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Effects of collisions on conservation laws in gyrokinetic field theory

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Effects of collisions on conservation laws for toroidal plasmas are investigated based on the gyrokinetic field theory. Associating the collisional system with a corresponding collisionless system at a given time such that the two systems have the same distribution functions and electromagnetic fields instantaneously, it is shown how the collisionless conservation laws derived from Noether's theorem are modified by the collision term. Effects of the external source term added into the gyrokinetic equation can be formulated similarly with the collisional effects. Particle, energy, and toroidal momentum balance equations including collisional and turbulent transport fluxes are systematically derived using a novel gyrokinetic collision operator, by which the collisional change rates of energy and canonical toroidal angular momentum per unit volume in the gyrocenter space can be given in the conservative forms. The ensemble-averaged transport equations of particles, energy, and toroidal momentum given in the present work are shown to include classical, neoclassical, and turbulent transport fluxes which agree with those derived from conventional recursive formulations.

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I. INTRODUCTION

Gyrokinetic theories and simulations are powerful means to investigate microinstabilities and turbulent transport processes in magnetically confined plasmas.[1–4] Originally, gyrokinetic equations are derived by recursive techniques combined with the WKB or ballooning representation.[5–10] On the other hand, modern derivations of the gyrokinetic equations are based on the Lagrangian and/or Hamiltonian formulations,[11] in which conservation laws for the phase-space volume and the magnetic moment are automatically ensured by Liouville's theorem and Noether's theorem, respectively.[12] Besides, conservation of the total energy and momentum is naturally obtained in the gyrokinetic field theory, where all governing equations for the distribution functions and the electromagnetic fields are derived from the Lagrangian which describes the whole system consisting of particles and fields.[13–17] A subtle point regarding the Lagrangian/Hamiltonian gyrokinetic formulations is that they basically treat collisionless systems so that Noether's theorem and conservation laws do not hold directly for collisional systems. In this paper, we examine how the collision and external source terms added into the gyrokinetic equations influence the conservation laws derived from Noether's theorem in the gyrokinetic field theory for collisionless systems.

For a given collisional kinetic system, we can imagine a corresponding collisionless kinetic system such that the two systems have the same distribution functions and electromagnetic fields instantaneously. As an example of two such systems, the Boltzmann-Poisson-Ampère system and the Vlasov-Poisson-Ampère system are considered in Sec. II, where we express the action integral for the latter collisionless system in terms of the distribution functions and the electromagnetic fields for the former collisional system to show how the conservation laws derived from Noether's theorem in the collisionless system are modified in the collisional system with external sources of particles, energy, and momentum. There, we confirm the natural result that, when adding no external sources but only the collision term that conserves the energy and momentum, the energy and momentum conservation laws for the Boltzmann-Poisson-Ampère system take the same forms as those for the Vlasov-Poisson-Ampère system. The above-mentioned procedures are repeated in Sec. III to treat the collisional and collisionless gyrokinetic systems. In our previous work,[18] using the gyrokinetic Vlasov-Poisson-Ampère system of equations, conservation laws of particles, energy, and toroidal angular momentum are obtained for collisionless toroidal plasmas, in which the slow temporal variation of the background magnetic field is taken into account in order to enable self-consistent treatment of physical processes on transport time scales. Based on these results, the particle, energy, and toroidal angular momentum balance equations for the collisional plasma are derived from the gyrokinetic Boltzmann-Poisson-Ampère system of equations in Secs. IV and V. In Sec. VI, it is shown by taking the ensemble average of these balance equations that the particle, energy, and toroidal angular momentum transport fluxes are given by the sum of the conventional expressions of the classical, neoclassical, and turbulent transport fluxes to the lowest order in the normalized gyroradius parameter. Conclusions are given in Sec. VII and formulas for transformation from particle to gyrocenter coordinates are presented in Appendix A.

Regarding the collision operator for the gyrokinetic equation, several works have been done, which take account of finite-gyroradius effects to modify the Landau collision operator defined in the particle coordinates.[19–27] The relation of the collision operator in the gyrocenter coordinates to that in the particle coordinates is explained in Appendix B. The Landau operator for Coulomb collisions conserves particles' number, kinetic energy, and momentum locally at the particle position although, in the gyrocenter position space, collisions induce transport fluxes of particles, energy, and momentum. Besides, it is emphasized in this work that the collisional change rates of the gyrocenter Hamiltonian (which includes not only the kinetic energy but also the potential energy) and of the canonical momentum (instead of the kinetic momentum) per unit volume in the gyrocenter space should take the conservative (or divergence) forms in order to properly derive the energy and momentum conservation laws for the collisional gyrokinetic system. The approximate collision operator which keeps these conservation properties of the gyrocenter energy and canonical toroidal angular momentum is shown in Appendix C. [It is noted that another form of the gyrokinetic collision operator, which satisfies the energy and momentum conservation laws, has recently been presented by Burby *et al.*[26]] Appendix D is given to describe how to derive the formula for the toroidal angular momentum transport flux due to the collision term.

II. BOLTZMANN-POISSON-AMPÈRE SYSTEM

In this section, conservation laws are investigated for the Boltzmann-Poisson-Ampère system of equations which provide the basis of approximate description by the collisional electromagnetic gyrokinetic system of equations for strongly magnetized plasmas considered in the subsequent sections. Time evolution of the distribution function $f_a(\mathbf{x}, \mathbf{v}, t)$ for particle species a is described by the Boltzmann kinetic equation,

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_a}{m_a}\left\{\mathbf{E}(\mathbf{x},t) + \frac{1}{c}\mathbf{v}\times\mathbf{B}(\mathbf{x},t)\right\}\cdot\frac{\partial}{\partial \mathbf{v}}\right] f_a(\mathbf{x},\mathbf{v},t) = \mathcal{K}_a(\mathbf{x},\mathbf{v},t), \tag{1}$$

where $\mathcal{K}_a(\mathbf{x},\mathbf{v},t)$ denotes the rate of change in the distribution function f_a due to Coulomb collisions and it may also include other parts representing external particle, momentum, and/or energy sources if any. The electromagnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ are written as $\mathbf{E} = -\nabla\phi - c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$, where the electrostatic potential ϕ and the vector potential $\mathbf{A}$ are determined by Poisson's equation,

$$\nabla^2\phi(\mathbf{x},t) = -4\pi\sum_a e_a\int f_a(\mathbf{x},\mathbf{v},t)d^3\mathbf{v} \equiv -4\pi\sigma, \tag{2}$$

and Ampère's law,

$$\nabla^2\mathbf{A}(\mathbf{x},t) = -\frac{4\pi}{c}\mathbf{j}_T, \tag{3}$$

respectively. Here, the Coulomb (or transverse) gauge condition $\nabla\cdot\mathbf{A} = 0$ is used and the current density $\mathbf{j} \equiv \sum_a e_a n_a \mathbf{u}_a \equiv \sum_a e_a \int f_a(\mathbf{x},\mathbf{v},t)\mathbf{v}d^3\mathbf{v}$ (or any vector field) is written as $\mathbf{j} = \mathbf{j}_L + \mathbf{j}_T$, where $\mathbf{j}_L \equiv -(4\pi)^{-1}\nabla\int d^3\mathbf{x}'(\nabla'\cdot\mathbf{j})/|\mathbf{x}-\mathbf{x}'|$ and $\mathbf{j}_T \equiv (4\pi)^{-1}\nabla\times(\nabla\times\int d^3\mathbf{x}'\,\mathbf{j}/|\mathbf{x}-\mathbf{x}'|)$ represent the longitudinal (or irrotational) and transverse (or solenoidal) parts, respectively.[28] Equations (1), (2), and (3) are the governing equations for the Boltzmann-Poisson-Ampère system.

Suppose that f_a, ϕ, and $\mathbf{A}$ which satisfy Eqs. (1)–(3) are given. Then, for the electromagnetic fields $\mathbf{E} = -\nabla\phi - c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$ given from ϕ and $\mathbf{A}$, we consider the distribution function f_a^V which is the solution of the Vlasov equation,

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_a}{m_a}\left\{\mathbf{E}(\mathbf{x},t) + \frac{1}{c}\mathbf{v}\times\mathbf{B}(\mathbf{x},t)\right\}\cdot\frac{\partial}{\partial \mathbf{v}}\right] f_a^V(\mathbf{x},\mathbf{v},t) = 0. \tag{4}$$

We also assume f_a^V to coincide instantaneously with f_a at a given time t_0 so that $f_a^V(\mathbf{x},\mathbf{v},t_0) = f_a(\mathbf{x},\mathbf{v},t_0)$. Therefore, equations obtained from Eqs. (2) and (3) with f_a replaced by f_a^V also hold at t_0. In other words, f_a^V, ϕ, and $\mathbf{A}$ satisfy the Vlasov-Poisson-Ampère system of equations at t_0. Note that the Vlasov-Poisson-Ampère system of equations can be derived from the variational principle using the action $\mathcal{I}$ defined by Eq. (1) in Ref. 29 where its variation $\delta\mathcal{I}$ associated with infinitesimal transformations of independent and dependent variables [see Eqs. (10) and (15) in Ref. 29] are explicitly shown in order to apply Noether's theorem for obtaining conservation laws of energy and momentum. Now, let us use f_a^V, ϕ, and $\mathbf{A}$ to define the action integral $\mathcal{I}$ over a small time interval, $t_0 - h/2 \le t \le t_0 + h/2$, during which the Vlasov-Poisson-Ampère system of equations are approximately satisfied by f_a^V, ϕ, and $\mathbf{A}$ within the errors of order h. Then, neglecting the errors of higher order in h, the variation $\delta\mathcal{I}$ can be written in the same form as in Eq. (15) of Ref. 29,

$$\delta\mathcal{I} = -\int_{t_0-h/2}^{t_0+h/2} dt \int d^3\mathbf{x}\,\left[\frac{\partial}{\partial t}\delta G_0^V(\mathbf{x},t) + \nabla\cdot\delta\mathbf{G}^V(\mathbf{x},t)\right], \tag{5}$$

where δG_0^V and $\delta\mathbf{G}^V$ are written as

$$\begin{aligned}
\delta G_0^V(\mathbf{x},t) &= \mathcal{E}_c^V\,\delta t_E - \mathbf{P}_c^V\cdot\delta\mathbf{x}_E,\\
\delta\mathbf{G}^V(\mathbf{x},t) &= \mathbf{Q}_c^V\,\delta t_E - \mathbf{\Pi}_c^V\cdot\delta\mathbf{x}_E + \mathbf{S}_\phi\,\delta\phi - \mathbf{\Sigma}_A\cdot\delta\mathbf{A}.
\end{aligned} \tag{6}$$

Here, t_E, $\delta\mathbf{x}_E$, $\delta\phi$, and $\delta\mathbf{A}$ represent variations of t, $\mathbf{x}$, ϕ, and $\mathbf{A}$, respectively, while $\mathcal{E}_c^V$, $\mathbf{P}_c^V$, $\mathbf{Q}_c^V$, $\mathbf{\Pi}_c^V$, $\mathbf{S}_\phi$, and $\mathbf{\Sigma}_A$ are defined by

$$\begin{aligned}
\mathcal{E}_c^V &= \sum_a\int d^3\mathbf{v}\; f_a^V(\mathbf{x},\mathbf{v},t)\left(\frac{1}{2}m_a|\mathbf{v}|^2 + e_a\phi\right)\\
&\quad + \frac{1}{8\pi}\left(-|\nabla\phi|^2 + |\mathbf{B}|^2\right),\\
\mathbf{P}_c^V &= \sum_a\int d^3\mathbf{v}\; f_a^V(\mathbf{x},\mathbf{v},t)\left(m_a\mathbf{v} + \frac{e_a}{c}\mathbf{A}\right),\\
\mathbf{Q}_c^V &= \sum_a\int d^3\mathbf{v}\; f_a^V(\mathbf{x},\mathbf{v},t)\left(\frac{1}{2}m_a|\mathbf{v}|^2 + e_a\phi\right)\mathbf{v}\\
&\quad + \frac{1}{4\pi}\left(\frac{\partial\phi}{\partial t}\nabla\phi + \frac{\lambda}{c}\frac{\partial\mathbf{A}}{\partial t} - \frac{\partial\mathbf{A}}{\partial t}\times\mathbf{B}\right),
\end{aligned}$$

$$\mathbf{\Pi}_c^V = \sum_a \int d^3\mathbf{v}\, f_a^V(\mathbf{x},\mathbf{v},t)\mathbf{v}\left(m_a\mathbf{v} + \frac{e_a}{c}\mathbf{A}\right) + \frac{1}{8\pi}\left(|\nabla\phi|^2 - B^2\right)\mathbf{I} + \frac{1}{4\pi}\Big[-(\nabla\phi)(\nabla\phi) + ((\nabla\mathbf{A}) - (\nabla\mathbf{A})^T)\cdot(\nabla\mathbf{A})^T - \frac{\lambda}{c}(\nabla\mathbf{A})^T\Big],$$

$$\mathbf{S}_\phi = -\frac{1}{4\pi}\nabla\phi, \text{ and } \mathbf{\Sigma}_A = \frac{1}{4\pi}\left(\mathbf{B}\times\mathbf{I} + \frac{\lambda}{c}\mathbf{I}\right), \quad (7)$$

respectively, where the superscript T represents the transpose of the tensor and $\mathbf{I}$ denotes the unit tensor. The field variable λ which appears in Eq. (7) is introduced in Ref. 29 as the Lagrange undetermined multiplier to derive the Coulomb gauge condition $\nabla\cdot\mathbf{A} = 0$ and it is shown from Eq. (8) in Ref. 29 that

$$\nabla^2\lambda = 4\pi\nabla\cdot\mathbf{j}. \quad (8)$$

Suppose that the variations t_E, $\delta\mathbf{x}_E$, $\delta\phi$, and $\delta\mathbf{A}$ are such that $\delta\mathcal{I} = 0$ holds for an arbitrary $\mathbf{x}$-integral domain in Eq. (5). Then, taking the small time interval limit $h \to +0$ in Eq. (6), we find that the conservation law,

$$\left[\frac{\partial}{\partial t}\delta G_0^V(\mathbf{x},t)\right]_{t=t_0} + \nabla\cdot\delta\mathbf{G}^V(\mathbf{x},t_0) = 0, \quad (9)$$

should be satisfied. This is the so-called "Noether's theorem. "Recalling that $f_a^V(\mathbf{x},\mathbf{v},t_0) = f_a(\mathbf{x},\mathbf{v},t_0)$ and comparing Eqs. (1) and (4) at $t = t_0$, we have

$$\left[\frac{\partial f_a^V(\mathbf{x},\mathbf{v},t)}{\partial t}\right]_{t=t_0} = \left[\frac{\partial f_a(\mathbf{x},\mathbf{v},t)}{\partial t}\right]_{t=t_0} - \mathcal{K}_a(\mathbf{x},\mathbf{v},t_0). \quad (10)$$

We now define $\mathcal{E}_c$, $\mathbf{P}_c$, $\mathbf{Q}_c$, and $\mathbf{\Pi}_c$ from $\mathcal{E}_c^V$, $\mathbf{P}_c^V$, $\mathbf{Q}_c^V$, and $\mathbf{\Pi}_c^V$, respectively, by replacing f_a^V with f_a in Eq. (7). Correspondingly, δG_0 and $\delta\mathbf{G}$ are defined from δG_0^V and $\delta\mathbf{G}^V$ by replacing $\mathcal{E}_c^V$, $\mathbf{P}_c^V$, $\mathbf{Q}_c^V$, and $\mathbf{\Pi}_c^V$ with $\mathcal{E}_c$, $\mathbf{P}_c$, $\mathbf{Q}_c$, and $\mathbf{\Pi}_c$, respectively, in Eq. (7). These definitions immediately yield $\delta\mathbf{G}^V(\mathbf{x},t_0) = \delta\mathbf{G}(\mathbf{x},t_0)$ and

$$\left[\frac{\partial\delta G_0^V(\mathbf{x},t)}{\partial t}\right]_{t=t_0} = \left[\frac{\partial\delta G_0(\mathbf{x},t))}{\partial t}\right]_{t=t_0} - \delta K_{G0}(\mathbf{x},t_0), \quad (11)$$

where Eq. (10) is used and δK_{G0} is defined by

$$\delta K_{G0} = K_{\mathcal{E}c}\,\delta t_E - \mathbf{K}_{Pc}\cdot\delta\mathbf{x}_E,$$
$$K_{\mathcal{E}c} = \sum_a \int d^3\mathbf{v}\,\mathcal{K}_a\left(\frac{1}{2}m_a|\mathbf{v}|^2 + e_a\phi\right),$$
$$\mathbf{K}_{Pc} = \sum_a \int d^3\mathbf{v}\,\mathcal{K}_a\left(m_a\mathbf{v} + \frac{e_a}{c}\mathbf{A}\right). \quad (12)$$

Substituting Eq. (11) into Eq. (9), we find that the conservation law is modified for the Boltzmann-Poisson-Ampère system as

$$\frac{\partial}{\partial t}\delta G_0(\mathbf{x},t) + \nabla\cdot\delta\mathbf{G}(\mathbf{x},t) = \delta K_{G0}, \quad (13)$$

where t_0 is rewritten as t because t_0 is an arbitrarily chosen time. Equation (13) shows that δK_{G0} represents effects of $\mathcal{K}_a$ in Eq. (1) on the conservation law. If $\mathcal{K}_a$ is given by the Coulomb collision term only, δK_{G0} defined by Eq. (12) vanishes because the collision term conserves particles' number, momentum, and energy.

Energy and momentum balance equations can be derived from Eq. (13) using symmetries of the system under infinitesimal time and space translations as shown later. Before deriving them, we first consider the equation for the particle number density $n_a \equiv \int f_a d^3\mathbf{v}$ which is obtained by taking the velocity-space integral of Eq. (1) as

$$\frac{\partial n_a}{\partial t} + \nabla\cdot(n_a\mathbf{u}_a) = \int \mathcal{K}_a d^3\mathbf{v}. \quad (14)$$

We hereafter assume that $\sum_a e_a \int \mathcal{K}_a d^3\mathbf{v} = 0$, which means that the source terms $\mathcal{K}_a$ conserve electric charge even if $\int \mathcal{K}_a d^3\mathbf{v} \neq 0$ for each species a. This seems a reasonable assumption in consistency with Eqs. (2) and (3) in which no external source terms are included. Then, multiplying Eq. (14) with the electric charge e_a and performing the summation over species result in the charge conservation law,

$$\frac{\partial\sigma}{\partial t} + \nabla\cdot\mathbf{j} = 0. \quad (15)$$

We also find from $\sum_a e_a \int \mathcal{K}_a d^3\mathbf{v} = 0$ that, in Eq. (12), the terms including ϕ and $\mathbf{A}$ vanish and make no contribution to $K_{\mathcal{E}c}$ and $\mathbf{K}_{Pc}$. As seen from Eqs. (2), (8), and (15), we can put $\lambda = \partial\phi/\partial t$ which is also used in Ref. 29 to derive energy and momentum conservation laws for the Vlasov-Poisson-Ampère system.

We now note that the action integral is invariant, namely, $\delta\mathcal{I} = 0$ under the infinitesimal translations in space and time represented by $\delta t_E = \epsilon_0$, $\delta\mathbf{x}_a = \delta\mathbf{x}_E = \boldsymbol{\epsilon}$, $\delta\mathbf{v}_a = 0$, $\delta\phi = 0$, and $\delta\mathbf{A} = 0$, where ϵ_0 and $\boldsymbol{\epsilon}$ are constant in time and space. These invariance properties hold because the integrands in the action integral $\mathcal{I}$ depend on $(\mathbf{x},t)$ only through variational variables [see Eq. (1) in Ref. 29]. Using the time translational symmetry, Eq. (13) reduces to the canonical energy balance equation,

$$\frac{\partial\mathcal{E}_c}{\partial t} + \nabla\cdot\mathbf{Q}_c = K_{\mathcal{E}c}, \quad (16)$$

where the canonical energy density and flux $(\mathcal{E}_c, \mathbf{Q}_c)$ are given by replacing f_a^V with f_a in the definitions of $(\mathcal{E}_c^V, \mathbf{Q}_c^V)$ in Eq. (7). In the same way as in Ref. 29, we use the kinetic energy density and flux, $(\mathcal{E}_p, \mathbf{Q}_p)$, defined by

$$\mathcal{E}_p = \sum_a \int d^3\mathbf{v}\, f_a(\mathbf{x},\mathbf{v},t)\frac{1}{2}m_a|\mathbf{v}|^2,$$
$$\mathbf{Q}_p = \sum_a \int d^3\mathbf{v}\, f_a(\mathbf{x},\mathbf{v},t)\frac{1}{2}m_a|\mathbf{v}|^2\mathbf{v}, \quad (17)$$

to modify Eq. (16) into more familiar forms. Then, the energy balance equation is finally written as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\mathcal{E}_p + \frac{|\mathbf{E}_L|^2 + |\mathbf{B}|^2}{8\pi}\right) \\
&\quad + \nabla\cdot\left(\mathbf{Q}_p + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} - \frac{1}{4\pi}\frac{\partial\phi}{\partial t}\mathbf{E}_T\right) \\
&= \frac{\partial}{\partial t}\left(\mathcal{E}_p + \frac{|\mathbf{E}_L|^2 + 2\mathbf{E}_L\cdot\mathbf{E}_T + |\mathbf{B}|^2}{8\pi}\right) \\
&\quad + \nabla\cdot\left(\mathbf{Q}_p + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} + \frac{1}{4\pi}\phi\frac{\partial\mathbf{E}_T}{\partial t}\right) \\
&= K_{\mathcal{E}c} \equiv \sum_a \int d^3\mathbf{v}\, \mathcal{K}_a \frac{1}{2}m_a|\mathbf{v}|^2, \qquad (18)
\end{aligned}$$

where $\mathbf{E}_L \equiv -\nabla\phi$ and $\mathbf{E}_T \equiv -c^{-1}\partial\mathbf{A}/\partial t$ are the longitudinal and transverse parts of the electric field, respectively.

Next, from the space translational symmetry and Eq. (13), we obtain the canonical momentum balance equation,

$$\frac{\partial\mathbf{P}_c}{\partial t} + \nabla\cdot\mathbf{\Pi}_c = \mathbf{K}_{Pc}, \qquad (19)$$

where the canonical momentum density and tensor $(\mathbf{P}_c, \mathbf{\Pi}_c)$ are given by replacing f_a^V with f_a in the definitions of $(\mathbf{P}_c^V, \mathbf{\Pi}_c^V)$ in Eq. (7). Furthermore, in the same way as in Ref. 29, the invariance of I under the infinitesimal rotation is shown to give the equation for the angular momentum, which is used to modify Eq. (19) into the momentum balance equation,

$$\frac{\partial}{\partial t}(\mathbf{P}_p + \mathbf{P}_f) + \nabla\cdot(\mathbf{\Pi}_p + \mathbf{\Pi}_f) = \mathbf{K}_{Pc} \equiv \sum_a \int d^3\mathbf{v}\, \mathcal{K}_a m_a \mathbf{v}. \qquad (20)$$

Here, the particle parts $(\mathbf{P}_p, \mathbf{\Pi}_p)$ of the momentum density and the pressure tensor are defined by

$$\begin{aligned}
\mathbf{P}_p &= \sum_a \int d^3\mathbf{v}\, f_a(\mathbf{x},\mathbf{v},t) m_a\mathbf{v}, \\
\mathbf{\Pi}_p &= \sum_a \int d^3\mathbf{v}\, f_a(\mathbf{x},\mathbf{v},t) m_a\mathbf{v}\mathbf{v}, \qquad (21)
\end{aligned}$$

and the field parts $(\mathbf{P}_f, \mathbf{\Pi}_f)$ are given by

$$\begin{aligned}
\mathbf{P}_f &= \frac{\mathbf{E}_L\times\mathbf{B}}{4\pi c}, \\
\mathbf{\Pi}_f &= \frac{1}{8\pi}(|\mathbf{E}_L|^2 + 2\mathbf{E}_L\cdot\mathbf{E}_T + |\mathbf{B}|^2)\mathbf{I} \\
&\quad - \frac{1}{4\pi}(\mathbf{E}_L\mathbf{E}_L + \mathbf{E}_L\mathbf{E}_T + \mathbf{E}_T\mathbf{E}_L + \mathbf{B}\mathbf{B}). \qquad (22)
\end{aligned}$$

Equations (18) and (20) take physically familiar forms of energy and momentum balance equations including external source terms. As mentioned earlier, if $\mathcal{K}_a$ is given by the Coulomb collision term only, $K_{\mathcal{E}c}$ and $\mathbf{K}_{Pc}$ vanish so that the energy and momentum balance equations for the Boltzmann-Poisson-Ampère system take the same forms as those for the Vlasov-Poisson-Ampère system.[29]

III. GYROKINETIC BOLTZMANN-POISSON-AMPÈRE SYSTEM

Let us start from the gyrokinetic Boltzmann equation written as

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \frac{d\mathbf{Z}_a}{dt}\cdot\frac{\partial}{\partial\mathbf{Z}}\right) F_a(\mathbf{Z},t) \\
&\quad = \sum_b C_{ab}^g[F_a, F_b](\mathbf{Z},t) + \mathcal{S}_a(\mathbf{Z},t), \qquad (23)
\end{aligned}$$

where $F_a(\mathbf{Z},t)$ is the gyrocenter distribution function for species a, $C_{ab}^g[F_a,F_b](\mathbf{Z},t)$ represents the rate of change in $F_a(\mathbf{Z},t)$ due to Coulomb collisions between particle species a and b, and $\mathcal{S}_a(\mathbf{Z},t)$ denotes other parts including external particle, momentum, and/or energy sources if any. The gyrocenter coordinates are written as $\mathbf{Z}_a = (\mathbf{X}_a, U_a, \mu_a, \xi_a)$, where $\mathbf{X}_a$, U_a, μ_a, and ξ_a represent the gyrocenter position, parallel velocity, magnetic moment, and gyrophase angle, respectively. Appendix A shows the relation of the gyrocenter coordinates to the particle coordinates in detail. The perturbation expansion parameter in the gyrokinetic theory is denoted by δ which represents the ratio of the gyroradius ρ to the macroscopic scale length L of the background field. It is shown in Appendix B how the collision operator $C_{ab}^g[F_a,F_b]$ for the gyrocenter distribution functions F_a and F_b is given from the collision operator $C_{ab}^p[f_a,f_b]$ for the particle distribution functions f_a and f_b.

The deviation of each distribution function from the local Maxwellian is regarded as of $\mathcal{O}(\delta)$, and accordingly the collision term C_{ab}^g is considered to be of $\mathcal{O}(\delta)$. We assume that the source term $\mathcal{S}_a$ is of $\mathcal{O}(\delta^2)$ so that its effect appears only in the transport time scale. We also assume that $\sum_a e_a \int dU \int d\mu \int d\xi \;\; D_a\mathcal{S}_a(\mathbf{Z},t) = 0$ in order to prevent the source term from affecting the charge conservation laws [see Eq. (60)]. Here, D_a denotes the Jacobian for the gyrocenter coordinates, $D_a \equiv \det[\partial(\mathbf{x}_a,\mathbf{v}_a)/\partial(\mathbf{X}_a,U_a,\xi_a,\mu_a)]$, where $(\mathbf{x}_a,\mathbf{v}_a)$ represent the particle coordinates consisting of the particle's position and velocity vectors.

We treat toroidal systems, for which the equilibrium magnetic field is given in the axisymmetric form as

$$\mathbf{B}_0 = \nabla\times\mathbf{A}_0 = I\nabla\zeta + \nabla\zeta\times\nabla\chi, \qquad (24)$$

where I and χ are constant on toroidal flux surfaces labeled by an arbitrary radial coordinate s and ζ is the toroidal angle. We note that I and χ represent the covariant toroidal component of the equilibrium field $\mathbf{B}_0$ and the poloidal magnetic flux divided by 2π, respectively. The equilibrium field $\mathbf{B}_0$ is allowed to be dependent on time. Then, following Ref. 18, the gyrocenter motion equations are written as

$$\frac{d\mathbf{Z}_a}{dt} = \{\mathbf{Z}_a, H_a\} + \{\mathbf{Z}_a, \mathbf{X}_a\}\cdot\frac{e_a}{c}\frac{\partial\mathbf{A}_a^*}{\partial t}, \qquad (25)$$

where the gyrocenter Hamiltonian, which is independent of ξ_a, is defined by

$$H_a = \frac{1}{2} m_a U_a^2 + \mu_a B_0 + e_a \Psi_a, \tag{26}$$

and $\mathbf{A}_a^*$ is given by $\mathbf{A}_a^* = \mathbf{A}_0(\mathbf{X}_a, t) + (m_a c/e_a) U_a \mathbf{b}(\mathbf{X}_a, t)$. Using the nonvanishing components of the Poisson brackets for pairs of the gyrocenter coordinates given by

$$\{\mathbf{X}_a, \mathbf{X}_a\} = \frac{c}{e_a B_{a\|}^*} \mathbf{b} \times \mathbf{I}, \quad \{\mathbf{X}_a, U_a\} = \frac{\mathbf{B}_a^*}{m_a B_{a\|}^*},$$
$$\{\xi_a, \mu_a\} = \frac{e_a}{m_a c}, \tag{27}$$

the gyrocenter motion equations in Eq. (25) are rewritten as

$$\frac{d\mathbf{X}_a}{dt} = \frac{1}{B_{a\|}^*} \left[\left(U_a + \frac{e_a}{m_a} \frac{\partial \Psi_a}{\partial U_a} \right) \mathbf{B}_a^* + c\mathbf{b} \times \left(\frac{\mu_a}{e_a} \nabla B_0 + \nabla \Psi_a + \frac{1}{c} \frac{\partial \mathbf{A}_a^*}{\partial t} \right) \right], \tag{28}$$

$$\frac{dU_a}{dt} = -\frac{\mathbf{B}_a^*}{m_a B_{a\|}^*} \cdot \left[\mu_a \nabla B_0 + e_a \left(\nabla \Psi_a + \frac{1}{c} \frac{\partial \mathbf{A}_a^*}{\partial t} \right) \right], \tag{29}$$

$$\frac{d\mu_a}{dt} = 0, \tag{30}$$

and

$$\frac{d\xi_a}{dt} = \Omega_a + \frac{e_a^2}{m_a c} \frac{\partial \Psi_a}{\partial \mu_a}. \tag{31}$$

Here, $\Omega_a \equiv e_a B_0/(m_a c)$, $\mathbf{b} = \mathbf{B}_0/B_0$, $B_{a\|}^* = \mathbf{B}_a^* \cdot \mathbf{b}$, and $\mathbf{B}_a^* = \nabla \times \mathbf{A}_a^*$. The field variable Ψ_a is defined by

$$\Psi_a = \langle \psi_a(\mathbf{Z}_a, t) \rangle_{\xi_a} + \frac{e_a}{2 m_a c^2} \langle |\mathbf{A}_1(\mathbf{X}_a + \boldsymbol{\rho}_a, t)|^2 \rangle_{\xi_a} - \frac{e_a}{2B_0} \frac{\partial}{\partial \mu} \langle [\widetilde{\psi}_a(\mathbf{Z}_a, t)]^2 \rangle_{\xi_a}, \tag{32}$$

where the field variable ψ_a is defined in terms of the electrostatic potential ϕ and the perturbation part of the vector potential $\mathbf{A}_1$ as

$$\psi_a(\mathbf{Z}_a, t) = \phi(\mathbf{X}_a + \boldsymbol{\rho}_a, t) - \frac{1}{c} \mathbf{v}_{a0}(\mathbf{Z}_a, t) \cdot \mathbf{A}_1(\mathbf{X}_a + \boldsymbol{\rho}_a, t). \tag{33}$$

The gyroradius vector is given by $\boldsymbol{\rho}_a = \mathbf{b}(\mathbf{X}_a, t) \times \mathbf{v}_{a0}(\mathbf{Z}_a, t)/\Omega_a(\mathbf{X}_a, t)$ and the zeroth-order particle velocity $\mathbf{v}_{a0}$ is written in terms of the gyrocenter coordinates as $\mathbf{v}_{a0}(\mathbf{Z}_a, t) = U_a \mathbf{b}(\mathbf{X}_a, t) - [2\mu_a B_0(\mathbf{X}_a)/m_a]^{1/2} [\sin \xi_a \mathbf{e}_1(\mathbf{X}_a, t) + \cos \xi_a \mathbf{e}_2(\mathbf{X}_a, t)]$, where the unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$ form a right-handed orthogonal system. The gyrophase-average and gyrophase-dependent parts of an arbitrary periodic function $Q(\xi_a)$ of the gyrophase ξ_a are written as

$$\langle Q \rangle_{\xi_a} \equiv \oint \frac{d\xi_a}{2\pi} Q(\xi_a) \quad \text{and} \quad \widetilde{Q} \equiv Q - \langle Q \rangle_{\xi_a}, \tag{34}$$

respectively. In the gyrocenter motion equations, effects of the time-dependent background magnetic field and those of the fluctuating electromagnetic fields appear through $\partial \mathbf{A}_a^*/\partial t$ and Ψ_a, respectively. It should be noted that $d\mathbf{Z}_a/dt$ on the left-hand side of Eq. (23) is regarded as a function of $(\mathbf{Z}, t)$ which is given by the right-hand side of Eq. (25).

We find from Eqs. (A3)–(A4) in Appendix A and Eq. (B1) in Appendix B that the gyrophase-dependent part of the right-hand side of Eq. (23) appears from C_{ab}^g and it is of $\mathcal{O}(\delta)$. Using $\Omega_a = \mathcal{O}(\delta^{-1})$, the gyrophase-dependent part of the left-hand side of Eq. (23) is written as $\Omega_a \partial \widetilde{F}_a/\partial \xi$ to the lowest order in δ. Then, it is concluded that $\widetilde{F}_a = \mathcal{O}(\delta^2)$. Taking the gyrophase average of Eq. (23), we obtain

$$\left(\frac{\partial}{\partial t} + \frac{d\mathbf{Z}_a}{dt} \cdot \frac{\partial}{\partial \mathbf{Z}} \right) F_a(\mathbf{Z}, t) = \sum_a \langle C_{ab}^g [F_a, F_b](\mathbf{Z}, t) \rangle_\xi + \mathcal{S}_a(\mathbf{Z}, t), \tag{35}$$

where $F_a(\mathbf{Z}, t)$ and $\mathcal{S}_a(\mathbf{Z}, t)$ are both regarded as independent of the gyrophase ξ and $\langle \cdots \rangle_\xi$ are omitted from them for simplicity. It is seen from Eq. (B1) that effects of $\widetilde{F}_a = \mathcal{O}(\delta^2)$ on $\langle C_{ab}^g [F_a, F_b] \rangle_\xi$ in the right-hand side of Eq. (35) are estimated as of $\mathcal{O}(\delta^3)$. Here and hereafter, we neglect $\widetilde{F}_a = \mathcal{O}(\delta^2)$ in both sides of the gyrokinetic Boltzmann equation given by Eq. (35). Even so, its moment equations can correctly include the collisional transport fluxes of particles, energy, and toroidal momentum up to the leading order, that is $\mathcal{O}(\delta^2)$, as confirmed later. In Appendix C, Eq. (C1) combined with Eqs. (C2), (C10), (C15), and (C16) presents the approximate gyrokinetic collision operator, which has favorable conservation properties and correctly describes collisional transport of energy and toroidal angular momentum.

The gyrokinetic Poisson equation and the gyrokinetic Ampère's law are written as[18]

$$\nabla^2 \phi(\mathbf{x}, t) = -4\pi \sum_a e_a \int d^6\mathbf{Z}\, D_a(\mathbf{Z}, t) \delta^3(\mathbf{X} + \boldsymbol{\rho}_a - \mathbf{x}) \times \left[F_a(\mathbf{Z}, t) + \frac{e_a \widetilde{\psi}_a}{B_0} \frac{\partial F_a}{\partial \mu} \right], \tag{36}$$

and

$$\nabla^2 (\mathbf{A}_0 + \mathbf{A}_1) = -\frac{4\pi}{c} (\mathbf{j}_G)_T, \tag{37}$$

respectively, where $(\mathbf{j}_G)_T$ is the transverse part of the gyrokinetic current density $\mathbf{j}_G$ defined by

$$\mathbf{j}_G \equiv \sum_a e_a \int d^6\mathbf{Z} D_a(\mathbf{Z}) \delta^3[\mathbf{X} + \boldsymbol{\rho}_a(\mathbf{Z}) - \mathbf{x}] \times \left(F_a(\mathbf{Z}, t) \left[\mathbf{v}_{a0}(\mathbf{Z}) - \frac{e_a}{m_a c} \mathbf{A}_1(\mathbf{X} + \boldsymbol{\rho}_a(\mathbf{Z}), t) \right] + \frac{e_a \widetilde{\psi}_a}{B_0} \frac{\partial F_a}{\partial \mu} \mathbf{v}_{a0}(\mathbf{Z}) \right). \tag{38}$$

It should also be noted that the Coulomb gauge conditions $\nabla \cdot \mathbf{A}_0 = 0$ and $\nabla \cdot \mathbf{A}_1 = 0$ for the equilibrium and perturbation parts of the vector potential are used here. The equilibrium vector potential $\mathbf{A}_0$ is given by $\mathbf{A}_0 = -\chi\nabla\zeta + \nabla\zeta\times\nabla\eta$, where $\eta = \eta(R, Z)$ is the solution of $\Delta_*\eta \equiv R^2\nabla\cdot(R^{-2}\nabla\eta) = I$. In Ref. 18, additional governing equations are derived in order to self-consistently determine I and χ for the time-dependent axisymmetric background field $\mathbf{B}_0 = \nabla\times\mathbf{A}_0 = I\nabla\zeta + \nabla\zeta\times\nabla\chi$. They are given by

$$I = \oint \frac{d\theta}{2\pi}\left[\frac{4\pi}{c}\overline{M_\zeta} + \overline{(B^{(\mathrm{gc})})_\zeta} - \overline{B_{1\zeta}}\right], \tag{39}$$

and

$$\begin{aligned}\Delta_*\chi = &\overline{\left(\frac{4\pi}{c}\left[(\mathbf{j}^{(\mathrm{gc})})_T + \nabla\times\mathbf{M}\right] - \nabla\times\mathbf{B}_1\right)\cdot R^2\nabla\zeta} \\ &+ \frac{\partial I}{\partial\chi}\overline{\Lambda_\zeta},\end{aligned} \tag{40}$$

where the toroidal-angle average is represented by $\overline{\cdots} \equiv (2\pi)^{-1}\oint\cdots d\zeta$, the poloidal angle is denoted by θ, and $\overline{\Lambda_\zeta} = I - (4\pi/c)\overline{M_\zeta} - \overline{(B^{(\mathrm{gc})})_\zeta} + \overline{B_{1\zeta}}$. Here, the covariant toroidal component of an arbitrary vector $\mathbf{V}$ is written as V_ζ. The turbulent part of the magnetic field is given by $\mathbf{B}_1 = \nabla\times\mathbf{A}_1$ and $\mathbf{B}^{(\mathrm{gc})}$ is defined by $\nabla\times\mathbf{B}^{(\mathrm{gc})} = (4\pi/c)(\mathbf{j}^{(\mathrm{gc})})_T$. The magnetization $\mathbf{M}$ can be obtained from the turbulent fields and the distribution functions for all species using Eqs. (41)–(43) in Ref. 18. Thus, Eqs. (35), (36), (37), (39), and (40) constitute the closed system of governing equations which determine F_a, ϕ, $\mathbf{A}_1$, I, and χ.

For the gyrocenter coordinates which have Poisson brackets given by Eq. (27), the Jacobian is given by $D_a = B^*_{a\parallel}/m_a$. It is important to note that the Jacobian D_a satisfies the gyrocenter phase-space conservation law,

$$\frac{\partial D_a}{\partial t} + \frac{\partial}{\partial\mathbf{Z}}\cdot\left(D_a\frac{d\mathbf{Z}_a}{dt}\right) = 0. \tag{41}$$

Then, using Eq. (41), the gyrokinetic Boltzmann equation in Eq. (35) can be rewritten as

$$\frac{\partial}{\partial t}(D_aF_a) + \frac{\partial}{\partial\mathbf{Z}}\cdot\left(D_aF_a\frac{d\mathbf{Z}_a}{dt}\right) = D_a\mathcal{K}_a, \tag{42}$$

where $\mathcal{K}_a$ is the gyrophase-independent function given by the right-hand side of Eq. (35),

$$\mathcal{K}_a(\mathbf{Z}, t) = \sum_a\langle C^g_{ab}[F_a, F_b](\mathbf{Z}, t)\rangle_\xi + \mathcal{S}_a(\mathbf{Z}, t). \tag{43}$$

We hereafter derive conservation laws for the gyrokinetic Boltzmann-Poisson-Ampère system of equations following the procedures similar to those shown in Sec. II. For that purpose, suppose that F_a, ϕ, $\mathbf{A}_1$, I, and χ satisfy Eqs. (35), (36), (37), (39), and (40). Then, we consider the gyrocenter distribution function F^V_a which obeys the gyrokinetic Vlasov equation,

$$\left(\frac{\partial}{\partial t} + \frac{d\mathbf{Z}_a}{dt}\cdot\frac{\partial}{\partial\mathbf{Z}}\right)F^V_a = 0, \tag{44}$$

where $d\mathbf{Z}_a/dt$ is evaluated by using the above-mentioned fields $(\phi, \mathbf{A}_1, I, \chi)$ obtained from the solution of the gyrokinetic Boltzmann-Poisson-Ampère system of equations. Here, it should be noted that, if the distribution functions F_a and F^V_a, which are given as the solutions of Eqs. (35) and (44), respectively, are initially gyrophase-independent, they are gyrophase-independent at any time. Besides, F^V_a is assumed to coincide instantaneously with F_a at at a given time t_0. Therefore, Eqs. (36), (37), (39), and (40) are all satisfied at that moment even if F_a is replaced with F^V_a in these equations. Thus, the gyrokinetic Vlasov-Poisson-Ampère system of equations are instantaneously satisfied by $(F^V_a, \phi, \mathbf{A}_1, I, \chi)$ at $t = t_0$. In Ref. 18, the action integral $\mathcal{I}$ is defined to derive all the governing equations for the gyrokinetic Vlasov-Poisson-Ampère system based on the variational principle, and its variation $\delta\mathcal{I}$ associated with the infinitesimal variable transformations are given to obtain conservation laws from Noether's theorem. Here, the action integral $\mathcal{I}$ can be expressed in terms of $(F^V_a, \phi, \mathbf{A}_1, I, \chi)$ over a small time interval, $t_0 - h/2 \le t \le t_0 + h/2$, during which the gyrokinetic Vlasov-Poisson-Ampère system of equations are approximately satisfied by them within the errors of order h. Then, neglecting the errors of higher order in h, we can write the variation $\delta\mathcal{I}$ in the same form as in Eq. (77) of Ref. 18,

$$\delta\mathcal{I} = -\int_{t_0-h/2}^{t_0+h/2}dt\int d^3\mathbf{X}\left[\frac{\partial}{\partial t}\delta G^V_0(\mathbf{X}, t) + \nabla\cdot\delta\mathbf{G}^V(\mathbf{X}, t)\right], \tag{45}$$

with the functions δG^V_0 and $\delta\mathbf{G}^V$ defined by

$$\begin{aligned}\delta G^V_0(\mathbf{X}, t) &= \mathcal{E}^V_c\;\delta t_E - \mathbf{P}^V_c\cdot\delta\mathbf{x}_E, \\ \delta\mathbf{G}^V(\mathbf{X}, t) &= \mathbf{Q}^V_c\;\delta t_E - \mathbf{\Pi}^V_c\cdot\delta\mathbf{x}_E + \mathbf{S}_\phi\;\delta\phi - \mathbf{\Sigma}_{A1}\cdot\delta\mathbf{A}_1 \\ &\quad - \mathbf{\Sigma}^V_{A0}\cdot\delta\mathbf{A}_0 + \mathbf{S}_\chi\delta\chi + \delta\mathbf{T}^V,\end{aligned} \tag{46}$$

where $\mathcal{E}^V_c$ and $\mathbf{P}^V_c$ are defined by

$$\begin{aligned}\mathcal{E}^V_c &= \sum_a\int dU\int d\mu\int d\xi\; D_aF^V_aH_a \\ &\quad + \frac{1}{8\pi}\left(-|\nabla\phi|^2 + |\mathbf{B}_0 + \mathbf{B}_1|^2\right), \\ \mathbf{P}^V_c &= \sum_a\int dU\int d\mu\int d\xi\; D_aF^V_a\left(m_aU\mathbf{b} + \frac{e_a}{c}\mathbf{A}_0\right),\end{aligned} \tag{47}$$

and definitions of other variables $\mathbf{Q}^V_c$, $\mathbf{\Pi}^V_c$, $\mathbf{S}_\phi$, $\mathbf{\Sigma}_{A1}$, $\mathbf{\Sigma}^V_{A0}$ $\mathbf{S}_\chi$, and $\delta\mathbf{T}^V$ are shown in Eq. (79) of Ref. 18. The superscript V in the variables $(\mathcal{E}^V_c, \mathbf{P}^V_c, \cdots)$ implies that they are defined using the distribution function F^V_a instead of F_a.

As explained in Ref. 18, the integral domain of Eq. (45) is not an arbitrary local one in the $\mathbf{X}$-space but it can be local only in the radial direction in order for Eq. (45) to be valid. Then, if the variations $\delta t_E, \delta\mathbf{x}_E, \cdots$ in Eq. (46) are such that $\delta\mathcal{I} = 0$ holds for a spatiotemporal integral domain defined by $[t_0 - h/2, t_0 + h/2] \times [s_1, s_2]$ where $[s_1, s_2]$ represents an arbitrary spatial volume region sandwiched between two flux surfaces labeled by s_1 and s_2, then the conservation law is derived as

$$\begin{aligned}&\left[\left\langle \frac{\partial}{\partial t}\delta G_0^V(\mathbf{X},t) + \nabla\cdot\delta\mathbf{G}^V(\mathbf{X},t_0)\right\rangle\right]_{t=t_0}\\ &= \left[\left\langle \frac{\partial}{\partial t}\delta G_0^V(\mathbf{X},t)\right\rangle + \frac{1}{V'}\frac{\partial}{\partial s}\left(V'\left\langle\delta\mathbf{G}^V\cdot\nabla s\right\rangle\right)\right]_{t=t_0}\\ &= 0. \end{aligned} \tag{48}$$

This is Noether's theorem for the gyrokinetic Vlasov-Poisson-Ampère system. In Eq. (48), $V' \equiv \partial V/\partial s$ represents the derivative of $V(s,t)$ with respect to s and $V(s,t)$ denotes the volume enclosed by the flux surface with the label s at the time t.

Using $F_a^V(\mathbf{Z},t_0) = F_a(\mathbf{Z},t_0)$ and comparing Eq. (35) with Eq. (44), we find

$$\left[\frac{\partial F_a^V(\mathbf{Z},t)}{\partial t}\right]_{t=t_0} = \left[\frac{\partial F_a(\mathbf{Z},t)}{\partial t}\right]_{t=t_0} - \mathcal{K}_a(\mathbf{Z},t_0), \tag{49}$$

where $\mathcal{K}$ is defined by Eq. (43). Let us also define δG_0 and $\delta\mathbf{G}$ from δG_0^V and $\delta\mathbf{G}^V$ by replacing F_a^V with F_a. Then, we have $\mathbf{G}^V(\mathbf{X},t_0) = \mathbf{G}(\mathbf{X},t_0)$ and

$$\left[\frac{\partial\delta G_0^V(\mathbf{X},t)}{\partial t}\right]_{t=t_0} = \left[\frac{\partial\delta G_0(\mathbf{X},t))}{\partial t}\right]_{t=t_0} - \delta K_{G0}(\mathbf{X},t_0), \tag{50}$$

where

$$\begin{aligned}\delta K_{G0} &= K_{\mathcal{E}c}\delta t_E - \mathbf{K}_{Pc}\cdot\delta\mathbf{x}_E,\\ K_{\mathcal{E}c} &= \sum_a e_a\int dU\int d\mu\int d\xi\ D_a\mathcal{K}_a H_a,\\ \mathbf{K}_{Pc} &= \sum_a e_a\int dU\int d\mu\int d\xi\ D_a\mathcal{K}_a\mathbf{p}_a^c. \end{aligned}\tag{51}$$

Here, $\mathbf{p}_a^c$ denotes the canonical momentum for species a defined by

$$\mathbf{p}_a^c = \frac{e_a}{c}\mathbf{A}_a^* = \frac{e_a}{c}\mathbf{A}_0 + m_a U\mathbf{b}. \tag{52}$$

Substituting Eq. (50) into Eq. (48) and rewriting the arbitrarily chosen time t_0 as t, we obtain the conservation law for the gyrokinetic Boltzmann-Poisson-Ampère system,

$$\begin{aligned}&\left\langle\frac{\partial}{\partial t}\delta G_0(\mathbf{X},t) + \nabla\cdot\delta\mathbf{G}(\mathbf{X},t)\right\rangle\\ &= \left\langle\frac{\partial}{\partial t}\delta G_0(\mathbf{X},t)\right\rangle + \frac{1}{V'}\frac{\partial}{\partial s}\left(V'\left\langle\delta\mathbf{G}\cdot\nabla s\right\rangle\right)\\ &= \left\langle\delta K_{G0}\right\rangle, \end{aligned}\tag{53}$$

where $\langle\delta K_{G0}\rangle$ represents effects of the collision and source terms on the conservation law. Under the nonstationary background field $\mathbf{B}_0$, flux surfaces may change their shapes and the grid of the flux coordinates moves. Then, Eq. (53) is rewritten as

$$\frac{\partial}{\partial t}\left(V'\left\langle\delta G_0\right\rangle\right) + \frac{\partial}{\partial s}\left(V'\left\langle(\delta\mathbf{G} - \delta G_0\mathbf{u}_s)\cdot\nabla s\right\rangle\right) = V'\left\langle\delta K_{G0}\right\rangle, \tag{54}$$

where $\mathbf{u}_s\cdot\nabla s$ represents the radial velocity of the flux surface labeled by s and $\mathbf{u}_s$ is defined by $\mathbf{u}_s = \partial\mathbf{x}(s,\theta,\zeta,t)/\partial t$ with the flux coordinates (s,θ,ζ) [see Eq. (2.35) in Ref. 31]. In Sec. V, gyrokinetic energy and toroidal angular momentum balance equations are derived from Eq. (54).

IV. EQUATIONS FOR GYROCENTER DENSITIES AND POLARIZATION

In this section, we take the velocity-space integral of the gyrokinetic Boltzmann equation in Eq. (35) to consider the particle transport before treating the energy and toroidal angular momentum transport in Sec. V. We define the gyrocenter density $n_a^{(\mathrm{gc})}$ by

$$n_a^{(\mathrm{gc})}(\mathbf{X},t) = \int dU\int d\mu\int d\xi\ D_a F_a, \tag{55}$$

and the gyrocenter flux $\mathbf{\Gamma}_a^{(\mathrm{gc})}$ by

$$\mathbf{\Gamma}_a^{(\mathrm{gc})} = n_a^{(\mathrm{gc})}\mathbf{u}_a^{(\mathrm{gc})} = \int dU\int d\mu\int d\xi\ D_a F_a\mathbf{v}_a^{(\mathrm{gc})}, \tag{56}$$

where $\mathbf{u}_a^{(\mathrm{gc})}$ represents the gyrocenter fluid velocity and the gyrocenter drift velocity $\mathbf{v}_a^{(\mathrm{gc})} = d\mathbf{X}_a/dt$ is given by evaluating the right-hand side of Eq. (28) at $(\mathbf{X}, U, \mu)$. Then, integrating the gyrokinetic Boltzmann equation, Eq. (42), with respect to the gyrocenter velocity-space coordinates (U,μ,ξ) and using Eq. (43), we obtain

$$\frac{\partial n_a^{(\mathrm{gc})}}{\partial t} + \nabla\cdot\left(\mathbf{\Gamma}_a^{(\mathrm{gc})} + \mathbf{\Gamma}_a^{\mathrm{C}}\right) = \int dU\int d\mu\int d\xi\ D_a\mathcal{S}_a. \tag{57}$$

Using the approximate collision operator given by Eq. (C1) in Appendix C, the particle flux $\mathbf{\Gamma}_a^{\mathrm{C}}$ due to collisions and finite gyroradii is defined by Eq. (C6) with putting $\mathcal{A}_a^p(\mathbf{z}) = 1$. [If the collision operator given by Eq. (B1) in Appendix B is employed, $\mathbf{\Gamma}_a^{\mathrm{C}}$ is defined by Eq. (B8).]

As shown in Ref. 18, the gyrokinetic Poisson equation in Eq. (36) is rewritten as

$$\sum_a e_a n_a^{(\mathrm{gc})} = \nabla\cdot\left(\frac{\mathbf{E}_L}{4\pi} + \mathbf{P}^{(\mathrm{pol})}\right), \tag{58}$$

where $\mathbf{E}_L = -\nabla\phi$, $\nabla = \partial/\partial\mathbf{X}$, and $\mathbf{P}^{(\mathrm{pol})}$ represents the polarization density defined by

$$\begin{aligned}\mathbf{P}^{(\mathrm{pol})} &= \sum_a e_a\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\sum_{i_1,\cdots,i_n}\int dU\int d\mu\int d\xi\\ &\quad\times\frac{\partial^{n-1}(D_a F_a^*\boldsymbol{\rho}_a\rho_{ai_1}\cdots\rho_{ai_{n-1}})}{\partial X_{i_1}\cdots\partial X_{i_{n-1}}}. \end{aligned}\tag{59}$$

Here, ρ_{ai} denotes the ith Cartesian component of $\boldsymbol{\rho}_a = \mathbf{b}(\mathbf{X},t) \times \mathbf{v}_{a0}(\mathbf{Z},t)/\Omega_a(\mathbf{X},t)$, and $F_a^* = F_a + (e_a\widetilde{\psi}_a/B_0)(\partial F_a/\partial\mu)$.

As mentioned before Eq. (24) in Sec. III, $\sum_a e_a \int dU \int d\mu \int d\xi\ D_a\mathcal{S}_a(\mathbf{Z},t) = 0$ is assumed. Then, using Eq. (57), we can obtain the charge conservation law,

$$\frac{\partial}{\partial t}\left(\sum_a e_a n_a^{(\mathrm{gc})}\right) + \nabla\cdot(\mathbf{j}^{(\mathrm{gc})} + \mathbf{j}^{\mathrm{C}}) = 0, \tag{60}$$

where the current density due to the gyrocenter drift and that due to the collisional particle transport are given by $\mathbf{j}^{(\mathrm{gc})} = \sum_a e_a \boldsymbol{\Gamma}_a^{(\mathrm{gc})} = \sum_a e_a n_a^{(\mathrm{gc})}\mathbf{u}_a^{(\mathrm{gc})}$ and $\mathbf{j}^{\mathrm{C}} = \sum_a e_a\boldsymbol{\Gamma}_a^{\mathrm{C}}$, respectively. Note that the magnetization current is solenoidal and, accordingly, it does not contribute to the charge conservation law in Eq. (60). Equation (58) is substituted into Eq. (60) to show

$$\mathbf{j}_L^{(\mathrm{gc})} + \mathbf{j}_L^{\mathrm{C}} = -\frac{\partial}{\partial t}\left(\frac{\mathbf{E}_L}{4\pi} + \mathbf{P}_L^{(\mathrm{pol})}\right), \tag{61}$$

where the subscript L is used to represent the longitudinal part of the vector variable. Then, using Eqs. (58), (60) and (61), we find that the useful formula,

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}\left(\mathcal{A}\sum_a e_a n_a^{(\mathrm{gc})}\right)\right\rangle + \left\langle \nabla\cdot\left(\mathcal{A}\mathbf{j}_L^{(\mathrm{gc})}\right)\right\rangle \\
&= \left\langle \frac{\partial\mathcal{A}}{\partial t}\sum_a e_a n_a^{(\mathrm{gc})}\right\rangle + \left\langle \mathbf{j}_L^{(\mathrm{gc})}\cdot\nabla\mathcal{A}\right\rangle - \left\langle \mathcal{A}\left(\nabla\cdot\mathbf{j}_L^{\mathrm{C}}\right)\right\rangle \\
&= \left\langle \nabla\cdot\left[\frac{\partial\mathcal{A}}{\partial t}\left(\frac{\mathbf{E}_L}{4\pi} + \mathbf{P}_L^{(\mathrm{pol})}\right) - \mathcal{A}\mathbf{j}_L^{\mathrm{C}}\right]\right\rangle \\
&\quad - \left\langle \frac{\partial}{\partial t}\left[\left(\frac{\mathbf{E}_L}{4\pi} + \mathbf{P}_L^{(\mathrm{pol})}\right)\cdot\nabla\mathcal{A}\right\rangle\right],
\end{aligned} \tag{62}$$

holds for any function $\mathcal{A}(\mathbf{X},t)$. The relation in Eq. (62) is used in Sec. V.B to derive Eq. (74).

V. GYROKINETIC ENERGY AND TOROIDAL ANGULAR MOMENTUM BALANCE EQUATIONS

In this section, energy and toroidal angular momentum balance equations for gyrokinetic systems including collisional processes are derived by using the results shown in Secs. III–IV.

A. Energy balance equation

The variation $\delta\mathcal{I}$ of the action given in Sec. III vanishes under the infinitesimal time translation represented by $\delta t_E = \epsilon$ where ϵ is an infinitesimally small constant. Here, all other infinitesimal variations $\delta\mathbf{x}_E$, $\delta\phi$, $\cdots$ are regarded as zero. Then, δG_0 and $\delta\mathbf{G}$ are determined by these conditions for the infinitesimal time translation and they satisfy Eq. (54) which, in the same manner as in Ref. 18, leads to the energy balance equation,

$$\frac{\partial}{\partial t}\left(V'\langle\mathcal{E}\rangle\right) + \frac{\partial}{\partial s}\left(V'\langle(\mathbf{Q}_c^* + \mathbf{Q}_R^* - \mathcal{E}\mathbf{u}_s)\cdot\nabla s\rangle\right) = V'\langle K_{\mathcal{E}c}\rangle. \tag{63}$$

Here, the energy density $\mathcal{E}$ is defined by

$$\begin{aligned}
\mathcal{E} &= \mathcal{E}_c + \nabla\cdot\left(\frac{1}{4\pi}\phi\nabla\phi - \boldsymbol{\Phi}_R\right) \\
&= \sum_a \int dU \int d\mu \int d\xi\ D_aF_a\left(\frac{m_a}{2}\left|\mathbf{v}_{a0} - \frac{e_a}{m_ac}\mathbf{A}_1\right|^2\right. \\
&\quad \left. + \frac{e_a}{2B_0}\frac{\partial}{\partial\mu}\left\langle\widetilde{\psi}_a\left(2\widetilde{\phi} - \widetilde{\psi}_a\right)\right\rangle_\xi\right) \\
&\quad + \frac{1}{8\pi}\left(|\nabla\phi|^2 + |\mathbf{B}_0 + \mathbf{B}_1|^2\right),
\end{aligned} \tag{64}$$

and the energy fluxes $\mathbf{Q}_c^*$ and $\mathbf{Q}_R^*$ are given by

$$\begin{aligned}
\mathbf{Q}_c^* &= \mathbf{Q}_c - \frac{1}{4\pi}\frac{\partial}{\partial t}(\phi\nabla\phi) \\
&= \sum_a \int dU \int d\mu \int d\xi\ D_aF_a\left[H_a\mathbf{v}_a^{(\mathrm{gc})} + \frac{\partial\mathbf{A}_0}{\partial t}\right. \\
&\quad \left.\times\left(-\mu\mathbf{b} + \frac{m_aU}{B_0}(\mathbf{v}_a^{(\mathrm{gc})})_\perp - \mathbf{N}_a\right)\right] - \frac{1}{4\pi}\phi\nabla\frac{\partial\phi}{\partial t} \\
&\quad - \frac{1}{4\pi}\frac{\partial(\mathbf{A}_0 + \mathbf{A}_1)}{\partial t}\times(\mathbf{B}_0 + \mathbf{B}_1) + \frac{1}{4\pi c}\left(\lambda\frac{\partial\mathbf{A}_1}{\partial t}\right. \\
&\quad \left. + \alpha\frac{\partial\mathbf{A}_0}{\partial t}\right) - \frac{1}{4\pi}\boldsymbol{\Lambda}\times\left(\frac{\partial\mathbf{A}_0}{\partial t} + \frac{\partial\chi}{\partial t}\nabla\zeta\right)
\end{aligned} \tag{65}$$

and

$$\mathbf{Q}_R^* = \mathbf{Q}_R + \frac{\partial\boldsymbol{\Phi}_R}{\partial t}, \tag{66}$$

respectively, where $\mathbf{N}_a$, $\mathbf{Q}_R$, and $\boldsymbol{\Phi}_R$ are defined by Eqs. (43), (85), and (88) in Ref. 18, respectively.

Recalling Eq. (51) and using Eq. (C5) into which we substitute $\mathcal{A}_a^g(\mathbf{Z}) = H_a(\mathbf{Z})$ and the approximate collision operator given by Eq. (C1) in Appendix C, we rewrite the right-hand side of Eq. (63) as

$$\begin{aligned}
V'\langle K_{\mathcal{E}c}\rangle &= -\frac{\partial}{\partial s}\left(V'\langle\mathbf{Q}^{\mathrm{C}}\cdot\nabla s\rangle\right) \\
&\quad + V'\sum_a\left\langle\int dU \int d\mu \int d\xi\ D_a\mathcal{S}_aH_a\right\rangle,
\end{aligned} \tag{67}$$

where the energy flux $\mathbf{Q}^{\mathrm{C}}$ due to collisions and finite gyroradii is defined by taking the summation of Eq. (C6) over species a with putting $\mathcal{A}_a^p(\mathbf{z}) = \frac{1}{2}m_av_{\|a}^2 + \mu_{0a}B_0(\mathbf{x}_a) + e_a\phi(\mathbf{x}_a)$. The derivation of Eq. (67) requires Eq. (C7) which is satisfied by the collision operator in the form of Eq.(C1) with appropriately choosing $\Delta\mathbf{x}_a^{(2)}$, $\Delta v_{\|a}$, and $\Delta\mu_{0a}$ as described in Appendix C.

Substituting Eq. (67) into Eq. (63), the energy balance equation is rewritten as

$$\frac{\partial}{\partial t}\left(V'\langle\mathcal{E}\rangle\right)+\frac{\partial}{\partial s}\left(V'\langle(\mathbf{Q}-\mathcal{E}\mathbf{u}_s)\cdot\nabla s\rangle\right) = V'\sum_a\left\langle\int dU\int d\mu\int d\xi\ D_a\mathcal{S}_aH_a\right\rangle, \qquad (68)$$

where the energy flux $\mathbf{Q}$ is given by

$$\mathbf{Q}=\mathbf{Q}_c^*+\mathbf{Q}_R^*+\mathbf{Q}^{\mathrm{C}}. \qquad (69)$$

The right-hand side of Eq. (68) represents the external energy source. It is confirmed later in Sec. VI.B that the ensemble average of $\langle\mathbf{Q}\cdot\nabla s\rangle$ coincides with the well-known expression of the radial energy transport to the lowest order in the δ-expansion.

B. Toroidal angular momentum balance equation

The toroidal angular momentum balance equation is derived from the fact that $\delta\mathcal{I}=0$ under the infinitesimal toroidal rotation represented by $\delta\mathbf{x}_E=\epsilon\mathbf{e}_\zeta(\mathbf{X})$. Here, ϵ is again an infinitesimally small constant, and $\mathbf{e}_\zeta(\mathbf{X})$ is defined by $\mathbf{e}_\zeta(\mathbf{X})=\partial\mathbf{X}(R,z,\zeta)/\partial\zeta=R^2\nabla\zeta$ where the right-handed cylindrical spatial coordinates (R,z,ζ) are used. We also define $\hat{\mathbf{z}}$ by $\hat{\mathbf{z}}=R\nabla\zeta\times\nabla R$ which represents the unit vector in the z-direction. Then, if putting the origin of the position vector $\mathbf{X}$ at $(R,z)=(0,0)$, we have $\mathbf{e}_\zeta(\mathbf{X})=\mathbf{X}\times\hat{\mathbf{z}}$. Under the infinitesimal toroidal rotation, the variations of the vector variables are given as $\delta\mathbf{A}_1=\epsilon\mathbf{A}_1\times\hat{\mathbf{z}}$ and $\delta\mathbf{A}_0=\epsilon\mathbf{A}_0\times\hat{\mathbf{z}}$ although the other variations δt_E, $\delta\phi$, $\cdots$, are all regarded as zero. Then, using these variations of the variables associated with the infinitesimal toroidal rotation, the canonical momentum balance equation is derived from Eq. (53) as

$$\left\langle\frac{\partial(\mathbf{P}_c\cdot\mathbf{e}_\zeta)}{\partial t}\right\rangle+\frac{1}{V'}\frac{\partial}{\partial s}\left[V'\langle\nabla s\cdot(\mathbf{\Pi}_c\cdot\mathbf{e}_\zeta + (\mathbf{\Sigma}_{A1}\times\mathbf{A}_1+\mathbf{\Sigma}_{A0}\times\mathbf{A}_0)\cdot\hat{\mathbf{z}}+\mathbf{P}_{R\zeta}]\rangle\right) = \langle K_{Pc\zeta}\rangle. \qquad (70)$$

Here, the density of the canonical toroidal angular momentum is defined by

$$\mathbf{P}_c\cdot\mathbf{e}_\zeta=\sum_a\int dU\int d\mu\int d\xi\ D_aF_a(p_a^c)_\zeta, \qquad (71)$$

with the toroidal component of the canonical momentum denoted by

$$(p_a^c)_\zeta=\frac{e_a}{c}A_{a\zeta}^*=\frac{e_a}{c}A_{0\zeta}+m_aUb_\zeta, \qquad (72)$$

where $b_\zeta\equiv I/B_0$ represents the covariant toroidal component of $\mathbf{b}\equiv\mathbf{B}_0/B_0$. Definitions of $\mathbf{\Sigma}_{A1}$, $\mathbf{\Sigma}_{A0}$, and $\mathbf{P}_{R\zeta}$ on the left-hand side of Eq. (70) are given by Eqs. (79) and (100) in Ref. 18. On the right-hand side of Eq. (70), the variation of the canonical toroidal angular momentum due to collisions and external sources is given by

$$K_{Pc\zeta}=\mathbf{K}_{Pc}\cdot\mathbf{e}_\zeta=\sum_a\int dU\int d\mu\int d\xi\ D_a\mathcal{K}_a(p_a^c)_\zeta. \qquad (73)$$

We follow the same procedures as shown in Sec. V.B of Ref. 18 and use Eqs. (70)–(73) and Eq. (62) with $\mathcal{A}=A_{0\zeta}=-\chi$ to write the toroidal angular momentum balance equation as

$$\begin{aligned}&\left\langle\frac{\partial}{\partial t}\left[P_{\|\zeta}-\frac{1}{c}\left(\mathbf{P}_L^{(\mathrm{pol})}+\frac{\mathbf{E}_L}{4\pi}\right)\cdot\nabla A_{0\zeta}\right]\right\rangle\\&+\frac{1}{V'}\frac{\partial}{\partial s}\left[V'\left\{\Pi_{\|\zeta}^s+\Pi_{R\zeta}^s-\frac{1}{4\pi}\langle A_{1\zeta}(\nabla\times\mathbf{B}_1)\cdot\nabla s\rangle\right.\right.\\&-\frac{1}{4\pi}\langle E_{L\zeta}E_L^s+B_{1\zeta}B_1^s\rangle+\frac{1}{4\pi c}\left\langle\frac{\partial\lambda}{\partial\zeta}A_1^s\right\rangle\\&\left.\left.+\frac{1}{c}\left\langle\frac{\partial A_{0\zeta}}{\partial t}\left(\mathbf{P}_L^{(\mathrm{pol})}+\frac{\mathbf{E}_L}{4\pi}\right)\cdot\nabla s\right\rangle\right\}\right]\\&=\langle K_{Pc\zeta}\rangle+\frac{1}{c}\left\langle\nabla\cdot\left(A_{0\zeta}\mathbf{j}_L^{\mathrm{C}}\right)\right\rangle,\end{aligned} \qquad (74)$$

where

$$\begin{aligned}P_{\|\zeta}&=\sum_a\int dU\int d\mu\int d\xi\ D_aF_am_aUb_\zeta,\\\Pi_{\|\zeta}^s&=\sum_a\int dU\int d\mu\int d\xi\ D_aF_am_aUb_\zeta\mathbf{v}_a^{(\mathrm{gc})}\cdot\nabla s,\\\Pi_{R\zeta}^s&=\mathbf{P}_{R\zeta}\cdot\nabla s.\end{aligned} \qquad (75)$$

Using Eqs. (73) and (C5) in Appendix C with putting $\mathcal{A}_a^g=(p_a^c)_\zeta$, the right-hand side of Eq. (74) is rewritten as

$$\begin{aligned}&\langle K_{Pc\zeta}\rangle+\frac{1}{c}\left\langle\nabla\cdot\left(A_{0\zeta}\mathbf{j}_L^{\mathrm{C}}\right)\right\rangle\\&=-\frac{1}{V'}\frac{\partial}{\partial s}\left[V'\left\langle\left(\mathbf{J}_{p\zeta}^{\mathrm{C}}+\frac{\chi}{c}\mathbf{j}_L^{\mathrm{C}}\right)\cdot\nabla s\right\rangle\right]\\&\quad+\sum_a\left\langle\int dU\int d\mu\int d\xi\ D_a\mathcal{S}_am_aUb_\zeta\right\rangle,\end{aligned} \qquad (76)$$

where $\mathbf{J}_{p\zeta}^{\mathrm{C}}$ is defined by taking the summation of Eq. (C6) over species a with putting $\mathcal{A}_a^p(\mathbf{z})=(e_a/c)[A_{0\zeta}(\mathbf{x})+A_{1\zeta}(\mathbf{x})]+m_av_\zeta$. In deriving Eq. (76), $\sum_a e_a\int dU\int d\mu\int d\xi\ D_a\mathcal{S}_a(\mathbf{Z},t)=0$ and Eq. (C12) are used. The approximate collision operator which satisfies Eq. (C12) is presented in Appendix C.

Substituting Eq. (76) into Eq. (74), the toroidal angular momentum balance equation is rewritten as

$$\frac{\partial}{\partial t}\left(V'\left\langle P_{\parallel\zeta}-\frac{1}{c}\left(\mathbf{P}_L^{(\mathrm{pol})}+\frac{\mathbf{E}_L}{4\pi}\right)\cdot\nabla A_{0\zeta}\right\rangle\right)$$
$$+\frac{1}{V'}\frac{\partial}{\partial s}\left[V'\left\{\Pi^s_{\parallel\zeta}+\Pi^s_{R\zeta}+(\Pi^{\mathrm{C}})^s-\frac{1}{4\pi}\right.\right.$$
$$\times\langle A_{1\zeta}(\nabla\times\mathbf{B}_1)\cdot\nabla s\rangle-\frac{1}{4\pi}\langle E_{L\zeta}E_L^s+B_{1\zeta}B_1^s\rangle$$
$$+\frac{1}{4\pi c}\left\langle\frac{\partial\lambda}{\partial\zeta}A_1^s\right\rangle+\frac{1}{c}\left\langle\frac{\partial A_{0\zeta}}{\partial t}\left(\mathbf{P}_L^{(\mathrm{pol})}+\frac{\mathbf{E}_L}{4\pi}\right)\cdot\nabla s\right\rangle$$
$$\left.\left.-\left\langle\left[P_{\parallel\zeta}-\frac{1}{c}\left(\mathbf{P}_L^{(\mathrm{pol})}+\frac{\mathbf{E}_L}{4\pi}\right)\cdot\nabla A_{0\zeta}\right](\mathbf{u}_s\cdot\nabla s)\right\rangle\right\}\right]$$
$$=\sum_a\left\langle\int dU\int d\mu\int d\xi\; D_a\mathcal{S}_a m_a U b_\zeta\right\rangle, \tag{77}$$

where the right-hand side represents the external source of the toroidal angular momentum and

$$(\Pi^{\mathrm{C}})^s=\left(\mathbf{J}^{\mathrm{C}}_{p\zeta}+\frac{\chi}{c}\mathbf{j}^{\mathrm{C}}_L\right)\cdot\nabla s \tag{78}$$

is the radial flux of the toroidal angular momentum due to collisions and finite gyroradii. In Sec. VI.C, we derive the ensemble-averaged toroidal angular momentum balance equation from Eq. (77) in order to confirm that it is consistent with the conventional result up to the second order in δ.

VI. ENSEMBLE-AVERAGED BALANCE EQUATIONS FOR PARTICLES, ENERGY, AND TOROIDAL ANGULAR MOMENTUM

In this section, the particle, energy and toroidal angular momentum balance equations derived in Secs. IV and V are ensemble-averaged for the purpose of verifying their consistency with those obtained by conventional recursive formulations.[30,33–35] In the same way as shown in Sec. VI of Ref. 18, we divide an arbitrary physical variable $\mathcal{Q}$ into the average and turbulent parts as

$$\mathcal{Q}=\langle Q\rangle_{\mathrm{ens}}+\hat{\mathcal{Q}}, \tag{79}$$

where $\langle\cdots\rangle_{\mathrm{ens}}$ represents the ensemble average, and we immediately find $\langle\hat{\mathcal{Q}}\rangle_{\mathrm{ens}}=0$. We identify the zeroth fields $\mathbf{A}_0$ and $\mathbf{B}_0$ with the ensemble-averaged parts to write $\mathbf{A}_0=\langle\mathbf{A}\rangle_{\mathrm{ens}}$, $\mathbf{A}_1=\hat{\mathbf{A}}$, $\mathbf{B}_0=\langle\mathbf{B}\rangle_{\mathrm{ens}}$, and $\mathbf{B}_1=\hat{\mathbf{B}}$. Regarding the electrostatic potential ϕ, it is written as the sum of the average and fluctuation parts, $\phi(\mathbf{x},t)=\langle\phi(\mathbf{x},t)\rangle_{\mathrm{ens}}+\hat{\phi}(\mathbf{x},t)$. Here, assuming that $\langle\phi(\mathbf{x},t)\rangle_{\mathrm{ens}}\neq 0$, the background $\mathbf{E}\times\mathbf{B}$ flow is retained and its velocity is regarded as $\mathcal{O}(\delta v_T)$, where δ and v_T represent the drift ordering parameter and the thermal velocity, respectively. Then, using Eq. (33), we have $\psi_a=\langle\psi_a\rangle_{\mathrm{ens}}+\hat{\psi}_a$, where

$$\langle\psi_a\rangle_{\mathrm{ens}}=\langle\phi\rangle_{\mathrm{ens}},\qquad \hat{\psi}_a=\hat{\phi}-\frac{\mathbf{v}_0}{c}\cdot\hat{\mathbf{A}}. \tag{80}$$

We assume that the ensemble average $\langle\mathcal{Q}\rangle_{\mathrm{ens}}$ of any variable $\mathcal{Q}$ considered here has a slow temporal variation subject to the so-called transport ordering, $\partial\ln\langle\mathcal{Q}\rangle_{\mathrm{ens}}/\partial t=\mathcal{O}(\delta^2 v_T/L)$, and that it has a gradient scale length L which is on the same order as gradient scale lengths of the equilibrium field and pressure profiles. We also impose the constraint of axisymmetry on $\langle\mathcal{Q}\rangle_{\mathrm{ens}}$ that is written as $\partial\langle\mathcal{Q}\rangle_{\mathrm{ens}}/\partial\zeta=0$ even though $\mathcal{Q}$ itself is not axisymmetric. On the other hand, the turbulent part $\hat{\mathcal{Q}}$ of $\mathcal{Q}$ is assumed to vary with a characteristic frequency $\omega=\mathcal{O}(v_T/L)$ and have gradient scale lengths L and ρ in the directions parallel and perpendicular to the equilibrium magnetic field $\mathbf{B}_0$, respectively.

The ensemble-averaged part $\langle F_a\rangle_{\mathrm{ens}}$ of the distribution function F_a for species a consists of the local Maxwellian part and the deviation from it,

$$\langle F_a\rangle_{\mathrm{ens}}=F_{aM}+\langle F_{a1}\rangle_{\mathrm{ens}}. \tag{81}$$

The local Maxwellian distribution function is written as $F_{aM}=n_{a0}[m_a/(2\pi T_{a0})]^{3/2}\exp[-(m_aU^2+\mu B_0)/(2T_{a0})]$ where the equilibrium density n_{a0} and temperature T_{a0} are regarded as uniform on flux surfaces. The first-order ensemble-averaged distribution function $\langle F_{a1}\rangle_{\mathrm{ens}}$ is determined by the drift kinetic equation, which can be derived by substituting Eq. (81) into the ensemble average of Eq. (23). The derived equation agrees, to $\mathcal{O}(\delta)$, with the well-known linearized drift kinetic equation, on which the neoclassical transport theory is based.[31,32]

The fluctuation part $\hat{F}_a$ is written as

$$\hat{F}_a=-F_{aM}\frac{e_a\langle\hat{\psi}_a\rangle_\xi}{T_a}+\hat{h}_a. \tag{82}$$

Substituting Eq. (82) into the fluctuation part of the gyrokinetic equation in Eq. (23) yields

$$\frac{\partial\hat{h}_a}{\partial t}+\{\hat{h}_a,H_a\}$$
$$=F_{aM}\left[\frac{e_a}{T_{a0}}\frac{\partial\langle\hat{\psi}_a\rangle_\xi}{\partial t}-\hat{\mathbf{v}}_a^{(\mathrm{gc})}\cdot\left(\nabla\ln p_{a0}+\frac{e_a}{T_{a0}}\nabla\langle\phi\rangle_{\mathrm{ens}}\right.\right.$$
$$\left.\left.+\left(\frac{\frac{1}{2}m_aU^2+\mu B_0}{T_{a0}}-\frac{5}{2}\right)\nabla\ln T_{a0}\right)\right]+C_a^L \tag{83}$$

where C_a^L represents the linear collision term defined by

$$C_a^L(\mathbf{X})=\sum_b\langle[C^p_{ab}(h_a(\mathbf{x}-\boldsymbol{\rho}_a),F_{bM})$$
$$+C^p_{ab}(F_{aM},h_b(\mathbf{x}-\boldsymbol{\rho}_b))]_{\mathbf{x}=\mathbf{X}+\boldsymbol{\rho}_a}\rangle_{\xi_a}. \tag{84}$$

Equation (83) is valid to the lowest order in δ and agrees with the conventional gyrokinetic equation for the nonadiabatic part $\hat{h}_a$ of the perturbed distribution function derived from using the WKB representation.[10,30] On the right-hand side of Eq. (83), the turbulent part $\hat{\mathbf{v}}_a^{(\mathrm{gc})}$ of the gyrocenter drift velocity $\mathbf{v}_a^{(\mathrm{gc})}=d\mathbf{X}_a/dt=\{\mathbf{X}_a,H_a\}$ is written as

$$\hat{\mathbf{v}}_a^{(\mathrm{gc})}=\frac{c}{B_0}\mathbf{b}\times\nabla\langle\hat{\psi}_a(\mathbf{X}+\boldsymbol{\rho}_a,t)\rangle_\xi+\mathcal{O}(\delta^2). \tag{85}$$

It is shown by using Eq. (82) and the WKB representation that, to the lowest order in δ, the turbulent parts of Eqs. (36) and (37) agree with the gyrokinetic Poisson equation and the gyrokinetic Ampère's law derived by conventional recursive formulations.[8,30]

A. Ensemble-averaged particle balance equation

Taking the ensemble average of Eq. (57) and subsequently its flux surface average, we obtain

$$\left\langle \frac{\partial \langle n_a^{(\mathrm{gc})}\rangle_{\mathrm{ens}}}{\partial t} \right\rangle + \frac{1}{V'}\frac{\partial}{\partial s}\left(V' \left\langle\left\langle \left(\boldsymbol{\Gamma}_a^{(\mathrm{gc})} + \boldsymbol{\Gamma}_a^{\mathrm{C}} \right)\cdot \nabla s \right\rangle\right\rangle \right) = \left\langle \int dU \int d\mu \int d\xi \; D_a \mathcal{S}_a \right\rangle, \tag{86}$$

where $\langle n_a^{(\mathrm{gc})}\rangle_{\mathrm{ens}} = n_{a0} + \mathcal{O}(\delta)$, and $\langle\langle \cdots \rangle\rangle$ represents a double average over the flux surface and the ensemble. Here, n_{a0} is the equilibrium density which is a flux-surface function and characterizes the Maxwellian distribution function F_{aM}. On the right-hand side, the source term $\mathcal{S}_a$ is regarded as of $\mathcal{O}(\delta^2)$ as well as all other terms in Eq. (86), and it is assumed to have no turbulent component so that $\mathcal{S}_a = \langle \mathcal{S}_a \rangle_{\mathrm{ens}}$.

It is shown in Ref. 18 that the radial gyrocenter particle flux is given by

$$(\Gamma_a^{(\mathrm{gc})})^s \equiv \left\langle\left\langle \boldsymbol{\Gamma}_a^{(\mathrm{gc})} \cdot \nabla s \right\rangle\right\rangle = (\Gamma_a^{\mathrm{NA}})^s + (\Gamma_a^{\mathrm{A}})^s, \tag{87}$$

where the nonturbulent part $(\Gamma_a^{\mathrm{NA}})^s$ and the turbulence-driven part $(\Gamma_a^{\mathrm{A}})^s$ are written as

$$\begin{aligned} (\Gamma_a^{\mathrm{NA}})^s &\equiv \left\langle \int dU \int d\mu \int d\xi \; D_a \langle F_a \rangle_{\mathrm{ens}} \langle \mathbf{v}_a^{(\mathrm{gc})} \rangle_{\mathrm{ens}} \cdot \nabla s \right\rangle \\ &= \left\langle \frac{c}{e_a B_0} \left[\mathbf{b} \times \left(\nabla \cdot \mathbf{P}_{a1}^{\mathrm{CGL}} \right) \right] \cdot \nabla s \right\rangle \\ &\quad + n_{a0} \left\langle \frac{c}{B_0} \left(\langle \mathbf{E} \rangle_{\mathrm{ens}} \times \mathbf{b} \right) \cdot \nabla s \right\rangle + \mathcal{O}(\delta^3), \end{aligned} \tag{88}$$

and

$$\begin{aligned} (\Gamma_a^{\mathrm{A}})^s &\equiv \left\langle \int dU \int d\mu \int d\xi \; D_a \langle \hat{F}_a \hat{\mathbf{v}}_a^{(\mathrm{gc})} \rangle_{\mathrm{ens}} \cdot \nabla s \right\rangle \\ &= -\left\langle\left\langle \frac{c}{B_0} \int dU \int d\mu \int d\xi \; D_a \hat{h}_a (\nabla \hat{\psi}_a \times \mathbf{b}) \cdot \nabla s \right\rangle\right\rangle \\ &\quad + \mathcal{O}(\delta^3), \end{aligned} \tag{89}$$

respectively. On the right-hand side of Eq. (88), $\mathbf{P}_{a1}^{\mathrm{CGL}}$ represents the first-order part of the pressure tensor in the Chew-Goldberger-Low (CGL) form[31] defined by $\mathbf{P}_{a1}^{\mathrm{CGL}} = \int dU \int d\mu \int d\xi \;\; D_a \langle F_{a1} \rangle_{\mathrm{ens}} \;\; \left[m_a U^2 \mathbf{bb} + \mu B_0 \left(\mathbf{I} - \mathbf{bb} \right) \right]$, and the ensemble-averaged electric field is given by $\langle \mathbf{E} \rangle_{\mathrm{ens}} = -\nabla \langle \phi \rangle_{\mathrm{ens}} - c^{-1} \partial \mathbf{A}_0 / \partial t$. Thus, Eq. (88) expresses the neoclassical radial particle flux and the radial $\mathbf{E} \times \mathbf{B}$ drift which are well-known by the neoclassical transport theory.[31] We also find that Eq. (89) agrees with the turbulent radial particle flux derived from the conventional gyrokinetic theory based on the WKB formalism.[30]

The radial classical particle flux is given by

$$\begin{aligned} (\Gamma_a^{\mathrm{C}})^s &\equiv \left\langle\left\langle \boldsymbol{\Gamma}_a^{\mathrm{C}} \cdot \nabla s \right\rangle\right\rangle \\ &= \sum_b \left\langle \frac{m_a c}{e_a B_0} \int dU \int d\mu \int d\xi \; D_a [(\mathbf{v} \times \mathbf{b}) \cdot \nabla s] \right. \\ &\quad \left. \times \left[C_{ab}^p (\widetilde{f}_{a1}, F_{bM}) + C_{ab}^p (F_{aM}, \widetilde{f}_{b1}) \right] \right\rangle + \mathcal{O}(\delta^3) \\ &= \left\langle \frac{c}{e_a B_0} (\mathbf{F}_{a1} \times \mathbf{b}) \cdot \nabla s \right\rangle + \mathcal{O}(\delta^3), \end{aligned} \tag{90}$$

where $\mathbf{F}_{a1} \equiv \int d^3\mathbf{v} m_a \mathbf{v} C_a^p$ is the collisional friction force.

It is well-known that the classical transport equation relating $(\Gamma_a^{\mathrm{C}})^s$ to the gradient forces is immediately derived from Eq. (90) because the first-order gyrophase-dependent part of the particle distribution function in Eq. (90) is expressed in terms of the gradient of the background Maxwellian distribution function as $\widetilde{f}_{a1} = -\boldsymbol{\rho}_a \cdot \nabla F_{aM}$ with the gradient operator ∇ taken for the fixed energy variable $\varepsilon = \frac{1}{2} m_a v^2 + e \langle \phi \rangle_{\mathrm{ens}}$.

In the same manner as in deriving Eq. (54) from Eq. (53), the ensemble-averaged particle transport equation can be obtained from Eq. (86) as

$$\frac{\partial}{\partial t} \left(V' n_{a0} \right) + \frac{\partial}{\partial s} \left(V' \left[(\Gamma_a)^s - n_{a0} \langle \mathbf{u}_s \cdot \nabla s \rangle \right] \right) = \left\langle \int dU \int d\mu \int d\xi \; D_a \mathcal{S}_a \right\rangle, \tag{91}$$

where the total radial particle flux is given by the sum of the classical, neoclassical, and turbulent parts as

$$\begin{aligned} (\Gamma_a)^s &= (\Gamma_a^{(\mathrm{gc})})^s + (\Gamma_a^{\mathrm{C}})^s \\ &= (\Gamma_a^{\mathrm{NA}})^s + (\Gamma_a^{\mathrm{A}})^s + (\Gamma_a^{\mathrm{C}})^s. \end{aligned} \tag{92}$$

As shown above, the well-known expressions of the classical, neoclassical and turbulent particle fluxes are included in $(\Gamma_a^{\mathrm{C}})^s$, $(\Gamma_a^{\mathrm{NA}})^s$, and $(\Gamma_a^{\mathrm{A}})^s$, respectively. The latter two fluxes are evaluated by the solutions $\langle F_{a1} \rangle_{\mathrm{ens}}$ and $\hat{h}_a$ of the first-order drift kinetic and gyrokinetic equations, respectively.

B. Ensemble-averaged energy balance equation

The ensemble average of the energy density defined by Eq. (64) is written as

$$\langle \mathcal{E} \rangle_{\mathrm{ens}} = \frac{3}{2} \sum_a n_{a0} T_{a0} + \frac{B_0^2}{8\pi} + \mathcal{O}(\delta), \tag{93}$$

where the energy density of the electric field is neglected as a small quantity of $\mathcal{O}(\delta^2)$. It is shown in Ref. 18 that the radial components of the first two terms on the right-hand side of Eq. (65) are double-averaged over the ensemble and the flux surface to give

$$\sum_a \left\langle\left\langle \int dU \int d\mu \int d\xi\; D_a F_a \left(H_a \mathbf{v}_a^{(\mathrm{gc})} - \mu \frac{\partial \mathbf{A}_0}{\partial t} \times \mathbf{b}\right) \cdot \nabla s \right\rangle\right\rangle$$
$$= \sum_a \left[(q_a^{(\mathrm{gc})})^s + \frac{5}{2} T_{a0} (\Gamma_a^{(\mathrm{gc})})^s \right] + \mathcal{O}(\delta^3). \tag{94}$$

Here, the radial particle flux $(\Gamma_a^{(\mathrm{gc})})^s$ is given by Eqs. (87)–(89) and the radial heat flux $(q_a)^s$ is written as

$$(q_a^{(\mathrm{gc})})^s = (q_a^{\mathrm{NA}})^s + (q_a^{\mathrm{A}})^s \tag{95}$$

which consists of the nonturbulent part,

$$(q_a^{\mathrm{NA}})^s = \left\langle \int dU \int d\mu \int d\xi\; D_a \langle F_{a1}\rangle_{\mathrm{ens}} \langle \mathbf{v}_a^{(\mathrm{gc})}\rangle_{\mathrm{ens}} \cdot \nabla s \times \left(\frac{1}{2} m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0} \right) \right\rangle$$
$$= T_{a0} \left\langle \frac{c}{e_a B_0} \left[\mathbf{b} \times \left(\nabla \cdot \mathbf{\Theta}_a^{\mathrm{CGL}} \right) \right] \cdot \nabla s \right\rangle + \mathcal{O}(\delta^3), \tag{96}$$

and the turbulence-driven part,

$$(q_a^{\mathrm{A}})^s = -\left\langle\left\langle \frac{c}{B_0} \int dU \int d\mu \int d\xi\; D_a \hat{h}_a (\nabla \hat{\psi}_a \times \mathbf{b}) \cdot \nabla s \times \left(\frac{1}{2} m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0} \right) \right\rangle\right\rangle + \mathcal{O}(\delta^3). \tag{97}$$

In Eq. (96), the heat stress tensor $\mathbf{\Theta}_a^{\mathrm{CGL}}$ is defined by $T_{a0}\mathbf{\Theta}_a^{\mathrm{CGL}} = \int dU \int d\mu \int d\xi\; D_a \langle F_{a1}\rangle_{\mathrm{ens}} \left(\frac{1}{2} m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0}\right)[m_a U^2 \mathbf{bb} + \mu B_0 (\mathbf{I} - \mathbf{bb})]$. The expression of Eq. (96) coincides with that of the neoclassical radial heat flux in terms of the heat stress tensor.[31] The turbulent heat flux in Eq. (97) takes the same form as that given by the conventional gyrokinetic theory.[30]

The radial component of $\mathbf{Q}^{\mathrm{C}}$ in Eq. (69) is ensemble-averaged to yield

$$\langle\langle \mathbf{Q}^{\mathrm{C}} \cdot \nabla s \rangle\rangle = (q_a^{\mathrm{C}})^s + \frac{5}{2} T_{a0} (\Gamma_a^{\mathrm{C}})^s + \mathcal{O}(\delta^3), \tag{98}$$

where the radial classical heat flux is given by

$$(q_a^{\mathrm{C}})^s = \sum_b \left\langle \frac{m_a c}{e_a B_0} \int dU \int d\mu \int d\xi\; D_a [(\mathbf{v} \times \mathbf{b}) \cdot \nabla s] \times [C_{ab}^p(\widetilde{f}_{a1}, F_{bM}) + C_{ab}^p(F_{aM}, \widetilde{f}_{b1})] \times \left(\frac{1}{2} m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0} \right) \right\rangle$$
$$= T_{a0} \left\langle \frac{c}{e_a B_0} (\mathbf{F}_{a2} \times \mathbf{b}) \cdot \nabla s \right\rangle. \tag{99}$$

Here, $\mathbf{F}_{a2} \equiv \int d^3\mathbf{v} (m_a v^2/2T_a - 5/2) m_a \mathbf{v} C_a^p$ is the collisional heat friction. The expression of the classical heat flux $(q_a^{\mathrm{C}})^s$ in Eq. (99) agrees with the conventional one[31] and it immediately gives the classical heat transport equation relating $(q_a^{\mathrm{C}})^s$ to the gradient forces in the same way as mentioned after Eq. (90) for the classical particle flux $(\Gamma_a^{\mathrm{C}})^s$.

Now, Eq. (68) is rewritten as

$$\frac{\partial}{\partial t} \left(V' \left[\frac{3}{2} \sum_a n_{a0} T_{a0} + \frac{B_0^2}{8\pi} \right] \right) + \frac{\partial}{\partial s} \left(V' \left[\sum_a \left((q_a)^s + \frac{5}{2} T_{a0} (\Gamma_a)^s \right) + \left\langle \mathbf{S}^{(\mathrm{Poynting})} \cdot \nabla s \right\rangle - \left(\frac{3}{2} \sum_a n_{a0} T_{a0} + \frac{B_0^2}{8\pi} \right) \langle \mathbf{u}_s \cdot \nabla s \rangle \right] \right)$$
$$= V' \sum_a \left\langle \int dU \int d\mu \int d\xi\; D_a \mathcal{S}_a \left(\frac{1}{2} m_a U^2 + \mu B_0 \right) \right\rangle + \mathcal{O}(\delta^3), \tag{100}$$

where the total radial heat flux is given by the sum of the classical, neoclassical, and turbulent parts as

$$(q_a)^s = (q_a^{(\mathrm{gc})})^s + (q_a^{\mathrm{C}})^s = (q_a^{\mathrm{NA}})^s + (q_a^{\mathrm{A}})^s + (q_a^{\mathrm{C}})^s, \tag{101}$$

and $\mathbf{S}^{(\mathrm{Poynting})} \equiv (c/4\pi)\langle \mathbf{E}\rangle_{\mathrm{ens}} \times \mathbf{B}_0$ represents the nonturbulent part of the Poynting vector. Using the relation $\langle \partial(B_0^2/8\pi)/\partial t\rangle = -(V')^{-1}\partial(V'\langle \mathbf{S}^{(\mathrm{Poynting})} \cdot \nabla s\rangle)/\partial s - \langle \mathbf{J}_0 \cdot \langle \mathbf{E}\rangle_{\mathrm{ens}}\rangle$ shown in Ref. 18, we also obtain

$$\frac{\partial}{\partial t} \left(V' \frac{3}{2} \sum_a n_{a0} T_{a0} \right) + \frac{\partial}{\partial s} \left(V' \left[\sum_a \left((q_a)^s + \frac{5}{2} T_{a0} (\Gamma_a)^s \right) - \frac{3}{2} \sum_a n_{a0} T_{a0} \langle \mathbf{u}_s \cdot \nabla s \rangle \right] \right)$$
$$= V' \langle \mathbf{J}_0 \cdot \langle \mathbf{E}\rangle_{\mathrm{ens}} \rangle + V' \sum_a \left\langle \int dU \int d\mu \int d\xi\; D_a \mathcal{S}_a \times \left(\frac{1}{2} m_a U^2 + \mu B_0 \right) \right\rangle + \mathcal{O}(\delta^3). \tag{102}$$

Equations (100) and (102) take the well-known forms of the energy balance equations[32] except that the terms associated with the electric field energy and the kinetic energies due to the fluid velocities are neglected here as small quantities of higher order in δ.

C. Ensemble-averaged toroidal angular momentum balance equation

The ensemble-averaged toroidal angular momentum balance equation is written as

$$\frac{\partial}{\partial t}\left(V'\left\langle\left[\sum_a n_{a0}m_a(u_{a\|}\mathbf{b}+\mathbf{u}_E)+\frac{\mathbf{S}^{(\mathrm{Poynting})}}{c^2}\right]\cdot\mathbf{e}_\zeta\right\rangle\right)$$
$$+\frac{\partial}{\partial s}\left(V'\left[\sum_a\left\{(\Pi_a^{\mathrm{NA}})^s+(\Pi_a^{\mathrm{A}})^s+(\Pi_a^{\mathrm{C}})^s\right.\right.\right.$$
$$\left.-\left\langle\left[\sum_a n_{a0}m_a(u_{a\|}\mathbf{b}+\mathbf{u}_E)+\frac{\mathbf{S}^{(\mathrm{Poynting})}}{c^2}\right]\cdot\mathbf{e}_\zeta(\mathbf{u}_s\cdot\nabla s)\right\rangle\right\}$$
$$\left.\left.-\frac{1}{4\pi}\left\langle\left\langle\nabla s\cdot\left[\hat{\mathbf{E}}_L\hat{\mathbf{E}}_L+\hat{\mathbf{B}}\hat{\mathbf{B}}+(\nabla\times\hat{\mathbf{B}})\hat{\mathbf{A}}\right]\cdot\mathbf{e}_\zeta\right\rangle\right\rangle\right]\right)$$
$$=\sum_a\left\langle\int dU\int d\mu\int d\xi\ D_a\mathcal{S}_a m_a U b_\zeta\right\rangle+\mathcal{O}(\delta^3),\qquad(103)$$

where $u_{a\|}$ represents the nonturbulent part of the parallel fluid velocity for particle species a defined by $n_{a0}u_{a\|}\equiv\int dU\int d\mu\int d\xi\langle F_{a1}\rangle_{\mathrm{ens}}U$ and $\mathbf{u}_E\equiv c\langle\mathbf{E}\rangle_{\mathrm{ens}}\times\mathbf{b}/B_0$ is the nonturbulent part of the $\mathbf{E}\times\mathbf{B}$ drift velocity. Equation (103) is derived from Eq. (77) following the same procedures as shown in Ref. 18 except that the additional transport flux Π_a^{C} defined in Eq. (107) and the external momentum source are newly included in the present case.

On the left-hand side of Eq. (103), the terms including $(u_{a\|}\mathbf{b}+\mathbf{u}_E)$ and $\mathbf{S}^{(\mathrm{Poynting})}$ are of $\mathcal{O}(\delta^3)$ although they are written down to explicitly show the inertia-term part. The nonturbulent and turbulence-driven parts of the radial flux of the toroidal angular momentum are defined by

$$(\Pi_a^{\mathrm{NA}})^s=\left\langle\int dU\int d\mu\int d\xi\ D_a\langle F_{a1}\rangle_{\mathrm{ens}}\right.$$
$$\left.\times m_a U b_\zeta\langle\mathbf{v}_a^{(\mathrm{gc})}\rangle_{\mathrm{ens}}\cdot\nabla s\right\rangle,\qquad(104)$$

and

$$(\Pi_a^{\mathrm{A}})^s=\left\langle\left\langle\int dU\int d\mu\int d\xi\ D_a\hat{h}_a\right.\right.$$
$$\left.\left.\times\, m_a(U\mathbf{b}+\mathbf{v}_{a0\perp})\cdot\mathbf{e}_\zeta(\hat{\mathbf{v}}_a^{(\mathrm{gc})}\cdot\nabla s)\right\rangle\right\rangle,\ (105)$$

respectively. It is shown in Appendix D that Eq. (78) is ensemble-averaged to give

$$\langle\langle(\Pi^{\mathrm{C}})^s\rangle\rangle=\sum_a\Pi_a^{\mathrm{C}}+\mathcal{O}(\delta^3),\qquad(106)$$

where the radial transport flux of the toroidal angular momentum for species a due to the collision term and finite gyroradii is defined by

$$\Pi_a^{\mathrm{C}}=-\sum_b\left\langle\frac{m_a c|\nabla s|^2}{2e_aB_0}\frac{\partial\chi}{\partial s}\int dU\int d\mu\int d\xi\ D_a\mu\right.$$
$$\left.\times\left[C_{ab}^p(\langle F_{a1}\rangle_{\mathrm{ens}},F_{bM})+C_{ab}^p(F_{aM},\langle F_{b1}\rangle_{\mathrm{ens}})\right]\right\rangle.\qquad(107)$$

The expressions for the toroidal momentum fluxes shown in Eqs. (104)–(107) agree with those given by conventional recursive formulations in Refs. 33–35. [Since the so-called high-flow ordering is used in Refs. 33 and 34, the expressions for the toroidal momentum fluxes in it reduce to those in the present work in the low-flow-speed limit.] As argued in Refs. 18 and 35, when there exists the up-down symmetry of the background magnetic field, all toroidal momentum fluxes vanish to $\mathcal{O}(\delta^2)$ and the nontrivial toroidal momentum balance equation is of $\mathcal{O}(\delta^3)$. In this case, gyrokinetic systems equations of higher-order accuracy in δ are required for the correct derivation of this $\mathcal{O}(\delta^3)$ toroidal momentum balance equation to determine the profile of the radial electric field[36] although we should note, at the same time, that the radial electric field is not necessary to determine the particle and energy transport fluxes to the lowest order in δ.[35]

VII. CONCLUSIONS

In this paper, particle, energy, and toroidal momentum balance equations including collisional and turbulent transport fluxes are systematically derived from the gyrokinetic Boltzmann-Poisson-Ampère system of equations. Considering an imaginary collisionless system, for which the distribution functions and electromagnetic fields coincide instantaneously with those for the considered collisional system, and expressing the action integral for the collisionless system in terms of the solution to the governing equations for the collisional system clarify effects of the collision and external source terms on the collisionless conservation laws derived from Noether's theorem. The gyrokinetic collision operator is newly presented, by which the collisional changes in the velocity-space integrals of the gyrocenter Hamiltonian and the canonical toroidal angular momentum can be written in the conservative (or divergence) forms. It is confirmed that, to the lowest order in the normalized gyroradius, the ensemble-averaged fluxes in the derived particle, energy, and toroidal angular momentum balance equations can be written by the sum of conventional expressions of classical, neoclassical, and turbulent transport fluxes. The extension of the present work to the case of the high-flow ordering remains as a future task.

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Appendix A: COORDINATE TRANSFORMATION

We consider the transformation of the phase-space coordinates in this Appendix, where the subscript representing the particle species is omitted as far as it is unnecessary. In terms of the position $\mathbf{x}$ and the velocity $\mathbf{v}$ of a given particle, we define the parallel velocity $v_\parallel = \mathbf{v}\cdot\mathbf{b}(\mathbf{x},t)$, the perpendicular velocity $\mathbf{v}_\perp = \mathbf{v} - v_\parallel \mathbf{b}$, and the zeroth-order magnetic moment,

$$\mu_0 = \frac{m v_\perp^2}{2B_0(\mathbf{x},t)} \tag{A1}$$

where the equilibrium field at position $\mathbf{x}$ and time t is denoted by $\mathbf{B}_0(\mathbf{x},t) = B_0(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)$. We also define the zeroth-order gyrophase by $\xi_0 = \tan^{-1}[(\mathbf{v}\cdot\mathbf{e}_1)/(\mathbf{v}\cdot\mathbf{e}_2)$ where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$ are unit vectors which form a right-handed orthogonal system at $(\mathbf{x},t)$. Then, the gyrocenter coordinates $\mathbf{Z} = (\mathbf{X}, U, \mu, \xi)$ are represented in terms of the particle coordinates $\mathbf{z} = (\mathbf{x}, v_\parallel, \mu_0, \xi_0)$ as

$$(\mathbf{X}, U, \mu, \xi) = (\mathbf{x}, v_\parallel, \mu_0, \xi_0) + (\Delta\mathbf{x}, \Delta v_\parallel, \Delta\mu_0, \Delta\xi_0), \tag{A2}$$

where

$$\begin{aligned}
\Delta\mathbf{x} &= -\boldsymbol{\rho} + \mathcal{O}(\delta^2) \\
\Delta v_\parallel &= -v_\parallel \mathbf{b}\cdot\nabla\mathbf{b}\cdot\boldsymbol{\rho} - \frac{1}{4}(3\boldsymbol{\rho}\cdot\nabla\mathbf{b}\cdot\mathbf{v}_\perp - \mathbf{v}_\perp\cdot\nabla\mathbf{b}\cdot\boldsymbol{\rho}) \\
&\quad + \frac{e}{mc}A_{1\parallel} + \mathcal{O}(\delta^2) \\
\Delta\mu_0 &= \frac{m}{B_0}\left[v_\parallel^2 \mathbf{b}\cdot\nabla\mathbf{b}\cdot\boldsymbol{\rho} + \frac{v_\parallel}{4}(3\boldsymbol{\rho}\cdot\nabla\mathbf{b}\cdot\mathbf{v}_\perp - \mathbf{v}_\perp\cdot\nabla\mathbf{b}\cdot\boldsymbol{\rho}) \right. \\
&\quad \left. + \frac{v_\perp^2}{2B_0}\boldsymbol{\rho}\cdot\nabla B_0\right] + \frac{e}{cB_0}\mathbf{v}_\perp\cdot\mathbf{A}_{1\perp} + \frac{e}{B_0}\widetilde{\psi} \\
&\quad + \mathcal{O}(\delta^2) \\
\Delta\xi_0 &= \frac{1}{\Omega B_0}(\mathbf{v}_\perp\cdot\nabla B_0) - \frac{\Omega v_\parallel}{4 v_\perp^2}\boldsymbol{\rho}\cdot\nabla\mathbf{b}\cdot\boldsymbol{\rho} \\
&\quad + \frac{v_\parallel}{4\Omega v_\perp^2}\mathbf{v}_\perp\cdot\nabla\mathbf{b}\cdot\mathbf{v}_\perp + \frac{v_\parallel^2}{\Omega v_\perp^2}(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{v}_\perp) \\
&\quad - \frac{\Omega}{v_\perp^2}\boldsymbol{\rho}\cdot\nabla\mathbf{v}_\perp\cdot\boldsymbol{\rho} + \frac{e}{cmv_\perp^2}\mathbf{b}\cdot(\mathbf{A}_1\times\mathbf{v}) \\
&\quad - \frac{e}{B_0}\left(\int\frac{\partial\widetilde{\psi}}{\partial\mu_0}d\xi_0\right) + \mathcal{O}(\delta^2).
\end{aligned} \tag{A3}$$

The formulas for $\Delta v_\parallel$ $\Delta\mu_0$, and $\Delta\xi_0$ in Eq. (A3) are obtained by combining the guiding center and gyrocenter coordinate transformations.[11,13,37] Here, effects of the background electric field and turbulent electromagnetic fields are included through $\widetilde{\psi}$ [see Eq. (80)] and $\mathbf{A}_1$. When the background electric field and turbulent electromagnetic fields vanish, $\Delta v_\parallel$ $\Delta\mu_0$, and $\Delta\xi_0$ in Eq. (A3) agree with the results in Ref. 37.

Denoting the coordinate transformation by $\mathcal{T}$, Eq. (A2) is rewritten as

$$\mathbf{Z} = \mathcal{T}(\mathbf{z}) = \mathbf{z} + \Delta\mathbf{z}. \tag{A4}$$

An arbitrary scalar field $\mathcal{A}$ on the phase space can be expressed in terms of either the gyrocenter coordinates $\mathbf{Z} = (\mathbf{X}, U, \mu, \xi)$ or the particle coordinates $\mathbf{z} = (\mathbf{x}, v_\parallel, \mu_0, \xi_0)$ as

$$\mathcal{A}^g(\mathbf{Z}) = \mathcal{A}^p(\mathbf{z}). \tag{A5}$$

Using Eqs. (A4), (A5), and the Taylor series expansion, we obtain

$$\begin{aligned}
\mathcal{A}^p(\mathbf{z}) &= (\mathcal{T}^*\mathcal{A}^g)(\mathbf{z}) \equiv \mathcal{A}^g(\mathcal{T}(\mathbf{z})) = \mathcal{A}^g(\mathbf{z}+\Delta\mathbf{z}) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1,\cdots,i_n}\Delta z^{i_1}\cdots\Delta z^{i_n}\frac{\partial^n\mathcal{A}^g(\mathbf{z})}{\partial z^{i_1}\cdots\partial z^{i_n}},
\end{aligned} \tag{A6}$$

where $\mathcal{T}^*\mathcal{A}^g$ denotes the pullback transformation of $\mathcal{A}^g$ by $\mathcal{T}$. Using the inverse transformation $\mathcal{T}^{-1}$, we also have $\mathcal{A}^g(\mathbf{Z}) = (\mathcal{T}^{-1*}\mathcal{A}^p)(\mathbf{Z}) \equiv \mathcal{A}^p(\mathcal{T}^{-1}(\mathbf{Z}))$.

The Jacobians D^p and D^g for the two coordinate systems $\mathbf{z}$ and $\mathbf{Z}$ are related to each other by

$$D^p(\mathbf{z}) = \det\left[\frac{\partial(\mathbf{Z})}{\partial(\mathbf{z})}\right] D^g(\mathbf{Z}), \tag{A7}$$

where $\partial(\mathbf{Z})/\partial(\mathbf{z})$ denotes the Jacobian matrix. Then, we use the following formula,

$$\delta^6(\mathbf{z}+\Delta\mathbf{z}-\mathbf{Z}) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1,\cdots,i_n}\Delta z_{i_1}\cdots\Delta z_{i_n}\frac{\partial^n\delta^6(\mathbf{z}-\mathbf{Z})}{\partial z_{i_1}\cdots\partial z_{i_n}}, \tag{A8}$$

and partial integrals to derive the relation between the expressions of the scalar density $\mathcal{D}\mathcal{A}$ in the gyrocenter and particle coordinate systems as

$$\begin{aligned}
&D^g(\mathbf{Z})\mathcal{A}^g(\mathbf{Z}) \\
&= \int d^6\mathbf{Z}'\ \delta^6(\mathbf{Z}'-\mathbf{Z})D^g(\mathbf{Z}')\mathcal{A}^g(\mathbf{Z}') \\
&= \int d^6\mathbf{z}\ \delta^6(\mathbf{z}+\Delta\mathbf{z}-\mathbf{Z})D^p(\mathbf{z})\mathcal{A}^p(\mathbf{z}) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{i_1,\cdots,i_n}\left[\frac{\partial^n\left[\Delta z^{i_1}\cdots\Delta z^{i_n}D^p(\mathbf{z})\mathcal{A}^p(\mathbf{z})\right]}{\partial z^{i_1}\cdots\partial z^{i_n}}\right]_{\mathbf{z}=\mathbf{Z}},
\end{aligned} \tag{A9}$$

where the replacement of $\mathbf{z}$ with $\mathbf{Z}$ is represented by $[\cdots]_{\mathbf{z}=\mathbf{Z}} \equiv \int d^6\mathbf{z}\ \delta^6(\mathbf{z}-\mathbf{Z})\cdots$.

Appendix B: COLLISION OPERATOR IN GYROCENTER COORDINATES

We can regard the collision term as a scalar field C on the phase space. When using the particle coordinates, we represent the collision term for collisions between species a and b by C^p_{ab}. A well-established collision operator $C^p_{ab}(f_a, f_b)$ for the particle distribution functions f_a and f_b is known as the Landau operator [see, for example, Eq. (3.22) in Ref. 32]. Then, the collision term C^g_{ab} represented in the gyrocenter coordinates is related to C^p_{ab} by

$$C^g_{ab}(F_a, F_b) = \mathcal{T}_a^{-1*} C^p_{ab}(\mathcal{T}_a^* F_a, \mathcal{T}_b^* F_b), \tag{B1}$$

where the distribution function for species a (b) in the particle coordinates is written as the pullback $f_a = \mathcal{T}_a^* F_a$ ($f_b = \mathcal{T}_b^* F_b$) of that in the gyrocenter coordinates F_a (F_b) by the coordinate transformation $\mathcal{T}_a$ ($\mathcal{T}_b$) described in Appendix A, and $\mathcal{T}_a^{-1*}$ transforms the collision term as a function of the particle coordinates into that of the gyrocenter coordinates.

In order to see collisional effects on conservation laws, it is convenient to represent the collision term in the gyrocenter coordinate using the transformation formula for the scalar density $D_a C_{ab}$ rather than that for the scalar C_{ab} shown in Eq. (B1). Using Eq. (A9), we can derive

$$\begin{aligned} & D^g_a(\mathbf{Z}_a) C^g_{ab}[F_a, F_b](\mathbf{Z}_a) \mathcal{A}^g_a(\mathbf{Z}_a) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1, \cdots, i_n} \\ &\times \left[\frac{\partial^n \left[\Delta z_a^{i_1} \cdots \Delta z_a^{i_n} D^p_a(\mathbf{z}_a) C^p_{ab}[f_a, f_b](\mathbf{z}_a) \mathcal{A}^p_a(\mathbf{z}_a) \right]}{\partial z_a^{i_1} \cdots \partial z_a^{i_n}} \right]_{\mathbf{z}_a = \mathbf{Z}_a}, \end{aligned} \tag{B2}$$

where $\mathcal{A}_a$ is an arbitrary scalar field depending on particle species and $f_a = \mathcal{T}_a^* F_a$ is rewritten by using Eq. (A6) as

$$f_a(\mathbf{z}_a) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, \cdots, i_n} \Delta z_a^{i_1} \cdots \Delta z_a^{i_n} \frac{\partial^n F_a(\mathbf{z}_a)}{\partial z_a^{i_1} \cdots \partial z_a^{i_n}}. \tag{B3}$$

Then, the gyrocenter representation of the collision operator C^g_{ab} acting on F_a and F_b is obtained by Eq. (B2) with putting $\mathcal{A}^g = \mathcal{A}^p = 1$ and using Eq. (B3) to express f_a and f_b in terms of F_a and F_b, respectively. Integrating Eq. (B2) with respect to (U, μ, ξ) and taking the summation over species b yield

$$\begin{aligned} & \int dU \int d\mu \int d\xi \; D^g_a(\mathbf{Z}) C^g_a(\mathbf{Z}) \mathcal{A}^g_a(\mathbf{Z}) \\ &= \left[\int d^3\mathbf{v} \; C^p_a(\mathbf{z}) \mathcal{A}^p_a(\mathbf{z}) \right]_{\mathbf{z} = \mathbf{Z}} - \nabla \cdot \mathbf{J}^{\mathrm{C}}_{Aa}, \end{aligned} \tag{B4}$$

where $C^g_a = \sum_b C^g_{ab}$ and $\nabla = \partial/\partial \mathbf{X}$ are used and $\int d^3\mathbf{v} = \int dv_\parallel \int d\mu_0 \int d\xi_0 \; D^p_a(\mathbf{z})$ denotes the velocity-space integral using the particle coordinates. Here, the transport flux $\mathbf{J}^{\mathrm{C}}_{Aa}$ of the quantity $\mathcal{A}_a$ due to collisions and finite gyroradii of particles is defined by

$$\begin{aligned} & \mathbf{J}^{\mathrm{C}}_{Aa}(\mathbf{X}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \sum_{i_1, \cdots, i_n} \frac{\partial^n}{\partial X^{i_1} \cdots \partial X^{i_n}} \\ &\quad \times \left[\int d^3\mathbf{v} \; \Delta\mathbf{x}_a \Delta x_a^{i_1} \cdots \Delta x_a^{i_n} C^p_a(\mathbf{z}) \mathcal{A}^p_a(\mathbf{z}) \right]_{\mathbf{x} = \mathbf{X}} \\ &= \left[\int d^3\mathbf{v} \; \Delta\mathbf{x}_a C^p_a(\mathbf{z}) \mathcal{A}^p_a(\mathbf{z}) \right]_{\mathbf{x} = \mathbf{X}} + \cdots . \end{aligned} \tag{B5}$$

The integral of an arbitrary scalar field $\mathcal{A}_a$ over the whole phase space is written in either the gyrocenter or particle coordinate system as

$$\int d^6\mathbf{Z} \; D^g_a(\mathbf{Z}) C^g_a(\mathbf{Z}) \mathcal{A}^g_a(\mathbf{Z}) = \int d^6\mathbf{z} \; D^p_a(\mathbf{z}) C^p_a(\mathbf{z}) \mathcal{A}^p_a(\mathbf{z}). \tag{B6}$$

For the case of $\mathcal{A}_a = 1$, Eqs. (B4) and (B5) reduce to

$$\int dU \int d\mu \int d\xi \; D^g_a(\mathbf{Z}) C^g_a(\mathbf{Z}) = -\nabla \cdot \mathbf{\Gamma}^{\mathrm{C}}_a(\mathbf{X}), \tag{B7}$$

and

$$\begin{aligned} & \mathbf{\Gamma}^{\mathrm{C}}_a(\mathbf{X}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \sum_{i_1, \cdots, i_n} \frac{\partial^n}{\partial X^{i_1} \cdots \partial X^{i_n}} \\ &\quad \times \left[\int d^3\mathbf{v} \; \Delta\mathbf{x}_a \Delta x_a^{i_1} \cdots \Delta x_a^{i_n} C^p_a(\mathbf{z}) \right]_{\mathbf{z} = \mathbf{Z}} \\ &= \left[\int d^3\mathbf{v} \; \Delta\mathbf{x}_a C^p_a(\mathbf{z}) \right]_{\mathbf{x} = \mathbf{X}} + \cdots , \end{aligned} \tag{B8}$$

respectively, where $\int d^3\mathbf{v} \; C^p_a(\mathbf{z}) = 0$ is used. Here, $\mathbf{\Gamma}^{\mathrm{C}}_a$ is regarded as the classical particle flux which occurs due to collisions and finite gyroradii. In fact, using $\Delta\mathbf{x}_a \simeq -\boldsymbol{\rho}_a$, we see that the primary term of $\mathbf{\Gamma}^{\mathrm{C}}_a$ shown in the last line of Eq. (B8) is identical to the conventional definition of the classical particle flux $\mathbf{\Gamma}^{\mathrm{cl}}_a \equiv (c/e_a B_0) \mathbf{F}_{a1} \times \mathbf{b}$, where $\mathbf{F}_{a1} \equiv \int d^3\mathbf{v} m_a \mathbf{v} C^p_a$ is the collisional friction force. Thus, we have $\mathbf{\Gamma}^{\mathrm{C}}_a = \mathbf{\Gamma}^{\mathrm{cl}}_a [1 + \mathcal{O}(\delta)]$.

Let us take the kinetic energy of the particle as $\mathcal{A}_a$ and put $\mathcal{A}^p_a = \frac{1}{2} m_a v_a^2 = \frac{1}{2} m_a v_{\parallel a}^2 + \mu_{0a} B_0(\mathbf{x}_a)$. Then, it is written in terms of the gyrocenter coordinates as

$$\begin{aligned} \mathcal{A}^g_a &= \mathcal{T}_a^{-1*} \left(\frac{1}{2} m_a v_a^2 \right) \\ &= \frac{1}{2} m_a U_a^2 + \mu_a B_0(\mathbf{X}_a) + \mu_a \boldsymbol{\rho}_a \cdot \nabla B_0(\mathbf{X}_a) \\ &\quad - m_a U_a (\Delta v_{\parallel a})_{\mathbf{z}_a = \mathbf{Z}_a} - (\Delta \mu_{0a})_{\mathbf{z}_a = \mathbf{Z}_a} B_0(\mathbf{X}_a) + \cdots \\ &= \frac{1}{2} m_a U_a^2 + \mu_a B_0(\mathbf{X}_a) + e_a \langle \psi_a(\mathbf{Z}_a) \rangle_{\xi_a} - e_a (\mathcal{T}_a^{-1*} \phi)(\mathbf{Z}_a) \\ &\quad + \mathcal{O}(\delta^2), \end{aligned} \tag{B9}$$

where the inverse $\mathcal{T}_a^{-1}$ of the transformation $\mathcal{T}_a$ given by Eq. (A3) is used. In this case, taking the summation of Eq. (B4) over species a and using the conservation property $\sum_a \int d^3\mathbf{v}\; C_a^p \frac{1}{2} m_a v^2 = 0$, we have

$$\sum_a \int dU \int d\mu \int d\xi\; D_a^g(\mathbf{Z}) C_a^g(\mathbf{Z}) T_a^{-1*} \left(\frac{1}{2} m_a v^2\right) = -\nabla\cdot\mathbf{Q}^{\mathrm{C}}, \tag{B10}$$

where $\mathbf{Q}^{\mathrm{C}}$ represents the transport flux of the total kinetic energy due to collisions and finite gyroradii defined by

$$\mathbf{Q}^{\mathrm{C}}(\mathbf{X}) = \sum_a \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \sum_{i_1,\cdots,i_n} \frac{\partial^n}{\partial X^{i_1}\cdots\partial X^{i_n}} \times \left[\int d^3\mathbf{v}\; \Delta\mathbf{x}_a \Delta x_a^{i_1}\cdots\Delta x_a^{i_n} C_a^p(\mathbf{z}) \frac{1}{2} m_a v^2\right]_{\mathbf{x}=\mathbf{X}} = \sum_a \left[\int d^3\mathbf{v}\; \Delta\mathbf{x}_a C_a^p(\mathbf{z}) \frac{1}{2} m_a v^2\right]_{\mathbf{x}=\mathbf{X}} + \cdots. \tag{B11}$$

To the lowest order in δ, the collisional energy flux $\mathbf{Q}^{\mathrm{C}}$ is approximately written as $\mathbf{Q}^{\mathrm{C}} \simeq \sum_a (\mathbf{q}_a^{\mathrm{cl}} + \frac{5}{2} T_a \mathbf{\Gamma}_a^{\mathrm{cl}})$. Here, the classical heat flux for species a is defined by $\mathbf{q}_a^{\mathrm{cl}} \equiv (cT_a/e_a B_0)\mathbf{F}_{a2} \times \mathbf{b}$, where $\mathbf{F}_{a2} \equiv \int d^3\mathbf{v}(m_a v^2/2T_a - 5/2) m_a \mathbf{v} C_a^p$ is the collisional heat friction. We note from Eq. (B9) that the expression of the kinetic energy in the gyrocenter coordinates should be generally given by the infinite series expansion in δ in order for the gyrocenter velocity-space integral of the collisional rate of change in the kinetic energy to take the form of the divergence of the energy flux without any local source or sink terms. In fact, this energy conservation property is broken if we keep only the lowest order terms in Eq. (B9) and evaluate the gyrocenter velocity-space integral $\sum_a \int dU \int d\mu \int d\xi\; D_a^g(\mathbf{Z}) C_a^g(\mathbf{Z}) \left(\frac{1}{2} m_a U^2 + \mu B_0(\mathbf{X})\right)$.

The above-mentioned subtle relation between expressions of the collisional energy conservation properties in the particle and gyrocenter coordinate systems is also found when considering the collisional momentum conservation. It should be recalled that the perturbative expansions in δ are truncated up to finite orders in deriving gyrokinetic equations as shown in Sec. III although the conservative form of equations for the energy and the toroidal angular momentum are obtained even from these approximate equations for the collisionless case since they are constructed based on the variational principle. Thus, from the viewpoint of practical applications, it is desirable for the approximate collision operator in the gyrocenter coordinates to keep the conservation properties. More rigorously speaking, we want the gyrokinetic collisional velocity-space integrals $\sum_a \int dU \int d\mu \int d\xi\; D_a^g(\mathbf{Z}) C_a^g(\mathbf{Z}) H_a(\mathbf{Z})$ and $\sum_a \int dU \int d\mu \int d\xi\; D_a^g(\mathbf{Z}) C_a^g(\mathbf{Z}) (p_\zeta^c)_a^g(\mathbf{Z})$ to take the divergence forms and include no local source or sink terms where $H_a(\mathbf{Z})$ and $(p_\zeta^c)_a^g(\mathbf{Z}) \equiv (e_a/c) A_{a\zeta}^*(\mathbf{Z})$ are the gyrocenter Hamiltonian and the canonical toroidal angular momentum defined by Eqs. (26) and (72), respectively. Here, it should be noted that not only kinetic parts of energy and toroidal momentum but also contributions from scalar and vector potentials are included in $H_a(\mathbf{Z})$ and $(e_a/c) A_{a\zeta}^*(\mathbf{Z})$. In Appendix C, we find how to construct the approximate gyrokinetic collision operator, by which the two integrals mentioned above are written in the divergence forms.

We now consider the entropy per unit volume defined in terms of the gyrocenter distribution functions as $S^g \equiv -\sum_a \int dU \int d\mu \int d\xi D_a^g(\mathbf{Z}) \log F_a(\mathbf{Z})$, in which the rate of change is given by $dS^g/dt = -\sum_a \int dU \int d\mu \int d\xi\;\; D_a^g(\mathbf{Z})\;\; [\log F_a(\mathbf{Z}) + 1]\;\; (dF_a/dt)$. Then, the rate of change in S^g due to collisions is obtained by putting $\mathcal{A}_a^g = -[\log F_a(\mathbf{Z}) + 1]$ in Eq. (B4) and taking the summation over species a as

$$-\sum_a \int dU \int d\mu \int d\xi\; D_a^g(\mathbf{Z}) C_a^g(\mathbf{Z}) [\log F_a(\mathbf{Z}) + 1] = -\sum_a \left[\int d^3\mathbf{v}\; C_a^p(\mathbf{z}) \log f_a(\mathbf{z})\right]_{\mathbf{x}=\mathbf{X}} - \nabla\cdot\mathbf{J}_S^{\mathrm{C}}, \tag{B12}$$

where $\int d^3\mathbf{v}\; C_a^p(\mathbf{z}) = 0$ is used although we should recall that $\int dU \int d\mu \int d\xi D_a^g(\mathbf{Z}) C_a^g(\mathbf{Z})$ does not vanish generally as seen from Eq. (B7). It is well-known that, when Landau's collision operator is used for C_a^p, the collisional entropy production rate given by the first term on the right-hand side of Eq. (B12) is nonnegative. This is Boltzmann's H-theorem which proves the second law of thermodynamics. The collisional transport flux $\mathbf{J}_S^{\mathrm{C}}$ of the entropy in Eq. (B12) is defined by

$$\mathbf{J}_S^{\mathrm{C}}(\mathbf{X}) = \sum_a \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} \sum_{i_1,\cdots,i_n} \frac{\partial^n}{\partial X^{i_1}\cdots\partial X^{i_n}} \times \left[\int d^3\mathbf{v}\; \Delta\mathbf{x}_a \Delta x_a^{i_1}\cdots\Delta x_a^{i_n} C_a^p(\mathbf{z}) [\log f_a(\mathbf{z}) + 1]\right]_{\mathbf{x}=\mathbf{X}} = -\sum_a \left[\int d^3\mathbf{v}\; \Delta\mathbf{x}_a C_a^p(\mathbf{z}) [\log f_a(\mathbf{z}) + 1]\right]_{\mathbf{x}=\mathbf{X}} + \cdots. \tag{B13}$$

It is shown that, to the lowest order in δ, the collisional entropy transport flux is written as $\mathbf{J}_S^{\mathrm{C}} = \sum_a (S_{a0}\mathbf{u}_a^{\mathrm{cl}} + \mathbf{q}_a^{\mathrm{cl}}/T_a)$ where the lowest-order entropy density S_{a0} for species a is given in terms of the local Maxwellian distribution function F_{aM} as $S_{a0} \equiv -\int dU \int d\mu \int d\xi\;\; F_{aM} \log F_{aM}$, and $\mathbf{u}_a^{\mathrm{cl}}$ is defined by $\mathbf{u}_a^{\mathrm{cl}} \equiv \mathbf{\Gamma}_a^{\mathrm{cl}}/n_a$. Here, we note again that the infinite series expansion in δ as given by Eq. (B2) is used in deriving Eq. (B12). When the expansion is truncated to finite order, the collisional entropy production term is represented by $-\sum_a \int d^3\mathbf{v}\; C_a^p \log f_a$ plus residual error terms of higher order in δ, and thus the H-theorem is only approximately satisfied.

Appendix C: COLLISION OPERATOR RELEVANT FOR GYROKINETIC CONSERVATION LAWS

In this Appendix, we consider an approximate gyrokinetic collision operator instead of the one given by Eq. (B1) [or Eq. (B2) with $\mathcal{A}^g = \mathcal{A}^p = 1$] in order to get the gyrokinetic collisional velocity-space integrals of energy and canonical toroidal momentum to take desirable conservative (or divergence) forms. The approximate collision operator is written in the gyrocenter coordinates as

$$
\begin{aligned}
&D_a^g(\mathbf{Z}_a)\langle C_{ab}^g[F_a,F_b](\mathbf{Z}_a)\rangle_{\xi_a} \\
&= \left\langle \left[\sum_{n=0}^{\infty} \frac{1}{n!} \right.\right. \\
&\quad \times \sum_{i_1,\cdots,i_n} \frac{\partial^n \left[\rho_a^{i_1}\cdots\rho_a^{i_n} D_a^p(\mathbf{z}_a) C_{ab}^p[f_a,f_b](\mathbf{z}_a)\right]}{\partial x_a^{i_1}\cdots\partial x_a^{i_n}} \\
&\quad - \frac{\partial}{\partial \mathbf{x}_a}\cdot\left[\Delta\mathbf{x}_a^{(2)} D_a^p(\mathbf{z}_a) C_{ab}^p[f_a,f_b](\mathbf{z}_a)\right] \\
&\quad - \frac{\partial}{\partial v_{\|a}}\left[\Delta v_{\|a} D_a^p(\mathbf{z}_a) C_{ab}^p[f_a,f_b](\mathbf{z}_a)\right] \\
&\quad \left.\left. - \frac{\partial}{\partial \mu_{0a}}\left[\Delta\mu_{0a} D_a^p(\mathbf{z}_a) C_{ab}^p[f_a,f_b](\mathbf{z}_a)\right]\right]_{\mathbf{z}_a=\mathbf{Z}_a}\right\rangle_{\xi_a},
\end{aligned}
\tag{C1}
$$

where $\Delta v_{\|a}$ and $\Delta\mu_{0a}$ are written as

$$
\begin{aligned}
&\Delta\mathbf{x}_a = -\boldsymbol{\rho}_a + \Delta\mathbf{x}_a^{(2)}, \qquad \Delta v_{\|a} = \Delta v_{\|a}^{(1)} + \Delta v_{\|a}^{(2)}, \\
&\Delta\mu_{0a} = \Delta\mu_{0a}^{(1)} + \Delta\mu_{0a}^{(2)},
\end{aligned}
\tag{C2}
$$

and f_a is given from F_a by $f_a(\mathbf{z}_a) = F_a(\mathbf{x}_a + \Delta\mathbf{x}_a, v_{\|0a} + \Delta v_{\|a}, \mu_{0a} + \Delta\mu_{0a})$. Here, $\Delta v_{\|a}^{(1)}$, and $\Delta\mu_{0a}^{(1)}$ are the $\mathcal{O}(\delta)$ parts of $\Delta v_{\|a}$ and $\Delta\mu_{0a}$ given in Eq. (A3). In this Appendix, we do not derive expressions for the $\mathcal{O}(\delta^2)$ parts $\Delta\mathbf{x}_a^{(2)}$, $\Delta v_{\|a}^{(2)}$, and $\Delta\mu_{0a}^{(2)}$ by the Lie perturbation expansion method which is used to define the gyrocenter coordinates with the well-conserved magnetic moment because it would unnecessarily give higher-order accuracy to the coordinate transformation than the accuracy of the gyrocenter motion equations themselves shown in Eqs. (28)–(31). Instead, we determine these $\mathcal{O}(\delta^2)$ terms from the conditions that the collisional change rates of energy and canonical toroidal angular momentum per unit volume in the gyrocenter space can be given in the conservative forms as shown below. Thus, the $\mathcal{O}(\delta^2)$ terms are introduced not for accuracy of higher order in δ but for satisfying the conservation property of the collision operator.

In Eq. (C1), the expansions in $(\Delta\mathbf{x}_a^{(2)}, \Delta v_{\|a}, \Delta\mu_{0a})$ are truncated to the first order while the infinite series expansion in $\Delta\mathbf{x}_a^{(1)} \equiv -\boldsymbol{\rho}_a$ is retained because fluctuations' wavelengths in the directions perpendicular to the equilibrium magnetic field can be of order of the gyroradius ρ_a. In the WKB (or ballooning) representation, the above-mentioned infinite series expansion can be treated using the Bessel functions of the gyroradius normalized by the perpendicular wavelength.[6,8,30] We should also note that the gyrophase average $\langle\cdots\rangle_{\xi_a}$ is taken so that the gyrokinetic equation with the collision term is solved only for the gyrophase-averaged part of the gyrocenter distribution function.

For an arbitrary function $\mathcal{A}_a^g(\mathbf{Z}_a)$ which is independent of the gyrophase ξ_a, we obtain the following formula,

$$
\begin{aligned}
&D_a^g(\mathbf{Z}_a)\langle C_{ab}^g[F_a,F_b](\mathbf{Z}_a)\rangle_{\xi_a}\mathcal{A}_a^g(\mathbf{Z}_a) \\
&= \left\langle \left[D_a^p(\mathbf{z}_a) C_{ab}^p[f_a,f_b](\mathbf{z}_a)\mathcal{A}_a^p(\mathbf{z}_a) + \sum_{n=1}^{\infty} \frac{1}{n!} \right.\right. \\
&\quad \times \sum_{i_1,\cdots,i_n} \frac{\partial^n \left[\rho_a^{i_1}\cdots\rho_a^{i_n} D_a^p(\mathbf{z}_a) C_{ab}^p[f_a,f_b](\mathbf{z}_a)\mathcal{A}_a^{p0}(\mathbf{z}_a)\right]}{\partial x_a^{i_1}\cdots\partial x_a^{i_n}} \\
&\quad - \frac{\partial}{\partial \mathbf{x}_a}\cdot\left[\Delta\mathbf{x}_a^{(2)} D_a^p(\mathbf{z}_a) C_{ab}^p[f_a,f_b](\mathbf{z}_a)\mathcal{A}_a^g(\mathbf{z}_a)\right] \\
&\quad - \frac{\partial}{\partial v_{\|a}}\left[\Delta v_{\|a} D_a^p(\mathbf{z}_a) C_{ab}^p[f_a,f_b](\mathbf{z}_a)\mathcal{A}_a^g(\mathbf{z}_a)\right] \\
&\quad \left.\left. - \frac{\partial}{\partial \mu_{0a}}\left[\Delta\mu_{0a} D_a^p(\mathbf{z}_a) C_{ab}^p[f_a,f_b](\mathbf{z}_a)\mathcal{A}_a^g(\mathbf{z}_a)\right]\right]_{\mathbf{z}_a=\mathbf{Z}_a}\right\rangle_{\xi_a},
\end{aligned}
\tag{C3}
$$

where $\mathcal{A}_a^p(\mathbf{z}_a)$ and $\mathcal{A}_a^{p0}(\mathbf{z}_a)$ are defined by

$$
\begin{aligned}
\mathcal{A}_a^p(\mathbf{z}_a) &= \mathcal{A}_a^{p0}(\mathbf{z}_a) + \left(\Delta\mathbf{x}_a^{(2)}\cdot\frac{\partial}{\partial\mathbf{x}_a} + \Delta v_{\|a}\frac{\partial}{\partial v_{\|a}}\right. \\
&\qquad \left. + \Delta\mu_{0a}\frac{\partial}{\partial\mu_{0a}}\right)\mathcal{A}_a^g(\mathbf{z}_a), \\
\mathcal{A}_a^{p0}(\mathbf{z}_a) &= \mathcal{A}_a^g(\mathbf{x}_a - \boldsymbol{\rho}_a, v_{\|a}, \mu_{0a}).
\end{aligned}
\tag{C4}
$$

We should note that the function $\mathcal{A}^p(\mathbf{z}_a)$ defined from $\mathcal{A}_a^g(\mathbf{Z}_a)$ in Eq. (C4) does not exactly coincide with that given in Eq. (A6) in Appendix A by the second- and higher-order terms in the series expansion with respect to $\Delta\mathbf{z}_a$. Integrating Eq. (C3) over the gyrocenter velocity space, we immediately obtain

$$
\begin{aligned}
&\int dU \int d\mu \int d\xi\; D_a^g(\mathbf{Z}) C_a^g(\mathbf{Z})\mathcal{A}_a^g(\mathbf{Z}) \\
&= \left[\int d^3\mathbf{v}\; C_a^p(\mathbf{z})\mathcal{A}_a^p(\mathbf{z})\right]_{\mathbf{x}=\mathbf{X}} - \nabla\cdot\mathbf{J}_{Aa}^{\mathrm{C}},
\end{aligned}
\tag{C5}
$$

where the transport flux $\mathbf{J}_{Aa}^{\mathrm{C}}$ due to collisions and finite gyroradii is defined by

$$
\begin{aligned}
&\mathbf{J}_{Aa}^{\mathrm{C}}(\mathbf{X}) \\
&= \left[\int d^3\mathbf{v}\; [-\boldsymbol{\rho}_a + \Delta\mathbf{x}_a^{(2)}] C_a^p(\mathbf{z})\mathcal{A}_a^p(\mathbf{z})\right]_{\mathbf{x}=\mathbf{X}} \\
&\quad + \sum_{n=1}^{\infty} \frac{-1}{(n+1)!} \sum_{i_1,\cdots,i_n} \frac{\partial^n}{\partial X^{i_1}\cdots\partial X^{i_n}} \\
&\quad \times \left[\int d^3\mathbf{v}\; \boldsymbol{\rho}_a \rho_a^{i_1}\cdots\rho_a^{i_n} C_a^p(\mathbf{z})\mathcal{A}_a^p(\mathbf{z})\right]_{\mathbf{x}=\mathbf{X}}.
\end{aligned}
\tag{C6}
$$

To the lowest order in δ, Eqs. (C3), (C5), and (C6) derived from the approximate collision operator in Eq. (C1) agree with Eqs. (B2), (B4), and (B5) given in Appendix B, respectively. The particle flux $\mathbf{\Gamma}_a^{\rm C}$ due to collisions and finite gyroradii is given from Eq. (C6) with putting $\mathcal{A}_a^p(\mathbf{z}) = 1$ in the same way as in Eqs. (B8).

Now, let us take $\mathcal{A}_a^g(\mathbf{Z}) = H_a(\mathbf{Z})$ in Eq. (C5). Here, $H_a(\mathbf{Z})$ denotes the gyrocenter Hamiltonian defined by Eq. (26). It is desirable that the gyrocenter velocity-space integral $\sum_a \int dU \int d\mu \int d\xi \; D_a^g(\mathbf{Z}) C_a^g(\mathbf{Z}) H_a(\mathbf{Z})$ takes the conservative form, which implies that the integral is expressed by the divergence term only and

$$\sum_a \int d^3\mathbf{v}\; C_a^p(\mathbf{z}) H_a^p(\mathbf{z}) = 0 \tag{C7}$$

holds. Here, using Eqs. (26) and (C4), $H_a^p(\mathbf{z})$ is given by

$$H_a^p(\mathbf{z}_a) = \frac{1}{2} m_a v_{\parallel a}^2 + \mu B_0(\mathbf{x}_a) + e_a \phi(\mathbf{x}_a) + \Delta H_a(\mathbf{z}_a), \tag{C8}$$

where

$$\begin{aligned}\Delta H_a(\mathbf{z}_a) \equiv & \left(\Delta \mathbf{x}_a^{(2)} \cdot \frac{\partial}{\partial \mathbf{x}_a} + \Delta v_{\parallel a} \frac{\partial}{\partial v_{\parallel a}} + \Delta \mu_{0a} \frac{\partial}{\partial \mu_{0a}} \right) H_a(\mathbf{z}_a) \\ & + \mu_{0a} \left[B_0(\mathbf{x}_a - \boldsymbol{\rho}_a) - B_0(\mathbf{x}_a) \right] \\ & + e_a \left[\Psi(\mathbf{x}_a - \boldsymbol{\rho}_a, v_{\parallel a}, \mu_{0a}) - \phi(\mathbf{x}_a) \right]. \end{aligned} \tag{C9}$$

It is easily seen that Eq. (C7) is satisfied if $\Delta H_a(\mathbf{z}_a) = 0$. Then, substituting Eq. (C2) into Eq. (C9) and using $\Delta H_a(\mathbf{z}_a) = 0$, we have

$$\begin{aligned} & \left(\Delta \mathbf{x}_a^{(2)} \cdot \frac{\partial}{\partial \mathbf{x}_a} + \Delta v_{\parallel a}^{(2)} \frac{\partial}{\partial v_{\parallel a}} + \Delta \mu_{0a}^{(2)} \frac{\partial}{\partial \mu_{0a}} \right) H_a(\mathbf{z}_a) \\ & = -\mu_{0a} \left[B_0(\mathbf{x}_a - \boldsymbol{\rho}_a) - B_0(\mathbf{x}_a) + (\boldsymbol{\rho}_a \cdot \nabla B_0) \left(1 + \frac{e_a}{B_0} \right. \right. \\ & \quad \left. \left. \times \frac{\partial \Psi}{\partial \mu_{0a}} \right) \right] - \frac{e_a^2}{B_0} \left[\phi(\mathbf{x}_a) - \langle \psi(\mathbf{X}_a + \boldsymbol{\rho}_a) \rangle_{\xi_a} \right] \frac{\partial \Psi}{\partial \mu_{0a}} \\ & \quad - \frac{e_a^2}{2 m_a c^2} \left\langle |\mathbf{A}_1(\mathbf{X}_a + \boldsymbol{\rho}_a)|^2 \right\rangle_{\xi_a} + \frac{e_a^2}{2B_0} \frac{\partial}{\partial \mu} \langle [\widetilde{\psi}_a(\mathbf{X}_a + \boldsymbol{\rho}_a)]^2 \rangle_{\xi_a} \\ & \quad + e_a \left[v_{\parallel a} \mathbf{b} \cdot \nabla \mathbf{b} \cdot \boldsymbol{\rho}_a + \frac{1}{4} (3 \boldsymbol{\rho}_a \cdot \nabla \mathbf{b} \cdot \mathbf{v}_\perp - \mathbf{v}_\perp \nabla \mathbf{b} \cdot \boldsymbol{\rho}_a) \right. \\ & \quad \left. - \frac{e_a}{m_a c} A_{1\parallel} \right] \left(\frac{\partial}{\partial v_{\parallel a}} - \frac{m_a v_{\parallel a}}{B_0} \frac{\partial}{\partial \mu_{0a}} \right) \Psi. \end{aligned} \tag{C10}$$

We find that the right-hand side of Eq. (C10) is of $\mathcal{O}(\delta^2)$. Then, as remarked after Eq. (C16), we can choose $\Delta \mathbf{x}_a^{(2)}$, $\Delta v_{\parallel a}^{(2)}$, and $\Delta \mu_{0a}^{(2)}$ which satisfy Eq. (C10) and are of $\mathcal{O}(\delta^2)$ so as to be consistent with Eq. (A3).

When we use $\Delta \mathbf{x}^{(2)} = 0$, $\Delta v_{\parallel a} = \Delta v_{\parallel a}^{(1)}$, and $\Delta \mu_{0a} = \Delta \mu_{0a}^{(1)}$ for Eq. (C1) by putting $\Delta v_{\parallel a}^{(2)} = \Delta \mu_{0a}^{(2)} = 0$, we have $\Delta H_a(\mathbf{z}_a) = \mathcal{O}(\delta^2)$ and $\sum_a \int d^3\mathbf{v}\; C_a^p(\mathbf{z}) H_a^p(\mathbf{z}) = \mathcal{O}(\delta^3)$ because $C_a^p(\mathbf{z}) = \mathcal{O}(\delta)$ holds for the distribution function, the zeroth order of which is given by the local Maxwellian. Therefore, even for this case where $\sum_a \int dU \int d\mu \int d\xi \; D_a^g(\mathbf{Z}) C_a^g(\mathbf{Z}) H_a(\mathbf{Z})$ is not completely given in the conservative form, the residual term $\sum_a \int d^3\mathbf{v}\; C_a^p(\mathbf{z}) H_a^p(\mathbf{z}) = \mathcal{O}(\delta^3)$ is smaller by a factor of δ than other transport terms of $\mathcal{O}(\delta^2)$ in the lowest-order energy balance equation given by Eq. (100) in Sec. VI.B.

We next put $\mathcal{A}_a^g(\mathbf{Z}) = (p_\zeta^c)_a^g(\mathbf{Z})$ in Eq. (C5). Here, $(p_\zeta^c)_a^g(\mathbf{Z})$ denotes the canonical toroidal angular momentum defined by

$$(p_\zeta^c)_a^g(\mathbf{Z}) \equiv \frac{e_a}{c} A_\zeta^*(\mathbf{Z}) \equiv \frac{e_a}{c} A_{0\zeta}(\mathbf{X}) + m_a U b_\zeta(\mathbf{X}), \tag{C11}$$

where $A_{0\zeta} = -\chi$ and $b_\zeta = I/B_0$. We now see that $\sum_a \int dU \int d\mu \int d\xi \; D_a^g(\mathbf{Z}) C_a^g(\mathbf{Z}) (p_\zeta^c)_a^g(\mathbf{Z})$ takes the conservative form if

$$\sum_a \int d^3\mathbf{v}\; C_a^p(\mathbf{z}) (p_\zeta^c)_a^p(\mathbf{z}) = 0. \tag{C12}$$

Here, using Eqs. (C4) and (C11), $(p_\zeta^c)_a^p(\mathbf{z})$ is given by

$$(p_\zeta^c)_a^p(\mathbf{z}) = \frac{e_a}{c} [A_{0\zeta}(\mathbf{x}) + A_{1\zeta}(\mathbf{x})] + m_a v_\zeta + \Delta(p_\zeta^c)_a(\mathbf{z}), \tag{C13}$$

where

$$\begin{aligned}\Delta(p_\zeta^c)_a(\mathbf{z}_a) = & \left(\Delta \mathbf{x}_a^{(2)} \frac{\partial}{\partial \mathbf{x}_a} + \Delta v_{\parallel a} \frac{\partial}{\partial v_{\parallel a}} \right) (p_\zeta^c)_a^g(\mathbf{x}_a, v_{\parallel a}) \\ & + (p_\zeta^c)_a^g(\mathbf{x}_a - \boldsymbol{\rho}_a, v_{\parallel a}) \\ & - \frac{e_a}{c} [A_{0\zeta}(\mathbf{x}_a) + A_{1\zeta}(\mathbf{x}_a)] - m_a v_{\zeta a}. \end{aligned} \tag{C14}$$

Again, we easily see that Eq. (C12) is satisfied if $\Delta(p_\zeta^c)_a(\mathbf{z}_a) = 0$. The $\mathcal{O}(\delta^2)$ variables, $\Delta \mathbf{x}_a^{(2)}$ and $\Delta v_{\parallel a}^{(2)}$, which meet the condition that $\Delta(p_\zeta^c)_a(\mathbf{z}_a) = 0$, are given by

$$\begin{aligned} & \Delta \mathbf{x}_a^{(2)} \cdot \nabla (p_\zeta^c)_a^g(\mathbf{z}_a) \\ & \quad = -\frac{e_a}{c} (\boldsymbol{\rho}_a \cdot \nabla \chi) \left[\frac{v_{\parallel a}}{\Omega_a} \mathbf{b} \cdot (\nabla \times \mathbf{b}) - \frac{1}{2B_0} \boldsymbol{\rho}_a \cdot \nabla B_0 \right] \\ & \qquad - \frac{m_a c}{e_a} \mu_{0a} W_\zeta, \end{aligned} \tag{C15}$$

and

$$\begin{aligned} & \Delta v_{\parallel a}^{(2)} \frac{\partial}{\partial v_{\parallel a}} (p_\zeta^c)_a^g(\mathbf{z}_a) = m_a b_\zeta(\mathbf{x}_a) \Delta v_{\parallel a}^{(2)} \\ & \quad = -m_a v_{\parallel a} \left[b_\zeta(\mathbf{x}_a - \boldsymbol{\rho}_a) - b_\zeta(\mathbf{x}_a) + \boldsymbol{\rho}_a \cdot \nabla b_\zeta \right] \\ & \quad + \frac{e_a}{c} \left[\chi(\mathbf{x}_a - \boldsymbol{\rho}_a) - \chi(\mathbf{x}_a) + \boldsymbol{\rho}_a \cdot \nabla \chi - \frac{1}{2} \boldsymbol{\rho}_a \boldsymbol{\rho}_a : \nabla \nabla \chi \right], \end{aligned} \tag{C16}$$

where $\nabla \equiv \partial / \partial \mathbf{x}_a$ and $W_\zeta \equiv -(\nabla R \cdot \nabla \chi)/(R B_0) + \frac{1}{2} b_\zeta \mathbf{b} \cdot (\nabla \times \mathbf{b})$. As a solution to Eq. (C15), we can assume $\Delta \mathbf{x}_a^{(2)}$ to be given in the form $\Delta \mathbf{x}_a^{(2)} = \Delta x_{a\chi}^{(2)} \nabla \chi$. We should note that $\nabla \chi \cdot \nabla (p_\zeta^c)_a^g = \mathcal{O}(\delta^{-1})$ and $\partial (p_\zeta^c)_a^g / \partial v_{\parallel a} = m_a b_\zeta = \mathcal{O}(\delta^0)$ while the right-hand sides of Eqs. (C15) and (C16) are of $\mathcal{O}(\delta)$ and $\mathcal{O}(\delta^2)$, respectively. Therefore, Eqs. (C15) and (C16) give $\Delta \mathbf{x}_a^{(2)}$ and $\Delta v_{\parallel a}^{(2)}$, which are both of $\mathcal{O}(\delta^2)$, consistently with Eq. (A3). Then, these $\Delta\mathbf{x}_a^{(2)}$ and $\Delta v_{\|a}^{(2)}$ are substituted into Eq. (C10) to determine $\Delta\mu_{0a}^{(2)}$ of $\mathcal{O}(\delta^2)$ as well.

Thus, the collision operator, which has the desired conservation properties as well as the accuracy required for correct description of collisional transport of the energy and the toroidal angular momentum, is given by Eq. (C1), in which $\Delta\mathbf{x}_a^{(2)}$ $\Delta v_{\|a}$, and $\Delta\mu_{0a}$ are defined by Eqs. (C2), (C10), (C15), and (C16). Using this collision operator, putting $\mathcal{A}^p(\mathbf{z}) = \frac{1}{2}m_a v_\|^2 + \mu_0 B_0(\mathbf{x}) + e_a\phi(\mathbf{x})$ and $\mathcal{A}_a^p(\mathbf{z}) = (e_a/c)[A_{0\zeta}(\mathbf{x}) + A_{1\zeta}(\mathbf{x})] + m_a v_\zeta$ in Eq. (C6) and taking their summation over species a define the transport fluxes $\mathbf{Q}^{\mathrm{C}}$ and $\mathbf{J}_{p\zeta}^{\mathrm{C}}$ of the energy and the canonical toroidal angular momentum, respectively, which appear in the energy and toroidal angular momentum balance equations in Secs. V.A and B [see Eqs. (68), (69), (77), and (78)]. In the definition of $\mathbf{Q}^{\mathrm{C}}$ mentioned above, the contribution of the potential energy part $e_a\phi$ is written as $\phi\sum_a e_a\mathbf{\Gamma}_a^{\mathrm{C}}$ which is smaller than the contribution of the kinetic energy part by a factor of δ because the classical particle fluxes represented by the lowest-order part of $\mathbf{\Gamma}_a^{\mathrm{C}}$ are intrinsically ambipolar. Therefore, the energy flux $\mathbf{Q}^{\mathrm{C}}$ defined here agrees with Eq. (B11) to the lowest order in δ. Regarding the entropy production discussed in Appendix B, the positive definiteness of the entropy production rate [corresponding to the first term on the left-hand side of Eq. (B12)] is only approximately shown by using the present model collision operator in Eq. (C5) with $\mathcal{A}_a^g(\mathbf{Z}) = -[\log F_a(\mathbf{Z}) + 1]$ because $\mathcal{A}_a^p(\mathbf{z}) = -[\log f_a(\mathbf{z}) + 1]$ is not rigorously derived from Eq. (C4) without the infinite series expansion in $\Delta\mathbf{z}_a$ as in Eq. (A6).

Appendix D: Derivation of Eqs. (106) and (107)

In this Appendix, it is shown how to derive Eqs. (106) and (107) by using the collision operator given in Appendix C. On the right-hand side of Eq. (78) where the radial flux $(\Pi^{\mathrm{C}})^s$ of the toroidal angular momentum due to collisions and finite gyroradii is defined, the two types of fluxes $\mathbf{j}^{\mathrm{C}} = \sum_a e_a\mathbf{\Gamma}_a^{\mathrm{C}}$ and $\mathbf{J}_{p\zeta}^{\mathrm{C}} = \sum_a \mathbf{J}_{p\zeta a}^{\mathrm{C}}$ are evaluated by taking the summation of Eq. (C6) over species a with putting $\mathcal{A}_a = e_a$ and $\mathcal{A}_a = (p_\zeta^c)_a^p(\mathbf{z})$, respectively. Here, $(p_\zeta^c)_a^p(\mathbf{z}) = (e_a/c)[A_{0\zeta}(\mathbf{x}) + A_{1\zeta}(\mathbf{x})] + m_a v_\zeta$ is used for the collision operator which conserves the toroidal angular momentum as explained in Appendix C. Consequently, the ensemble average of $(\Pi^{\mathrm{C}})^s$ is expressed explicitly up to $\mathcal{O}(\delta^2)$ as

$$
\begin{aligned}
\langle(\Pi^{\mathrm{C}})^s\rangle_{\mathrm{ens}} &= \left\langle\left(\mathbf{J}_{p\zeta}^{\mathrm{C}} + \frac{\chi}{c}\mathbf{j}_L^{\mathrm{C}}\right)\cdot\nabla s\right\rangle_{\mathrm{ens}} \\
&= \sum_a \int d^3\mathbf{v}\langle C_a^p(\mathbf{z})\rangle_{\mathrm{ens}}\left[\frac{e_a}{2c}\boldsymbol{\rho}_a\boldsymbol{\rho}_a : \nabla\chi\nabla s\right. \\
&\quad \left. - m_a\boldsymbol{\rho}_a\mathbf{v}_\perp : \mathbf{e}_\zeta\nabla s\right] + \mathcal{O}(\delta^3) \\
&= -\sum_{a,b}\frac{m_a c|\nabla s|^2}{2e_a B_0}\frac{\partial\chi}{\partial s}\int dU\int d\mu\int d\xi\, D_a\mu \\
&\quad \times\left[C_{ab}^p(\langle F_{a1}\rangle_{\mathrm{ens}}, F_{bM}) + C_{ab}^p(F_{aM}, \langle F_{b1}\rangle_{\mathrm{ens}})\right] \\
&\quad + \mathcal{O}(\delta^3),
\end{aligned}
\tag{D1}
$$

from which Eqs. (106) and (107) are immediately obtained. It is noted that the $\mathcal{O}(\delta^2)$ part of $\langle(\Pi^{\mathrm{C}})^s\rangle_{\mathrm{ens}}$ has no contribution from the gyrophase-dependent part of the distribution function, the lowest-order part of which is given by $\widetilde{f}_{a1} = -\boldsymbol{\rho}_a\cdot\nabla F_{aM}$ with the gradient operator ∇ taken for the fixed energy variable $\varepsilon = \frac{1}{2}m_a v^2 + e\langle\phi\rangle_{\mathrm{ens}}$.

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