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RESEARCH ARTICLE

Some Inequalities for the First General Zagreb Index of Graphs and Line Graphs

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Abstract The first general Zagreb index $M_1^\alpha(G)$ of a graph G is equal to the sum of the αth powers of the vertex degrees of G. For $\alpha \geq 0$ and $k \geq 1$, we obtain the lower and upper bounds for $M_1^\alpha(G)$ and $M_1^\alpha(L(G))$ in terms of order, size, minimum/maximum vertex degrees and minimal non-pendant vertex degree using some classical inequalities and majorization technique, where $L(G)$ is the line graph of G. Also, we obtain some bounds and exact values of $M_1^\alpha(J(G))$ and $M_1^\alpha(L^k(G))$, where $J(G)$ is a jump graph (complement of a line graph) and $L^k(G)$ is an iterated line graph of a graph G.

Keywords First Zagreb index · General first Zagreb index · Line graph · Jump graph · Iterated line graph

Mathematics Subject Classification 05C05 · 05C07 · 05C35

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1 Introduction

The graphs considered here are finite, undirected, without loops and multiple edges. Let $G = (V, E)$ be a connected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The degree $d_G(v)$ of a vertex v is the number of vertices adjacent to v. The edge e connecting the vertices u and v is denoted by $e = uv$. The degree of an edge $e = uv$ is denoted by $d_G(e)$ and similarly defined by $d_G(e) = d_G(u) + d_G(v) - 2$. The vertices and edges of a graph are said to be its elements, [1, 2].

A molecular graph is a graph in which the vertices correspond to the atoms and the edges to the bonds of a molecule. A single number that can be computed from the molecular graph and used to characterize some property of the underlying molecule is said to be a topological index or molecular structure descriptor. Numerous such descriptors have been considered in theoretical chemistry and have found some applications, especially in QSPR/QSAR research, [3, 4].

The first Zagreb index was introduced by Gutman and Trinajstić [5]. It is an important molecular descriptor and has been closely correlated with many chemical properties. The first Zagreb index of G is defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \text{ or}$$

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$

For their history, applications and mathematical properties, see [6–13] and the references cited therein.

Li et al. [14, 15] introduced the generalized version of the first Zagreb index defined as

$$M_1^{\alpha}(G) = \sum_{u \in V(G)} d_G(u)^{\alpha},$$

where α is an arbitrary real number. This graph invariant is also known as "general zeroth-order Randić index," [16, 17].

Following $M_1(G)$, the forgotten topological index (also called F-index) was introduced by Furtula and Gutman [18] and is defined as

$$F(G) = \sum_{v \in V(G)} d_G(u)^3 = \sum_{uv \in E(G)} \left[d_G(u)^2 + d_G(v)^2\right].$$

Shirdel et al. [19] introduced the first hyper-Zagreb index of G, which is defined as

$$HM_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2.$$

Clearly, we note that $M_1^0(G) = n$, $M_1^1(G) = 2m$, $M_1^2(G) = M_1(G)$ and $M_1^3(G) = F(G)$ and $M_1^4(G) = HM_1(G)$.

2 Bounds on $M_1^{\alpha}(G)$

First, we prove a lower bound for $M_1^{\alpha}(G)$ in terms of order n and size m. For this purpose, we recall the following facts.

If a real-valued function $f(x)$ defined on an interval has a second derivative $f''(x)$, then a necessary and sufficient condition for it to be strictly convex on that interval is that $f''(x) > 0$. For a positive integer k, if $f(x)$ is strictly convex, then by Jensen's inequality, we have $f\left(\sum_{i=1}^{k} \frac{x_i}{k}\right) \leq \frac{1}{k}\sum_{i=1}^{k} f(x_i)$ with equality if and only if $x_1 = x_2 = \cdots = x_k$, and if $-f(x)$ is strictly convex, then the inequality is reversed.

Theorem 1 *Let G be a simple graph with $n \geq 3$ vertices and m edges. Then, for all $\alpha \geq 1$,*

$$M_1^{\alpha}(G) \geq \frac{2^{\alpha} m^{\alpha}}{n^{\alpha-1}}.$$

Further, equality holds if and only if G is regular.

Proof Let G be a simple graph with $n \geq 3$ vertices and m edges. We have $M_1^{\alpha}(G) = \sum_{v \in V(G)} d_G(v)^{\alpha}$, for all $\alpha \geq 1$. Since $f(x) = x^{\alpha}$ is strictly convex for $x > 0$. By Jensen's inequality, we have

$$\sum_{i=1}^{k} f(x_i) \geq kf\left(\sum_{i=1}^{k} \frac{x_i}{k}\right).$$

Therefore,

$$\begin{aligned} M_1^{\alpha}(G) &\geq n\left(\sum_{u \in V(G)} \frac{d_G(u)}{n}\right)^{\alpha} \\ &\geq n\left(\frac{2m}{n}\right)^{\alpha} \\ &\geq \frac{2^{\alpha} m^{\alpha}}{n^{\alpha-1}}. \end{aligned}$$

□

Corollary 1 *For any (n, m)-simple graph G with $n \geq 3$ vertices and $\alpha = 2$,*

$$M_1^2(G) = M_1(G) \geq \frac{4m^2}{n}.$$

Theorem 2 *Let G be a connected graph with $n \geq 3$ vertices and m edges. Then, for all $\alpha \geq 1$, we have*

$$M_1^{\alpha}(G) \geq n\left(\prod_{u \in V(G)} (d_G(u))^{\alpha}\right)^{\frac{1}{n}}.$$

Proof Let $f(x) = \log x$. Clearly, $\log(x)$ is concave in $(0, \infty)$. By Jensen's inequality, we have

$$\begin{aligned} \log\left(\sum_{u \in V(G)} \frac{(d_G(u))^{\alpha}}{n}\right) &\geq \frac{1}{n}\sum_{u \in V(G)} \log(d_G(u))^{\alpha} \\ &\geq \frac{1}{n} \log\left(\prod_{u \in V(G)} d_G(u)^{\alpha}\right) \\ &\geq \log\left(\prod_{u \in V(G)} d_G(u)^{\alpha}\right)^{\frac{1}{n}}. \end{aligned}$$

Therefore,

$$\frac{1}{n}\sum_{u \in V(G)} d_G(u)^{\alpha} \geq \left(\prod_{u \in V(G)} d_G(u)^{\alpha}\right)^{\frac{1}{n}}$$

$$\sum_{u \in V(G)} (d_G(u))^{\alpha} \geq n\left(\prod_{u \in V(G)} d_G(u)^{\alpha}\right)^{\frac{1}{n}}.$$

Thus, the upper bound follows. □

In order to prove our next result giving lower and upper bounds for $M_1^{\alpha}(G)$ in terms of order n, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$ of a graph G, we apply the well-known Diaz-Metcalf inequality.

Theorem 3 [20] *Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two n-tuples consisting of positive real numbers satisfying the condition*

$$ra_i \leq b_i \leq Ra_i$$

for $1 \le i \le n$. *Then,*

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i^2 \le (r+R) \sum_{i=1}^{n} a_i b_i.$$

Equality holds if and only if $b_i = Ra_i$ or $b_i = ra_i$ *for* $1 \le i \le n$.

Theorem 4 *Let G be a simple graph with* $n \ge 3$ *vertices and m edges. Then, for all* $\alpha \ge 1$,

$$n\,\delta(G)^{\alpha} \le M_1^{\alpha}(G) \le n\,\Delta(G)^{\alpha}.$$

Further, both left- and right-hand side equality holds if and only if G is regular.

Proof Let $b_i = d_G(u)^{\frac{\alpha}{2}}$; $a_i = 1$ and $r = \delta(G)^{\frac{\alpha}{2}}$; $R = \Delta(G)^{\frac{\alpha}{2}}$. Clearly,

$$\delta(G)^{\frac{\alpha}{2}} \le d_G(u)^{\frac{\alpha}{2}} \le \Delta(G)^{\frac{\alpha}{2}}.$$

By the Diaz–Metcalf inequality, we have

$$\sum_{u \in V(G)} d_G(u)^{a} + (\delta(G)\Delta(G))^{\frac{\alpha}{2}} \sum_{u \in V(G)} 1 \le (\delta(G)^{\frac{\alpha}{2}} + \Delta(G)^{\frac{\alpha}{2}}) \sum_{u \in V(G)} d_G(u)^{\frac{\alpha}{2}},$$

and hence,

$$M_1^{\alpha}(G) + n\,\delta(G)^{\frac{\alpha}{2}} \le (\delta(G)^{\frac{\alpha}{2}} + \Delta(G)^{\frac{\alpha}{2}}) n\,\Delta(G)^{\frac{\alpha}{2}}.$$

Therefore, we have $d_G(u)^{\frac{\alpha}{2}} \le \Delta(G)^{\frac{\alpha}{2}}$ for all $u \in V(G)$. Hence, we can write

$$M_1^{\alpha}(G) \le (\delta(G)^{\frac{\alpha}{2}} + \Delta(G)^{\frac{\alpha}{2}}) n\,\Delta(G)^{\frac{\alpha}{2}} - n\,\delta(G)^{\frac{\alpha}{2}}$$

and

$$M_1^{\alpha}(G) \le n\,\delta(G)^{\frac{\alpha}{2}} + n\,\Delta(G)^{\frac{\alpha}{2}} - n\,\delta(G)^{\frac{\alpha}{2}} = n\,\Delta(G)^{\alpha}.$$

Thus, the upper bound follows.

Let $b_i = d_G(u)^{\alpha}$; $a_i = 1$ and $r = \delta(G)^{\alpha}$; $R = \Delta(G)^{\alpha}$. By the Diaz–Metcalf inequality, we have

$$\sum_{u \in V(G)} d_G(u)^{2\alpha} + (\delta(G)\Delta(G))^{\alpha} \sum_{u \in V(G)} 1 \le (\delta(G)^{\alpha} + \Delta(G)^{\alpha}) \sum_{u \in V(G)} d_G(u)^{\alpha}$$

and

$$\begin{aligned} M_1^{\alpha}(G) &\ge \frac{1}{(\delta(G)^{\alpha} + \Delta(G)^{\alpha})} \left[\sum_{u \in V(G)} d_G(u)^{2a} + n\left(\delta(G)\Delta(G)\right)^{\alpha} \right] \\ &\ge \frac{1}{(\delta(G)^{\alpha} + \Delta(G)^{\alpha})} \left[n\,\delta(G)^{2\alpha} + n\,\delta(G)^{\alpha}\Delta(G)^{\alpha} \right] \\ &\ge \frac{1}{(\delta(G)^{\alpha} + \Delta(G)^{\alpha})} n\,\delta(G)^{\alpha}[\delta(G)^{\alpha} + \Delta(G)^{\alpha}] = n\,\delta(G)^{\alpha}. \end{aligned}$$

Thus, the lower bound follows.

Equality is attained if and only if G is regular. □

The following results are immediate from Theorem 4.

Corollary 2 *For any graph G with* $n \ge 3$ *vertices and* $\alpha = 2$,

$$M_1^2(G) = M_1(G) \le n\,\Delta(G)^2.$$

Corollary 3 *For any complete graph* K_n *with* $n \ge 3$ *vertices and* $\alpha = 2$,

$$M_1^2(K_n) = n(n-1)^2.$$

Our next result gives lower and upper bounds for $M_1^{\alpha}(G)$ in terms of size m, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$ of a graph G.

Theorem 5 *Let G be a simple graph with* $n \ge 3$ *vertices and m edges. Then, for all* $\alpha \ge 0$,

$$2m\,\delta(G)^{\alpha-1} \le M_1^{\alpha}(G) \le 2m\,\Delta(G)^{\alpha-1}.$$

Further, both left- and right-hand side equality holds if and only if G is regular.

Proof Let f be a positive function defined on nonnegative integers. Then,

$$\begin{aligned} X(G) &= \sum_{u \in V(G)} f(d_G(u)) = \sum_{u \in V(G)} \sum_{uv \in E(G)} \frac{f(d_G(u))}{d_G(u)} \\ &= \sum_{uv \in E(G)} \left(\frac{f(d_G(u))}{d_G(u)} + \frac{f(d_G(v))}{d_G(v)} \right), \end{aligned}$$

where $X(G)$ is any degree-based topological index of a graph G.

Defining $f(d_G(u)) = (d_G(u))^{\alpha}$, we have

$$M_1^{\alpha}(G) = \sum_{uv \in E(G)} \left[(d_G(u))^{\alpha-1} + (d_G(v))^{\alpha-1} \right].$$

Since for all $u \in V(G)$, $\delta(G) \le d_G(u) \le \Delta(G)$. Hence, we have

$$\sum_{uv \in E(G)} 2\ \delta(G)^{\alpha-1} \le M_1^{\alpha}(G)$$

$$2\ m\ \delta(G)^{\alpha-1} \le M_1^{\alpha}(G)$$

Equality is attained on both sides if and only if G is regular. □

Now, to obtain another result for the lower and upper bounds for $M_1^{\alpha}(G)$ in terms of order n, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$ of a graph G, we apply the Polya–Szego inequality as follows.

Theorem 6 [21] *Let* $a = (a_1, a_2, \ldots, a_n)$ *and* $b = (b_1, b_2, \ldots, b_n)$ *be two n-tuples of positive numbers. If*

$0<\gamma\leq a_i\leq A<\infty$ and $0<\beta\leq b_i\leq B<\infty$ for each $i\in\{1,2,\ldots,n\}$, then

$$\sum_{i=1}^{n} a_i^2\cdot\sum_{i=1}^{n} b_i^2\leq\frac{(\gamma\beta+AB)^2}{4\gamma\beta AB}\left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

The equality holds iff $p=n\frac{A}{\gamma}/\left(\frac{A}{\gamma}+\frac{B}{\beta}\right)$ and $q=n.\frac{B}{\beta}/\left(\frac{A}{\gamma}+\frac{B}{\beta}\right)$ are integers and p of the numbers $a_1,a_2,\ldots,a_n$ are equal to γ and q of these numbers are equal to A, and similarly if q of the corresponding numbers b_i are equal to B and and q of them are equal to β.

Theorem 7 *For any $(n,\,m)$-connected graph G with $n\geq 3$ vertices and $\alpha\geq 0$, we have*

$$\frac{2n\,\delta(G)^{2\alpha}}{(\delta(G)^{\alpha}+\Delta(G)^{\alpha})}\left(\frac{\Delta(G)}{\delta(G)}\right)^{\frac{\alpha}{2}}\leq M_1^{\alpha}(G)$$
$$\leq\frac{n}{4}\left(\frac{\Delta(G)}{\delta(G)}\right)^{\frac{\alpha}{2}}\left(\delta(G)^{\frac{\alpha}{2}}+\Delta(G)^{\frac{\alpha}{2}}\right)^2.$$

Further, both left- and right-hand side equalities hold if and only if G is regular.

Proof Let $a_i=d_G(u)^{\frac{\alpha}{2}}$ and $b_i=1$ in the Polya–Szego inequality. Clearly, $\gamma=\delta(G)^{\frac{\alpha}{2}}$, $A=\Delta(G)^{\frac{\alpha}{2}}$, $\beta=1$ and $B=1$. We have

$$\sum_{u\in V(G)} d_G(u)^a\,n\leq\frac{\left(\delta(G)^{\frac{\alpha}{2}}+\Delta(G)^{\frac{\alpha}{2}}\right)}{4\,\delta(G)^{\frac{\alpha}{2}}\Delta(G)^{\frac{\alpha}{2}}}\left[\sum_{u\in V(G)} d_G(u)^{\frac{\alpha}{2}}\right]^2$$
$$\leq\frac{\left(\delta(G)^{\frac{\alpha}{2}}+\Delta(G)^{\frac{\alpha}{2}}\right)}{4\,\delta(G)^{\frac{\alpha}{2}}\Delta(G)^{\frac{\alpha}{2}}}\left[n\,\Delta(G)^{\frac{\alpha}{2}}\right]^2.$$

Since $d_G(u)^{\frac{\alpha}{2}}\leq\Delta(G)^{\frac{\alpha}{2}}$ for all $u\in V(G)$, we have

$$M_1^{\alpha}(G)\leq\frac{\left(\delta(G)^{\frac{\alpha}{2}}+\Delta(G)^{\frac{\alpha}{2}}\right)}{4\,n\,\delta(G)^{\frac{\alpha}{2}}\Delta(G)^{\frac{\alpha}{2}}}\left[n^2\,\Delta(G)^{\alpha}\right]$$
$$\leq\frac{n}{4}\left(\frac{\Delta(G)}{\delta(G)}\right)^{\frac{\alpha}{2}}\left(\delta(G)^{\frac{\alpha}{2}}+\Delta(G)^{\frac{\alpha}{2}}\right)^2.$$

Thus, the upper bound follows.

Let $a_i=d_G(u)^{\alpha}$ and $b_i=1$ in the Polya–Szego inequality. Clearly, $\gamma=\delta(G)^{\alpha}$, $A=\Delta(G)^{\alpha}$, $\beta=1$ and $B=1$. We have

$$M_1^{\alpha}(G)\geq\frac{4\,n\,(\delta(G)\Delta(G))^{\alpha}}{(\delta(G)^{\alpha}+\Delta(G)^{\alpha})^2}\sum_{u\in V(G)} d_G(u)^{2\alpha}$$
$$\geq$$

Thus, the lower bound follows.

Equality is attained on both sides if and only if G is regular. □

In order to prove our next result giving another upper bound for $M_1^{\alpha}(G)$, this time in terms of the size m of the graph G. We make use of the following well-known majorization technique, [22].

Let $a=(a_1,a_2,\ldots,a_n)$ and $b=(b_1,b_2,\ldots,b_n)$ be two non-increasing n-tuples of real numbers. Then, we say that a is majorized by b denoted by $a\prec b$ if

1. $a_1+a_2+\cdots+a_k\leq b_1+b_2+\cdots+b_k$ for $1\leq k\leq n-1$
2. $a_1+a_2+\cdots+a_n=b_1+b_2+\cdots+b_n$.

Lemma 1 [23] *Suppose $a=(a_1,a_2,\ldots,a_n)$ and $b=(b_1,b_2,\ldots,b_n)$ are two non-increasing n-tuples of real numbers. If $a\prec b$, then for any convex function f,*

$$\sum_{j=1}^{n} f(a_j)\leq\sum_{j=1}^{n} f(b_j).$$

Theorem 8 *For any $(n,\,m)$-connected graph G with $n\geq 3$ vertices and for all $\alpha>1$ or $\alpha<0$, we have*

$$M_1^{\alpha}(G)\leq 2^{\alpha}\,m^{\alpha}.$$

Proof We prove this result by means of the majorization technique. Clearly, $(d_G(u_1),d_G(u_2),\ldots,d_G(u_n))\prec(2m,0,\ldots,0)$ and $f(x)=x^{\alpha}$ is convex for all $\alpha>1$ or $\alpha<0$ and $x>0$. By the above lemma, we have

$$\begin{aligned}&\left(d_G(u_1)^{\alpha}+d_G(u_2)^{\alpha}+\cdots+d_G(u_n)^{\alpha}\right)\\&\quad\leq((2m)^{\alpha}+0+\cdots+0)\end{aligned},$$

and hence,

$$\sum_{u\in V(G)} d_G(u)^{\alpha}\leq(2m)^{\alpha}.$$

Thus, the upper bound follows. □

Corollary 4 *For any (n,m)-connected graph G with $n\geq 3$ and $\alpha=2$,*

$$M_1^2(G)\leq 4\,m^2.$$

To prove our next result, we make use of the following definitions and observations:

A vertex $v\in V(T)$ is called a branching vertex of a tree T, if $\deg_T(v)\geq 3$. If $\deg_T(v)=1$, the vertex v is named a leaf (or a pendent vertex) of T. Further, the path with n vertices, of which exactly two are leaves, is denoted by P_n, and the star graph with exactly $n-1$ leaves and one branching vertex is denoted by S_n or $K_{1,n-1}$; $n\geq 2$ vertices.

It is easy to see that every tree on n vertices has at least two leaves and at most $\frac{n-2}{2}$ branching vertices.

Theorem 9 *For any nontrivial tree T of order n with $\alpha\geq 1$,*

$$M_1^{\alpha}(P_n)\leq M_1^{\alpha}(T)\leq M_1^{\alpha}(S_n).$$

Proof We prove the result by using the majorization technique. Clearly, for the degree sequence of any tree T, we have

$$\big(d_T(u_1), d_T(u_2), \ldots, d_T(u_n)\big) \prec (n-1, \underbrace{1, 1, \ldots, 1}_{(n-1)-\text{times}}).$$

Let $f(x) = x^{\alpha}$. Clearly, $f(x)$ is convex in $(0, \infty)$. Then,

$$\sum_{u \in V(T)} d_T(u)^{\alpha} \le (n-1)^{\alpha} + \underbrace{1 + 1 + \ldots + 1}_{(n-1)-\text{times}} \le (n-1) + (n-1)^{\alpha}.$$

But, for a tree, we have $m = n-1$, and therefore, we have $M_1^{\alpha}(T) \le m + m^{\alpha}$. Thus, the upper bound follows.

Now, the degree sequence of any tree T satisfies

$$\big(1, 2, 2, \ldots 2, 1\big) \prec \big(d_T(u_1), d_T(u_2), \ldots, d_T(u_n)\big).$$

Therefore,

$$1^{\alpha} + 2^{\alpha} + \cdots + 2^{\alpha} + 1^{\alpha} \le \sum_{u \in V(T)} d_T(u)^{\alpha},$$

where $f(x) = x^{\alpha}$ is convex for all $\alpha \ge 1$. Hence,

$$M_1^{\alpha}(P_n) = 2 + (n-2)2^{\alpha} \le M_1^{\alpha}(T).$$

Thus, the lower bound follows. □

Now, we obtain some lower and upper bounds for $M_1^{\alpha}(G)$ in terms of the number of pendent vertices and minimal non-pendent vertices of G.

Theorem 10 *For any (n, m)-connected graph G with η-pendent vertices, minimal non-pendent vertex degree $\delta_1(G)$ and $\alpha \ge 0$,*

$$\eta + (m-\eta)\,\delta_1(G)^{\alpha} \le M_1^{\alpha}(G) \le \eta + (m-\eta)\,\Delta(G)^{\alpha}.$$

Further, equality in both lower and upper bounds are attained if and only if G is regular.

Proof For any $\alpha \ge 0$, we have

$$M_1^{\alpha}(G) = \sum_{u \in V(G)} [d_G(u)]^{\alpha} = \sum_{u \in V(G); d_G(u)=1} 1^{\alpha} + \sum_{u \in V(G); d_G(u) \ne 1} [d_G(u)]^{\alpha} \le \eta + (m-\eta)\,\Delta(G)^{\alpha}.$$

Similarly,

$$M_1^{\alpha}(G) \ge \eta + (m-\eta)\,\delta_1(G)^{\alpha}.$$

Hence, the desired result follows.

Further, the equality in both lower and upper bounds is attained if and only if G is regular. □

Nordhaus and Gaddum [24] gave tight bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then, such type of results have been derived for several other graph invariants. Here, we derive such a relation for $M_1^{\alpha}(G)$.

Theorem 11 *For any (n, m)-connected graph G with $\alpha \ge 2$ and $n \ge 4$ vertices having a connected $\overline{G}$, we have*

1. $2^{1-\alpha}\, n\,(n-1)^{\alpha} \le M_1^{\alpha}(G) + M_1^{\alpha}(\overline{G}) \le n\,(n-1)^{\alpha}$,
2. $n^2 \le M_1^{\alpha}(G) \times M_1^{\alpha}(\overline{G}) \le n^2 (n-1)^{2\alpha}$.

Proof

1. Since $m + \overline{m} = \frac{n(n-1)}{2}$, we have $d_G(u) + d_{\overline{G}}(u) = n-1$ and $d_G(v) + d_{\overline{G}}(v) = n-1$. Hence, we have

$$\begin{aligned} M_1^{\alpha}(G) + M_1^{\alpha}(\overline{G}) &= \sum_{uv \in E(G)} d_G(u)^{\alpha-1} + d_G(v)^{\alpha-1} \\ &\quad + \sum_{uv \notin E(\overline{G})} d_{\overline{G}}(u)^{\alpha-1} + d_{\overline{G}}(v)^{\alpha-1} \\ &\le m\left[2(n-1)^{\alpha-1}\right] + \overline{m}\left[2(n-1)^{\alpha-1}\right] \\ &\le 2(m+\overline{m})\,(n-1)^{\alpha-1}. \end{aligned}$$

Thus, the upper bound follows. By Theorem 1, we have $M_1^{\alpha}(G) \ge \frac{2^{\alpha} m^{\alpha}}{n^{\alpha-1}}$ and $M_1^{\alpha}(\overline{G}) \ge \frac{2^{\alpha} (\overline{m})^{\alpha}}{n^{\alpha-1}}$. Therefore,

$$M_1^{\alpha}(G) + M_1^{\alpha}(\overline{G}) \ge \frac{2^{\alpha}}{n^{\alpha-1}}\,[m^{\alpha} + \overline{m}^{\alpha}].$$

Since m^{α} is convex for all $\alpha \ge 2$, then by Jensen's inequality, we have

$$\begin{aligned} M_1^{\alpha}(G) + M_1^{\alpha}(\overline{G}) &\ge \frac{2^{\alpha}}{n^{\alpha-1}}\; 2\left(\frac{m+\overline{m}}{2}\right)^{\alpha} \\ &\ge \frac{2^{\alpha+1}}{n^{\alpha-1}}\; \frac{n^{\alpha}\,(n-1)^{\alpha}}{2^{2\alpha}}. \end{aligned}$$

Thus, the lower bound follows.

2. Since $1 \le \delta(G) \le \Delta(G) \le n-1$ and $1 \le \delta(\overline{G}) \le \Delta(\overline{G}) \le n-1$ for both G and $\overline{G}$ are connected with $n \ge 4$ vertices, by Theorem 4, we have the desired results.

□

3 Line Graphs

The line graph $L(G)$ is the graph with vertex set $V(L(G)) = E(G)$, and whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common. As $L(G)$ is defined on the edge set of a graph G, isolated vertices of G, if there is any, play no role in $L(G)$, [8, 25, 26].

Ranjini et al. [27] obtained the first and second Zagreb indices of the line graphs of the subdivision graphs. Here, we obtain some lower and upper bounds for $M_1^{\alpha}(L(G))$ in terms of the size m, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$ of a graph G.

Theorem 12 *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then, for all $\alpha \geq 0$, we have*

$$2^{\alpha} m \left(\delta(G) - 1\right)^{\alpha} \leq M_1^{\alpha}(L(G)) \leq 2^{\alpha} m \left(\Delta(G) - 1\right)^{\alpha}.$$

Further, both left- and right-hand side equalities holds if and only if G is regular.

Proof Let G be a simple connected (n, m)-graph with $\alpha \geq 0$. Then,

$$M_1^{\alpha}(L(G)) = \sum_{uv \in E(G)} [d_G(u) + d_G(v) - 2]^{\alpha}.$$

Since for all $u, v \in V(G)$, $\delta(G) \leq \{d_G(u), d_G(v)\} \leq \Delta(G)$, we have

$$\sum_{uv \in E(G)} (\delta(G) + \delta(G) - 2)^{\alpha} \leq M_1^{\alpha}(L(G)) \leq \sum_{uv \in E(G)} (\Delta(G) + \Delta(G) - 2)^{\alpha},$$

and therefore,

$$2^{\alpha} m \left(\delta(G) - 1\right)^{\alpha} \leq M_1^{\alpha}(L(G))$$

Further, both left- and right-hand side equalities hold if and only if G is regular. □

We now give our next result giving a lower bound for $M_1^{\alpha}(L(G))$ in terms of the size m and the first Zagreb index $M_1(G)$.

Theorem 13 *Let G be an (n, m)-connected graph with $n \geq 3$ vertices. Then, for all $\alpha \geq 1$, we have*

$$M_1^{\alpha}(L(G)) \geq \frac{\left(M_1(G) - 2m\right)^{\alpha}}{m^{\alpha - 1}}.$$

Further, equality holds if and only if G is regular.

Proof Let G be a (n, m)-connected graph with $n \geq 3$ vertices and m edges. Clearly, $|V(L(G))| = m$ and $|E(L(G))| = \frac{1}{2}M_1(G) - m$. Hence, we have $M_1^{\alpha}(L(G)) = \sum_{v \in V(L(G))} d_{L(G)}(v)^{\alpha}$. For all $\alpha \geq 1$ and $x > 0$, $f(x) = x^{\alpha}$ is strictly convex. Hence, by Jensen's inequality, we have

$$\begin{aligned} M_1^{\alpha}(L(G)) &\geq m \left(\sum_{u \in V(L(G))} \frac{d_{L(G)}(u)}{n} \right)^{\alpha} \\ &\geq m \left(\frac{1}{m} 2 \left(\frac{1}{2} M_1(G) - m \right) \right)^{\alpha} \\ &\geq \frac{\left(M_1(G) - 2m\right)^{\alpha}}{m^{\alpha - 1}}. \end{aligned}$$

Further, equality holds if and only if G is regular. □

Corollary 5 *Let G be an r-regular graph with $n \geq 3$ vertices. Then, for all $\alpha \geq 1$, we have*

$$M_1^{\alpha}(L(G)) = 2^{\alpha - 1} n\, r\, (r - 1)^{\alpha}.$$

Let G be an (n, m)-connected graph. Then, $G \cong L(G)$ if and only if $G \cong C_n$; $n \geq 3$. Hence, we have

Corollary 6 *For any cycle C_n with $n \geq 3$ vertices,*

$$M_1^{\alpha}(C_n) = M_1^{\alpha}(L(C_n)) = n\, 2^{\alpha}.$$

The following is an inequality for $M_1^{\alpha}(L(G))$ in terms of $M_1^{\alpha - i}(L(G))$'s.

Theorem 14 *For any (n, m)-connected graph G with $n \geq 3$ vertices and $\alpha \geq 0$,*

$$M_1^{\alpha}(L(G)) \geq \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i 2^i m^{-a+i+1} (M_1^{\alpha - i}(G)).$$

Further, equality holds if and only if G is regular.

Proof Let G be a (n, m)-connected graph with $n \geq 3$ and $\alpha \geq 0$. Then,

$$\begin{aligned} M_1^{\alpha}(L(G)) &= \sum_{uv \in E(G)} [d_G(u) + d_G(v) - 2]^{\alpha} \\ &= \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i [d_G(u) + d_G(v)]^{\alpha - i}\, 2^i \\ &= \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i 2^i \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^{\alpha - i} \\ &= (-1)^{\alpha} 2^{\alpha} m \\ &\quad + \sum_{i=0}^{\alpha - 1} \binom{\alpha}{i} (-1)^i 2^i \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^{\alpha - i}. \end{aligned}$$

By Jensen's inequality as x^{α} is a convex function for $\alpha \geq 1$ and $x > 0$, we have

$$\begin{aligned} M_1^{\alpha}(L(G)) &\geq (-1)^{\alpha} 2^{\alpha} m \\ &\quad + \sum_{i=0}^{\alpha - 1} \binom{\alpha}{i} (-1)^i 2^i m \sum_{uv \in E(G)} \left[\frac{d_G(u) + d_G(v)}{m} \right]^{\alpha - i} \\ &\geq (-1)^{\alpha} 2^{\alpha} m \\ &\quad + \sum_{i=0}^{\alpha - 1} \binom{\alpha}{i} (-1)^i 2^i m^{i - \alpha + 1} [M_1^{\alpha - 1}(G)] \\ &\geq \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i 2^i m^{-a+i+1} (M_1^{\alpha - i}(G)). \end{aligned}$$

Further, equality holds if and only if G is regular. □

Corollary 7 *For any connected graph G with $n \geq 3$ vertices and $\alpha = 2$, we have*

$$M_1^2(L(G)) = M_1(L(G)) \geq \frac{(M_1(G))^2}{m} - M_1(G) + 4m.$$

Theorem 15 *Let G be a r-regular connected graph with $\alpha \geq 0$. Then,*

$$M_1^{\alpha}(L(G)) = \sum_{i=0}^{\alpha}\sum_{j=0}^{\alpha-i}\binom{\alpha}{i}\binom{\alpha-i}{j}(-1)^i(2)^{i-1}\, M_1^{\alpha-i+1}(G).$$

Proof Let G be an r-regular connected graph with $\alpha \geq 0$. Then,

$$\begin{aligned} M_1^{\alpha}(L(G)) &= \sum_{uv\in E(G)}[d_G(u)+d_G(v)-2]^{\alpha} \\ &= \sum_{i=0}^{\alpha}\binom{\alpha}{i}(-1)^i\,2^i\sum_{uv\in E(G)}[d_G(u)+d_G(v)]^{\alpha-i} \\ &= \sum_{i=0}^{\alpha}\binom{\alpha}{i}(-1)^i\,2^i\sum_{j=0}^{\alpha-i}\binom{\alpha-i}{j} \\ &\quad \sum_{uv\in E(G)}(d_G(u))^{\alpha-i-j}\,(d_G(v))^j. \end{aligned}$$

Since G is an r-regular connected graph as $d_G(u) = d_G(v) = r$, we have

$$\begin{aligned} M_1^{\alpha}(L(G)) &= \sum_{i=0}^{\alpha}\binom{\alpha}{i}(-1)^i\,2^i\sum_{j=0}^{\alpha-i}\binom{\alpha-i}{j}\sum_{uv\in E(G)}(d_G(u))^{\alpha-i} \\ &= \sum_{i=0}^{\alpha}\sum_{j=0}^{\alpha-i}\binom{\alpha}{i}(-1)^i\,2^i\binom{\alpha-i}{j}\,\frac{1}{2}\sum_{u\in V(G)}(d_G(u))^{\alpha-i+1} \\ &= \sum_{i=0}^{\alpha}\sum_{j=0}^{\alpha-i}\binom{\alpha}{i}\binom{\alpha-i}{j}(-1)^i 2^{i-1}\, M_1^{\alpha-i+1}(G). \end{aligned}$$

□

Corollary 8 *Let G be an r-regular connected graph with $\alpha = 2$. Then,*

$$\begin{aligned} M_1^2(L(G)) &= M_1(L(G)) \\ &= \sum_{i=0}^{2}\sum_{j=0}^{2-i}\binom{2}{i}\binom{2-i}{j}(-1)^i 2^{i-1}\, M_1^{3-i}(G) \\ &= 2nr(r-1)^2. \end{aligned}$$

In order to prove our next result on $M_1^{\alpha}(L(G))$, we make use of the following trinomial theorem. For more details, we refer to [28].

Theorem 16 *Let x, y, z be three real numbers and n be a whole number. Then,*

$$(x+y+z)^n = \sum_{i,j,k\geq 0; i+j+k=n}\frac{n!}{i!\,j!\,k!}\,x^i y^j z^k.$$

Theorem 17 *Let G be an r-regular connected graph and let α be a whole number. Then,*

$$M_1^{\alpha}(L(G)) = \frac{1}{2}\sum_{i,j,k\geq 0; i+j+k=\alpha}\frac{\alpha!}{i!\,j!\,k!}(-2)^k\, M_1^{i+j+1}(G).$$

Proof Let G be an r-regular connected graph and let α be a whole number. Then,

$$\begin{aligned} M_1^{\alpha}(L(G)) &= \sum_{uv\in E(G)}[d_G(u)+d_G(v)-2]^{\alpha} \\ &= \sum_{uv\in E(G)} \\ &\quad \sum_{i,j,k\geq 0; i+j+k=\alpha}\frac{\alpha!}{i!\,j!\,k!}(d_G(u))^i\,(d_G(v))^j\,(-2)^k \\ &= \sum_{i,j,k\geq 0; i+j+k=\alpha}\frac{\alpha!}{i!\,j!\,k!}(-2)^k \\ &\quad \sum_{uv\in E(G)}(d_G(u))^i\,(d_G(v))^j. \end{aligned}$$

Since G is r-regular, $d_G(u) = d_G(v) = r$, and hence, we have

$$\begin{aligned} M_1^{\alpha}(L(G)) &= \sum_{i,j,k\geq 0; i+j+k=\alpha}\frac{\alpha!}{i!\,j!\,k!}(-2)^k\,\frac{1}{2}\sum_{u\in V(G)}[d_G(u)]^{i+j+1} \\ &= \frac{1}{2}\sum_{i,j,k\geq 0; i+j+k=\alpha}\frac{\alpha!}{i!\,j!\,k!}(-2)^k\, M_1^{i+j+1}(G). \end{aligned}$$

□

The following results are immediate from the above theorem.

Corollary 9 *For any r-regular connected graph G with $\alpha = 2$, we have*

$$\begin{aligned} M_1^2(L(G)) = M_1(L(G)) &= \frac{1}{2}\sum_{i,j,k\geq 0; i+j+k=2}\frac{2!}{i!\,j!\,k!}(-2)^k\, M_1^{i+j+1}(G) \\ &= 2nr(r-1)^2. \end{aligned}$$

Corollary 10 *Let G be an r-regular connected graph and let α be a whole number. Then,*

1. $M_1^2(L(G)) = \frac{1}{2}\,M_1^2(G)$ *for* $i = 2, j = 0, k = 0$ or $i = 0, j = 2, k = 0$,
2. $M_1^0(L(G)) = \frac{1}{2}\,M_1^0(G) = n$ *for* $i = 0, j = 0, k = 2$,
3. $M_1^1(L(G)) = -2M_1^1(G) = m$ for $i = 1, j = 0, k = 1$ *or* $i = 0, j = 1, k = 1$,
4. $M_1^2(L(G)) = M_1^2(G) = n$ *for* $i = 1, j = 1, k = 0$.

Theorem 18 *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$, and let α be a whole number. Then,*

$$M_1^{\alpha}(L(K_{r,s})) = \frac{1}{2}\sum_{i,j,k\geq 0; i+j+k=\alpha} \frac{\alpha!}{i!\,j!\,k!}\,(-2)^k\, r^{i+1} \times s^{j+1}.$$

Proof Proof is similar to that of previous theorem. □

The following results are immediate from the above theorem.

Corollary 11 *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ and let $\alpha = 2$. Then,*

$$M_1^2(L(K_{r,s})) = rs(r+s-2)^2.$$

Corollary 12 *Let G be an r-regular connected graph and α be a whole number. Then, the sum of all coefficients of $M_1^{i+j+1}(G)$ in the expansion of $M_1^{\alpha}(L(G))$ is equal to zero.*

Corollary 13 *Let G be an r-regular connected graph and α be a whole number. Then, the number of terms in the expansion of $M_1^{\alpha}(L(G))$ is $\binom{\alpha+2}{2}$.*

Let $\delta_1(G)$ be the minimal degree of non-pendant vertices. Now we obtain lower and upper bounds for $M_1^{\alpha}(L(G))$ in terms of the number of pendent vertices and $\delta_1(G)$.

Theorem 19 *For any (n, m)-connected graph G with η pendent vertices and minimal non-pendant vertex degree $\delta_1(G)$,*

$$(\delta_1(G)-1)^{\alpha}[\eta + (m-\eta)2^{\alpha}] \leq M_1^{\alpha}(L(G)) \leq (\Delta(G)-1)^{\alpha}[\eta + (m-\eta)2^{\alpha}].$$

Proof For any $\alpha \geq 0$, we have

$$\begin{aligned} M_1^{\alpha}(L(G)) &= \sum_{uv\in E(G)} [d_G(u)+d_G(v)-2]^{\alpha} \\ &= \sum_{uv\in E(G); d_G(u)=1, d_G(v)\neq 1} [d_G(v)-1]^{\alpha} \\ &\quad + \sum_{uv\in E(G); d_G(u)\neq 1, d_G(v)\neq 1} [d_G(u)+d_G(v)-2]^{\alpha} \\ &\leq \eta(\Delta(G)-1)^{\alpha} + (m-\eta)(2\Delta(G)-2)^{\alpha}. \end{aligned}$$

Thus, the upper bound follows. Similarly,

$$M_1^{\alpha}(L(G)) \geq (\delta_1(G)-1)^{\alpha}[\eta + (m-\eta)2^{\alpha}].$$

Hence, the desired result follows. □

Remark 1 The bounds in the above theorem are attained if and only if $d_G(u) = d_G(v) = \Delta(G) = \delta_1(G)$ for each $uv \in E(G)$ with $d_G(u) \neq 1$, $d_G(v) \neq 1$ and $d_G(v) = \Delta(G) = \delta_1(G)$ for each $uv \in E(G)$ with $d_G(u) = 1$.

4 Jump Graph

The complement of the line graph is called the jump graph $J(G)$ of a graph G, i.e., $J(G) = \overline{L(G)}$. For more details, we refer to [29, 30].

Theorem 20 *For an (n, m)-connected graph G with $n \geq 4$ and $\alpha \geq 0$, we have*

$$M_1^{\alpha}(J(G)) \geq \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i (m-1)^i M_1^{\alpha-i}(L(G)).$$

Proof Let G be an (n, m)-connected graph with $n \geq 4$ and $\alpha \geq 0$. Then,

$$\begin{aligned} M_1^{\alpha}(J(G)) &= \sum_{u\in V(J(G))} [d_{J(G)}(u)]^{\alpha} \\ &= \sum_{u\in V(L(G))} \left[(m-1) - d_{L(G)}(u)\right]^{\alpha}, \end{aligned}$$

since $d_{L(G)}(u) + d_{J(G)}(u) = m-1$. Hence,

$$\begin{aligned} M_1^{\alpha}(J(G)) &= \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i (m-1)^i \sum_{u\in V(L(G))} [d_{L(G)}(u)]^{\alpha-i} \\ &= \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i (m-1)^i M_1^{\alpha-i}(L(G)). \end{aligned}$$

□

By the above result, we reach to the following results.

Corollary 14 *For any path graph P_n with $n \geq 5$ vertices and $\alpha = 2$, we have*

$$M_1^2(J(P_n)) = n^3 - 9n^2 + 28n - 30.$$

Theorem 21 *Let G be a connected r-regular graph with $\alpha \geq 0$. Then,*

$$M_1^{\alpha}(J(G)) = nr\sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i\, 2^{\alpha-2i-1}\, (r-1)^{\alpha-i}\, (nr-2)^i.$$

Proof Let G be a connected r-regular graph with $\alpha \geq 0$. Then,

$$\begin{aligned} M_1^{\alpha}(J(G)) &= \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i (m-1)^i \frac{nr}{2} (2r-2)^{\alpha-i} \\ &= \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i \left(\frac{nr}{2}-1\right)^i \frac{nr}{2}\, 2^{\alpha-i} (r-1)^{\alpha-i} \\ &= \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i \frac{(nr-2)^i}{2^i} \frac{nr}{2}\, 2^{\alpha-i} (r-1)^{\alpha-i} \\ &= nr\sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i\, 2^{\alpha-2i-1} (r-1)^{\alpha-i} (nr-2)^i. \end{aligned}$$

□

By the above results, we obtain the followings.

Corollary 15 *For any complete graph K_n with $n \geq 3$ vertices and $\alpha = 2$, we have*

$$M_1^2(J(K_n)) = \frac{n(n-1)}{8}\left[n^4 - 10n^3 + 37n^2 - 60n + 36\right].$$

Corollary 16 *For any cycle C_n with $n \geq 3$ vertices and $\alpha = 2$, we have*

$$M_1^2(J(C_n)) = n^3 - 6n^2 + 9n.$$

Corollary 17 *For any complete bipartite graph $K_{r,s}$ with $2 \leq r \leq s$ and $\alpha = 2$, we have*

$$M_1^2(J(K_{r,s})) = rs \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i (rs-1)^i (r+s-2)^{\alpha-i}.$$

5 Iterated Line Graph

If G is a graph and $L(G) = L^1(G)$ is its line graph, then $L^k(G)$, $k = 2, 3, \ldots$, defined recursively as $L^k(G) = L(L^{k-1}(G))$, are named as the iterated line graphs of G. For more details, we refer to [31–33].

Theorem 22 *For any complete bipartite graph $K_{r,s}$ with $1 \leq r \leq s$ and $\alpha \geq 0$, we have*

$$M_1^\alpha(L^k(K_{r,s})) = \begin{cases} rs(r+s-2)^\alpha & \text{for } k = 1; \\ rs\left[\prod_{i=0}^{k-2}\left(2^{i-1}(r+s-2) - 2^i + 1\right)\right] \\ \quad \times \left(2^{k-1}(r+s-2) - 2^k + 2\right)^\alpha & \text{for } k > 1. \end{cases}$$

Proof Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ and $\alpha \geq 0$. By combinatorial methods, we find that $|V(L(K_{r,s}))| = rs$ and $|E(L(K_{r,s}))| = \frac{rs}{2}(r+s-2)$. Since the kth-iterated line graph of $K_{r,s}$ is a $(2^{k-1}(r+s-2) - 2^k + 2)$-regular graph, we also obtain

$$V(L^k(K_{r,s})) = \begin{cases} rs & \text{for } k = 1; \\ rs\left[\prod_{i=0}^{k-2}\left(2^{i-1}(r+s-2) - 2^i + 1\right)\right] & \text{for } k > 1 \end{cases}$$

and

$$E(L^k(K_{r,s})) = \begin{cases} \frac{rs}{2}(r+s-2) & \text{for } k = 1; \\ \frac{rs}{2^k}\left[\prod_{i=0}^{k-1}\left(2^i(r+s-2) - 2^i + 1\right)\right] & \text{for } k > 1 \end{cases}$$

Therefore, $M_1^\alpha(L^k(K_{r,s})) = \sum_{u \in V(L^k(K_{r,s}))}\left[d_{L^k(K_{r,s})}(u)\right]^\alpha$. Hence, the desired result follows. □

By the above results, we reach to the following.

Corollary 18 *For any star graph $K_{1,s}$ with $s \geq 1$ and $\alpha \geq 2$,*

$$M_1^\alpha(L^k(K_{1,s})) = \begin{cases} s(s-1)^\alpha & \text{for } k = 1; \\ s\left[\prod_{i=0}^{k-2}\left(2^{i-1}(s-1) - 2^i + 1\right)\right] \\ \quad \times \left(2^{k-1}(s-1) - 2^k + 2\right)^\alpha & \text{for } k > 1. \end{cases}$$

Theorem 23 *Let G be a connected r-regular graph with $\alpha \geq 0$ and $k \geq 1$. Then,*

$$M_1^\alpha(L^k(G)) = n\left[\prod_{i=0}^{k-1}\left(2^{i-1}r - 2^i + 1\right)\right]\left(2^k(r-2) + 2\right)^\alpha.$$

Proof Let G be a connected r-regular graph with $\alpha \geq 0$ and $k \geq 1$. Then,

$$M_1^\alpha(L^k(G)) = \sum_{u \in V(L^k(G))}\left[d_{L^k(G)}(u)\right]^\alpha.$$

For a connected r-regular graph G with $|V(G)| = n$, we have

$$|V(L^k(G))| = n\prod_{i=0}^{k-1}\left(2^{i-1}r - 2^i + 1\right),$$

and

$$|E(L^k(G))| = \frac{n}{2^{k+1}}\prod_{i=0}^{k-1}\left(2^i r - 2^{i+1} + 2\right).$$

Further, the kth-iterated line graph of a connected r-regular graph is $\left(2^k r - 2^{k+1} + 2\right)$-regular graph. Therefore,

$$\begin{aligned} M_1^\alpha(L^k(G)) &= \sum_{u \in V(L^k(G))}\left[d_{L^k(G)}(u)\right]^\alpha \\ &= \sum_{u \in V(L^k(G))}\left[2^k r - 2^{k+1} + 2\right]^\alpha \\ &= \sum_{u \in V(L^k(G))}\left[2^k(r-2) + 2\right]^\alpha \\ &= n\left[\prod_{i=0}^{k-1}\left(2^{i-1}r - 2^i + 1\right)\right]\left(2^k(r-2) + 2\right)^\alpha. \end{aligned}$$

□

By the above results, we finally obtain the following results.

Corollary 19 *For any cycle C_n with $n \geq 3$ and $\alpha \geq 2$, we have*

$$M_1^\alpha(L^k(C_n)) = n\,2^\alpha.$$

Corollary 20 *For any complete graph K_n with $n \geq 3$ and $\alpha \geq 2$, we have*

$$M_1^{\alpha}(L^k(K_n)) = n\left[\prod_{i=0}^{k-1}\left(2^{i-1}(n-3)+1\right)\right]\left(2^k(n-3)+2\right)^{\alpha}.$$

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