{ab}[\delta f_b] \cdot C^{TS}_{ab}(f_{aM} m_a \mathbf{v}/T_a)$$
$$- W_{ab}[\delta f_b] C^{TS}_{ab}(f_{aM} x_a^2), \tag{24}$$

where

$$\mathbf{V}_{ab}[\delta f_b] \equiv \frac{T_b}{\gamma_{ab}} \int d^3v \frac{\delta f_b}{f_{bM}} C^{TS}_{ba}(f_{bM} m_b \mathbf{v}/T_b), \tag{25}$$

and

$$W_{ab}[\delta f_b] \equiv \frac{T_b}{\eta_{ab}} \int d^3v \frac{\delta f_b}{f_{bM}} C^{TS}_{ba}(f_{bM} x_b^2). \tag{26}$$

In Eqs. (25)–(26),

$$\gamma_{ab} \equiv T_a \int d^3v (m_a v_\parallel/T_a) C^{TS}_{ab}(f_{aM} m_a v_\parallel/T_a)$$
$$= -\frac{n_a m_a}{\tau_{ab}} \frac{\alpha_{ab}}{(1+\alpha^2_{ab})^{3/2}} \left(\frac{T_a}{T_b} + \alpha^2_{ab} \right)$$
$$= -\frac{16\sqrt{\pi}}{3} \frac{n_a n_b e_a^2 e_b^2 \ln\Lambda}{(v^2_{Ta} + v^2_{Tb})^{3/2}} \left(\frac{1}{m_a} + \frac{1}{m_b} \right), \tag{27}$$

and

$$\eta_{ab} \equiv T_a \int d^3v x_a^2 C^{TS}_{ab}(f_{aM} x_a^2)$$
$$= -\frac{n_a T_a}{\tau_{ab}} \frac{3\alpha_{ab}}{(1+\alpha^2_{ab})^{5/2}} \left(\frac{T_a}{T_b} + \alpha^2_{ab} \right)$$
$$= -8\sqrt{\pi} \ln\Lambda \frac{n_a n_b e_a^2 e_b^2 v^2_{Ta} v^2_{Tb}}{(v^2_{Ta} + v^2_{Tb})^{5/2}} \left(\frac{1}{m_a} + \frac{1}{m_b} \right) \tag{28}$$

are used. We see $\gamma_{ab} = \gamma_{ba}$ and $\eta_{ab} = \eta_{ba}$ from Eqs. (27) and (28), respectively. It can be easily verified that the test-particle operator C^{TS}_{ab} and the field particle part C^{FS}_{ab} defined in Eqs. (19) and (24) obey conservation laws for particles, momentum, and energy. In addition, C^{FS}_{ab} satisfies the adjointness relation,

$$T_a \int d^3v\, \frac{\delta f_a}{f_{aM}} C^{FS}_{ab}(\delta f_b) = T_b \int d^3v\, \frac{\delta f_b}{f_{bM}} C^{FS}_{ba}(\delta f_a). \tag{29}$$

It is shown in Ref.[23] that the Sugama operator satisfies Boltzmann's H-theorem,

$$T_a \int d^3v\, \frac{\delta f_a}{f_{aM}} [C^{TS}_{ab}(\delta f_a) + C^{FS}_{ab}(\delta f_b)]$$
$$+ T_b \int d^3v\, \frac{\delta f_b}{f_{bM}} [C^{TS}_{ba}(\delta f_b) + C^{FS}_{ba}(\delta f_a)] \leq 0. \tag{30}$$

We also find that, for the case of $m_a/m_b \ll 1$, C_{ab}^{TS} and C_{ab}^{FS} coincide with C_{ab}^{T} and C_{ab}^{F} of the linearized Landau collision operator to the lowest order in $(m_a/m_b)^{1/2}$. For the case of $m_a/m_b \gg 1$, C_{ab}^{FS} equals C_{ab}^{F} to the lowest order in $(m_b/m_a)^{1/2}$ while, for $T_a \neq T_b$, C_{ab}^{TS} differs from C_{ab}^{T} by the non-self-adjoint part which remains to the lowest order [see the paragraph including Eq. (B16) in Appendix B].

The matrix elements $M_{ab}^{(S)jk}$ and $N_{ab}^{(S)jk}$ $(j,k = 0,1,2,\cdots)$ are defined by replacing C_{ab}^{T} and C_{ab}^{F} with C_{ab}^{TS} and C_{ab}^{FS}, respectively, in Eq. (11). Similarly, the friction coefficients $l_{ij}^{(S)ab}$ $(i,j = 1,2,\cdots)$ are defined using $M_{ab}^{(S)jk}$ and $N_{ab}^{(S)jk}$ in Eq. (13). Then, from the momentum conservation law satisfied by C_{ab}^{TS} and C_{ab}^{FS}, we obtain

$$M_{ab}^{(S)0j} + \frac{T_a v_{Ta}}{T_b v_{Tb}} N_{ba}^{(S)0j} = 0 \quad (j = 0,1,2,\cdots), \tag{31}$$

and

$$\sum_a l_{1j}^{(S)ab} = 0 \quad (j = 1,2,\cdots). \tag{32}$$

We also have

$$\begin{aligned} M_{ab}^{00} &= -N_{ab}^{00} = M_{ab}^{(S)00} = -N_{ab}^{(S)00}, \\ l_{11}^{ab} &= l_{11}^{(S)ab}. \end{aligned} \tag{33}$$

From the adjointness relations shown in Eqs. (23) and (29), the symmetry properties of the matrix elements and the friction coefficients are derived as

$$\begin{aligned} &M_{ab}^{(S)ij} = M_{ab}^{(S)ji}, \quad \frac{N_{ab}^{(S)ij}}{T_a v_{Ta}} = \frac{N_{ba}^{(S)ji}}{T_b v_{Tb}} \quad (i,j = 0,1,2,\cdots), \\ &l_{ij}^{(S)ab} = l_{ji}^{(S)ba} \quad (i,j = 1,2,\cdots). \end{aligned} \tag{34}$$

In addition, it is found from replacing (C_{ab}^{T}, C_{ab}^{F}) with $(C_{ab}^{TS}, C_{ab}^{FS})$ in Eq. (11) and using Eqs. (24), (31), (33), and (34) that $N_{ab}^{(S)ij}$ $(i,j = 0,1,2,\cdots)$ are represented by

$$N_{ab}^{(S)ij} = \frac{M_{ab}^{(S)0i} N_{ab}^{(S)0j}}{M_{ab}^{00}} = \frac{N_{ab}^{(S)i0} N_{ab}^{(S)0j}}{N_{ab}^{00}}. \tag{35}$$

In Appendix C, we find the detailed expressions of the matrix elements $(M_{ab}^{(S)ij}, N_{ab}^{(S)ij})$ as well as $(M_{ab}^{ij}, N_{ab}^{ij})$.

In the moment method,[3,50] the neoclassical transport coefficients, with which the radial particle and heat fluxes and the parallel current are linearly related to the radial density and temperature gradients and the parallel electric field, can be expressed in terms of the viscosity coefficients and the friction coefficients. The friction coefficients $l_{ij}^{(S)ab}$ derived from the Sugama operator do not all coincide with l_{ij}^{ab} given by the Landau operator even for the case of $T_a = T_b$ (see Sec. III.A below). The dependence of the neoclassical transport coefficients on the friction coefficients becomes relatively strong in the highly collisional regime where accurate values of the friction coefficients need to be derived from the model collision operator for correctly describing the neoclassical transport. In Sec. IV, the improved Sugama operator is presented to reproduce such accurate friction coefficients.

A. Equal temperature case

When $T_a = T_b$, the test particle part of the Sugama operator is equivalent to that of the Linearized Landau collision operator,

$$C_{ab}^{TS} = C_{ab}^{T}, \tag{36}$$

which can be easily verified from Eq. (19) with $\theta_{ab} = 1$, $\mathcal{Q}_{ab}(g) = g$, and $C_{ab}^{T} = C_{ab}^{T0}$ for that case. In this equal temperature case, we have

$$\begin{aligned} M_{ab}^{(S)ij} &= M_{ab}^{ij}, \\ N_{ab}^{(S)ij} &= \frac{N_{ab}^{i0} N_{ab}^{0j}}{N_{ab}^{00}} \quad (i,j = 0,1,2,\cdots), \end{aligned} \tag{37}$$

from which we see

$$\begin{aligned} N_{ab}^{(S)i0} &= N_{ab}^{i0} \quad (i = 0,1,2,\cdots), \\ N_{ab}^{(S)0j} &= N_{ab}^{0j} \quad (j = 0,1,2,\cdots), \end{aligned} \tag{38}$$

and

$$\begin{aligned} l_{i1}^{(S)ab} &= l_{i1}^{ab} \quad (i = 1,2,\cdots), \\ l_{1j}^{(S)ab} &= l_{1j}^{ab} \quad (j = 1,2,\cdots). \end{aligned} \tag{39}$$

We find from Eqs. (33) and (39) that the deviations of the friction coefficients $l_{ij}^{(S)ab}$ evaluated by the Sugama operator from l_{ij}^{ab} by the Landau operator appear only for $i \geq 2$ and $j \geq 2$. It is also noted that, for collisions between particles of like species $(a = b)$, the Sugama operator is equivalent to the linearized model collision operator given in Refs.[18,19,22].

IV. IMPROVED SUGAMA OPERATOR

The improved Sugama operator $C_{ab}^{LS(\mathrm{imp})}$ is defined by adding the correction part ΔC_{ab}^{LS} to the original one C_{ab}^{LS},

$$C_{ab}^{LS(\mathrm{imp})}(\delta f_a, \delta f_b) \equiv C_{ab}^{LS}(\delta f_a, \delta f_b) + \Delta C_{ab}^{LS}(\delta f_a, \delta f_b). \tag{40}$$

In order for $C_{ab}^{LS(\mathrm{imp})}$ to reproduce the same friction-flow relations and friction coefficients as those in Eqs. (12) and (13) derived from the linearized Landau collision operator C_{ab}^{L}, the correction term $\Delta C_{ab}^{LS}(\delta f_a, \delta f_b)$ is defined by

$$\Delta C_{ab}^{LS}(\delta f_a, \delta f_b) \equiv f_{aM} \frac{m_a}{T_a} \mathbf{v} \cdot \sum_{j=0}^{\infty} \Delta \mathbf{C}_{abj}^{L}[\delta f_a, \delta f_b] L_j^{(3/2)}(x_a^2), \tag{41}$$

with

$$\Delta \mathbf{C}^L_{abj}[\delta f_a, \delta f_b] \equiv \frac{c_j}{\tau_{ab}} \sum_{k=0}^{\infty} \Big(\Delta M^{jk}_{ab}\ \mathbf{u}_{ak}[\delta f_a] + \Delta N^{jk}_{ab}\ \mathbf{u}_{bk}[\delta f_b] \Big) \ \ (j = 0, 1, 2, \cdots), \quad (42)$$

where $\mathbf{u}_{ak}[\delta f_a]$ and $\mathbf{u}_{bk}[\delta f_b]$ are evaluated using Eq. (8). The corrections ΔM^{jk}_{ab} and ΔN^{jk}_{ab} of the matrix elements are defined by

$$\begin{aligned} \Delta M^{jk}_{ab} &\equiv M^{jk}_{ab} - M^{(S)jk}_{ab}, \\ \Delta N^{jk}_{ab} &\equiv N^{jk}_{ab} - N^{(S)jk}_{ab} \\ &= \frac{N^{00}_{ab} N^{jk}_{ab} - N^{(S)j0}_{ab} N^{(S)0k}_{ab}}{N^{00}_{ab}}, \end{aligned} \quad (43)$$

where the matrix elements M^{jk}_{ab} and N^{jk}_{ab} ($M^{(S)jk}_{ab}$ and $M^{(S)jk}_{ab}$) are given using the test and field particle parts of the Landau operator (the original Sugama operator) in Eq. (11). From Eq. (33), we immediately find

$$\Delta M^{00}_{ab} = \Delta N^{00}_{ab} = 0. \quad (44)$$

Using the improved Sugama operator $C^{LS(\mathrm{imp})}_{ab}$ defined by Eqs. (40)–(42) instead of the linearized Landau collision operator C^L_{ab} to evaluate the matrix elements and friction coefficients in Eqs. (11) and (13), we can confirm that $C^{LS(\mathrm{imp})}_{ab}$ still gives the same values to M^{ij}_{ab}, N^{ij}_{ab}, and l^{ab}_{ij} as C^L_{ab} does, and accordingly, the improved operator correctly reproduces the friction-flow relations in Eq. (12) derived from the Landau operator as well as it retains conservation laws of particles, momentum, and energy. Therefore, we can expect that the classical and neoclassical transport fluxes are accurately evaluated using the improved operator up to the highly collisional regime. As shown in the literature,[4,51] in order to correctly describe the neoclassical transport for the case where all particle species belong to the Pfirsch-Schlüter collisionality regime, we need accurate values for at least the part of the friction coefficients l^{ab}_{ij} with $i, j = 1, 2, 3$. Accordingly, in this highly collisional case, truncation of the summation $\sum_j$ and $\sum_k$ in Eqs. (41) and (42) should not be done unless the terms with $j \leq 2$ and $k \leq 2$ are retained. When the truncation is done such that ΔM^{0k}_{ab} and ΔN^{0k}_{ba} with $k \leq k_{\max}$ ($k_{\max}$: an arbitrary integer number) are included, the matrix elements associated with the improved operator satisfy the relations given in the same form as in Eq. (14) or Eq. (31) so that the momentum conservation law still holds as well as the conservation laws of particles and energy.

We note here that the correction term $\Delta C^{LS}_{ab}(\delta f_a, \delta f_b)$ is given for only the $l = 1$ spherical harmonic component of the Sugama operator to correctly reproduce the friction-flow relations which are regarded as an important factor in accurate evaluation of collisional transport and flow profiles influencing turbulent transport. As pointed out in Ref. 3, since the spherical harmonic functions of degree l is an eigenfunction of the pitch-angle-scattering operator included in the test particle collision part with an eigenvalue proportional to $-l(l+1)$, the test particle part tends to be more dominant over the field particle part as l is larger. Also, in the highly collisional regime, anisotropic components of the distribution function represented by the spherical harmonic functions of higher l's are considered to be stronger damped. Thus, without correction terms in the $l \geq 2$ spherical harmonic components of the field particle part, the improved operator is expected to work accurately for describing the distribution function at high collisionality. Besides, in principle, we can extend our procedures to add correction terms to all other spherical harmonic components similarly with the approximation method of Hirshman and Sigmar.[16] Then, the Landau field particle operator is recovered by using spherical harmonic functions and Laguerre functions of all degrees.

As described in Appendix B, the adjointness relations, the resultant symmetry properties for M^{ij}_{ab}, N^{ij}_{ab}, and l^{ab}_{ij} in Eq. (17), and Boltzmann's H-theorem in the form of Eq. (30) are not exactly but only approximately satisfied by the linearized Landau operator and the improved Sugama operator for the case of unequal species temperatures $T_a \neq T_b$. Also, it should be recalled here that the two species need to have very different masses for their temperatures to be significantly different from each other.

When the summations $\sum_j$ and $\sum_k$ in Eqs. (41) and (42) are truncated at the same maximum number $j_{\max} = k_{\max}$, the adjointness relations of the improved Sugama operator are still satisfied for $T_a = T_b$ because the matrix elements M^{jk}_{ab} and N^{jk}_{ab} evaluated by the improved operator still keep the symmetry properties. On the other hand, the H-theorem is not guaranteed by this truncation even for $T_a = T_b$ because not all but only some fraction of the matrix elements N^{jk}_{ab} $(j, k = 0, 1, 2, \cdots)$ of the Sugama field particle operator are replaced with those of the Landau field particle operator. [Note that the friction-flow relations including all matrix elements M^{jk}_{ab} and N^{jk}_{ab} $(j, k = 0, 1, 2, \cdots)$ given by the linearized Landau operator is equivalent to the $l = 1$ spherical harmonic part of that Landau operator which satisfies the H-theorem for the $l = 1$ parts of the distribution functions.] It is shown in Ref.[23] that the H-theorem for the original Sugama operator can be derived from the fact that its field particle part can be completely expressed in terms of the test particle part although the same technique of the derivation of the H-theorem cannot be used for the truncated version of the improved Sugama operator. However, we can still expect that the H-theorem is approximately satisfied by the truncated model if the $l = 1$ parts of the distribution functions are well represented by the linear combinations of only low-order Laguerre polynomials. As shown in Ref.[4], sufficiently accurate evaluations of collisional (classical and neoclassical) transport fluxes can be made using the friction-flow relations including the Laguerre polynomial moments up to the order of $j = 2$. This appears to be because higher-order Laguerre-polynomial components of the distribution functions are stronger suppressed by the energy diffusion operator [see Eq. (B4) in Appendix B]. Therefore, except for the case where the field particle distribution takes a special form due to some external sources, we don't generally expect that higher ($j \geq 3$) moments need to be retained in Eq. (41).

It is easily found from the definition of the improved operator in Eqs. (40)–(42) that $C_{ab}^{LS}(\delta f_a, \delta f_b) = C_{ab}^{LS(\text{imp})}(\delta f_a, \delta f_b)$ holds if the perturbed distribution functions δf_s ($s = a, b$) include no $l = 1$ components ($\delta f_s^{(l=1)} = 0$) [see Eqs. (6) and (7)]. Therefore, if δf_s ($s = a, b$) are given by the perturbed Maxwellian with the perturbed densities δn_s and temperatures δT_s as

$$\delta f_s = f_{sM}\left[\frac{\delta n_s}{n_s} + \frac{\delta T_s}{T_s}\left(\frac{m_s v^2}{2T_s} - \frac{3}{2}\right)\right], \tag{45}$$

for which $\delta T_a/T_a = \delta T_b/T_b$ is assumed, then $C_{ab}^{LS(\text{imp})}(\delta f_a, \delta f_b)$ vanishes as $C_{ab}^{LS}(\delta f_a, \delta f_b)$ does.[23] However, when δf_s ($s = a, b$) are written as the shifted Maxwellian $\delta f_s = f_{sM}(m_a/T_s)(\mathbf{u}_s \cdot \mathbf{v})$ with the same flow velocity $\mathbf{u}_a = \mathbf{u}_b$ and the different equilibrium temperatures $T_a \neq T_b$, $C_{ab}^{LS}(\delta f_a, \delta f_b)$ vanishes although neither $C_{ab}^{L}(\delta f_a, \delta f_b)$ nor $C_{ab}^{LS(\text{imp})}(\delta f_a, \delta f_b)$ does exactly. This is related to the fact that the symmetry properties $l_{ab}^{ij} = l_{ba}^{ji}$ ($i, j = 1, 2, \cdots$) are slightly broken when $T_a \neq T_b$ (see Appendix D).

When the above-mentioned adjointness relations and resultant symmetry properties are satisfied, they provide useful techniques for calculating the neoclassical transport coefficients.[1–5,47–50] Therefore, it will be beneficial for such applications if we can have a linearized collision model which satisfies the adjoint relations exactly even for $T_a \neq T_b$ while giving small inaccuracies to the values of the matrix elements and the friction coefficients. Such a model is presented in Appendix D where the correction part of the improved Sugama operator is symmetrized.

A. Equal temperature case

When $T_a = T_b$, we use Eqs. (37) and (38) to obtain

$$\Delta M_{ab}^{ij} = 0,$$
$$\Delta N_{ab}^{ij} = \frac{N_{ab}^{00} N_{ab}^{ij} - N_{ab}^{i0} N_{ab}^{0j}}{N_{ab}^{00}} \quad (i, j = 0, 1, 2, \cdots), \tag{46}$$

from which we have

$$\Delta N_{ab}^{00} = \Delta N_{ab}^{i0} = \Delta N_{ab}^{0j} = 0 \quad (i, j = 1, 2, \cdots). \tag{47}$$

V. COLLISION OPERATOR FOR GYROKINETIC EQUATIONS

There are two types of gyrokinetic equations. One is the gyrokinetic equation derived by using the WKB representation for the perturbed distribution function[52–57] which has a high wavenumber in the direction perpendicular to the equilibrium magnetic field $\mathbf{B}$. The other is derived by using the Lie transform technique to properly define the gyrocenter coordinates for description of the total distribution function.[58,59] The collision operator for the former type of the gyrokinetic equation is considered in this section as well as in the literature.[20–23] On the other hand, several studies have been done to represent the collision operator for the latter type in the gyrocenter coordinates.[24–30]

When applying the improved Sugama operator to the gyrokinetic equation for the perpendicular wavenumber vector $\mathbf{k}_\perp$, the collision operator is transformed into the following form,

$$\begin{aligned}
&\oint \frac{d\xi}{2\pi} e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} C_{ab}^{LS(\text{imp})}(e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} h_{a\mathbf{k}_\perp}, e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_b} h_{b\mathbf{k}_\perp}) \\
&= \oint \frac{d\xi}{2\pi} e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} C_{ab}^{LS}(e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} h_{a\mathbf{k}_\perp}, e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_b} h_{b\mathbf{k}_\perp}) \\
&\quad + \oint \frac{d\xi}{2\pi} e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} \Delta C_{ab}^{LS}(e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} h_{a\mathbf{k}_\perp}, e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_b} h_{b\mathbf{k}_\perp}),
\end{aligned} \tag{48}$$

where $\boldsymbol{\rho}_a \equiv (\mathbf{b} \times \mathbf{v})/\Omega_a$ ($\mathbf{b} \equiv \mathbf{B}/B, \Omega_a \equiv e_a B/m_a c$) and $\oint d\xi/(2\pi)$ represent the gyroradius vector and the gyrophase average, respectively, and $h_{a\mathbf{k}_\perp}$ is given from the nonadiabatic part of the perturbed particle distribution function $\delta f_{a\mathbf{k}_\perp} = -(e\phi_{\mathbf{k}_\perp}/T_a) f_{aM} + e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} h_{a\mathbf{k}_\perp}$. The detailed expression of the first term on the right-hand of Eq. (48) is shown in Ref.[23] while the second term is written as

$$\begin{aligned}
&\Delta C_{ab}^{LS(GK)}(\delta f_{a\mathbf{k}_\perp}, \delta f_{b\mathbf{k}_\perp}) \\
&\equiv \oint \frac{d\xi}{2\pi} e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} \Delta C_{ab}^{LS}(\delta f_{a\mathbf{k}_\perp}, \delta f_{b\mathbf{k}_\perp}) \\
&\equiv \frac{m_a}{T_a} \frac{f_{aM}}{\tau_{ab}} \sum_{j=0}^{\infty} c_j L_j^{(3/2)}(x_a^2) \\
&\quad \times \sum_{k=0}^{\infty} \Big[\Delta M_{ab}^{jk} \left\{ \overline{u}_{\parallel ak}[h_{a\mathbf{k}_\perp}] J_{0a} v_\parallel + \overline{u}_{\perp ak}[h_{a\mathbf{k}_\perp}] J_{1a} v_\perp \right\} \\
&\quad + \Delta N_{ab}^{jk} \left\{ \overline{u}_{\parallel bk}[h_{b\mathbf{k}_\perp}] J_{0a} v_\parallel + \overline{u}_{\perp bk}[h_{b\mathbf{k}_\perp}] J_{1a} v_\perp \right\} \Big],
\end{aligned} \tag{49}$$

where $J_{0s} \equiv J_0(k_\perp v_\perp/\Omega_s)$ and $J_{1s} \equiv J_1(k_\perp v_\perp/\Omega_s)$ ($s = a, b$) denote the zeroth- and first-order Bessel functions of the normalized perpendicular wavenumber $k_\perp v_\perp/\Omega_s$, respectively, and

$$\begin{aligned}
\overline{u}_{\parallel sk}[h_{s\mathbf{k}_\perp}] &\equiv \frac{c_k}{n_s} \int d^3v \; L_k^{(3/2)}(x_s^2) h_{s\mathbf{k}_\perp} J_{0s} v_\parallel, \\
\overline{u}_{\perp sk}[h_{s\mathbf{k}_\perp}] &\equiv \frac{c_k}{n_s} \int d^3v \; L_k^{(3/2)}(x_s^2) h_{s\mathbf{k}_\perp} J_{1s} v_\perp,
\end{aligned} \tag{50}$$

are used.

In the case of application to the drift kinetic equation for studying neoclassical transport, we neglect the finite gyroradius effects and take the limit $\mathbf{k}_\perp \cdot \boldsymbol{\rho}_s \to 0$ ($s = a, b$). Then, we put $J_{0s} \to 1$, $J_{1s} \to 0$, and $\overline{u}_{\perp sk}[h_{s\mathbf{k}_\perp}] \to 0$ in Eqs. (49) and (50).

VI. CONCLUSIONS

In this paper, the improved linearized model collision operator which can be applied up to the highly collisional regime is presented. The improved operator is constructed by adding the correction part to the previous model by Sugama *et al.* so as to reproduce the same friction-flow relations as those given by the linearized Landau collision operator. In the improved model, conservation laws of particles, momentum, and energy are retained while the adjointness relations and Boltzmann's H-theorem are approximately valid for collisions between unlike particle species with unequal temperatures and very different masses. It is also shown that the improved operator can be modified to satisfy the adjointness relations exactly even in the unequal-temperature case. This modification causes the friction coefficients to deviate from those given by the Landau operator although the influence of the deviations is made small by the very different masses.

Performing the gyrophase average with keeping the finite gyroradius effect, the improved operator is represented in the suitable form for gyrokinetic equations. In the zero-gyroradius limit, the gyrophase-averaged improved operator can be used in drift kinetic equations to accurately evaluate neoclassical transport in all collisionality regimes. It is considered that only the terms with $j \leq 2$ in the Laguerre polynomial expansion of the correction part of the operator need to be kept even for the most collisional case where all particle species are in the Pfirsch-Schlüter collisionality regime. The present model is expected to be useful for simulation studies of neoclassical and turbulent transport processes in plasmas including multi-species of particles in various collisional regimes.

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Appendix A: COLLISIONAL ENERGY TRANSFER BETWEEN UNLIKE SPECIES WITH UNEQUAL TEMPERATURES

Using Eq. (4), the collisional energy transfer from species b to a, which equals the opposite sign of that from species a to b, is given by

$$\int d^3v \, C_{ab}(f_{aM}, f_{bM}) \frac{m_a v^2}{2} = -\int d^3v \, C_{ba}(f_{bM}, f_{aM}) \frac{m_b v^2}{2}$$
$$= -3 \frac{m_a \alpha_{ab}^3}{m_b (1+\alpha_{ab}^2)^{3/2}} \frac{n_a (T_a - T_b)}{\tau_{ab}}, \tag{A1}$$

where each species is assumed to be in the local equilibrium state represented by the Maxwellian distribution function. Thus, if $T_a \neq T_b$, collisions cause the temperatures of the two species to approach to each other, and the characteristic rate ν_{ab}^{et} of the collisional energy transfer from species b to a is given by

$$\nu_{ab}^{et} = \frac{m_a \alpha_{ab}^3}{m_b (1+\alpha_{ab}^2)^{3/2}} \frac{1}{\tau_{ab}} = \frac{e_b^2 n_b m_a \alpha_{ab}^3}{e_a^2 n_a m_b (1+\alpha_{ab}^2)^{3/2}} \frac{1}{\tau_{aa}}$$
$$= \frac{e_a^2 m_b}{e_b^2 m_a (1+\alpha_{ab}^2)^{3/2}} \frac{1}{\tau_{bb}}. \tag{A2}$$

We now assume that $|e_a/e_b| = \mathcal{O}(1)$, $n_a/n_b = \mathcal{O}(1)$, and $T_a/T_b = \mathcal{O}(1)$. Then, in the case where $m_a/m_b = \mathcal{O}(1)$, we have $\alpha_{ab} = \mathcal{O}(1)$ and accordingly $\nu_{ab}^{et} \sim 1/\tau_{aa} \sim 1/\tau_{bb}$ from Eq. (A2). This implies that, the relaxation toward the equal-temperature ($T_a = T_b$) state due to the unlike-species collisions and the thermalization toward the Maxwellian equilibrium are expected to have occurred on the same time scale and that the Maxwellian distribution functions f_{aM} and f_{bM} should have the same temperature $T_a = T_b$.

Next, we consider another case where $m_a/m_b \ll 1$ or $m_a/m_b \gg 1$ holds. Then, $\alpha_{ab} \gg 1$ and $\nu_{ab}^{et} \sim (m_a/m_b)/\tau_{aa} \sim (m_a/m_b)^{1/2}/\tau_{bb}$ are obtained for $m_a/m_b \ll 1$ while $\alpha_{ab} \ll 1$ and $\nu_{ab}^{et} \sim (m_b/m_a)^{1/2}/\tau_{aa} \sim (m_b/m_a)/\tau_{bb}$ for $m_a/m_b \gg 1$. Therefore, when $m_a/m_b \ll 1$ or $m_a/m_b \gg 1$, collisional energy exchange between species a and b is so slow that f_{aM} and f_{bM} are permitted to have unequal temperatures $T_a \neq T_b$.

We now consider the case where $m_a/m_b \gg 1$, $|e_a/e_b| \gg 1$, $n_a/n_b \ll 1$, and $T_a/T_b = \mathcal{O}(1)$. This can happen when a and b represent heavy minority impurity ions with high charge number and bulk hydrogen isotopes (or electrons), respectively. Then, we have $\alpha_{ab} \ll 1$ and

$$\nu_{ab}^{et} \sim \frac{e_b^2 n_b m_b^{1/2}}{e_a^2 n_a m_a^{1/2}} \frac{1}{\tau_{aa}} \sim \frac{e_a^2 m_b}{e_b^2 m_a} \frac{1}{\tau_{bb}}. \tag{A3}$$

For example, using Eq. (A3) in realistic cases as in the JET and ASDEX Upgrade experiments shown in Ref.[12], the characteristic rates ν_{Wi}^{et} and ν_{We}^{et} of the collisional energy transfer from tungsten impurities (W) to bulk hydrogen isotopes (i) and to electrons (e) are estimated to satisfy

$$\nu_{Wi}^{et} \sim 1/\tau_{WW}, \qquad \nu_{Wi}^{et} > 1/\tau_{ii}, \tag{A4}$$

and

$$\nu_{We}^{et} \ll 1/\tau_{WW}, \qquad \nu_{We}^{et} \ll 1/\tau_{ee}, \tag{A5}$$

respectively. Under these conditions, it is reasonable to assume $T_W = T_i$ although it is not to assume $T_W = T_e$.

Appendix B: EFFECTS OF UNEQUAL TEMPERATURES ON ADJOINTNESS RELATIONS

Based on the Landau collision operator defined in Eq. (1), the test and field particle parts are written as

$$C_{ab}^{T}(\delta f_a) \equiv C_{ab}^{T0}(\delta f_a) + C_{ab}^{TN}(\delta f_a),$$
$$C_{ab}^{F}(\delta f_b) \equiv C_{ab}^{F0}(\delta f_b) + C_{ab}^{FN}(\delta f_b), \tag{B1}$$

where

$$\begin{aligned}
C_{ab}^{T0}(\delta f_a) &\equiv \frac{2\pi e_a^2 e_b^2 \ln\Lambda}{m_a^2}\frac{\partial}{\partial \mathbf{v}}\cdot\left[f_{aM}(\mathbf{v})\frac{\partial}{\partial \mathbf{v}}\left(\frac{\delta f_a(\mathbf{v})}{f_{aM}(\mathbf{v})}\right)\cdot\int d^3v'\ f_{bM}(\mathbf{v}')\mathbf{U}(\mathbf{v}-\mathbf{v}')\right] \\
&\equiv \nu_D^{ab}(v)\mathcal{L}(\delta f_a) + \mathcal{C}_v^{ab}(\delta f_a), \\
C_{ab}^{TN}(\delta f_a) &\equiv \left(\frac{1}{T_b}-\frac{1}{T_a}\right)\frac{2\pi e_a^2 e_b^2 \ln\Lambda}{m_a}\frac{\partial}{\partial \mathbf{v}}\cdot\left[\delta f_a(\mathbf{v})\mathbf{v}\cdot\int d^3v'\ f_{bM}(\mathbf{v}')\mathbf{U}(\mathbf{v}-\mathbf{v}')\right] \\
&\equiv \left(\frac{1}{T_b}-\frac{1}{T_a}\right)\frac{m_a}{v^2}\frac{\partial}{\partial v}\left[\frac{\nu_\parallel^{ab}(v)}{2}v^5\delta f_a\right], \\
C_{ab}^{F0}(\delta f_b) &\equiv -\frac{2\pi e_a^2 e_b^2 \ln\Lambda}{m_a m_b}\frac{\partial}{\partial \mathbf{v}}\cdot\left[f_{aM}(\mathbf{v})\cdot\int d^3v'\ f_{bM}(\mathbf{v}')\mathbf{U}(\mathbf{v}-\mathbf{v}')\cdot\frac{\partial}{\partial \mathbf{v}'}\left(\frac{\delta f_b(\mathbf{v}')}{f_{bM}(\mathbf{v}')}\right)\right], \\
C_{ab}^{FN}(\delta f_b) &\equiv \left(\frac{1}{T_b}-\frac{1}{T_a}\right)\frac{2\pi e_a^2 e_b^2 \ln\Lambda}{m_a}\frac{\partial}{\partial \mathbf{v}}\cdot\left[f_{aM}(\mathbf{v})\mathbf{v}\cdot\int d^3v'\ \delta f_b(\mathbf{v}')\mathbf{U}(\mathbf{v}-\mathbf{v}')\right].
\end{aligned} \tag{B2}$$

Here, $C_{ab}^{T0}(\delta f_a)$ consists of the pitch-angle-scattering part $\nu_D^{ab}(v)\mathcal{L}(\delta f_a)$ and the energy diffusion part $\mathcal{C}_v^{ab}(\delta f_a)$. The pitch-angle-scattering operator $\mathcal{L}$ is defined by

$$\begin{aligned}
\mathcal{L}(\delta f_a) &\equiv \frac{1}{2}\frac{\partial}{\partial \mathbf{v}}\cdot\left[\left(v^2\mathbf{I}-\mathbf{vv}\right)\cdot\frac{\partial \delta f_a}{\partial \mathbf{v}}\right] \\
&= \frac{1}{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\delta f_a}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\delta f_a}{\partial\varphi^2}\right],
\end{aligned} \tag{B3}$$

where $\mathbf{I}$ denotes the unit tensor and (v,θ,φ) represent spherical coordinates in the velocity space. The energy diffusion operator $\mathcal{C}_v^{ab}$ is defined by

$$\mathcal{C}_v^{ab}(\delta f_a) \equiv \frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{\nu_\parallel^{ab}(v)}{2}v^4 f_{aM}\frac{\partial}{\partial v}\left(\frac{\delta f_a}{f_{aM}}\right)\right]. \tag{B4}$$

The collision frequencies for pitch-angle scattering and energy diffusion are given by $\nu_D^{ab}(v) \equiv (3\sqrt{\pi}/4)\tau_{ab}^{-1}[\Phi(x_b) - G(x_b)]/x_a^3$ and $\nu_\parallel^{ab}(v) \equiv (3\sqrt{\pi}/2)\tau_{ab}^{-1}G(x_b)/x_a^3$, respectively, where $(3\sqrt{\pi}/4)\tau_{ab}^{-1} \equiv 4\pi n_b e_a^2 e_b^2 \ln\Lambda/(m_a^2 v_{Ta}^3)$ ($\ln\Lambda$: The Coulomb logarithm), $\Phi(x) \equiv 2\pi^{-1/2}\int_0^x e^{-t^2}dt$, $G(x) \equiv [\Phi(x) - x\Phi'(x)]/(2x^2)$, $x_s \equiv v/v_{Ts}$, and $v_{Ts} \equiv (2T_s/m_s)^{1/2}$ $(s = a, b)$. We can easily confirm that $\nu_D\mathcal{L}$, $\mathcal{C}_v^{ab}$, and accordingly C_{ab}^{T0} are all self-adjoint so that

$$\int d^3v\ \frac{\delta f_a}{f_{aM}}C_{ab}^{T0}(\delta g_a) = \int d^3v\ \frac{\delta g_a}{f_{aM}}C_{ab}^{T0}(\delta f_a) \tag{B5}$$

holds for arbitrary functions δf_a and δg_a of $\mathbf{v}$. It can also be shown that C_{ab}^{F0} satisfies the adjointness relation written as

$$\int d^3v\ \frac{\delta f_a}{f_{aM}}C_{ab}^{F0}(\delta f_b) = \int d^3v\ \frac{\delta f_b}{f_{bM}}C_{ba}^{F0}(\delta f_a). \tag{B6}$$

The remaining test and field particle operators C_{ab}^{TN} and C_{ab}^{FN} do not keep adjoint relations such as Eqs. (B5) and (B6) satisfied by C_{ab}^{T0} and C_{ab}^{F0}, respectively, although C_{ab}^{TN} and C_{ab}^{FN} vanish for $T_a = T_b$.

We also note that the two pairs of the operators $(C_{ab}^{T0}, C_{ab}^{F0})$ and $(C_{ab}^{TN}, C_{ab}^{FN})$ independently satisfy the particle, momentum and energy conservation laws, which are written as

$$\int d^3v\ C_{ab}^{TA}(\delta f_a) = \int d^3v\ C_{ab}^{FA}(\delta f_b) = 0 \quad (A = 0, N), \tag{B7}$$

$$\int d^3v\ m_a\mathbf{v}C_{ab}^{TA}(\delta f_a) + \int d^3v\ m_b\mathbf{v}C_{ba}^{FA}(\delta f_a) = 0 \quad (A = 0, N), \tag{B8}$$

and

$$\int d^3v\ \frac{1}{2}m_a v^2 C_{ab}^{TA}(\delta f_a) + \int d^3v\ \frac{1}{2}m_b v^2 C_{ba}^{FA}(\delta f_a) = 0 \quad (A = 0, N), \tag{B9}$$

respectively.

From the Galilean invariance and spherical symmetry of the Landau collision operator, we have an identity, $\int d^3v\ \ m_a(\mathbf{v}-\mathbf{u})C_{ab}[f_{aM}(\mathbf{v}-\mathbf{u}), f_{bM}(\mathbf{v}-\mathbf{u})] = \int d^3v\ m_a\mathbf{v}C_{ab}[f_{aM}(\mathbf{v}), f_{bM}(\mathbf{v})] = 0$, for an arbitrary vector $\mathbf{u}$ which is independent of $\mathbf{v}$. Then, taking the $\mathbf{u} \to 0$ limit of the above identity and using the particle and momentum conservation laws, we can derive another type of relations,

$$\begin{aligned}
&\int d^3v\ m_a\mathbf{v}\ C_{ab}^{T}(f_{aM}m_a\mathbf{v}/T_a) \\
&= \int d^3v\ m_b\mathbf{v}\ C_{ba}^{T}(f_{bM}m_b\mathbf{v}/T_b) \\
&= -\int d^3v\ m_a\mathbf{v}\ C_{ab}^{F}(f_{bM}m_b\mathbf{v}/T_b) \\
&= -\int d^3v\ m_b\mathbf{v}\ C_{ba}^{F}(f_{aM}m_a\mathbf{v}/T_a),
\end{aligned} \tag{B10}$$

We should note that the symmetry properties shown in Eq. (B10) are valid even when $T_a \neq T_b$ although they are not satisfied in the same way as Eqs. (B7)–(B9) are separately satisfied by the two pairs of the operators $(C_{ab}^{TA}, C_{ab}^{FA})$ $(A = 0, N)$ for $T_a \neq T_b$.

Using Eq. (B1), the matrix elements M_{ab}^{ij} and N_{ab}^{ij}, which are defined by Eq. (11), are written as

$$\begin{aligned} M_{ab}^{ij} &= M_{ab}^{(0)ij} + M_{ab}^{(N)ij}, \\ N_{ab}^{ij} &= N_{ab}^{(0)ij} + N_{ab}^{(N)ij}, \end{aligned} \tag{B11}$$

where $M_{ab}^{(A)ij}$ and $N_{ab}^{(A)ij}$ $(A = 0, N)$ are defined by

$$\begin{aligned} \frac{n_a}{\tau_{ab}} M_{ab}^{(A)ij} &\equiv \int d^3v \; v_\parallel L_i^{(3/2)}(x_a^2) C_{ab}^{TA}\left(f_{aM} L_j^{(3/2)}(x_a^2) \frac{m_a v_\parallel}{T_a} \right), \\ \frac{n_a}{\tau_{ab}} N_{ab}^{(A)ij} &\equiv \int d^3v \; v_\parallel L_i^{(3/2)}(x_a^2) C_{ab}^{FA}\left(f_{bM} L_j^{(3/2)}(x_b^2) \frac{m_b v_\parallel}{T_b} \right). \end{aligned} \tag{B12}$$

Then, the momentum conservation law shown in Eq. (B8) is used to find

$$M_{ab}^{(A)0j} + \frac{T_a v_{Ta}}{T_b v_{Tb}} N_{ba}^{(A)0j} = 0 \quad (A = 0, N; j = 0, 1, 2, \cdots), \tag{B13}$$

The symmetry properties of $M_{ab}^{(0)ij}$ and $N_{ab}^{(0)ij}$ are derived from the adjointness relations given by Eqs. (B5) and (B6) as

$$M_{ab}^{(0)ij} = M_{ab}^{(0)ji} \quad \text{and} \quad \frac{N_{ab}^{(0)ij}}{T_a^2 v_{Ta}} = \frac{N_{ba}^{(0)ji}}{T_b^2 v_{Tb}}, \tag{B14}$$

respectively. Also from Eq. (B10), we obtain

$$M_{ab}^{00} = -N_{ab}^{00}, \qquad \frac{N_{ab}^{00}}{T_a v_{Ta}} = \frac{N_{ba}^{00}}{T_b v_{Tb}}. \tag{B15}$$

It should be noted that the symmetry properties of $N_{ab}^{(0)ij}$ and N_{ab}^{00} take different forms with respect to the way the temperatures enter.

In the case of $m_a/m_b = \mathcal{O}(1)$, the temperatures T_a and T_b are expected to be close to each other because of the relatively fast energy exchange due to collisions. Therefore, only when $m_a \ll m_b$ or $m_a \gg m_b$, T_a can be significantly different from T_b. In the limiting case $m_a \ll m_b$, it is shown that the pitch-angle-scattering term $\nu_D(v)\mathcal{L}(\delta f_a)$ is dominant in the test particle operator $C_{ab}^T(\delta f_a)$ where the energy scattering term $C_v^{ab}(\delta f_a)$ and the non-adjoint part $C_{ab}^{TN}(\delta f_a)$ are negligible in the lowest order of the expansion with respect to $(m_a/m_b)^{1/2}$. However, when $T_a \neq T_b$, $C_{ab}^{FN}(\delta f_b)$ is not negligible but it is necessary to keep contributions from both $C_{ab}^{F0}(\delta f_b)$ and $C_{ab}^{FN}(\delta f_b)$ for accurately evaluating collisional momentum transfer. Then, it can be shown that, to the lowest order in $(m_a/m_b)^{1/2}$, the test and field particle parts of the Sugama operator $C_{ab}^S(\delta f_a, \delta f_b) = C_{ab}^{TS}(\delta f_a) + C_{ab}^{FS}(\delta f_b)$ correctly approximate $C_{ab}^T(\delta f_a)$ and $C_{ab}^F(\delta f_b)$ of the linearized Landau operator, respectively.

We next consider the case in which $m_a \gg m_b$ and $T_a \neq T_b$ hold. In this case, $C_{ab}^{TN}(\delta f_a)$ is not negligibly small compared with $C_{ab}^{T0}(\delta f_a)$ while $C_{ab}^{FN}(\delta f_b)$ does not contribute to $C_{ab}^F(\delta f_b)$ in the lowest order of the expansion with respect to $(m_b/m_a)^{1/2}$. Then, $C_{ab}^F(\delta f_b)$ is well approximated by either $C_{ab}^{F0}(\delta f_b)$ or $C_{ab}^{FS}(\delta f_b)$ although the difference of $C_{ab}^T(\delta f_a)$ from $C_{ab}^{T0}(\delta f_a)$ or $C_{ab}^{TS}(\delta f_a)$ is significant. However, this difference doesn't cause serious errors in solving the kinetic equation for δf_a as far as $C_{ab}^T(\delta f_a)/C_{aa}^T(\delta f_a) \sim (e_b/e_a)^2(n_b/n_a)(m_b/m_a)^{1/2}$ becomes very small. [This ratio $(e_b/e_a)^2(n_b/n_a)(m_b/m_a)^{1/2}$ can be large in such a case of tungsten impurity as mentioned in Appendix A although, for that case, $T_a = T_b$ is expected so that $C_{ab}^T(\delta f_a) = C_{ab}^{T0}(\delta f_a) = C_{ab}^{TS}(\delta f_a)$ holds.] Except for this limiting case of $m_a \gg m_b$ and $T_a \neq T_b$, we can suppose that the matrix elements M_{ab}^{ij} evaluated by $C_{ab}^T(\delta f_a) = C_{ab}^{T0}(\delta f_a) + C_{ab}^{TN}(\delta f_a)$ satisfy the symmetry relations of the same form as those for $M_{ab}^{(0)ij}$ shown in Eq. (B14),

$$M_{ab}^{ij} = M_{ab}^{ji} \;\; (i, j = 0, 1, 2, \cdots). \tag{B16}$$

It is recalled that contributions of $C_{ab}^{TN}(\delta f_a)$ to the collisional momentum transfer are taken into account in defining $C_{ab}^{TS}(\delta f_a)$ such that $C_{ab}^{TS}(\delta f_a)$ and $C_{ab}^T(\delta f_a) = C_{ab}^{T0}(\delta f_a) + C_{ab}^{TN}(\delta f_a)$ give the same matrix element M_{ab}^{00} even when $m_a \gg m_b$ and $T_a \neq T_b$. Also, $C_{ab}^{TS}(\delta f_a)$ is constructed so as to yield the matrix elements $M_{ab}^{(S)ij}$ which satisfy symmetry relations of the same form as in Eq. (B16).

When $m_a/m_b \ll 1$ and $T_a \neq T_b$, $C_{ab}^{FN}(\delta f_a)$ makes a significant contribution to $C_{ab}^F(\delta f_a) = C_{ab}^{F0}(\delta f_a) + C_{ab}^{FN}(\delta f_a)$. In this case, we can show that, to the lowest order in $(m_a/m_b)^{1/2}$,

$$\frac{N_{ab}^{ij}}{T_a v_{Ta}} = \frac{N_{ba}^{ji}}{T_b v_{Tb}} \;\; (i, j = 0, 1, 2, \cdots) \tag{B17}$$

are satisfied by the matrix elements N_{ab}^{ij} associated with $C_{ab}^F(\delta f_a)$. Note that the second relation in Eq. (B15), which holds exactly, is included as a special case in the symmetry relations shown by Eq. (B17) and that they take a different form from those for $N_{ab}^{(0)ij}$ in Eq. (B14). Also, the matrix elements $N_{ab}^{(S)ij}$ evaluated by $C_{ab}^{FS}(\delta f_a)$ satisfy symmetry relations of the same form as in Eq. (B17).

In summary, the adjointness relations of the linearized Landau operator C_{ab}^L are not satisfied rigorously in collisions between unlike species with unequal temperatures although significantly different temperatures occur in the case where the two species have so different masses that the adjointness relations and symmetry properties of the matrix elements and the friction coefficients can still be used as approximately valid formulas. On the other hand, the Sugama operator C_{ab}^{LS} in Sec. III and the operator $C_{ab}^{LS*(\text{imp})}$ in Appendix D are constructed so as to exactly keep the adjointness relations which can be useful in formulating efficient methods of evaluating Onsager symmetric collisional transport coefficients.[1–5,47–50]

Appendix C: MATRIX ELEMENTS ASSOCIATED WITH THE LINEARIZED LANDAU OPERATOR AND THE SUGAMA OPERATOR

This Appendix shows how the matrix elements $M_{ab}^{ij} = M_{ab}^{(0)ij} + M_{ab}^{(N)ij}$ and $M_{ab}^{ij} = M_{ab}^{(0)ij} + M_{ab}^{(N)ij}$ [see Eqs. (B11) and (B12) in Appendix B] which are associated with the test part $C_{ab}^{T} = C_{ab}^{T0} + C_{ab}^{TN}$ and the field part $C_{ab}^{F} = C_{ab}^{F0} + C_{ab}^{FN}$ of the linearized Landau operator $C_{ab}^{L} = C_{ab}^{T} + C_{ab}^{F}$ are expressed in terms of $\alpha_{ab} \equiv v_{Ta}/v_{Tb}$, T_a/T_b, and m_a/m_b. In addition, it is shown how to evaluate $M_{ab}^{(S)ij}$ and $N_{ab}^{(S)ij}$ defined from the Sugama operator $C_{ab}^{LS} = C_{ab}^{TS} + C_{ab}^{FS}$ (see Sec. III).

First, the 00 elements of the matrices $M_{ab}^{(A)ij}$ and $N_{ab}^{(A)ij}$ $(A = 0, N)$ are written as follows:

$$
\begin{aligned}
M_{ab}^{(0)00} &= -\frac{\alpha_{ab}}{(1+\alpha_{ab}^2)^{1/2}}, \\
M_{ab}^{(N)00} &= \left(1 - \frac{T_a}{T_b}\right) \frac{\alpha_{ab}}{(1+\alpha_{ab}^2)^{3/2}}, \\
N_{ab}^{(0)00} &= -\frac{T_a}{T_b} M_{ab}^{(0)00} = \frac{T_a}{T_b} \frac{\alpha_{ab}}{(1+\alpha_{ab}^2)^{1/2}}, \\
N_{ab}^{(N)00} &= \alpha_{ab}^2 M_{ab}^{(N)00} = \left(1 - \frac{T_a}{T_b}\right) \frac{\alpha_{ab}^3}{(1+\alpha_{ab}^2)^{3/2}}.
\end{aligned} \tag{C1}
$$

Then, the 00 elements, $M_{ab}^{00} = M_{ab}^{(0)00} + M_{ab}^{(N)00}$ and $N_{ab}^{00} = N_{ab}^{(0)00} + N_{ab}^{(N)00}$, which equal $M_{ab}^{(S)00}$ and $N_{ab}^{(S)00}$, respectively, are given by

$$
\begin{aligned}
M_{ab}^{00} = -N_{ab}^{00} = M_{ab}^{(S)00} &= -N_{ab}^{(S)00} \\
&= -\left(1 + \frac{m_a}{m_b}\right) \frac{\alpha_{ab}^3}{(1+\alpha_{ab}^2)^{3/2}}.
\end{aligned} \tag{C2}
$$

Next, the $0i$ elements $M_{ab}^{(A)0i}$, $N_{ab}^{(A)0i}$ $(A = 0, N)$, M_{ab}^{0i}, and N_{ab}^{0i} $(i = 1, 2)$ are given by

$$
\begin{aligned}
M_{ab}^{(0)01} &= -\frac{3\alpha_{ab}^3}{2(1+\alpha_{ab}^2)^{3/2}}, \\
M_{ab}^{(0)02} &= -\frac{15\alpha_{ab}^5}{8(1+\alpha_{ab}^2)^{5/2}}, \\
M_{ab}^{(N)01} &= (\theta_{ab}^2 - 1) M_{ab}^{(0)01} = \left(1 - \frac{T_a}{T_b}\right) \frac{3\alpha_{ab}^3}{2(1+\alpha_{ab}^2)^{5/2}}, \\
M_{ab}^{(N)02} &= (\theta_{ab}^2 - 1) M_{ab}^{(0)02} = \left(1 - \frac{T_a}{T_b}\right) \frac{15\alpha_{ab}^5}{8(1+\alpha_{ab}^2)^{7/2}}, \\
M_{ab}^{01} &= \theta_{ab}^2 M_{ab}^{(0)01} = -\frac{3\alpha_{ab}^5}{2(1+\alpha_{ab}^2)^{5/2}} \left(1 + \frac{m_a}{m_b}\right), \\
M_{ab}^{02} &= \theta_{ab}^2 M_{ab}^{(0)02} = -\frac{15\alpha_{ab}^7}{8(1+\alpha_{ab}^2)^{7/2}} \left(1 + \frac{m_a}{m_b}\right), \\
N_{ab}^{(0)01} &= -\frac{T_a}{T_b} \alpha_{ab} M_{ba}^{(0)01} = \frac{T_a}{T_b} \frac{3\alpha_{ab}}{2(1+\alpha_{ab}^2)^{3/2}}, \\
N_{ab}^{(0)02} &= -\frac{T_a}{T_b} \alpha_{ab} M_{ba}^{(0)02} = \frac{T_a}{T_b} \frac{15\alpha_{ab}}{8(1+\alpha_{ab}^2)^{5/2}}, \\
N_{ab}^{(N)01} &= -\frac{T_a}{T_b} \alpha_{ab} M_{ba}^{(N)01} = \left(1 - \frac{T_a}{T_b}\right) \frac{3\alpha_{ab}^3}{2(1+\alpha_{ab}^2)^{5/2}}, \\
N_{ab}^{(N)02} &= -\frac{T_a}{T_b} \alpha_{ab} M_{ba}^{(N)02} = \left(1 - \frac{T_a}{T_b}\right) \frac{15\alpha_{ab}^3}{8(1+\alpha_{ab}^2)^{7/2}}, \\
N_{ab}^{01} &= \frac{3\alpha_{ab}^3}{2(1+\alpha_{ab}^2)^{5/2}} \left(1 + \frac{m_a}{m_b}\right), \\
N_{ab}^{02} &= \frac{15\alpha_{ab}^3}{8(1+\alpha_{ab}^2)^{7/2}} \left(1 + \frac{m_a}{m_b}\right),
\end{aligned} \tag{C3}
$$

where θ_{ab} defined in Eq. (21) and the momentum conservation law shown in Eq. (B13) are used.

The $i0$ elements $M_{ab}^{(A)i0}$ and $N_{ab}^{(A)i0}$ $(A = 0, N; i = 1, 2)$ are given by

$$
\begin{aligned}
M_{ab}^{(0)10} &= M_{ab}^{(0)01}, \qquad M_{ab}^{(0)20} = M_{ab}^{(0)02}, \\
M_{ab}^{(N)10} &= \left(\frac{T_a}{T_b} - 1\right) \frac{\alpha_{ab}(10 + \alpha_{ab}^2)}{2(1+\alpha_{ab}^2)^{5/2}}, \\
M_{ab}^{(N)20} &= \left(\frac{T_a}{T_b} - 1\right) \frac{3\alpha_{ab}^3(28 + 3\alpha_{ab}^2)}{8(1+\alpha_{ab}^2)^{7/2}}, \\
N_{ab}^{(0)10} &= \frac{T_a^2}{T_b^2} \alpha_{ab} N_{ba}^{(0)01} = -\frac{T_a}{T_b} M_{ab}^{(0)01}, \\
N_{ab}^{(0)20} &= \frac{T_a^2}{T_b^2} \alpha_{ab} N_{ba}^{(0)02} = -\frac{T_a}{T_b} M_{ab}^{(0)02}, \\
N_{ab}^{(N)10} &= \left(1 - \frac{T_a}{T_b}\right) \frac{3\alpha_{ab}^3(-2 + \alpha_{ab}^2)}{2(1+\alpha_{ab}^2)^{5/2}}, \\
N_{ab}^{(N)20} &= \left(1 - \frac{T_a}{T_b}\right) \frac{15\alpha_{ab}^5(-4 + \alpha_{ab}^2)}{8(1+\alpha_{ab}^2)^{7/2}},
\end{aligned} \tag{C4}
$$

where the relations shown in Eqs. (B13) and (B14) are used. Using Eqs. (C3) and (C4), we can immediately evaluate $M_{ab}^{i0} = M_{ab}^{(0)i0} + M_{ab}^{(N)i0}$ and $N_{ab}^{i0} = N_{ab}^{(0)i0} + N_{ab}^{(N)i0}$ $(i = 1, 2)$.

The ij elements $M_{ab}^{(A)ij}$ and $N_{ab}^{(A)ij}$ $(A = 0, N; i = 1, 2)$ are written as

$$
\begin{aligned}
M_{ab}^{(0)11} &= -\frac{\alpha_{ab}(30+16\alpha_{ab}^2+13\alpha_{ab}^4)}{4(1+\alpha_{ab}^2)^{5/2}}, \\
M_{ab}^{(0)12} &= M_{ab}^{(0)21} = -\frac{3\alpha_{ab}^3(84+32\alpha_{ab}^2+23\alpha_{ab}^4)}{16(1+\alpha_{ab}^2)^{7/2}}, \\
M_{ab}^{(0)22} &= -\frac{\alpha_{ab}}{64(1+\alpha_{ab}^2)^{9/2}} \\
&\quad \times (1400+1792\alpha_{ab}^2+3672\alpha_{ab}^4+1088\alpha_{ab}^6+433\alpha_{ab}^8), \\
M_{ab}^{(N)11} &= \left(1-\frac{T_a}{T_b}\right)\frac{3\alpha_{ab}(10-2\alpha_{ab}^2+3\alpha_{ab}^4)}{4(1+\alpha_{ab}^2)^{7/2}}, \\
M_{ab}^{(N)12} &= \left(1-\frac{T_a}{T_b}\right)\frac{3\alpha_{ab}^3(84-2\alpha_{ab}^2+19\alpha_{ab}^4)}{16(1+\alpha_{ab}^2)^{9/2}}, \\
M_{ab}^{(N)21} &= -\left(1-\frac{T_a}{T_b}\right)\frac{\alpha_{ab}}{16(1+\alpha_{ab}^2)^{9/2}} \\
&\quad \times (280+84\alpha_{ab}^2+348\alpha_{ab}^4+19\alpha_{ab}^6), \\
M_{ab}^{(N)22} &= \left(1-\frac{T_a}{T_b}\right)\frac{\alpha_{ab}}{64(1+\alpha_{ab}^2)^{11/2}} \\
&\quad \times (1400-112\alpha_{ab}^2+2424\alpha_{ab}^4-556\alpha_{ab}^6+233\alpha_{ab}^8), \\
N_{ab}^{(0)11} &= \frac{T_a}{T_b}\frac{27\alpha_{ab}^3}{4(1+\alpha_{ab}^2)^{5/2}}, \\
N_{ab}^{(0)12} &= \frac{T_a}{T_b}\frac{225\alpha_{ab}^3}{16(1+\alpha_{ab}^2)^{7/2}}, \\
N_{ab}^{(0)21} &= \frac{T_a}{T_b}\frac{225\alpha_{ab}^5}{16(1+\alpha_{ab}^2)^{7/2}}, \\
N_{ab}^{(0)22} &= \frac{T_a}{T_b}\frac{2125\alpha_{ab}^5}{64(1+\alpha_{ab}^2)^{9/2}}, \\
N_{ab}^{(N)11} &= \left(1-\frac{T_a}{T_b}\right)\frac{9\alpha_{ab}^3(-2+3\alpha_{ab}^2)}{4(1+\alpha_{ab}^2)^{7/2}}, \\
N_{ab}^{(N)12} &= \left(1-\frac{T_a}{T_b}\right)\frac{45\alpha_{ab}^3(-2+5\alpha_{ab}^2)}{16(1+\alpha_{ab}^2)^{9/2}}, \\
N_{ab}^{(N)21} &= \left(1-\frac{T_a}{T_b}\right)\frac{75\alpha_{ab}^5(-4+3\alpha_{ab}^2)}{16(1+\alpha_{ab}^2)^{9/2}}, \\
N_{ab}^{(N)22} &= \left(1-\frac{T_a}{T_b}\right)\frac{525\alpha_{ab}^5(-4+5\alpha_{ab}^2)}{64(1+\alpha_{ab}^2)^{11/2}}. \qquad \text{(C5)}
\end{aligned}
$$

Then, $M_{ab}^{ij} = M_{ab}^{(0)ij} + M_{ab}^{(N)ij}$ and $N_{ab}^{ij} = N_{ab}^{(0)ij} + N_{ab}^{(N)ij}$ $(i, j = 1, 2)$ are evaluated from the results shown in Eq. (C5).

Now, we can use the matrix elements $M_{ab}^{(0)0i} = M_{ab}^{(0)i0}$ and $M_{ab}^{(0)ij}$ $(i, j = 1, 2, \cdots)$ to express the matrix elements $M_{ab}^{(S)0i} = M_{ab}^{(S)i0}$ and $M_{ab}^{(S)ij}$ $(i, j = 1, 2, \cdots)$ by

$$
\begin{aligned}
M_{ab}^{(S)0i} &= M_{ab}^{(S)i0} = \theta_{ab} M_{ab}^{(0)0i}, \\
M_{ab}^{(S)ij} &= M_{ab}^{(0)ij}, \qquad \text{(C6)}
\end{aligned}
$$

and write the matrix elements $N_{ab}^{(S)0i}$ and $N_{ab}^{(S)i0}$ $(i = 1, 2, \cdots)$ as

$$
\begin{aligned}
N_{ab}^{(S)0i} &= -\frac{T_a}{T_b}\alpha_{ab} M_{ba}^{(S)0i} = \theta_{ba} N_{ab}^{(0)0i}, \\
N_{ab}^{(S)i0} &= \frac{T_a}{T_b}\alpha_{ab} N_{ba}^{(S)0i} = -M_{ab}^{(S)0i}, \qquad \text{(C7)}
\end{aligned}
$$

where Eqs. (31), (34), (B13), and (C6) are used. Then, Eq. (35) can be used to evaluate $N_{ab}^{(S)ij}$ $(i, j = 1, 2, \cdots)$ from $N_{ab}^{(S)i0}$, $N_{ab}^{(S)0j}$ and N_{ab}^{00} [see Eq. (C2)].

Appendix D: IMPROVED SUGAMA OPERATOR MODIFIED BY SYMMETRIZING MATRIX ELEMENTS

In this Appendix, the improved Sugama operator defined in Eq. (40) is modified when $T_a \neq T_b$ as follows:

$$
C_{ab}^{LS*(\mathrm{imp})}(\delta f_a, \delta f_b) \equiv C_{ab}^{LS}(\delta f_a, \delta f_b) + \Delta C_{ab}^{F*}(\delta f_b) \qquad \text{(D1)}
$$

where $C_{ab}^{LS}(\delta f_a, \delta f_b)$ represents the original Sugama operator described in Sec. III and the new correction part $\Delta C_{ab}^{F*}(\delta f_b)$ is defined by

$$
\Delta C_{ab}^{F*}(\delta f_b) \equiv f_{aM}\frac{m_a}{T_a}\mathbf{v}\cdot\sum_{j=1}^{\infty}\Delta\mathbf{C}_{abj}^{F*}[\delta f_b] L_j^{(3/2)}(x_a^2). \qquad \text{(D2)}
$$

Here, $\Delta\mathbf{C}_{abj}^{F*}[\delta f_b]$ $(j = 1, 2, \cdots)$ are given by

$$
\Delta\mathbf{C}_{abj}^{F*}[\delta f_b] \equiv \frac{c_j}{\tau_{ab}}\sum_{k=1}^{\infty}\Delta N_{ab}^{*jk}\,\mathbf{u}_{bk}[\delta f_b] \qquad (j = 1, 2, \cdots), \qquad \text{(D3)}
$$

and

$$
\begin{aligned}
\Delta N_{ab}^{*jk} &\equiv N_{ab}^{*jk} - N_{ab}^{(S)jk} \\
&= \frac{N_{ab}^{00}N_{ab}^{*jk} - N_{ab}^{(S)i0}N_{ab}^{(S)0j}}{N_{ab}^{00}} \quad (j, k = 1, 2, \cdots), \qquad \text{(D4)}
\end{aligned}
$$

where

$$
N_{ab}^{*jk} \equiv \frac{T_a v_{Ta}}{2}\left(\frac{N_{ab}^{jk}}{T_a v_{Ta}} + \frac{N_{ba}^{kj}}{T_b v_{Tb}}\right) \quad (j, k = 1, 2, \cdots). \qquad \text{(D5)}
$$

We can now use the test and field particle part of $C_{ab}^{LS*(\mathrm{imp})}(\delta f_a, \delta f_b)$ to obtain the matrix elements M_{ab}^{*ij} and N_{ab}^{*ij} in the same way as shown in Eq. (11). Then, the friction coefficients l_{*ij}^{ab} can be derived from M_{ab}^{*ij} and N_{ab}^{*ij} [see Eq. (13)]. Since $\Delta C_{ab}^{F*}(\delta f_b)$ defined in Eq. (D2) gives the correction only in the field particle part, we immediately see that

$$
M_{ab}^{*ij} = M_{ab}^{(S)ij} \quad (i, j = 0, 1, 2, \cdots). \qquad \text{(D6)}
$$

We also find that

$$
N_{ab}^{*i0} = N_{ab}^{(S)i0}, \quad N_{ab}^{*0j} = N_{ab}^{(S)0j}, \quad (i, j = 0, 1, 2, \cdots), \qquad \text{(D7)}
$$

and N_{ab}^{*ij} $(i,j=1,2,\cdots)$ are given by Eq. (D5). It is confirmed from Eqs. (D5), (D6) and (D7) that the matrix elements M_{ab}^{*ij} and N_{ab}^{*ij} satisfy

$$M_{ab}^{*ij} = M_{ab}^{*ji}, \quad \frac{N_{ab}^{*ij}}{T_a v_{Ta}} = \frac{N_{ba}^{*ji}}{T_b v_{Tb}} \quad (i,j=0,1,2,\cdots), \tag{D8}$$

which leads to the symmetry of the friction coefficients l_{*ij}^{ab},

$$l_{*ij}^{ab} = l_{*ji}^{ba} \quad (i,j=1,2,\cdots). \tag{D9}$$

The modified operator $C_{ab}^{LS*(\mathrm{imp})}(\delta f_a, \delta f_b)$ exactly satisfies the adjointness relations in the same form as those in Eq. (16) and accordingly induces the Onsager symmetry of collisional transport coefficients.

When $T_a \neq T_b$, the values of M_{ab}^{*ij}, N_{ab}^{*ij}, and l_{*ij}^{ab} are different from those of M_{ab}^{ij}, N_{ab}^{ij}, and l_{ij}^{ab} given by the linearized Landau operator, respectively. However, as explained in Appendix B, the differences between these values are not expected to cause serious errors in solutions of kinetic equations because $m_a/m_b \ll 1$ or $m_a/m_b \gg 1$ are required if T_a and T_b differ significantly from each other.

Noting that $\Delta C_{ab}^{F*}(\delta f_b)$ never influences collisional momentum and energy transfer, we can confirm that $C_{ab}^{LS*(\mathrm{imp})}(\delta f_a, \delta f_b)$ keeps conservation laws of particles, momentum, and energy. Especially, the momentum conservation law imposes the constraints on the matrix elements and the friction coefficients as

$$\begin{aligned} & M_{ab}^{*0j} + \frac{T_a v_{Ta}}{T_b v_{Tb}} N_{ba}^{*0j} = 0 \quad (j=0,1,2,\cdots), \\ & \sum_a l_{*1j}^{ab} = 0 \quad (j=1,2,\cdots), \end{aligned} \tag{D10}$$

which are rewritten with the help of Eqs. (D8) and (D9) as

$$\begin{aligned} & M_{ab}^{*j0} + N_{ab}^{*j0} = 0 \quad (j=0,1,2,\cdots), \\ & \sum_b l_{*j1}^{ab} = 0 \quad (j=1,2,\cdots). \end{aligned} \tag{D11}$$

Then, if the perturbed functions are written as $\delta f_s = f_{sM}(m_a/T_s)(\mathbf{u}_s\cdot\mathbf{v})$ $(s=a,b)$ with the condition $\mathbf{u}_a = \mathbf{u}_b$, we find that $C_{ab}^{LS*(\mathrm{imp})}(\delta f_a, \delta f_b) = \tau_{ab}^{-1} f_{aM}(m_a/T_a)\mathbf{v}\cdot\sum_{j=0}^{\infty} c_j L_j^{(3/2)}(x_a^2)(M_{ab}^{*j0}\mathbf{u}_a + N_{ab}^{*j0}\mathbf{u}_b) = 0$ because of Eq. (D11) and $\mathbf{u}_a = \mathbf{u}_b$. Noting that $C_{ab}^{LS*(\mathrm{imp})}$ is also annihilated by the perturbed distribution functions δf_s $(s=a,b)$ given by Eq. (45) with $\delta T_a/T_a = \delta T_b/T_b$, it is now remarked that $C_{ab}^{LS*(\mathrm{imp})}(\delta f_a, \delta f_b)$ vanishes for the perturbed distribution functions given by the perturbed Maxwellian with the perturbed densities δn_s, temperatures δT_s, and flows $\mathbf{u}_s$ $(s=a,b)$ as

$$\delta f_s = f_{sM}\left[\frac{\delta n_s}{n_s} + \frac{m_s}{T_s}\mathbf{u}_s\cdot\mathbf{v} + \frac{\delta T_s}{T_s}\left(\frac{m_s v^2}{2T_s} - \frac{3}{2}\right)\right], \tag{D12}$$

where $\mathbf{u}_a = \mathbf{u}_b$ and $\delta T_a/T_a = \delta T_b/T_b$.

Using Eq. (D1), the collision operator for gyrokinetic equations is given by

$$\begin{aligned} & \oint \frac{d\xi}{2\pi} e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} C_{ab}^{LS*(\mathrm{imp})}(e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} h_{a\mathbf{k}_\perp}, e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_b} h_{b\mathbf{k}_\perp}) \\ = & \oint \frac{d\xi}{2\pi} e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} C_{ab}^{LS}(e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} h_{a\mathbf{k}_\perp}, e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_b} h_{b\mathbf{k}_\perp}) \\ & + \oint \frac{d\xi}{2\pi} e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} \Delta C_{ab}^{F*}(e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a} h_{a\mathbf{k}_\perp}, e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_b} h_{b\mathbf{k}_\perp}). \end{aligned} \tag{D13}$$

The detailed expression of the first term on the right-hand of Eq. (D13) is found in Ref.[23] while the second term is expressed by Eq. (49) with putting $\Delta M_{ab}^{jk} = 0$ and replacing ΔN_{ab}^{jk} by $\Delta N_{ab}^{*jk} \equiv N_{ab}^{*jk} - N_{ab}^{(S)jk}$.

Since the two colliding particle species need to have very different masses for their temperatures to be significantly different from each other, the improved Sugama operators presented in this Appendix and Sec. IV do not seem to show large quantitative differences from each other for the case of $T_a \neq T_b$ where $m_a/m_b \ll 1$ or $m_a/m_b \gg 1$ holds. It is not so clear how the adjointness properties of the linearized collision operator is crucial for accurate prediction of turbulent transport or for formulation of efficient turbulence simulation methods in comparison with their roles in neoclassical transport theory and simulation. Unless one can recognize merits of the adjointness properties for analytical or numerical calculations of turbulent transport, the operator presented in Sec. IV may seem more suitable for gyrokinetic simulation in the unequal temperature case than that in this Appendix because the former describes the friction-flow relations more accurately. However, we still note that there are several theoretical studies on the Onsager-type symmetry of the quasilinear turbulent transport matrix,[60–64] for which the collision operator given in this Appendix can be useful to study collisional effects.

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