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# Conservation laws for collisional and turbulent transport processes in toroidal plasmas with large mean flows

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A novel gyrokinetic formulation is presented by including collisional effects into the Lagrangian variational principle to yield the governing equations for background and turbulent electromagnetic fields and gyrocenter distribution functions which can simultaneously describe classical, neoclassical, and turbulent transport processes in toroidal plasmas with large toroidal flows on the order of the ion thermal velocity. Noether's theorem modified for collisional systems and the collision operator given in terms of Poisson brackets are applied to derivation of the particle, energy, and toroidal momentum balance equations in the conservative forms which are desirable properties for long-time global transport simulation.

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Profiles of background $\mathbf{E}\times\mathbf{B}$ and toroidal flows are regarded as key factors which influence magnetic plasma confinement although severe accuracy requirements for theoretically predicting those flow profiles are sometimes controversial among recent studies based on the low-flow ordering[1–5] in which the background flow velocity $V_0$ is assumed to be of $\mathcal{O}(\delta v_{Ti})$. Here, $v_{Ti}$ is the ion thermal velocity and $\delta \sim \rho_{Ti}/L \ll 1$ represents the ordering parameter defined by the ratio of the ion thermal gyroradius $\rho_{Ti}$ to the background gradient scale length $L$. On the other hand, under the high-flow ordering $V_0 = \mathcal{O}(v_{Ti})$[6–14], the toroidal momentum transport equation which determines the background radial electric field profile can be derived with the same-order accuracy as the particle and energy transport equations. As a modern theoretical technique, the Lagrangian variational principle[10,11,15,16] is used for deriving the gyrokinetic equation to investigate transport processes in magnetized plasmas. However, useful properties of this principle such as Noether's theorem are originally applied only to collisionless systems[15–17] although our previous work[18] clarifies how Noether's theorem can be modified to evaluate collisional effects on conservation laws. In this study, we present a novel formulation of collisional and turbulent transport in toroidal plasmas under the high-flow ordering by generalizing the previous study to derive governing equations for background and turbulent electromagnetic fields and gyrocenter distribution functions which satisfy conservation laws for particles, energy, and toroidal momentum.

We here use the gyrocenter coordinates denoted by $\mathbf{Z}_a = (Z_a^i)_{i=1,\cdots,6} = (\mathbf{X}_a, U_a, \mu_a, \xi_a)$ where $\mathbf{X}_a$, $U_a$, $\mu_a$, and $\xi_a$ represent the gyrocenter position, parallel velocity, magnetic moment, and gyrophase angle, respectively. The single-particle Lagrangian $L_a$ for species $a$ with the mass $m_a$ and the charge $e_a$ is written as

$$L_a = \mathbf{P}_a^c \cdot \dot{\mathbf{X}}_a + \frac{m_a c}{e_a}\mu_a \dot{\xi}_a - H_a, \tag{1}$$

where $\dot{} \equiv d/dt$ represents the time derivative and $H_a$ is the single-particle Hamiltonian [see Eq. (2)]. The canonical momentum is denoted by $\mathbf{P}_a^c \equiv (e_a/c)\mathbf{A}_a^*$ and $\mathbf{A}_a^*$ is defined by $\mathbf{A}_a^* \equiv \mathbf{A}_0 + (m_a c/e_a)(U_a\mathbf{b} + \mathbf{V}_0)$, where $\mathbf{A}_0$, $\mathbf{b} \equiv \mathbf{B}_0/B_0$, and $\mathbf{V}_0 \equiv V_\zeta \nabla\zeta \equiv V^\zeta \mathbf{e}_\zeta$ are the vector potential for the equilibrium magnetic field $\mathbf{B}_0 = \nabla\times\mathbf{A}_0$, the unit vector parallel to $\mathbf{B}_0$, and the background toroidal flow, respectively, and they are all regarded as functions of $(\mathbf{X}_a, t)$. The contravariant basis vector in the toroidal direction is given by $\mathbf{e}_\zeta \equiv R^2\nabla\zeta$ with the toroidal angle $\zeta$ and the major radius $R = |\nabla\zeta|^{-1}$. Under the high-flow ordering, we have $\mathbf{E}_0 + \mathbf{V}_0\times\mathbf{B}_0/c = 0$ where the zeroth-order electric field $\mathbf{E}_0$ is given by $\mathbf{E}_0 = -\nabla\Phi_0$ and the zeroth-order potential $\Phi_0$ is a flux-surface function[6]. Then, the component of $\mathbf{V}_0$ perpendicular to $\mathbf{B}_0$ is written as $(\mathbf{V}_0)_\perp = \mathbf{V}_{E0} \equiv (c/B_0)(\mathbf{E}_0\times\mathbf{b})$, and we obtain $V^\zeta = -c\partial\Phi_0/\partial\chi$ where $\chi = -A_{0\zeta}$ gives the poloidal flux of the equilibrium field $\mathbf{B}_0$ divided by $2\pi$.

The single-particle Hamiltonian $H_a$ is written as

$$H_a = e_a\Phi_0 + \frac{1}{2}m_a|U_a\mathbf{b} + \mathbf{V}_0|^2 + \mu_a B_0 + H_{a1}^V + e_a\Psi_a, \tag{2}$$

where $e_a\Phi_0$ is the dominant term of $\mathcal{O}(\delta^{-1})$ while $\frac{1}{2}m_a|U_a\mathbf{b}+\mathbf{V}_0|^2$ and $\mu_a B_0$ are of $\mathcal{O}(\delta^0)$. The toroidal flow $\mathbf{V}_0$ also induces the $\mathcal{O}(\delta)$ part of Hamiltonian defined by $H_{a1}^V \equiv (m_a c/e_a)\mu_a[\frac{1}{2}\mathbf{b}\cdot(\nabla\times\mathbf{V}_{E0}) + V^\zeta W_\zeta]$, where $W_\zeta \equiv -(RB_0)^{-1}(\nabla R\cdot\nabla\chi) + \frac{1}{2}b_\zeta\mathbf{b}\cdot(\nabla\times\mathbf{b})$ is included to take account of Littlejohn's gyro-gauge-dependent term[19] and reproduce the equations for the parallel acceleration $dU_a/dt$ and the change rate of the kinetic energy correctly up to $\mathcal{O}(\delta)$. The turbulent fields are included in $\Psi_a$ which is defined in terms of the first-order potential fields $\phi_1$ and $\mathbf{A}_1$ as

$$\Psi_a \equiv \langle\psi_a\rangle_{\xi_a} + \frac{e_a}{2m_a c^2}\left\langle|\mathbf{A}_1|^2\right\rangle_{\xi_a} - \frac{e_a}{2B_0}\frac{\partial}{\partial\mu}\langle(\widetilde{\psi}_a)^2\rangle_{\xi_a}, \tag{3}$$

where $\psi_a \equiv \phi_1(\mathbf{X}_a + \boldsymbol{\rho}_a, t) - c^{-1}(\mathbf{V}_0 + \mathbf{v}_a')\cdot\mathbf{A}_1(\mathbf{X}_a +$

$\boldsymbol{\rho}_a, t)$ and $\boldsymbol{\rho}_a \equiv \mathbf{b} \times \mathbf{v}'_a/\Omega_a$. The velocity $\mathbf{v}'_a$ observed from the rotating frame is written as $\mathbf{v}'_a = U_a\mathbf{b} - [2\mu_a B_0/m_a]^{1/2}[\sin\xi_a\,\mathbf{e}_1 + \cos\xi_a\,\mathbf{e}_2]$, where the unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$ form a right-handed orthogonal system at $(\mathbf{X}_a, t)$. The gyrophase-average and gyrophase-dependent parts of an arbitrary periodic function $Q(\xi_a)$ are denoted by $\langle Q\rangle_{\xi_a} \equiv \oint d\xi_a\, Q(\xi_a)/(2\pi)$ and $\widetilde{Q} \equiv Q - \langle Q\rangle_{\xi_a}$, respectively.

Now, the Lagrangian for the whole system is given by

$$L = \sum_a \int d^6 Z_0\, D_a(\mathbf{Z}_0, t_0) F_a(\mathbf{Z}_0, t_0) L_a|_{\mathbf{Z}_a = \mathbf{Z}_a(\mathbf{Z}_0, t_0; t)} + L_f, \tag{4}$$

where $\int d^6 Z_0 \equiv \int d^3 X_0 \int_{-\infty}^{\infty} dU_0 \int_0^{\infty} d\mu_0 \int_0^{2\pi} d\xi_0$ represents the integral with respect to the initial gyrocenter coordinates, $F_a(\mathbf{Z}_0, t_0)$ is the particle distribution function at the initial time $t_0$, and $L_a|_{\mathbf{Z}_a = \mathbf{Z}_a(\mathbf{Z}_0, t_0; t)}$ is given by Eq. (1) with $\mathbf{Z}_a = \mathbf{Z}_a(\mathbf{Z}_0, t_0; t)$ which represents the gyrocenter phase space trajectory at the time $t$ satisfying the initial condition, $\mathbf{Z}_a(\mathbf{Z}_0, t_0; t_0) = \mathbf{Z}_0$. The Jacobian is given by $D_a \equiv B^*_{a\parallel}/m_a$ where $B^*_{a\parallel} \equiv \mathbf{B}^*_a \cdot \mathbf{b}$ and $\mathbf{B}^*_a \equiv \nabla \times \mathbf{A}^*_a$. The Lagrangian associated with electromagnetic fields is defined by $L_f \equiv \int d^3 X\, \mathcal{L}_f$ with

$$\mathcal{L}_f = \frac{1}{8\pi}\left[|\nabla(\Phi_0 + \phi_1)|^2 - |\nabla \times (\mathbf{A}_0 + \mathbf{A}_1)|^2\right] + \frac{\mathbf{\Lambda}}{4\pi} \cdot (\mathbf{B}_0 - I\nabla\zeta - \nabla\zeta \times \nabla\chi) + \frac{\alpha}{4\pi c}\nabla \cdot \mathbf{A}_0 + \frac{\lambda}{4\pi c}\nabla \cdot \mathbf{A}_1. \tag{5}$$

The variational principle, $\delta\mathcal{I} \equiv \delta\int_{t_1}^{t_2} L dt = 0$, yields the gyrocenter motion equations and the equations for the background and turbulent electromagnetic fields. The Lagrangian undetermined multipliers $\mathbf{\Lambda}$, $\alpha$, and $\lambda$ are introduced in Eq. (5) to impose constraint conditions $\mathbf{B}_0 \equiv \nabla \times \mathbf{A}_0 = I\nabla\zeta + \nabla\zeta \times \nabla\chi$, $\nabla \cdot \mathbf{A}_0 = 0$, and $\nabla \cdot \mathbf{A}_1 = 0$, respectively. The gyrokinetic Poisson equation is obtained from the condition $\delta\mathcal{I}/\delta\phi_1 = 0$ as[17]

$$\nabla \cdot (\mathbf{E}_L + 4\pi\mathbf{P}^{(\mathrm{pol})}) = 4\pi \sum_a e_a n_a^{(\mathrm{gc})}, \tag{6}$$

where the longitudinal part of the electric field $\mathbf{E} \equiv -\nabla(\Phi_0 + \phi_1) - c^{-1}\partial(\mathbf{A}_0 + \mathbf{A}_1)/\partial t$ is denoted by $\mathbf{E}_L \equiv -\nabla(\Phi_0 + \phi_1) \equiv \mathbf{E}_0 + \mathbf{E}_{L1}$ (the subscript $L$ represents the longitudinal part of the vector), the gyrocenter density is given by $n_a^{(\mathrm{gc})}(\mathbf{X}, t) \equiv \int d^3 v^{(\mathrm{gc})}\, F_a(\mathbf{Z}, t)$, and the polarization density is written as[17]

$$\mathbf{P}^{(\mathrm{pol})} = \sum_a e_a \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \times \sum_{i_1,\cdots,i_n} \int d^3 v^{(\mathrm{gc})}\, \frac{\partial^n (D_a F^*_a \boldsymbol{\rho}_a \rho_{ai_1} \cdots \rho_{ai_n})}{\partial X_{i_1} \cdots \partial X_{i_n}}. \tag{7}$$

Here, $\int d^3 v^{(\mathrm{gc})} \equiv \int dU \int d\mu \int d\xi\, D_a(\mathbf{Z}, t)$ represents the integral over the gyrocenter velocity space, $\rho_{ai}$ is the $i$th Cartesian component of $\boldsymbol{\rho}_a$, and $F^*_a \equiv F_a + (e_a\widetilde{\psi}_a/B_0)(\partial F_a/\partial\mu)$. From $\delta\mathcal{I}/\delta\Phi_0 = 0$, we obtain the surface-averaged gyrokinetic Poisson equation,

$$\frac{\langle\nabla \cdot \mathbf{E}_L\rangle}{4\pi} = \sum_a e_a \left\langle n_a^{(\mathrm{gc})} - \nabla \cdot \left[\int d^3 v^{(\mathrm{gc})} F_a \left\{\frac{\mathbf{b}}{\Omega_a} \times \left(\dot{\mathbf{X}}_a - \mathbf{V}_0 - \frac{e_a}{m_a}\frac{\partial\Psi_a}{\partial\mathbf{V}_0}\right) + \frac{2c\mu}{e_a\Omega_a R}\nabla R\right\} - \frac{\nabla\chi}{|\nabla\chi|^2}\nabla \cdot \left(\int d^3 v^{(\mathrm{gc})} \frac{c\mu F_a}{2e_a\Omega_a}\nabla\chi\right)\right]\right\rangle, \tag{8}$$

where $\langle\cdots\rangle$ represents the flux-surface average. Equations (6) and (8) give two conditions to determine $\phi_1$ and $\Phi_0$. Instead of Eq. (8), we can also use the toroidal momentum balance given later in Eq. (21). The equations necessary for determining the other fields $\mathbf{A}_0$, $\mathbf{A}_1$, $I$, $\chi$, $\mathbf{\Lambda}$, $\alpha$, and $\lambda$ are derived from the conditions $\delta\mathcal{I}/\delta\mathbf{A}_0 = \delta\mathcal{I}/\delta\mathbf{A}_1 = 0$, $\delta\mathcal{I}/\delta I = \delta\mathcal{I}/\delta\chi = 0$, $\nabla \cdot \mathbf{A}_0 = \nabla \cdot \mathbf{A}_1 = 0$, and $\mathbf{B}_0 \equiv \nabla \times \mathbf{A}_0 = I\nabla\zeta + \nabla\zeta \times \nabla\chi$.[17]

The gyrocenter motion equations is derived from $\delta\mathcal{I}/\delta\mathbf{Z}_a = 0$ as

$$\frac{d\mathbf{Z}_a}{dt} = \{\mathbf{Z}_a, H_a\} + \{\mathbf{Z}_a, \mathbf{X}_a\} \cdot \frac{e_a}{c}\frac{\partial\mathbf{A}^*_a}{\partial t}. \tag{9}$$

Here, nonvanishing Poisson brackets between the gyrocenter coordinates are obtained from Eq. (1) as $\{\mathbf{X}_a, \mathbf{X}_a\} = c(\mathbf{b} \times \mathbf{I})/(e_a B^*_{a\parallel})$, $\{\mathbf{X}_a, U_a\} = \mathbf{B}^*_a/(m_a B^*_{a\parallel})$, and $\{\xi_a, \mu_a\} = e_a/(m_a c)$. We now consider the gyrokinetic Boltzmann equation for the distribution function $F_a(\mathbf{Z}, t)$,

$$\left(\frac{\partial}{\partial t} + \frac{d\mathbf{Z}_a}{dt} \cdot \frac{\partial}{\partial\mathbf{Z}}\right) F_a = \sum_a \langle C_{ab}[F_a, F_b]\rangle_\xi + \mathcal{S}_a \equiv \mathcal{K}_a, \tag{10}$$

where $C_{ab}[F_a, F_b]$ represents the rate of change in $F_a$ due to Coulomb collisions between particle species $a$ and $b$ and $\mathcal{S}_a$ is an external source term. Here, $F_a$ is assumed to be independent of the gyrophase $\xi$. When $\mathcal{K}_a = 0$, Eq. (10) reduces to the gyrokinetic Vlasov equation for which Noether's theorem can be applied to derive conservation laws of energy and toroidal momentum from symmetry properties[17]. However, even if $\mathcal{K}_a \neq 0$, we can still derive the energy and toroidal momentum balance equations from Noether's theorem modified using the correspondence relation between $\partial F_a^V/\partial t$ and $\partial F_a/\partial t - \mathcal{K}_a$ where $F_a^V$ and $F_a$ represent the solution of Eq. (10) for $\mathcal{K}_a = 0$ and that for $\mathcal{K}_a \neq 0$, respectively[18].

In Ref.[18], a gyrokinetic collision operator is constructed under the low-flow ordering such that collisional terms in the particle, energy and momentum equations are represented by the divergences of the classical transport fluxes. To obtain similar representations for the high-flow case, we here follow Burby *et al.*[20] and use Poisson brackets to write the collision operator as

$$C_{ab}[F_a, F_b] = -\alpha_{ab} \sum_{i=1}^{3} \{x_{ai}, \gamma_i^{ab}\}, \tag{11}$$

where $\alpha_{ab} \equiv 2\pi e_a^2 e_b^2 \ln\Lambda$ and $\ln\Lambda$ is the Coulomb logarithm. Here, $x_{ai}$ and $\gamma_i^{ab}$ are the $i$th Cartesian components of the particle position vector $\mathbf{x}_a = \mathbf{X}_a + \boldsymbol{\rho}_a$ and the vector $\boldsymbol{\gamma}^{ab}$, respectively, the latter of which is defined by

$$\boldsymbol{\gamma}^{ab}(\mathbf{Z}_a) \equiv \int d^6 Z_b\, D_b(\mathbf{Z}_b)\delta[\mathbf{x}_a(\mathbf{Z}_a) - \mathbf{x}_a(\mathbf{Z}_a)]\mathbf{U}(\mathbf{u}_{ab})\cdot\mathbf{A}_{ab}, \tag{12}$$

with $\mathbf{u}_{ab} \equiv \{\mathbf{x}_a, H_a\} - \{\mathbf{x}_b, H_b\}$, $\mathbf{U}(\mathbf{u}) \equiv (u^2\mathbf{I} - \mathbf{uu})/u^3$, and $\mathbf{A}_{ab} \equiv F_a(\mathbf{Z}_a)\{\mathbf{x}_b, F_b(\mathbf{Z}_b)\} - F_b(\mathbf{Z}_b)\{\mathbf{x}_a, F_a(\mathbf{Z}_a)\}$. Then, we can show that the integral of the collision operator with respect to the gyrocenter velocity variables $(U_a, \mu_a, \xi_a)$ at the fixed gyrocenter position $\mathbf{X}_a = \mathbf{X}$ does not vanish but it is given in the divergence form as

$$\int d^3 v_a^{(\mathrm{gc})}\, C_{ab}[F_a, F_b] = -\nabla\cdot\boldsymbol{\Gamma}_{ab}^{\mathrm{C}}(\mathbf{X}), \tag{13}$$

where $\nabla \equiv \partial/\partial\mathbf{X}$ and

$$\boldsymbol{\Gamma}_{ab}^{\mathrm{C}} \equiv -\alpha_{ab}\int d^3 v_a^{(\mathrm{gc})}\{\mathbf{X}_a, \mathbf{x}_a\}\cdot\boldsymbol{\gamma}^{ab} \tag{14}$$

represents the classical particle flux due to finite gyroradii and collisions between the species $a$ and $b$[18]. In addition, Eq. (11) can be used to derive the integral formulas representing the divergences of energy, toroidal momentum, and entropy fluxes at the gyrocenter position $\mathbf{X}_a = \mathbf{X}_b = \mathbf{X}$ as

$$\begin{aligned}
&\int d^3 v_a^{(\mathrm{gc})}\, C_{ab}H_a + \int d^3 v_b^{(\mathrm{gc})}\, C_{ba}H_b = -\nabla\cdot(\mathbf{Q}_{ab}^{\mathrm{C}} + \mathbf{Q}_{ba}^{\mathrm{C}}),\\
&\int d^3 v_a^{(\mathrm{gc})}\, C_{ab}P_{a\zeta}^c + \int d^3 v_b^{(\mathrm{gc})}\, C_{ba}P_{b\zeta}^c = -\nabla\cdot(\boldsymbol{\Pi}_{ab\zeta}^{\mathrm{C}} + \boldsymbol{\Pi}_{ba\zeta}^{\mathrm{C}}),\\
&-\int d^3 v_a^{(\mathrm{gc})}\, C_{ab}(\log F_a + 1) - \int d^3 v_b^{(\mathrm{gc})}\, C_{ba}(\log F_b + 1)\\
&\qquad = \sigma_{ab}^{\mathrm{C}} - \nabla\cdot(\mathbf{J}_{Sab}^{\mathrm{C}} + \mathbf{J}_{Sba}^{\mathrm{C}}).
\end{aligned} \tag{15}$$

Here, the energy flux $\mathbf{Q}_{ab}^{\mathrm{C}}$ is defined by

$$\begin{aligned}
\mathbf{Q}_{ab}^{\mathrm{C}} \equiv -\alpha_{ab}\Bigg[&\int d^3 v_a^{(\mathrm{gc})} H_a\{\mathbf{X}_a, \mathbf{x}_a\}\cdot\boldsymbol{\gamma}^{ab} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\\
&\times\sum_{i_1,\cdots,i_n}\frac{\partial^n\left(\int d^3 v_a^{(\mathrm{gc})}\rho_{ai_1}\cdots\rho_{ai_n}\boldsymbol{\rho}_a\boldsymbol{\gamma}^{ab}\cdot\{\mathbf{x}_a, H_a\}\right)}{\partial X_{ai_1}\cdots\partial X_{ai_n}}\Bigg].
\end{aligned} \tag{16}$$

The toroidal momentum flux $\boldsymbol{\Pi}_{ab\zeta}^{\mathrm{C}}$ and the entropy flux $\mathbf{J}_{Sab}^{\mathrm{C}}$ are defined by the right-hand side of Eq. (16) with $H_a$ replaced by $P_{a\zeta}^c$ and $-(\log F_a + 1)$, respectively, and the entropy production rate $\sigma_{ab}^{\mathrm{C}}$ is given by

$$\begin{aligned}
\sigma_{ab}^{\mathrm{C}}(\mathbf{X}) = \alpha_{ab}&\int d^6 Z_a\int d^6 Z_a D_a D_b\delta(\mathbf{x}_a - \mathbf{x}_b)\delta(\mathbf{x}_a - \mathbf{X})\\
&\times(F_aF_b)^{-1}\mathbf{A}_{ab}\cdot\mathbf{U}(\mathbf{u}_{ab})\cdot\mathbf{A}_{ab}.
\end{aligned} \tag{17}$$

With the formula $\mathbf{a}\cdot\mathbf{U}(\mathbf{u})\cdot\mathbf{a} = u^{-3}[a^2u^2 - (\mathbf{a}\cdot\mathbf{u})^2] \geq 0$, Eq. (17) proves $\sigma_{ab}^{\mathrm{C}} \geq 0$ which represents the second law of thermodynamics.

Now, using Eqs. (10) and (13), we obtain the particle balance equation,

$$\frac{\partial n_a^{(\mathrm{gc})}}{\partial t} + \nabla\cdot(\boldsymbol{\Gamma}_a^{(\mathrm{gc})} + \boldsymbol{\Gamma}_a^{\mathrm{C}}) = \int d^3 v^{(\mathrm{gc})}\,\mathcal{S}_a, \tag{18}$$

where $\boldsymbol{\Gamma}_a^{(\mathrm{gc})} \equiv n_a^{(\mathrm{gc})}\mathbf{u}_a^{(\mathrm{gc})} \equiv \int d^3 v^{(\mathrm{gc})}\, F_a\mathbf{v}_a^{(\mathrm{gc})}$, $\mathbf{v}_a^{(\mathrm{gc})} \equiv d\mathbf{X}_a/dt$, and $\boldsymbol{\Gamma}_a^{\mathrm{C}} \equiv \sum_b\boldsymbol{\Gamma}_{ab}^{\mathrm{C}}$. Flux-surface-averaging Eq. (18) gives

$$\begin{aligned}
&\frac{\partial}{\partial t}(V'\langle n_a^{(\mathrm{gc})}\rangle) + \frac{\partial}{\partial s}(V'\langle(\boldsymbol{\Gamma}_a^{(\mathrm{gc})} + \boldsymbol{\Gamma}_a^{\mathrm{C}} - n_a^{(\mathrm{gc})}\mathbf{u}_s)\cdot\nabla s\rangle)\\
&= V'\left\langle\int d^3 v^{(\mathrm{gc})}\,\mathcal{S}_a\right\rangle,
\end{aligned} \tag{19}$$

where $s$ is an arbitrary label of a flux surface, $V' \equiv \partial V(s,t)/\partial s$, $V(s,t)$ is the volume enclosed by the flux surface, and $\mathbf{u}_s$ is defined by $\mathbf{u}_s \equiv \partial\mathbf{X}(s,\theta,\zeta,t)/\partial t$ with the flux coordinates $(s,\theta,\zeta)$. In the same manner as in Ref.[18], we use the modified Noether's theorem and the collision term in Eq. (11) to derive the energy and toroidal momentum balance equations written as

$$\begin{aligned}
&\frac{\partial}{\partial t}(V'\langle\mathcal{E}^*\rangle) + \frac{\partial}{\partial s}(V'\langle(\mathbf{Q} - \mathcal{E}^*\mathbf{u}_s)\cdot\nabla s\rangle)\\
&= V'\sum_a\left\langle\int d^3 v^{(\mathrm{gc})}\,\mathcal{S}_a(H_a - e_a\Phi_0)\right\rangle
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(V'\left\langle P_{\parallel V\zeta} + \frac{1}{c}\left(\mathbf{P}_L^{(\mathrm{pol})} + \frac{\mathbf{E}_L}{4\pi}\right)\cdot\nabla\chi\right\rangle\right)\\
&+\frac{\partial}{\partial s}\Bigg[V'\Bigg\{\Pi_{\parallel V\zeta}^s + \Pi_{R\zeta}^s + (\Pi^{\mathrm{C}*})^s - \frac{1}{4\pi}\langle A_{1\zeta}(\nabla\times\mathbf{B}_1)\cdot\nabla s\rangle\\
&-\frac{1}{4\pi}\langle E_{L1\zeta}E_{L1}^s + B_{1\zeta}B_1^s\rangle + \frac{1}{4\pi c}\left\langle\frac{\partial\lambda}{\partial\zeta}A_1^s\right\rangle\\
&-\frac{1}{c}\frac{\partial\chi(s,t)}{\partial t}\left\langle\left(\mathbf{P}_L^{(\mathrm{pol})} + \frac{\mathbf{E}_L}{4\pi}\right)\cdot\nabla s\right\rangle - \langle P_{\parallel V\zeta}\mathbf{u}_s\cdot\nabla s\rangle\Bigg\}\Bigg]\\
&= V'\sum_a\left\langle\int d^3 v^{(\mathrm{gc})}\,\mathcal{S}_a m_a(Ub_\zeta + V_\zeta)\right\rangle,
\end{aligned} \tag{21}$$

respectively. In the particle, energy, and toroidal momentum balance equations given by Eqs. (19)–(21), effects of the time-evolving background magnetic field are included through $\mathbf{u}_s\cdot\nabla s$ which represents the radial motion velocity of the flux surface. The energy density $\mathcal{E}^*$ and the toroidal momentum density $P_{\parallel V\zeta}$ are defined by

$$\begin{aligned}
\mathcal{E}^* \equiv \sum_a\int d^3 v^{(\mathrm{gc})}\, F_a&\left(\frac{m_a}{2}\left|\mathbf{V}_0 + \mathbf{v}_a' - \frac{e_a}{m_a c}\mathbf{A}_1\right|^2 + H_{a1}^V\right.\\
&\left.+\frac{e_a^2}{2B_0}\frac{\partial}{\partial\mu}\left\langle\widetilde{\psi}_a\left(2\widetilde{\phi_1} - \widetilde{\psi}_a\right)\right\rangle_\xi\right) - \mathbf{P}^{(\mathrm{pol})}\cdot\nabla\Phi_0\\
&+\frac{1}{8\pi}\left(|\nabla(\Phi_0 + \phi_1)|^2 + |\mathbf{B}_0 + \mathbf{B}_1|^2\right),
\end{aligned} \tag{22}$$

and $P_{\parallel V\zeta} \equiv \sum_a\int d^3 v^{(\mathrm{gc})}\, F_a m_a(Ub_\zeta + V_\zeta)$, respectively. In Eq. (21), $\Pi_{\parallel V\zeta}^s \equiv \sum_a\langle\int d^3 v^{(\mathrm{gc})}\, F_a m_a(Ub_\zeta + V_\zeta)\mathbf{v}_a^{(\mathrm{gc})}\cdot\nabla s\rangle$ represents the radial flux of the toroidal momentum due to gyrocenter motion and contains both collisional and turbulent effects while the residual turbulent and collisional fluxes of the toroidal momentum are denoted by $\Pi_{R\zeta}^s$ and $(\Pi^{\mathrm{C}*})^s$, respectively, which are both caused by finite gyroradii[18]. In Eq. (20), $\mathbf{Q}$ contains collisional and turbulent energy transport fluxes as well as the Poynting energy flux. The classical energy flux $\mathbf{Q}^{C*}$ included in $\mathbf{Q}$ and the radial flux $(\Pi^{C*})^s$ of the toroidal momentum are written as $\mathbf{Q}^{C*} \equiv \sum_a (\mathbf{Q}_a^C - e_a \Phi_0 \mathbf{\Gamma}_a^C)$ and $(\Pi^{C*})^s \equiv \sum_a \langle [\mathbf{\Pi}_{a\zeta}^C + (e_a/c)\chi \mathbf{\Gamma}_a^C] \cdot \nabla s \rangle$, respectively, where $\mathbf{Q}_a^C \equiv \sum_b \mathbf{Q}_{ab}^C$ and $\mathbf{\Pi}_{a\zeta}^C \equiv \sum_b \mathbf{\Pi}_{ab\zeta}^C$. Equations (19), (20), and (21) have the external source terms on the right-hand sides and take the conservative forms on the left-hand sides where collisional effects are included.

To compare the present results with those from the conventional recursive and WKB techniques[6–8,12], we represent an arbitrary physical variable $\mathcal{Q}$ by the sum of the average and fluctuation parts, $\mathcal{Q} = \langle Q \rangle_{\rm ens} + \hat{\mathcal{Q}}$, where $\langle \cdots \rangle_{\rm ens}$ represents the ensemble average. We here write $\mathbf{A}_0 = \langle \mathbf{A} \rangle_{\rm ens}$, $\mathbf{A}_1 = \hat{\mathbf{A}}$, $\Phi_0 = \langle \Phi \rangle_{\rm ens}$, and $\phi_1 = \langle \phi \rangle_{\rm ens} + \hat{\phi}$. The lowest-order distribution function is given by $f_{a0} \equiv N_a (m_a/2\pi T_a)^{3/2} \exp(-\epsilon/T_a)$, where $N_a$ and $T_a$ are flux-surface functions and $\epsilon \equiv \frac{1}{2} m_a U^2 + \mu B_0 + e_a \widetilde{\langle \phi \rangle}_{\rm ens} - \frac{1}{2} m_a V_0^2$.[6–8] The fluctuation part of $F_a$ is written as $\hat{F}_a = -f_{a0} e_a \langle \hat{\psi}_a \rangle_\xi / T_a + \hat{h}_a$. Then, the fluctuation part of Eq. (10) is found to agree, to $\mathcal{O}(\delta)$, with the gyrokinetic equation for $\hat{h}_a$ obtained from using the WKB representation[12] while the linearized drift kinetic equation for the neoclassical transport theory [6–8] can be derived from the average part of Eq. (10).

The ensemble-averaged particle, energy, and toroidal momentum balance equations derived from Eqs. (19), (20) and (21) are all consistent with the results from the conventional recursive formulations[12]. As an example, the ensemble average of Eq. (21) is written as

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left( V' \left\langle \rho_m \left(1 + \frac{v_{PA}^2}{c^2}\right) V_\zeta \right\rangle \right) \\
&+ \frac{\partial}{\partial s}\left( V' \left[ \sum_a \left\{ \Pi_a - \left\langle \rho_m \left(1 + \frac{v_{PA}^2}{c^2}\right) V_\zeta (\mathbf{u}_s \cdot \nabla s) \right\rangle \right\} \right.\right. \\
&\left.\left. - \frac{1}{4\pi} \left\langle \left\langle \nabla s \cdot \left[ \langle \mathbf{E} \rangle_{\rm ens} \langle \mathbf{E} \rangle_{\rm ens} + \hat{\mathbf{E}}_L \hat{\mathbf{E}}_L + \hat{\mathbf{B}}\hat{\mathbf{B}} + (\nabla \times \hat{\mathbf{B}})\hat{\mathbf{A}} \right] \cdot \mathbf{e}_\zeta \right\rangle \right\rangle \right] \right) \\
&= V' \sum_a \left\langle \int d^3 v\, \mathcal{S}_a m_a (U b_\zeta + V_\zeta) \right\rangle,
\end{aligned}
\tag{23}
$$

where $\rho_m \equiv \sum_a n_{a0} m_a \equiv \sum_a m_a \int d^3 v\, f_{a0}$, $v_{PA} \equiv R^{-1} |\nabla \chi| / (4\pi \rho_m)^{1/2}$, and $\langle\langle \cdots \rangle\rangle$ represents a double average over the flux surface and the ensemble. The transport ordering $\partial/\partial t = \mathcal{O}(\delta^2)$ and $\mathcal{S}_a = \mathcal{O}(\delta^2)$ are used in Eq. (23) where all terms are of $\mathcal{O}(\delta^2)$ and other higher-order terms are neglected. The momentum flux $\Pi_a$ including collisional and turbulent effects is written as

$$
\begin{aligned}
\Pi_a = \frac{c m_a}{\chi'} \Bigg\langle &-n_a V_\zeta E_\zeta^{(A)} - \frac{m_a}{2 e_a} \int d^3 v\, C_a v_\zeta^2 \\
&+ \int d^3 v\, v_\zeta \left\langle \hat{h}_a \frac{\partial \hat{\psi}_a}{\partial \zeta} \right\rangle_{\rm ens} \Bigg\rangle,
\end{aligned}
\tag{24}
$$

where $E_\zeta^{(A)} \equiv -c^{-1} (\partial \mathbf{A}_0/\partial t) \cdot \mathbf{e}_\zeta$, $C_a \equiv \sum_b C_{ab}$, and $v_\zeta \equiv \mathbf{e}_\zeta \cdot (\mathbf{V}_0 + \mathbf{v}')$. The toroidal momentum balance given by Eqs. (23) with (24), which describes the evolutions of the toroidal flow and background radial electric field profiles, agrees with the result from the recursive method in Ref.[12] except that, in Ref.[12], the background field $\mathbf{B}_0$ is assumed to be stationary and $\mathbf{u}_s$ does not appear.

In summary, the Lagrangian variational principle and the collision operator represented in terms of Poisson brackets are combined for presenting the new gyrokinetic formulation to derive governing equations of background and turbulent electromagnetic fields and gyrocenter distribution functions for toroidally rotating plasmas. They satisfy the particle, energy, and toroidal momentum balance equations which, except for the external source terms, are written in the conservative forms suitable for long-time global transport simulation[21–23] to pursue evolutions of the background density, temperature, and flow profiles. These balance equations contain all classical, neoclassical, and turbulent transport fluxes which, in the scale-separation limit, coincide with those derived from conventional recursive formulations. Especially, in the present high-flow case, the background radial electric field can be determined from the toroidal momentum balance equation of the second order, which is in contrast with the low-flow axisymmetric case where higher-order accuracy is required to determine the radial electric field.

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