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Implication of cross-diffusion on the stability of double diffusive convection in an imposed magnetic field

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Abstract. The effects of cross-diffusion on linear and weak nonlinear stability of double diffusive convection in an electrically conducting horizontal fluid layer with an imposed vertical magnetic field are investigated. The criterion for the onset of stationary and oscillatory convection is obtained analytically by performing the linear instability analysis. Several noteworthy departures from those of doubly diffusive fluid systems are unveiled under certain parametric conditions. It is shown that (i) disconnected closed convex oscillatory neutral curve separated from the stationary neutral curve exists requiring three critical thermal Rayleigh numbers to completely specify the linear instability criteria instead of a usual single critical value, (ii) an electrically conducting fluid layer in the presence of magnetic field can be destabilized by stable solute concentration gradient, and (iii) a doubly diffusive conducting fluid layer can be destabilized in the presence of magnetic field. It is demonstrated that small variations in the off-diagonal elements enforce discrepancies in the instability criteria. A weak nonlinear stationary stability analysis has been performed using a perturbation method and a cubic Landau equation is derived and the stability of bifurcating equilibrium solution is discussed. It is found that subcritical bifurcation occurs depending on the choices of governing parameters.

Mathematics Subject Classification. 70K20, 70K50, 34A34, 76E06, 76R50, 76A10.

Keywords. Magnetic field, Cross-diffusion, Double diffusive convection, Nonlinear stability.

1. Introduction

Double diffusive convection in a fluid layer has been studied extensively both theoretically and experimentally due to its wide range of applications in various fields such as solidification of molten alloys, oceanography, high quality crystal production, astrophysics, geothermally heated lakes and magmas to mention a few. When compared to single component fluid systems, a variety of interesting convective phenomena can occur in double diffusive fluid systems wherein the instability sets in even when the basic state is hydrostatically stable. The instability sets in either via fingering or diffusive regime. Excellent reviews on this topic are reported by Tuner [1], Huppert and Turner [2] and Platten and Legros [3].

The gradient of one component contributing to the flux of the other component is recognizable in double diffusive fluid systems and such diffusion transport phenomena are commonly referred to as cross-diffusion. The following linear combination of driving gradients for heat ($\vec{J}_T$) and solute concentration ($\vec{J}_S$) fluxes hold when the system is not too far from the thermodynamic equilibrium

$$\vec{J}_T = -(D_{11}\nabla T + D_{12}\nabla S) \tag{1}$$

$$\vec{J}_S = -(D_{21}\nabla T + D_{22}\nabla S) \tag{2}$$

where T is the temperature, S is the solute concentration, D_{11} is the thermal diffusivity and D_{22} is the solute analog of D_{11}, while D_{12} and D_{21} are the cross diffusion diffusivities called Dufour and Soret coefficients, respectively. Since flux contributions from the gradient of each component are significant, both cross-diffusive terms are to be retained. It is observed that, in some instances, the character of the

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instability mechanism itself differs due to the presence of cross-diffusion terms. For some parameter combinations, predicted finger instability when cross-diffusion is not taken into account becomes a diffusive instability even when the cross diffusion coefficients are two orders of magnitude smaller than the main diagonal elements of the diffusivity matrix.

There exists gargantuan literature on double diffusive convection with Soret and/or Dufour effects. An elementary transition state approach was used by Mortimer and Eyring [4] to obtain a simple model theory for the Soret and Dufour effects. Both Soret and Dufour effects on thermosolutal convection was investigated by Knobloch [5] and showed that the equations are identical to the thermosolutal problem except for a relation between the thermal and solute Rayleigh numbers. An in-depth study of double diffusive convection caused by molecular diffusion in a solute-solute pair for which both Soret and Dufour effects are important was carried out by McDougall [6]. The stability of gravity modulated double cross diffusive fluid layer was taken up by Terrones et al. [7], while a priori bounds and structural stability for double diffusive convection incorporating the Soret effect was considered by Straughan and Hutter [8]. The simultaneous influence of Soret and Dufour effects on linear and nonlinear double diffusive convection in a layer of couple stress fluid was studied by Malashetty et al. [9].

Another deeply studied classical example of double diffusive convection is thermal convection in an electrically conducting fluid layer in the presence of a uniform vertical magnetic field, referred to as magnetoconvection. This is a classical problem and has been studied extensively because of its applications in astrophysical and geophysical problems as well as in many engineering applications [10–17]. In general, it has been shown that the presence of magnetic field is to introduce oscillatory convection as the preferred mode of instability if the magnetic diffusivity is less than the fluid viscosity and the Chandrasekhar number exceeds a threshold value. Moreover, the magnetic field is found to instill stabilizing effect on stationary and oscillatory convection. Harfash and Straughan [18] studied the effect of vertical magnetic field on convective movement of a reacting solute in a viscous incompressible fluid occupying a plane layer. The effect of vertical throughflow on the onset of magnetoconvection was considered by Reza and Gupta [19]. In his review article on magnetoconvection, Stein [20] emphasized the need for future studies pertaining to this field.

The fluid dynamical systems encountered in geophysics and astrophysics often involves compositional gradients in addition to gradients in magnetic field, angular momentum and temperature. As a result, multidiffusive effects may be expected but it becomes a formidable task to make a general study encompassing all these effects. To simplify the problem, particular combinations of effects have been isolated to make the studies manageable. One such study that has been undertaken in the past is the interaction between the magnetic field and double diffusive convection, known as double diffusive magnetoconvection. Lortz [21] was the first to study finite amplitude steady thermohaline magnetoconvection with the object of clarifying some of the mathematical aspects of the so-called relative stability criterion of Malkus and Veronis [22] and provided the behaviour of steady solutions. Rudraiah and Shivakumara [23] studied linear and nonlinear double diffusive magnetoconvection and the vigor of convection was analyzed by quantizing heat and mass transfer in terms of Nusselt numbers. It was further displayed by Rudraiah and Shivakumara [24] that a doubly diffusive conducting fluid layer can be destabilized with an imposed magnetic field and with the addition of a bottom-heavy solute gradient. In his review article, Rudraiah [25] principally dealt with the interaction between double-diffusive convection and an externally imposed vertical magnetic field in a Boussinesq fluid. Later, Shivakumara [26] showed the existence of disconnected oscillatory neutral curves indicating the requirement of three critical thermal Rayleigh numbers to specify the linear stability criteria of double diffusive magnetoconvection. Prakash et al. [27] mathematically established that oscillatory motions of growing amplitude in an initially bottom-heavy configuration cannot be manifested in magneto-hydrodynamic triply diffusive convection problem, while the onset of convection in a multicomponent fluid layer in the presence of a uniform magnetic field was analyzed by Prakash et al. [28]. Recently, Naveen Kumar et al. [29] investigated linear and nonlinear stability of double diffusive convection in an electrically conducting couple stress fluid layer in the presence of a uniform vertical

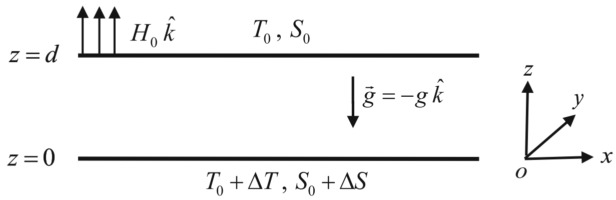

Fig. 1. Physical configuration

magnetic field. They showed that some of the unusual results observed in the case of Newtonian fluids will carry over to the electrically conducting couple stress fluids as well.

In all the aforementioned investigations, the effects of cross-diffusion (Soret and Dufour effects) are not considered despite their occurrence in noticeable magnitude and their decisive influence on the nature of convective instability. In this context, the study turns out to be of paramount importance not only in astrophysical and geophysical applications but also finds its relevance in crystal growth techniques wherein unwanted convection with compositional gradients has to be controlled to enhance crystal purity. These unavoidable convective motions can be effectively controlled with the help of magnetic field which influence convection via the Lorentz force to inhibit the motion perpendicular to the field. As a result, the overturning motions that are essential for convection are suppressed and convective energy transport from the interior to the surface is reduced. Under the circumstances, it will be intriguing to emphasize the impact of cross diffusion terms on linear and nonlinear double diffusive magnetoconvection and to examine their role on some of the novel results explored in their absence. We have initiated this fundamental study in the present paper by considering both Soret and Dufour effects. The linear instability problem is investigated using normal mode analysis and the condition for stationary and oscillatory convection is delineated. A weak nonlinear stationary stability analysis is carried out based on perturbation method and the stability of bifurcating solution is analyzed by deriving a cubic Landau equation.

2. Mathematical formulation

The schematic configuration of the problem under consideration is shown in Fig. 1. It consists of an incompressible electrically conducting binary fluid layer of depth d in the presence of a uniform, externally imposed magnetic field $\vec{H} = H_0\hat{k}$, where $\hat{k}$ is the unit vector in the vertical direction. The temperature T and the solute concentration S are taken as two diffusing components with cross diffusion contributions to the fluxes. A Cartesian coordinate system is chosen with the origin at the bottom of the fluid layer and the z-axis is pointing vertically upward. The gravity is acting in the negative vertical direction. On the bottom boundary $z = 0$, T is maintained at $T_0 + \Delta T$ and S at $S_0 + \Delta S$ while on the top boundary $z = d$, they are maintained at T_0 and S_0, where ΔT, $\Delta S > 0$.

The Alfvén speed is much smaller than the sound speed and the continuity equation is

$$\nabla \cdot \vec{q} = 0 \tag{3}$$

where $\vec{q}$ is the velocity. The fluid density depends on both temperature, solute concentration and also, in general, on the pressure. For sufficiently small isobaric changes in T and S the fluid obeys an Oberbeck-Boussinesq equation of state wherein the density depends linearly on temperature and solute concentration and we have approximately (Veronis [30])

$$\rho = \rho_0[1 - \beta_T(T - T_0) + \beta_S(S - S_0)] \tag{4}$$

where $\beta_T = -(\partial\rho/\partial T)_{S,p}/\rho_0$ is the isobaric thermal expansion coefficient, $\beta_S = (\partial\rho/\partial S)_{T,p}/\rho_0$ is the isobaric solute concentration expansion coefficient, p is the pressure and ρ_0 is the fluid density at reference temperature T_0 and solute concentration S_0. The validity of the Oberbeck–Boussinesq approximation has been proved by Rajagopal et al. [31]. Fluctuations in fluid density affect only the buoyancy term in the equation of motion, which then takes the form [10,23]

$$\rho_0 \left\{ \frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} \right\} = -\nabla P + \rho\vec{g} + \mu(\vec{H}\cdot\nabla)\vec{H} + \mu_f \nabla^2 \vec{q} \tag{5}$$

where $\vec{H}$is the magnetic field, $P = p + \mu H^2/2$ is the total pressure, $\vec{g}$ is the gravitational acceleration, μ is the magnetic permeability and μ_f is the fluid viscosity.

The magnetic field satisfies the equation

$$\nabla\cdot\vec{H} = 0 \tag{6}$$

and then the evolution of the magnetic field is governed by the induction equation

$$\frac{\partial \vec{H}}{\partial t} = \nabla\times(\vec{q}\times\vec{H}) + \nu_m \nabla^2 \vec{H} \tag{7}$$

where ν_m is the magnetic viscosity assumed to be uniform.

Following Fourier's law and Fick's law, the diffusion equations are, respectively

$$\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)T + \nabla\cdot\vec{J}_T = 0 \tag{8}$$

$$\frac{\partial S}{\partial t} + (\vec{q}\cdot\nabla)S + \nabla\cdot\vec{J}_S = 0. \tag{9}$$

On using Eqs. (1) and (2), Eqs. (8) and (9), can be written in the matrix form

$$\left\{ \frac{\partial}{\partial t} + (\vec{q}\cdot\nabla) \right\} \begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \nabla^2 T \\ \nabla^2 S \end{bmatrix}. \tag{10a,b}$$

The basic state is steady and quiescent, and the basic solution whose stability is under investigation is

$$\begin{aligned} &\vec{q}_b = 0,\ \vec{H}_b = H_0\hat{k},\ T_b = T_0 + \Delta T\left(1 - \frac{z}{d}\right),\ S_b = S_0 + \Delta S\left(1 - \frac{z}{d}\right) \\ &P_b = P_0 - \rho_0 g\left\{ z - \beta_T \Delta T\left(z - \frac{z^2}{2d}\right) + \beta_S \Delta S\left(z - \frac{z^2}{2d}\right) \right\} \end{aligned} \tag{11}$$

where the subscript b denotes the basic state and P_0 is the pressure at $z = 0$. To examine the stability of the above basic state solution, the variables are perturbed in the form

$$\vec{q} = \vec{q}_b + \vec{q}',\ \vec{H} = \vec{H}_b + \vec{H}',\ T = T_b + T',\ S = S_b + S',\ P = P_b + P',\ \rho = \rho_b + \rho' \tag{12}$$

where $\vec{q}'$, $\vec{H}'$, T', S', P' and ρ' are the perturbed velocity, magnetic field, temperature, solute concentration, pressure and density, respectively. Substituting Eq. (12) into the governing equations, we obtain (after neglecting the primes for simplicity)

$$\nabla\cdot\vec{q} = 0 \tag{13}$$

$$\rho_0 \left\{ \frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} \right\} = -\nabla P - (\beta_T T - \beta_S S)\vec{g} + \mu H_0 \frac{\partial \vec{H}}{\partial z} + \mu(\vec{H}\cdot\nabla)\vec{H} + \mu_f \nabla^2 \vec{q} \tag{14}$$

$$\left\{ \frac{\partial}{\partial t} + (\vec{q}\cdot\nabla) \right\} \begin{bmatrix} T \\ S \end{bmatrix} = \begin{bmatrix} \Delta T \\ \Delta S \end{bmatrix} \frac{w}{d} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \nabla^2 T \\ \nabla^2 S \end{bmatrix} \tag{15a,b}$$

$$\nabla\cdot\vec{H} = 0 \tag{16}$$

$$\frac{\partial \vec{H}}{\partial t} = \nabla\times(\vec{q}\times\vec{H}) + (\vec{H}_0\cdot\nabla)\vec{q} + \nu_m \nabla^2 \vec{H}. \tag{17}$$

Because of the one dimensionality of the basic state and the horizontal isotropy of the problem, the analysis is restricted to two-dimensional motions and the stream functions are introduced in the form

$$\vec{q} = \left(\frac{\partial \psi}{\partial z},\ 0\,,\ -\frac{\partial \psi}{\partial x}\right), \tag{18}$$

$$\vec{H} = \left(\frac{\partial A}{\partial z},\ 0\,,\ -\frac{\partial A}{\partial x}\right) \tag{19}$$

where $\psi(x,z,t)$ is the velocity stream function and $A(x,z,t)$ is the magnetic field stream function. Eliminating the pressure term from Eq. (14) by cross differentiation and non-dimensionalizing the quantities using d, d^2/D_{11}, D_{11}, $\nu D_{11}/\beta_T g d^3$, $\nu D_{11}/\beta_S g d^3$ and H_0/d as the units of length, time, velocity stream function, temperature, solute concentration and magnetic stream function, respectively, the resulting stability equations can then be written in the operator form

$$(\mathscr{L}_1 + \mathscr{L}_2)\Psi = \left[\, \tfrac{1}{Pr} J\left(\psi, \nabla^2\psi\right),\, -R_m J\left(A, \nabla^2 A\right),\, J\left(\psi, T\right),\, J\left(\psi, S\right),\, J\left(\psi, A\right)\right]^t \tag{20}$$

where $\Psi = \left[\,\psi,\, T,\, S,\, A\right]^t$, $J(\cdot\ ,\ \cdot)$ is the Jacobian, $\mathscr{L}_1$ and $\mathscr{L}_2$ are the linear differential operators given by

$$\mathscr{L}_1 = \begin{bmatrix} \frac{1}{Pr}\frac{\partial}{\partial t}\nabla^2 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial t} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial t} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial t} \end{bmatrix}, \mathscr{L}_2 = \begin{bmatrix} -\nabla^4 & \frac{\partial}{\partial x} & -\frac{\partial}{\partial x} & -R_m \frac{\partial}{\partial z}\nabla^2 \\ R_T \frac{\partial}{\partial x} & -\tau_{11}\nabla^2 & -\tau_{12}\nabla^2 & 0 \\ R_S \frac{\partial}{\partial x} & -\tau_{21}\nabla^2 & -\tau_{22}\nabla^2 & 0 \\ -\frac{\partial}{\partial z} & 0 & 0 & -\tau_m \nabla^2 \end{bmatrix}. \tag{21}$$

Here, $R_T = g\beta_T d^3 \Delta T/\nu D_{11}$ is the thermal Rayleigh number, $R_S = \beta_S g d^3 \Delta S/\nu D_{11}$is the solute Rayleigh number, $Q = \mu H_0^2 d^2/\rho_0 \nu \nu_m$ is the Chandrasekhar number, $\Pr = \nu/D_{11}$is the Prandtl number, $\tau_m = \nu_m/D_{11}$ is the ratio of diffusivities, $\tau_{12} = \beta_T D_{12}/\beta_S D_{11}$ is the Dufour number, $\tau_{21} = \beta_S D_{21}/\beta_T D_{11}$ is the Soret number, $\tau_{11} = 1$ and $\tau_{22} = D_{22}/D_{11}$ is the ratio of diffusivities. We note that $R_m (= Q\tau_m)$ is like thermal Rayleigh number has the product νD_{11} in the denominator and independent of ν_m. Thus, the problem considered can be identified as a triply diffusive convection problem with magnetic field as a third diffusing component.

We confine ourselves to the problem with stress free surfaces, which are considered to be flat and perfect conductors of heat and solute concentration. Also, the tangential component of the magnetic field is continuous across the boundaries. The required boundary conditions are [13]

$$\psi = \frac{\partial^2 \psi}{\partial z^2} = T = S = \frac{\partial A}{\partial z} = 0 \text{ at } z = 0, 1. \tag{22}$$

3. Linear instability analysis

We neglect the nonlinear terms in Eq. (21) and assume the solution satisfying Eq.(22) in the form

$$[\psi,\ T,\ S] = [B_1 \sin(\alpha x),\ B_2 \cos(\alpha x),\ B_3 \cos(\alpha x)] e^{\sigma\, t} \sin(\pi z), A = B_4 \sin(\alpha x)\ \cos(\pi z) \tag{23}$$

where α is the horizontal wave number, σ is the growth term and $B_1 - B_4$ are constants. Substituting Eq. (23) into the linearized version of Eq. (21), we obtain a system of algebraic equations which can be written in the following matrix form

$$\begin{bmatrix} \sigma\,\delta + \delta^2 Pr & \alpha Pr & -\alpha Pr & \pi\,\delta Pr R_m \\ \alpha R_T & \sigma + \delta\,\tau_{11} & \delta\,\tau_{12} & 0 \\ \alpha R_S & \delta\,\tau_{21} & \sigma + \delta\,\tau_{22} & 0 \\ -\pi & 0 & 0 & \sigma + \delta\,\tau_m \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

where $\delta = \alpha^2 + \pi^2$. A non-trivial solution to the above system of equations exists if and only if

$$\begin{vmatrix} \sigma\,\delta + \delta^2 Pr & \alpha Pr & -\alpha Pr & \pi\,\delta Pr R_m \\ \alpha R_T & \sigma + \delta\,\tau_{11} & \delta\,\tau_{12} & 0 \\ \alpha R_S & \delta\,\tau_{21} & \sigma + \delta\,\tau_{22} & 0 \\ -\pi & 0 & 0 & \sigma + \delta\,\tau_m \end{vmatrix} = 0. \tag{25}$$

For neutral solutions, we put $\sigma = i\omega$ in Eq. (25) and obtain an expression for R_T, after clearing the complex quantities from the denominator, in the form

$$R_T = \frac{1}{\mathrm{M}_3}\left(\mathrm{M}_1 + i\omega\delta\,\mathrm{M}_2\right) \tag{26}$$

where

$$\begin{aligned}
\mathrm{M}_1 &= \left(\delta^2\tau_m^2 + \omega^2\right)\left[Pr\left\{a_4\delta^2\left(a_3\alpha^2 R_S + a_2\delta^3\right) + \left(\alpha^2 R_S + a_5\delta^3\right)\omega^2\right\} - \delta\,\omega^2\left[\delta^2\left(a_3\tau_{21} + a_4\tau_{22}\right) + \omega^2\right]\right] \\
&\quad + Pr\pi^2\delta R_m\left[a_2 a_4\delta^4\tau_m + \delta^2\left(a_3\tau_{21} + a_4\tau_{22} + a_5\tau_m\right)\omega^2 + \omega^4\right] \\
\mathrm{M}_2 &= \left(\delta^2\tau_m^2 + \omega^2\right)\left\{\delta\left(a_2\,a_4\delta^4 + a_5\,\omega^2\right) + Pr\left(\alpha^2 R_S\left(a_4 - a_3\right) + \delta^2\left(a_3\tau_{21} + a_4\tau_{22}\right) + \delta\,\omega^2\right)\right\} \\
&\quad + Pr\pi^2\delta R_m\left[\delta^2\left(a_3\tau_{21}\tau_m + a_4\tau_{22}\tau_m - a_2 a_4\right) + \left(\tau_m - a_5\right)\omega^2\right] \\
\mathrm{M}_3 &= \alpha^2 Pr\left(\delta^2\tau_m^2 + \omega^2\right)\left(a_4^2\,\delta^2 + \omega^2\right)
\end{aligned}$$

with

$$\begin{aligned}
&b_1 = \delta; b_2 = \delta^2\left[Pr + \tau_m + a_1\right]; b_3 = \delta^3\left(a_2 + a_1\tau_m\right) + \alpha^2 Pr\left(R_S - R_T\right) + \pi^2 Pr\delta R_m + \delta^3 Pr\left(a_1 + \tau_m\right) \\
&b_4 = a_2\delta^4\tau_m + Pr\delta\left[\alpha^2 R_S\left(a_3 + \tau_m\right) - \alpha^2 R_T\left(a_4 + \tau_m\right) + \delta\left(a_1\pi^2 R_m + \delta^2\left(a_2 + a_1\tau_m\right)\right)\right] \\
&b_5 = Pr\delta^2\left[a_2\,\pi^2\delta R_m + \left(a_3\alpha^2 R_S - a_4\alpha^2 R_T + a_2\delta^3\right)\tau_m\right] \\
&a_1 = \tau_{11} + \tau_{22}; a_2 = \tau_{11}\tau_{22} - \tau_{12}\tau_{21}; a_3 = \tau_{11} + \tau_{12}; a_4 = \tau_{22} + \tau_{21}; a_5 = \tau_{11} - \tau_{21}.
\end{aligned}$$

The thermal Rayleigh number R_T being a physical quantity, it implies either $\omega = 0$ or $\mathrm{M}_2 = 0$ in Eq. (26). The condition $\omega = 0$ corresponds to stationary convection and the case $\mathrm{M}_2 = 0 (\omega \neq 0)$ corresponds to oscillatory convection.

3.1. Stationary convection ($\omega = 0$)

The stationary onset occurs at $R_T = R_T^s$, where

$$R_T^s = \frac{a_3}{a_4}R_S + \frac{a_2}{a_4}\left[\frac{\delta^3}{\alpha^2} + \frac{Q\pi^2\delta}{\alpha^2}\right]. \tag{27}$$

The above equation indicates that the cross-diffusion terms influence the stationary onset. The minimum value of R_T^s can be found by setting the derivative of Eq. (27) with respect to α equal to zero. Then, we get an expression for the critical wave number α_c as

$$2\alpha_c^6 + 3\pi^2\alpha_c^4 - \pi^4\left(Q + \pi^2\right) = 0. \tag{28}$$

It is worth noting that α_c is independent of not only additional diffusing component but also cross-diffusion terms and coincides with that of Chandrasekhar [10]. The critical value of R_T^s is denoted by R_{Tc}^s and determined numerically from Eq. (27) for various values of governing parameters. In the absence of cross-diffusion terms, Eq. (27) becomes

$$R_T^s = \frac{R_S}{\tau_{22}} + \frac{\delta^3}{\alpha^2} + \left(\frac{\pi^2\delta}{\alpha^2}\right)Q. \tag{29}$$

This result coincides with that of Rudraiah and Shivakumara [23]. In the absence of additional diffusing component (i.e. $R_S = 0$), the above equation coincides with Chandrasekhar [10] and Proctor and Weiss [13]. In the absence of magnetic field (i.e.,$Q = 0$), Eq. (23) reduces to

$$R_T^s = \frac{R_S}{\tau_{22}} + \frac{\delta^3}{\alpha^2} \tag{30}$$

which coincides with Veronis [30]. Equation (30) attains its critical value at $\alpha_c^2 = \pi^2/2$ and the critical value is

$$R_{Tc}^s = \frac{R_S}{\tau_{22}} + \frac{27\pi^4}{4}. \tag{31}$$

3.2. Oscillatory convection ($\omega \neq 0$, $\mathrm{M_2} = 0$)

The condition $\mathrm{M}_2 = 0$ gives a dispersion relation of the form

$$m_1(\omega^2)^2 + m_2(\omega^2) + m_3 = 0 \tag{32}$$

where

$$\begin{aligned}
m_1 &= \delta\,(Pr + a_5)\\
m_2 &= \delta^3\left[a_2a_4 + a_5\tau_m^2\right] + Pr\left\{\alpha^2R_S\,(a_4 - a_3) + \delta\left[\pi^2R_m\,(\tau_m - a_5) + \delta^2\left(a_3\tau_{21} + a_4\tau_{22} + \tau_m^2\right)\right]\right\}\\
m_3 &= \delta^2Pr\tau_m^2\left\{\alpha^2R_S\,(a_4 - a_3) + \frac{a_2a_4\,\delta^3}{Pr} + \delta^3\,(a_3\tau_{21} + a_4\tau_{22}) + \frac{\pi^2\delta R_m}{\tau_m^2}\left[(a_3\tau_{21} + a_4\tau_{22})\,\tau_m - a_2a_4\right]\right\}.
\end{aligned}$$

Equation (32) may produce two real positive values of ω^2 corresponding to two different onset frequencies at the same wave number α for some suitable combination of governing parameters. In that case, for each one of these frequency values ($\omega^2 > 0$), there is a corresponding real value of R_T on the oscillatory neutral curve which may have important consequences as far as the linear instability of the system is concerned.

The oscillatory thermal Rayleigh number can be obtained from

$$R_T^o = \frac{\mathrm{M}_1}{\mathrm{M}_3} \tag{33}$$

and ω^2 is given by Eq. (32). It is difficult to find the critical value of $\alpha$and the corresponding critical oscillatory number R_{Tc}^o analytically. The critical values of R_T^o are determined numerically for various values of governing parameters. For any chosen governing parameters, the value of R_T^o with respect to α is determined as follows. Equation (32) is solved first to determine the positive values of ω^2. If there are none, then no oscillatory convection is possible. If there is only one positive value of ω^2 then the critical value of R_T^o with respect to the wave number is computed numerically from Eq. (33) and is denoted by R_{Tc}^o. If there are two positive values of ω^2, then the least of R_T^o amongst the positive values of ω^2 is retained and the critical value of R_T^o with respect to the wave number is computed numerically. The smaller value amongst R_{Tc}^o and R_{Tc}^s is called the critical thermal Rayleigh number and denoted by R_{Tc}.

4. Weak nonlinear stability analysis

The linear instability theory gives us the condition for instability but do not predict the amplitude of convective motion and the stability of bifurcating solution (subcritical/supercritical). Because of these reasons, a weak nonlinear stationary stability analysis has been carried out using a well-known perturbation method [32,33]. The dependent variables ψ, T, C, A and R_D are expanded in powers of $\chi(\ll 1)$ as follows

$$R_T = R_{Tc} + \chi^2 R_{T2} + \cdots,\ \psi = \sum_{i=1}^{\infty} \chi^i \psi_i,\ T = \sum_{i=1}^{\infty} \chi^i T_i,\ S = \sum_{i=1}^{\infty} \chi^i S_i,\ A = \sum_{i=1}^{\infty} \chi^i A_i. \tag{34}$$

Also a slow time scale s is introduced in the form $s = \chi^2 t$ and the operator $\partial/\partial t$ is replaced by $\partial/\partial t = \chi^2 \partial/\partial s$. Substituting Eq. (34) into Eq. (21), we get a set of partial differential equations at each order in χ.

At the first-order in χ, we get a system of homogeneous linear partial differential equations

$$\mathscr{L}_2 \Psi_1 = \left[\,0\ 0\ 0\ 0\,\right]^t \tag{35}$$

where $\Psi_i = \left[\,\psi_i\ T_i\ S_i\ A_i\,\right]^t\ (i = 1, 2, 3, \cdots)$. This corresponds to linear instability problem for the stationary case. The eigenvalue and the eigenfunctions are given in the previous section. Let

$$\psi_1 = \psi,\ \ T_1 = T,\ \ S_1 = S,\ \ A_1 = A. \tag{36}$$

The undetermined amplitudes $B_1 - B_4$ are related by

$$B_2 = \frac{\alpha B_1}{\delta D}\left[\tau_{12} R_S - \tau_{22} R_T\right],\ B_3 = \frac{\alpha B_1}{\delta D}\left[\tau_{21} R_T - \tau_{11} R_S\right],\ B_4 = \frac{\pi B_1}{\delta\,\tau_m} \tag{37}$$

and the amplitude B_1 will be determined at a later stage.

At the second-order in χ, we get a system of inhomogeneous partial differential equations

$$\mathscr{L}_2 \Psi_2 = \left[\,0\ J(\psi_1, T_1)\ J(\psi_1, S_1)\ J(\psi_1, A_1)\,\right]^t, \tag{38}$$

where

$$\begin{bmatrix} J(\psi_1, T_1) \\ J(\psi_1, S_1) \\ J(\psi_1, A_1) \end{bmatrix} = \frac{\pi \alpha B_1}{2} \begin{bmatrix} B_2 \sin(2\pi z) \\ B_3 \sin(2\pi z) \\ B_4 \sin(2\alpha x) \end{bmatrix}.$$

The solution is

$$\begin{aligned} &\psi_2 = 0; \quad T_2 = \frac{\alpha^2 B_1^2}{8\pi\delta D^2}\left\{a_6 R_S - a_7 R_T\right\}\sin(2\pi z), \\ &S_2 = \frac{\alpha^2 B_1^2}{8\pi\delta D^2}\left\{a_8 R_T - a_9 R_S\right\}\sin(2\pi z); \quad A_2 = \frac{-\pi^2 B_1^2}{8\alpha\delta\tau_m^2}\sin(2\alpha x) \end{aligned} \tag{39}$$

where

$$a_6 = \tau_{12}\tau_{22} + \tau_{11}\tau_{12}; a_7 = \tau_{22}\tau_{22} + \tau_{21}\tau_{12}; a_8 = \tau_{11}\tau_{21} + \tau_{21}\tau_{22} \quad ; \quad a_9 = \tau_{11}\tau_{11} + \tau_{21}\tau_{12}.$$

At the third-order in χ, we obtain

$$\mathscr{L}_2 \Psi_3 = \left[\,\Delta_1\ \Delta_2\ \Delta_3\ \Delta_4\,\right]^t, \tag{40}$$

where

$$\begin{aligned} \Delta_1 &= \left\{\frac{\delta}{Pr}\frac{dB_1}{ds} - \frac{\pi^4 R_m}{8\,\delta\,\tau_m^3}\left[1 - \frac{4\alpha^2}{\delta}\right]B_1^3\right\}\sin(\alpha x)\sin(\pi z) + \cdots, \\ \Delta_2 &= -\left\{\frac{dB_2}{ds} + \alpha R_{T2} B_1 + \frac{\alpha^3}{8\delta D^2}\left\{a_6 R_S - a_7 R_T\right\}B_1^3\right\}\cos(\alpha x)\sin(\pi z) + \cdots \\ \Delta_3 &= -\left\{\frac{dB_3}{ds} + \frac{\alpha^3}{8\delta D^2}\left\{a_8 R_T - a_9 R_S\right\}B_1^3\right\}\cos(\alpha x)\sin(\pi z) + \cdots, \\ \Delta_4 &= -\left\{\frac{dB_4}{ds} + \frac{\pi^3 B_1^3}{8\delta\tau_m^2}\right\}\sin(\alpha x)\cos(\pi z) + \cdots. \end{aligned}$$

The procedure outlined in the works of Rosenblat [32] and Raghunatha et al. [33], yields a cubic-Landau equation of the form

$$\Gamma \frac{dB_1}{ds} = \frac{8\delta}{\alpha^2} R_{T2}\, B_1 - \Omega B_1^3 \tag{41}$$

where

$$\Gamma = \frac{8a_2\delta^3}{a_4\alpha^4 Pr} + \frac{8}{a_2\alpha^2}\left\{\left(\tau_{22} + \frac{a_3}{a_4}\tau_{21}\right)R_T^s - \left(\tau_{12} + \frac{a_3}{a_4}\tau_{11}\right)R_S\right\} - \frac{8a_2\delta\pi^2 R_m}{a_4\alpha^4\tau_m^2}$$
$$\Omega = \frac{R_T^s}{a_2^2}\left[a_7 + \frac{a_3}{a_4}a_8\right] - \frac{R_S}{a_2^2}\left[\frac{a_3}{a_4}a_9 + a_6\right] - \frac{2a_2\,\pi^4\delta\, R_m}{a_4\,\alpha^4\tau_m^3}\left(1 - \frac{2\alpha^2}{\delta}\right).$$

The coefficient of nonlinear term in Eq. (41) is Ω and its sign decides the nature of bifurcation (subcritical/supercritical) in the neighborhood of R_T^s. For the steady state, Eq. (41) becomes

$$B_1^2 \frac{\alpha^2}{8(\alpha^2+\pi^2)} = \frac{R_{T2}}{\Omega}. \tag{42}$$

The above equation suggests that R_{T2} and Ω must have the same sign, and $R_{T2} = \phi\,\Omega$, where ϕ is a positive constant. Without loss in generality, we can choose $\phi = 1$ and then we have

$$R_{T2} = \Omega. \tag{43}$$

The bifurcation is said to be stable if $R_{T2} > 0$ (supercritical) and unstable if $R_{T2} < 0$ (subcritical). Since the expression for R_{T2} is cumbersome, no definite conclusion can be drawn directly about the stability of bifurcating solution. Hence, the expression for R_{T2} has been evaluated numerically to analyze the stability of bifurcating solution for different values of governing parameters. In the absence of cross diffusion, the expression for R_{T2} reduces to

$$R_{T2} = \frac{\left(\tau_{22}^2 - 1\right)}{\tau_{22}^3} R_S + \frac{\delta^3}{\alpha^2} + \frac{\pi^2\delta}{\alpha^2}\frac{R_m}{\tau_m} + \frac{2\pi^4\left(\alpha^2-\pi^2\right)}{\alpha^4}\frac{R_m}{\tau_m^3}. \tag{44}$$

From Eq. (44) it is observed that if $\alpha^2 \geq \pi^2$ and $\tau_{22} > 1$, then the bifurcation turns out to be always supercritical. In the absence of additional diffusing component ($R_S = 0$), Eq. (44) becomes

$$R_{T2} = \frac{\delta^3}{\alpha^2} + \frac{\pi^2\delta}{\alpha^2}\frac{R_m}{\tau_m} + \frac{2\pi^4\left(\alpha^2-\pi^2\right)}{\alpha^4}\frac{R_m}{\tau_m^3} \tag{45}$$

and this expression coincides with that of Knobloch et al. [34] obtained using a modified perturbation method with self-adjoint operator technique. In the absence of magnetic field ($R_m = 0$), the above equation reduces to $R_{T2} = \delta^3/\alpha^2 > 0$ and note that the subcritical instability is not possible. In the absence of magnetic field, Eq. (44) reduces to

$$R_{T2} = \frac{\left(\tau_{22}^2 - 1\right)}{\tau_{22}^3} R_S + \frac{\delta^3}{\alpha^2} \tag{46}$$

and coincides with Nagata and Thomas [35] obtained using the functional analysis approach.

5. Results and discussion

The Soret (thermo-diffusion) and Dufour (diffusion-thermo) effects on the linear instability and a weak nonlinear stability of doubly diffusive electrically conducting fluid layer in the presence of a uniform vertical magnetic field are investigated. The results obtained by performing the above analyses are discussed separately in the following sub-sections.

5.1. Linear instability analysis

Analytical expression for the occurrence of stationary and oscillatory convection is delineated using the linear instability analysis. The presence of magnetic field acts like a third diffusing component and the study serves as an example of a triple diffusive fluid system. Some novel results are uncovered from the present analysis.

To validate the results, thermal Rayleigh numbers were numerically determined for the onset of stationary (R_T^s) and oscillatory (R_T^o) convection under the limiting conditions. The results obtained for different values of Q and τ_m when $R_S = 0$ and $\tau_{12} = \tau_{21} = \tau_{22} = 0$ (i.e. in the absence of solute concentration and cross-diffusion terms) for a fixed value of $\mathrm{Pr} = 1$ and $\alpha = \pi$ are compared with those of Weiss [11] in Table 1. The results are found to be in excellent agreement. Besides, the critical thermal Rayleigh number R_{Tc} computed numerically as detailed before for different values of R_S and Q for $\mathrm{Pr} = 7$, $\tau_m = 3$, $\tau_{11} = 1$, $\tau_{22} = 0.01$ in the absence of cross-diffusion ($\tau_{12} = \tau_{21} = 0$) are tabulated in Table 2 along with those of Rudraiah [25] (shown within the parenthesis). From the table it is seen that the results are in total agreement.

The variation of R_{Tc} as a function of solute Rayleigh number R_s for different values of $Q = 0$, 100 and 200 when $\mathrm{Pr} = 7$, $\tau_m = 3$ and $\tau_{22} = 0.01$ is exhibited in Fig. 2a–d. The results obtained in the absence of cross diffusion effect ($\tau_{12} = \tau_{21} = 0$) are illustrated in Fig. 2a. It is observed that there is a discontinuity in the curves of R_{Tc} for each value of Q very near to $R_s = 0$, called the threshold value (i.e. the value beyond which the oscillatory convection is preferred). The curve lying to the left and right of the discontinuity corresponds to stationary and oscillatory onset, respectively. The effect of increasing Q is to increase R_{Tc} and thus it has a stabilizing effect on the onset of stationary and oscillatory convection. The relative influence of only Soret and the combined Soret and Dufour effects on the instability characteristics of the system are displayed in Fig. 2b–d. Figure 2b shows the results for $\tau_{12} = 0$ and $\tau_{21} = 0.5$, while Fig. 2c, d, respectively, depict the results for the simultaneous presence of Soret as well as Dufour parameters are less than unity ($\tau_{12} = 0.2$, $\tau_{21} = 0.8$) and greater than unity ($\tau_{12} = 1.2$, $\tau_{21} = 1.8$). In all these cases, the threshold value of R_s increases with increasing Q significantly. Although the cross diffusion effects show marginal influence on the stationary onset, they affect the oscillatory onset prominently.

Figure 3a, b shows the variation of critical thermal Rayleigh number for the stationary onset R_{Tc}^s as a function of R_S for different values of Soret number$\tau_{21}$and Dufour number τ_{12}, respectively for $\mathrm{Pr} = 1$, $Q = 2000$, $\tau_m = 0.25$, and $\tau_{22} = 0.2$. We observe that R_{Tc}^s increases with increasing R_S indicating its effect is to delay the onset of stationary convection. From Fig. 3a, it is evident that increase in the value of τ_{21} is to hasten the onset of stationary convection. However, τ_{12} shows a dual behaviour on the stationary onset depending on the strength of the solute concentration gradient and this is evident from Fig. 3b. We note that there exists a value of R_S below which increase in τ_{12} hastens the onset of convection but beyond which an opposite trend could be seen. Thus, the cross diffusion coefficients influence the instability of the system.

The variation of R_{Tc}^s versus Chandrasekhar number Q for different values of τ_{21} and τ_{12} is shown in Fig. 4a, b, respectively for $\mathrm{Pr} = 1$, $R_S = 5000$, $\tau_m = 0.1$, and $\tau_{22} = 0.2$. We observe that R_{Tc}^s increases with an increase in the value of Q indicating that the effect of magnetic field is to stabilize the fluid motion. Whereas, increase in the value of τ_{21} decreases the value of R_{Tc}^s and thus it has a destabilizing effect on the system for all values of Q considered (see Fig. 4a). But the Dufour number τ_{12} exhibits a mixed behaviour on the stability of the system. That is, increase in τ_{12} leads to instability up to a certain value of Q and exceeding which a reverse trend is noticed (see Fig. 4b).

Nonetheless, the magnetic field and stable solute concentration in the presence of cross diffusion terms display some unusual behaviour on the onset of oscillatory convection and also on the nature of convective instability under certain parametric conditions and these are examined below:

5.1.1. Disconnected oscillatory neutral curves. The occurrence of disconnected oscillatory neutral curves is an integral feature of triply diffusive fluid systems. To account for this trend, a systematic study on the

Table 1. Comparison of values of R_T^s and R_T^o for different values of Q and τ_m for $\mathrm{Pr} = 1$, $\alpha = \pi$, $R_S = 0$, $\tau_{11} = 1$ and $\tau_{12} = \tau_{21} = \tau_{22} = 0$ with those of Weiss [11]

Q	τ_m	R_T^s		R_T^o	
		Weiss [11]	Present study	Weiss [11]	Present study
100	0.100	2753	2753.1936	1051	1051.4900
100	0.050	2753	2753.1936	911	910.9640
200	0.050	4727	4727.1144	963	962.7790
200	0.500	4727	4727.1144	3233	3233.8000
500	0.200	10,649	10,648.8771	2306	2306.5100
1000	0.100	20,518	20,518.4815	2029	2028.5800
2000	0.050	40,258	40,257.6903	1895	1895.4600
4000	0.025	79,736	79,736.1079	1830	1830.3600

TABLE 2. Comparison of values of R_{Tc} for different values of R_S and Q for $\mathrm{Pr} = 7$, $\tau_m = 3$, $\tau_{11} = 1$, $\tau_{22} = 0.01$ and $\tau_{12} = \tau_{21} = 0$ with those of Rudraiah [25] shown within the parenthesis

R_S	$Q = 0$ R_{Tc}	$Q = 100$ R_{Tc}	$Q = 500$ R_{Tc}	$Q = 1000$ R_{Tc}	Mode of instability
−1000	− 99,342.500 (− 99,342.49)	− 97,346.300 (− 97,346.29)	− 91,421.100 (− 91,421.21)	−84,793.000 (− 84,793.00)	Stationary
− 100	− 9342.490 (− 9342.49)	− 7346.290 (− 7346.29)	− 1421.110 (− 1421.12)	5207.010 (5207.00)	Stationary
− 10	− 342.489 (− 342.49)	1653.710 (1653.71)	7578.890 (7578.88)	14,207.000 (14,207.00)	Stationary
0	657.511 (657.51)	2653.710 (2653.71)	8578.890 (8578.88)	15207.000 (15207.00)	Stationary
10	673.798 (673.79)	2686.990 (2686.99)	8655.960 (8656.00)	15,331.400 (15,331.88)	Oscillatory
100	752.660 (752.66)	2786.510 (2786.51)	8763.460 (8763.46)	15,441.400 (15,441.41)	Oscillatory
1000	1541.290 (1541.30)	37,772.760 (3772.76)	9835.450 (9835.45)	16,539.700 (16,539.68)	Oscillatory
10,000	9427.540 (9427.54)	12946.2 (12,946.22)	20,219 (20,219.03)	27,344.5 (27,344.46)	Oscillatory

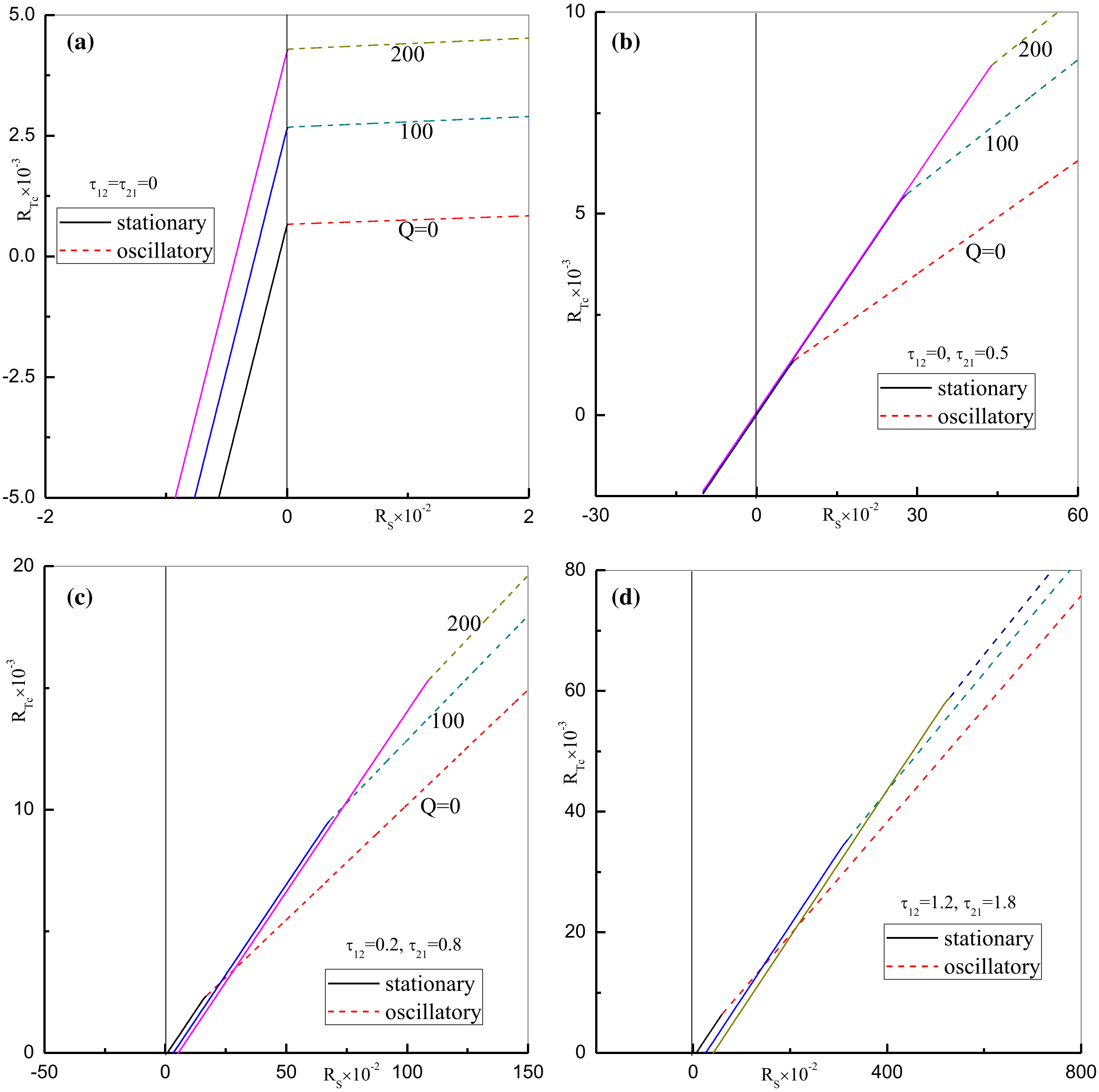

Fig. 2. Variation of R_{Tc} with R_S for different values of Q when Pr $= 7$, $\tau_m = 3$ and $\tau_{22} = 0.01$: **a** without cross diffusion terms ($\tau_{12} = \tau_{21} = 0$), **b** with Soret effect ($\tau_{12} = 0,\ \tau_{21} = 0.5$), **c** with cross diffusion terms ($\tau_{12} = 0.2,\ \tau_{21} = 0.8$) and **d** with cross diffusion terms ($\tau_{12} = 1.2,\ \tau_{21} = 1.8$)

evolution of neutral stability curves is taken up in the parameter space for which there is a possibility of existing two oscillatory neutral solutions. The evolution of neutral stability curves for different values of R_S is shown in Fig. 5a–d. The results presented here are for Pr $= 3$, $Q = 300$, $\tau_m = 5$, $\tau_{12} = 0.35$, $\tau_{21} = 0.7$ and $\tau_{22} = 8.9$ (i.e. $\tau_{22} > \tau_m > 1$). It may be noted that the oscillatory convection is possible even if τ_{22} and τ_m are greater than unity; a result of contrast not found when the solute concentration and magnetic field are present in isolation. Figure 5a exhibits the stationary and oscillatory neutral

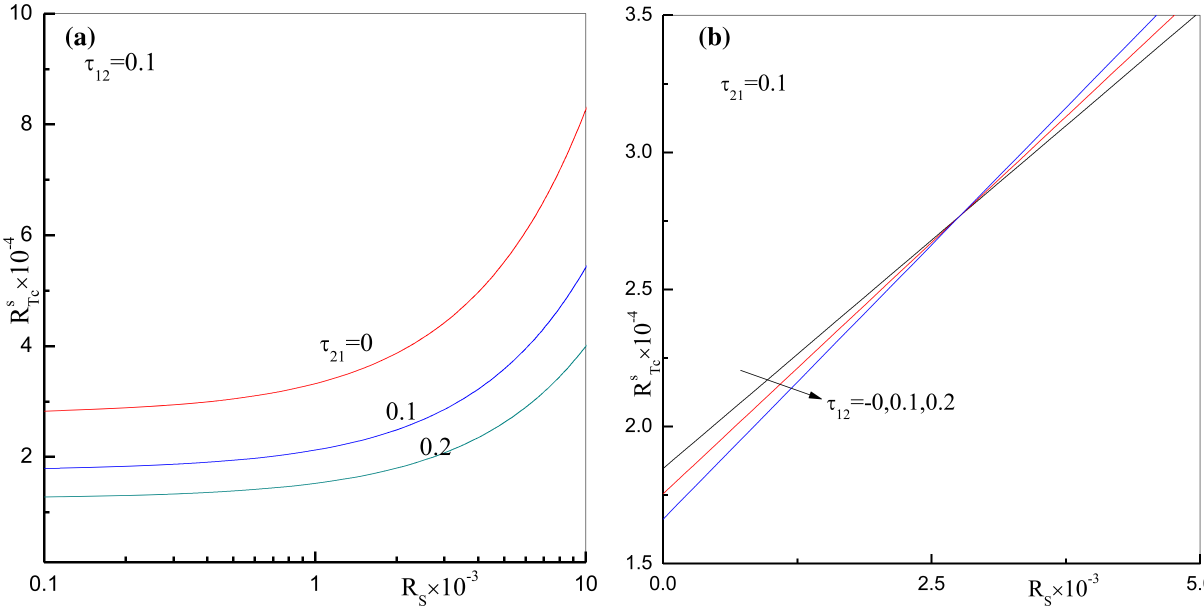

Fig. 3. Variation of R^s_{Tc} with R_S for different values of **a** τ_{21} with $\tau_{12} = 0.1$, and **b** τ_{12} with $\tau_{21} = 0.1$ when $\Pr = 1$, $Q = 2000$, $\tau_m = 0.25$ and $\tau_{22} = 0.2$

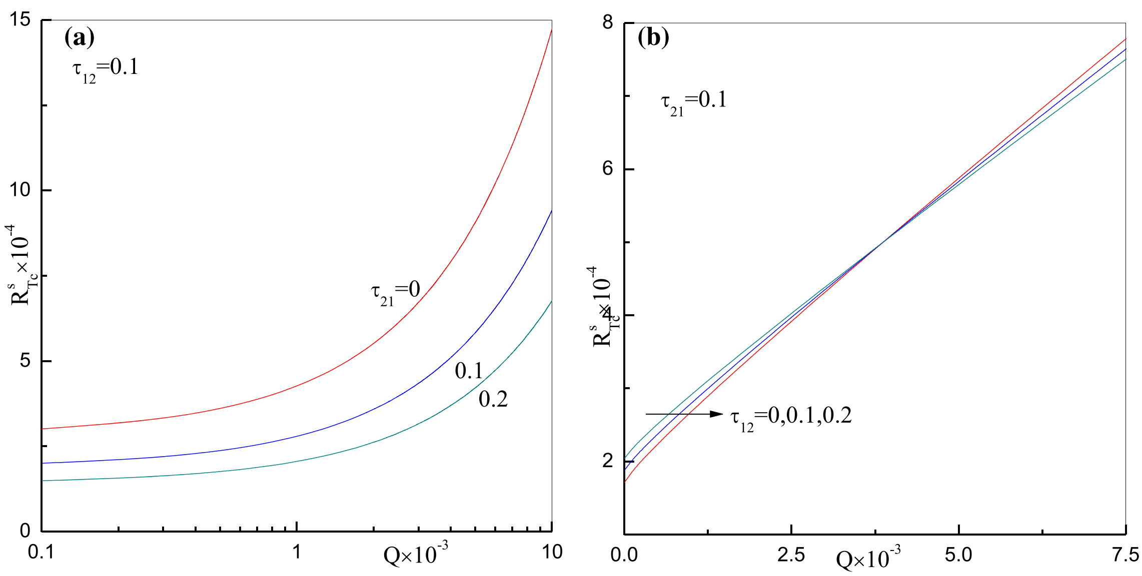

Fig. 4. Variation of R^s_{Tc} with Q for different values of **a** τ_{21} with $\tau_{12} = 0.1$, and **b** τ_{12} with $\tau_{21} = 0.1$ when $\Pr = 1$, $R_S = 5000$, $\tau_m = 0.1$ and $\tau_{22} = 0.2$

curves for $R_S = -56{,}000$ and it is seen that the oscillatory neutral curve is connected to the stationary neutral curve at two bifurcation points signifying a single critical thermal Rayleigh number to establish the linear instability criteria. When $R_S = -50{,}000$, the bifurcation points disappear and the oscillatory neutral curve becomes closed convex curve and detaches from the stationary neutral curve as shown in

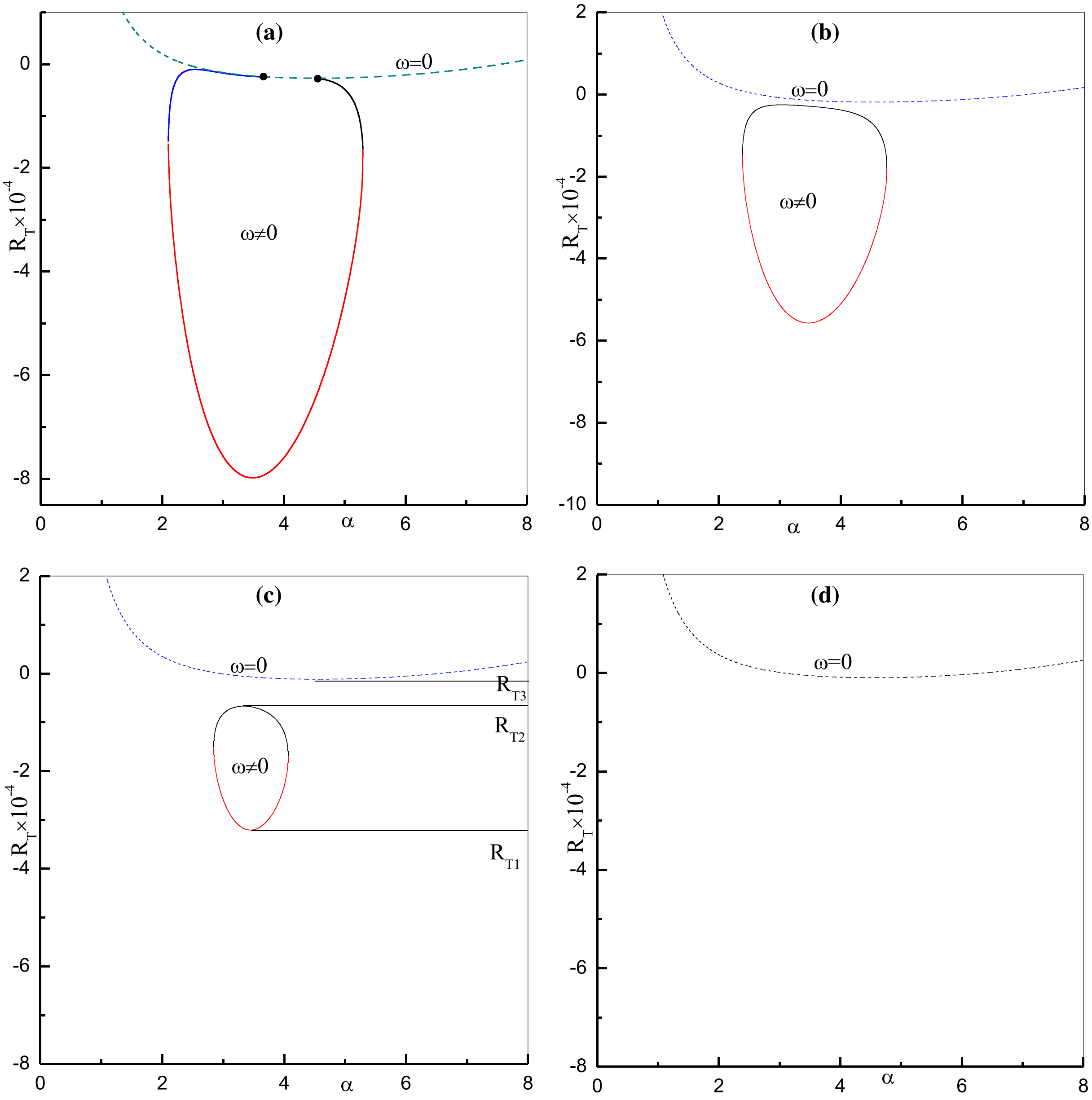

FIG. 5. Evolution of neutral stability curves for Pr = 3, Q = 300, τ_m = 5, τ_{12} = 0.35, τ_{21} = 0.7, and τ_{22} = 8.9: **a** $R_S = -56{,}000$, **b** $R_S = -50{,}000$, **c** $R_S = -45{,}000$, and **d** $R_S = -43{,}600$

Fig. 5b. The closed convex oscillatory neutral curve moves well below the stationary neutral curve for $R_S = -45{,}000$ as shown in Fig. 5c and this has important implication on the instability characteristics of the system. From this figure, it is evident that there exists a range of thermal Rayleigh numbers $R_{T2} < R_T < R_{T3}$ in which all solutions oscillatory or stationary are stable. Thus, the linear instability criteria involve three values of R_T and may be stated as follows. For $R_T < R_{T1}$ and $R_{T2} < R_T < R_{T3}$, the system is linearly stable. For $R_{T1} < R_T < R_{T2}$ and $R_T > R_{T3}$, the system is unstable. Thus, three

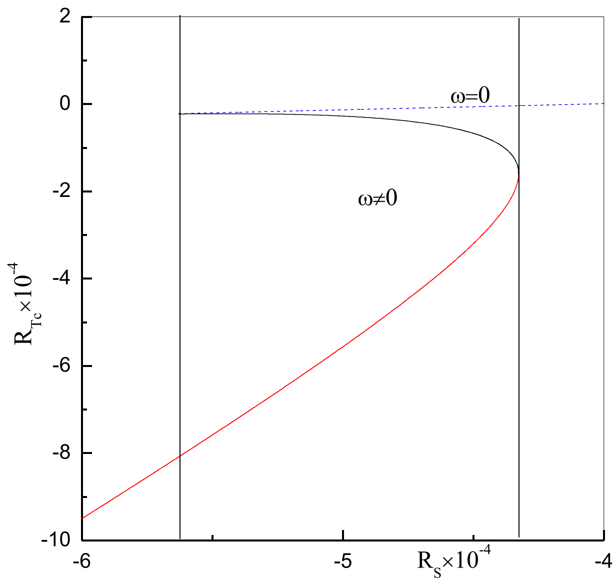

FIG. 6. Stability boundary for Pr = 3, $Q = 300$, $\tau_m = 5$, $\tau_{12} = 0.35$, $\tau_{21} = 0.7$, and $\tau_{22} = 8.9$

critical values of R_T are needed to specify completely the linear instability criteria instead of a usual single value as observed in doubly diffusive fluid systems. Interestingly, the closed convex oscillatory neutral curve reduces in its size with further decrease in the value of $|R_S|$ and subsequently disappears at $R_S = -43{,}600$, leaving behind only the stationary neutral curve (see Fig. 5d).

Figure 6 exemplifies the multivalued nature of the stability boundary in the (R_{Tc}, R_S) plane for the same parametric values considered in Fig. 5. From the figure it is evident that for $R_S < -56259.1815$ the onset is via oscillatory convection and which sets in at a lower value of R_T than does stationary instability with a single critical value of R_T. For $R_S > -43247.639$, the onset is via stationary convection and there is again a single critical value of R_T. But the interesting multivalued nature of the stability boundary portion lies in the range $-56259.1815 < R_S < -43247.639$. In this finite range of R_S, it is obvious that three values of R_{Tc} are needed to specify the linear instability criteria.

Figure 7a–d shows the evolution of neutral stability curves for different values of Q when Pr = 3, $R_S = -56{,}000$, $\tau_m = 5$, $\tau_{12} = 0.35$, $\tau_{21} = 0.7$, and $\tau_{22} = 8.9$. Figure 7a exhibits the stationary and oscillatory neutral curves for $Q = 300$ and note that the oscillatory neutral curve is connected to the stationary neutral curve at two bifurcation points and requiring only one value of R_T to establish the linear instability criteria. The closed oscillatory neutral curve becomes closed and detaches completely from the stationary neutral curve as shown in Fig. 7b when Q takes the value 380. The closed convex oscillatory neutral goes on shrinking with increasing Q (see Fig. 7c) and finally disappears at $Q = 470$ leaving only the stationary neutral curve (see Fig. 7d).

The implication of cross-diffusion terms on the nature of instability of the system is emphasized in Fig. 8a–d which show the evolution of neutral stability curves for Pr = 3, $Q = 300$, $R_S = -55{,}000$, $\tau_m = 5$, $\tau_{21} = 0.7$, and $\tau_{22} = 8.9$. Figure 8a shows the neutral curves when the cross-diffusion terms are absent ($\tau_{12} = \tau_{21} = 0$). For this case, it is observed that the oscillatory neutral curve is attached to the stationary neutral curve at two bifurcation points and hence single critical value of R_T is sufficient to establish the linear instability criteria. However, when the cross-diffusion effects are introduced with $\tau_{12} = 1$ and $\tau_{21} = 0.7$ the oscillatory neutral curve detaches from the stationary neutral curve as shown in Fig. 8b. Keeping the value of Soret number, τ_{21} fixed at 0.7 and increasing the Dufour number τ_{12} to

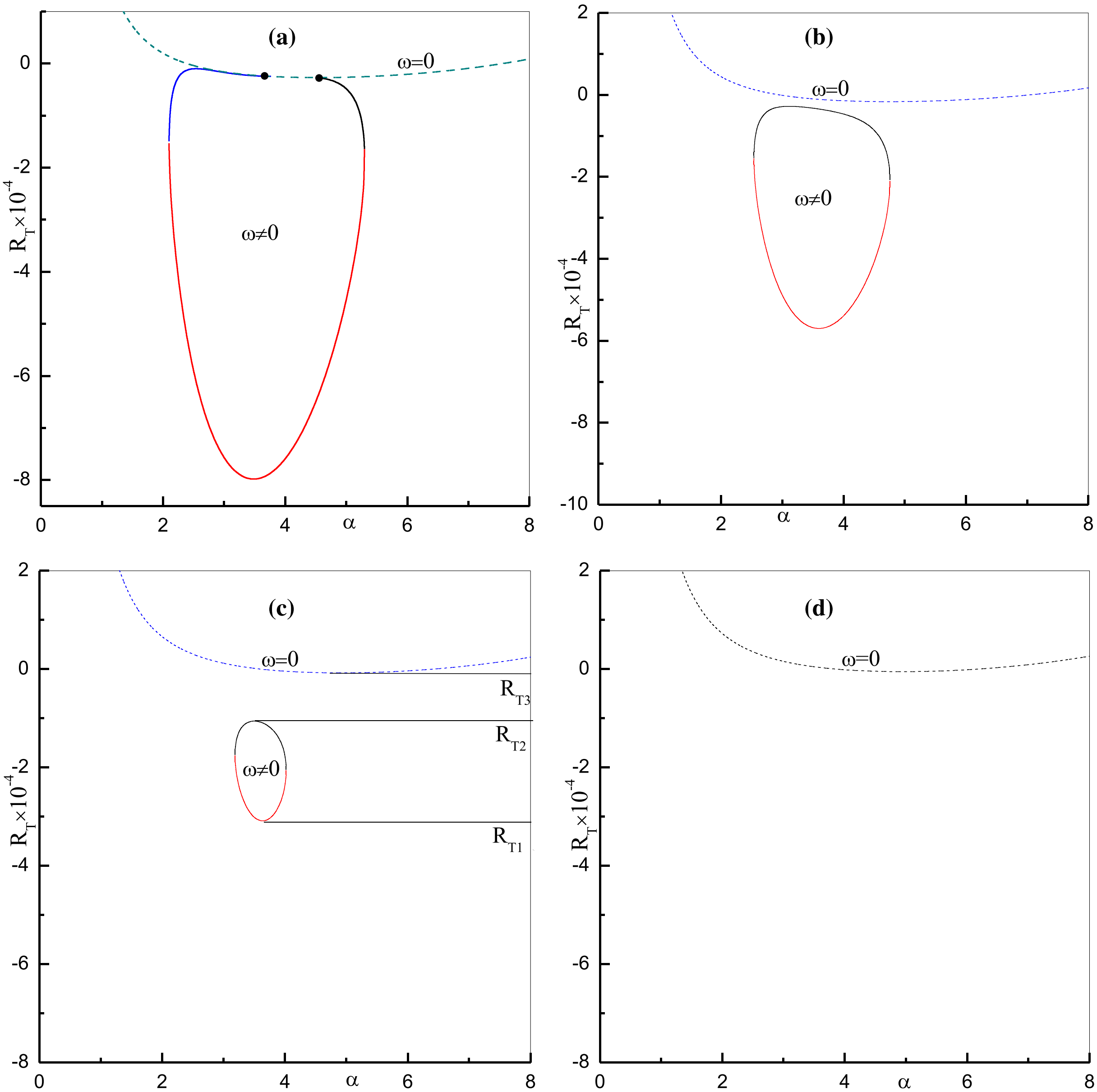

Fig. 7. Evolution of neutral stability curves for Pr = 3, $R_S = -56{,}000$, $\tau_m = 5$, $\tau_{12} = 0.35$, $\tau_{21} = 0.7$, and $\tau_{22} = 8.9$: **a** $Q = 300$, **b** $Q = 380$, **c** $Q = 450$, and **d** $Q = 470$

1.8 it shows that the closed convex oscillatory neutral curve shrinks and lies well below the stationary neutral curve indicating the requirement of three critical values of thermal Rayleigh number to dictate the linear instability criteria (Fig. 8c). Finally, at $\tau_{12} = 2.1$, the oscillatory neutral curve disappears and only the stationary neutral curve exists (Fig. 8d). It is thus evident that seemingly small variations in the cross diffusion terms exhibit profound effect on the nature of instability. It is thus crucial to consider the contribution of off-diagonal elements in analysing the instability of the system.

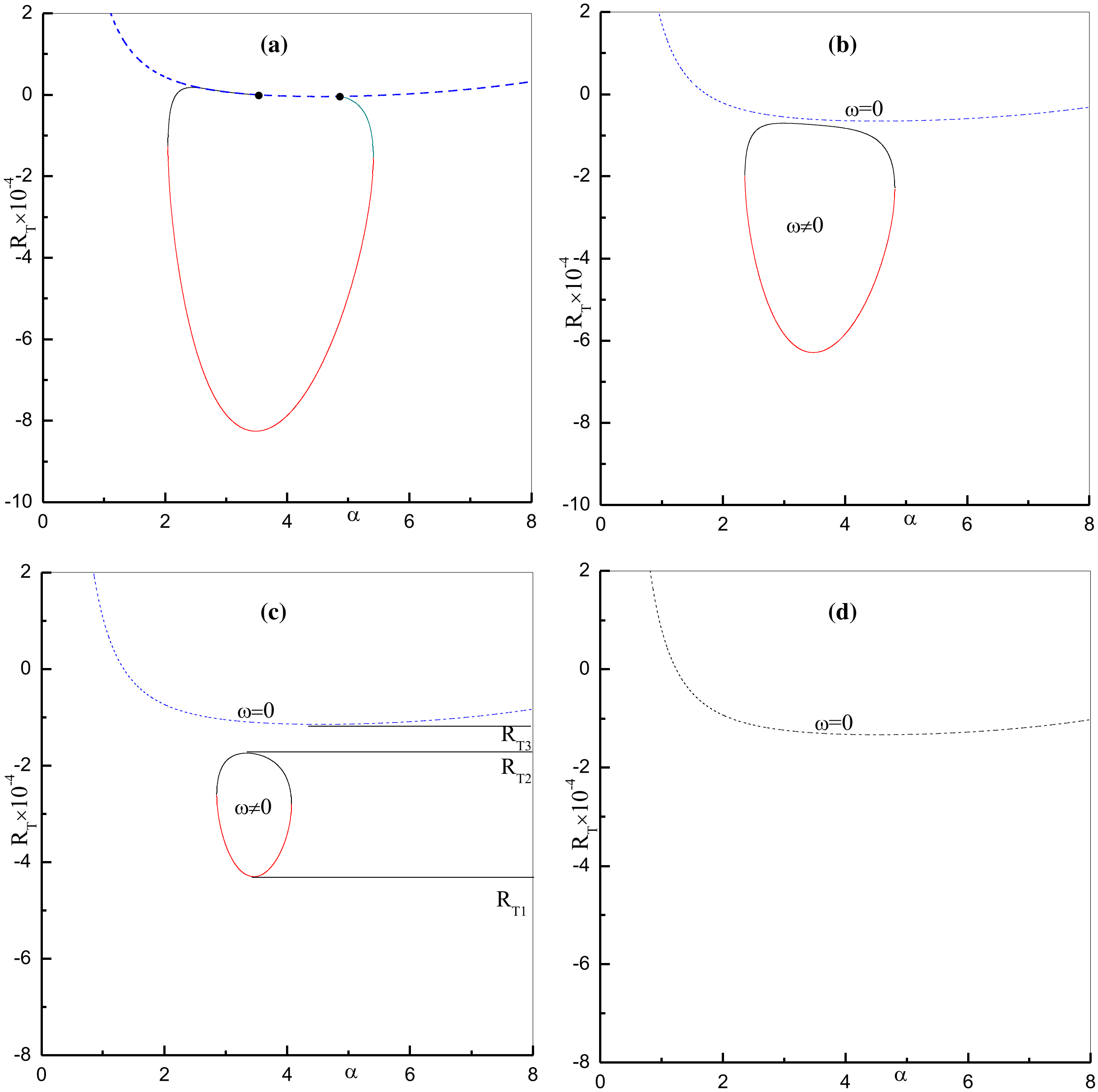

FIG. 8. Evolution of neutral stability curves for Pr = 3, , $Q = 300$, $R_S = -55{,}000$, $\tau_m = 5$, $\tau_{21} = 0.7$, and $\tau_{22} = 8.9$: **a** $\tau_{12} = 0$, **b** $\tau_{12} = 1$, **c** $\tau_{12} = 1.8$, and **d** $\tau_{12} = 2.1$

5.1.2. Destabilization due to stable solute concentration. In double diffusive convection, the effect of stable solute gradient is to increase the critical thermal oscillatory Rayleigh number and hence its effect is to delay the onset of oscillatory convection. To the contrary, the presence of heavy solute at the bottom of an electrically conducting fluid layer in the presence of magnetic field destabilizes the system under certain parametric conditions. Figure 9a, b demonstrates the variation of critical oscillatory Rayleigh number R^o_{Tc} and the corresponding frequency of oscillations ω_c^2 as a function of R_S when Pr $= 0.1$, $Q = 2000$, $\tau_m = 0.25$, $\tau_{22} = 0.01$ (i.e.$\tau_{22} < \tau_m < 1$) and for different values of Soret numberτ_{21}as well

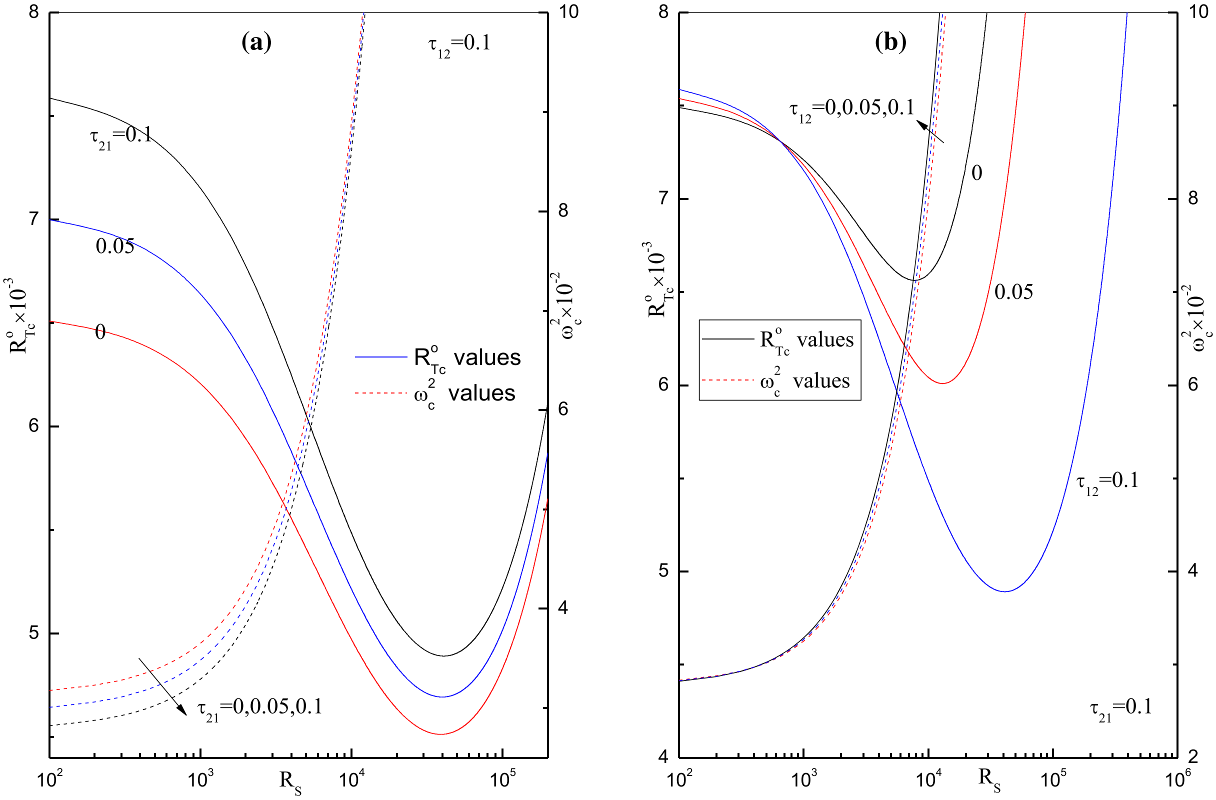

Fig. 9. Variation of R^o_{Tc} and ω_c^2 with R_S for different values of **a** τ_{21} with $\tau_{12} = 0.1$, and **b** τ_{12} with $\tau_{21} = 0.1$ when $\Pr = 0.1$, $Q = 2000$, $\tau_m = 0.25$, and $\tau_{22} = 0.01$

as Dufour numberτ_{12}. From these figures, it is obvious that there is some intermediate range of R_S in which the layer gets destabilized by increasing R_S. The destabilization manifests itself as a minimum in the R^o_{Tc}-R_S plane with increasing ω_c^2. The results shown in Fig. 9a for $\tau_{21} = 0$, 0.05 and 0.1 when $\tau_{12} = 0.1$ disclose that increase in τ_{21} is to delay the onset of oscillatory convection and the simultaneous presence of both Soret and Dufour effects is found to be more stabilizing compared to the presence of only Dufour effect. Figure 9b displays the results for $\tau_{12} = 0$, 0.05 and 0.1 when $\tau_{21} = 0.1$. There exists a threshold value of $R_S = 672.63$ beyond which increasing τ_{12} hastens the onset of oscillatory convection and prior to which it shows an opposite trend. The curves of ω_c^2 are also shown in Fig. 9a, b and it is observed that the presence of cross-diffusion terms is to reduce the frequency of oscillations and also the destabilization is associated with a monotonically increasing ω_c^2. This is because the diffusion of solute, for small values of τ_{22}, is so slow that substantial changes in the bobbing frequency of the fluid parcel can be produced by changes in R_S that have little stabilizing effect via solute diffusion. Thus, the frequency can be turned by adjusting R_S. If the frequency is too small, a bobbling parcel of fluid will always remain in approximate thermal equilibrium with its environment. If the frequency is too high, no significant heat transfer will occur into or out of the parcel in the first place. In either extreme, the basic overstability mechanism is operating at less than optimal efficiency. At some intermediate frequency, however, the maximum efficiency is achieved, and overstable oscillations set in at a lower value of R^o_{Tc} than is possible for larger or smaller frequencies.

Figure 10 depicts the implications of cross diffusion terms on the variation of R^o_{Tc} as a function of R_S for $\Pr = 0.1$, $Q = 2000$, $\tau_m = 0.25$ and $\tau_{22} = 0.01$. We note that the curves of R^o_{Tc} pass through a

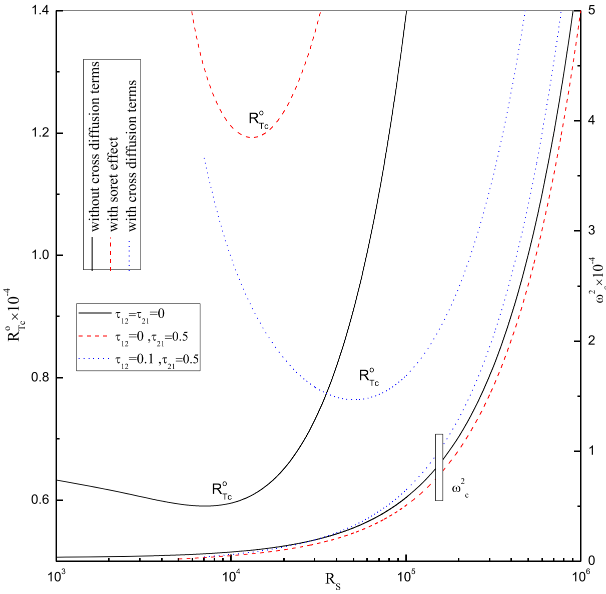

FIG. 10. Variation of R^o_{Tc} and ω_c^2 with R_S for different values of cross diffusion terms when $\Pr = 0.1$, $Q = 2000$, $\tau_m = 0.25$, and $\tau_{22} = 0.01$

minimum with increasing R_S and the range of R_S up to which the destabilization occurs depends on the values of τ_{12} and τ_{21}. For $(\tau_{12}, \tau_{21}) = (0, 0)$, $(0,\ 0.5)$ and $(0.1,\ 0.5)$, the stabilizing solute concentration instills instability up to the value of $R_S = 7127.84$, 13183.40 and 50622.40, respectively. Thus, Soret and Dufour effects increase the range of solute Rayleigh number significantly up to which the system gets destabilized compared to their absence and also in the presence of only Soret effect.

5.1.3. Destabilization due to magnetic field. In the study of magnetoconvection, it is a well-established fact that the magnetic field suppresses both steady and oscillatory instability [10]. In other words, increase in the value of Chandrasekhar number Q is to increase both steady and oscillatory critical thermal Rayleigh number. However, the presence of an additional diffusing component contradicts this result in which case it is witnessed that the magnetic field hastens the onset of oscillatory convection rather than suppressing it. Figure 11a, b shows the variation of R^o_{Tc} and ω_c^2 as a function of Q for different values of τ_{21} $(= 0,\ 0.1$ and 0.2 with $\tau_{12} = 0.2)$ and τ_{12} $(= 0,\ 0.1$ and 0.2 with $\tau_{21} = 0.2)$, respectively for $\Pr = 0.1$, $R_S = 4500$, $\tau_m = 0.1$ and $\tau_{22} = 0.28$ (i.e. $\tau_{22} > \tau_m < 1$). The figures demonstrate that R^o_{Tc} decreases with Q initially and with further increase in Q stabilizes the fluid layer again. The destabilization manifests itself as a minimum in the R^o_{Tc}-Q plot. Further examination of these figures reveals that increase in τ_{21} (Fig. 11a) and τ_{12} (Fig. 11b) is to delay and hasten the onset of oscillatory convection, respectively. The variation of critical frequency ω_c^2 as a function of Q is also illustrated in Fig. 11a, b which increases with increasing Q, while it decreases with increasing τ_{21} and decreasing τ_{12}. The destabilization due to magnetic field may have a physical basis similar to that of destabilizing effect due to a stable solute concentration gradient as observed previously. This can be explained using the concept of frequency phenomenon as before. We note that for small values of Q, ω_c^2 is relatively small so that the parcel of fluid can remain in approximate density equilibrium with its environment via the diffusion of heat and solute. As Q increases, however, so does the oscillation frequency. This has the effect of making it more difficult for the parcel to remain in the density equilibrium with its surroundings as it bobs up and down and the oscillations will grow. Of course, if ω_c^2 becomes too large, the basic overstability mechanism will fail because very little heat or solute will be transferred to the parcel of the fluid during a cycle. Thus, we

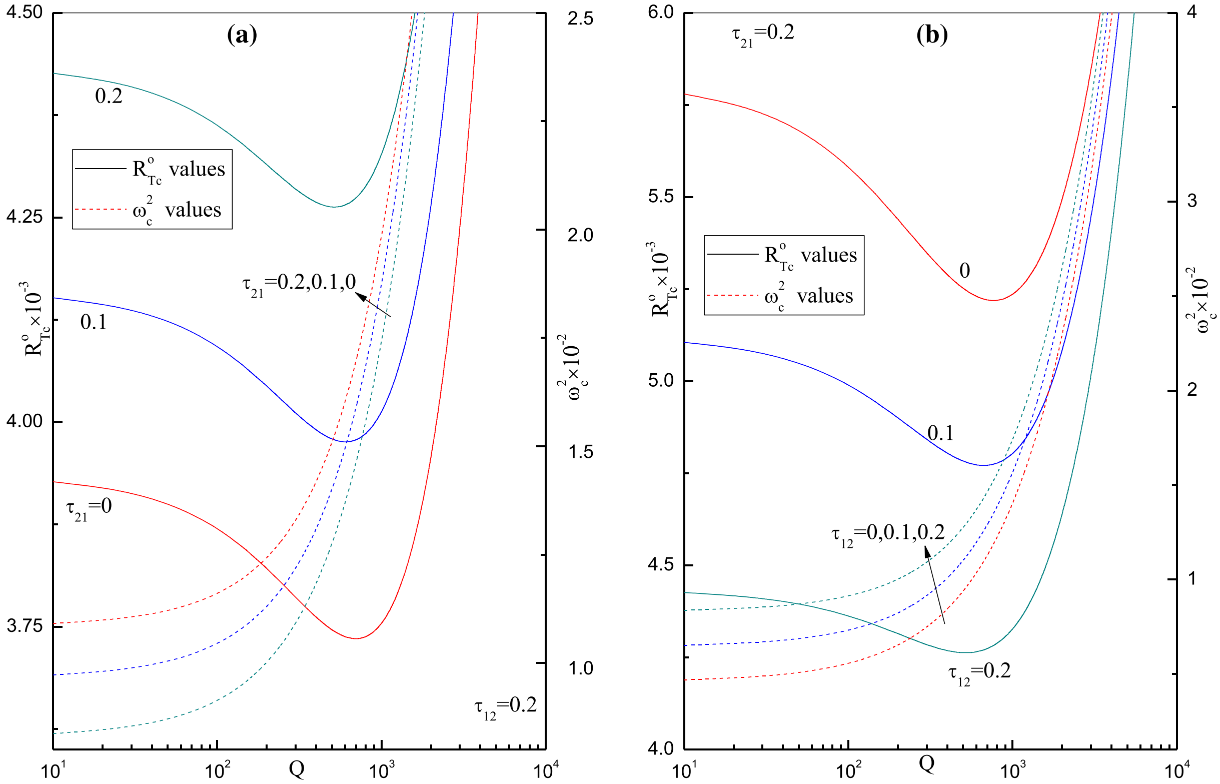

FIG. 11. Variation of R^o_{Tc} and ω_c^2 with Q for different values of **a** τ_{21} with $\tau_{12} = 0.2$, and **b** τ_{12} with $\tau_{21} = 0.2$ when $\Pr = 0.1$, $R_S = 4500$, $\tau_m = 0.1$, and $\tau_{22} = 0.28$

see that the instability is facilitated with increasing ω_c^2. The overstability mechanism achieves its optimal efficiency and convection becomes possible at a lower value of R^o_{Tc} than is possible for smaller or larger values of ω_c^2.

As before, the complexities of Soret and Dufour effects on the variation of R^o_{Tc} as a function of Q are illustrated in Fig. 12 for $\Pr = 0.1$, $R_S = 4500$, $\tau_m = 0.1$ and $\tau_{22} = 0.28$. The curves of R^o_{Tc} pass through a minimum with increasing Q and the range of Q up to which the destabilization occurs depends on the values of τ_{12} as well as τ_{21}. For values of $(\tau_{12}, \tau_{21}) = (0, 0)$, (0, 0.2) and (0.1, 0.2), the results reveal that an increase in the value of Chandrasekhar number is to enforce instability upto $Q = 810$, 780 and 680, respectively. We note that the presence of both Soret and Dufour effects is to decrease the range of destabilization due to magnetic field compared to their absence. Moreover, it is seen that R^o_{Tc} increases monotonically with increasing Q without showing any destabilization for $(\tau_{12}, \tau_{21}) = (0, 0.6)$. This is another instance showing the implication of cross-diffusion terms on the instability characteristics of the system.

5.2. Weak nonlinear stationary stability analysis

A weak nonlinear stability analysis has been carried out using a perturbation method. A cubic Landau equation is derived in terms of the amplitude function to identify the existence of supercritical or subcritical instability. The stability of stationary bifurcating solution is discussed in the neighborhood of its critical value for a range of parametric values. The nature of stationary bifurcation is determined

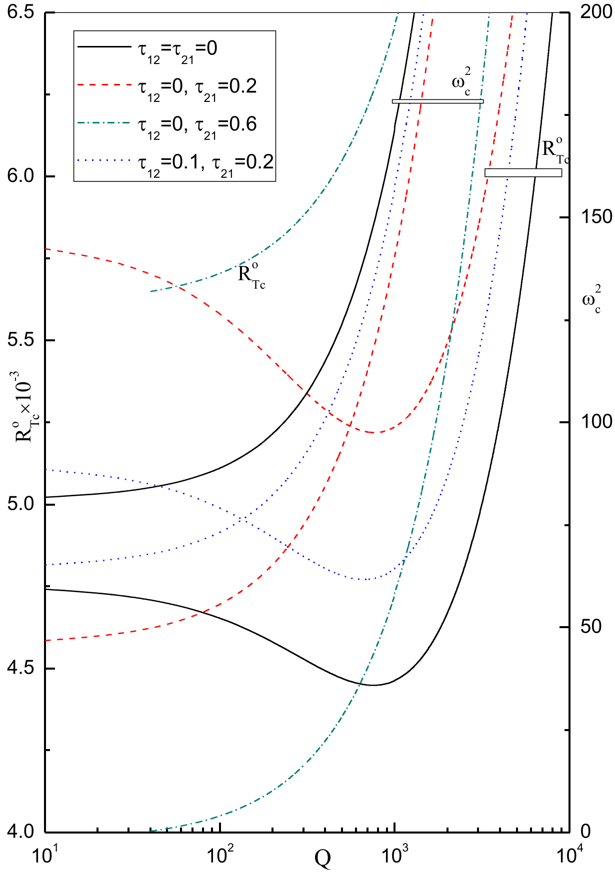

FIG. 12. Variation of R^o_{Tc} and ω_c^2 with Q for different values of cross diffusion terms when $\Pr = 0.1$, $R_S = 4500$, $\tau_m = 0.1$, and $\tau_{22} = 0.28$

from the sign of Landau constant R_{T2} (i.e. the bifurcation is supercritical if $R_{T2} > 0$ and subcritical if $R_{T2} < 0$). The point at which R_{T2}changes the sign is termed as the tricritical point. Figures 13 and 14a, b represent the computed values of R_{T2} as a function of R_S for different values of Q with and without cross diffusion terms. These figures demonstrate the possibility of occurring subcritical bifurcation for a range of parametric values indicating the occurrence of instability before the linear threshold is reached. This is believable because the linear instability analysis provides only adequate condition for instability. Besides, the tricritical point is shifted towards higher values of R_S with increasing Q and in the presence of Soret and Dufour effects. Moreover, the Dufour and Soret numbers exhibit opposing contributions on the range of R_S beyond which the subcritical bifurcation is possible.

6. Conclusions

The simultaneous presence of Soret and Dufour effects on linear and weak nonlinear stability of double diffusive magnetoconvection are investigated. The presence of magnetic field acts as an additional diffusing component and in overall the entire fluid system behaves like a triply diffusive one. The condition for the onset of stationary and oscillatory convection is obtained by accomplishing linear instability analysis, while a cubic Landau equation is derived by performing a weak nonlinear stability analysis. The critical

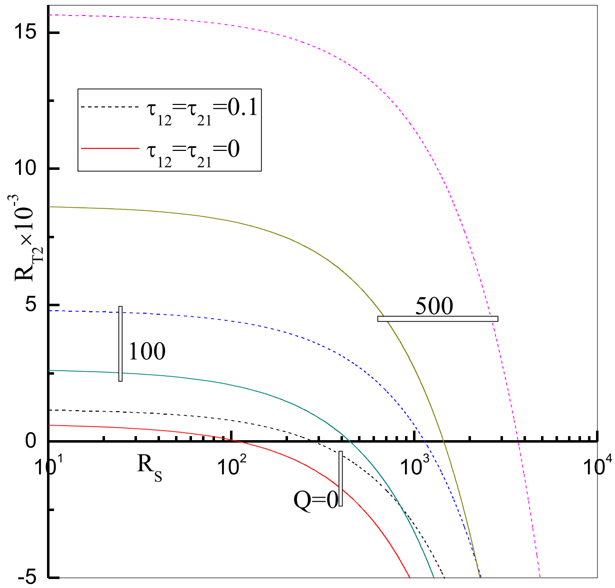

Fig. 13. Variation of R_{T2} with R_S for different values of τ_{12}, τ_{21} and Q when $\tau_m = 5$ and $\tau_{22} = 0.5$

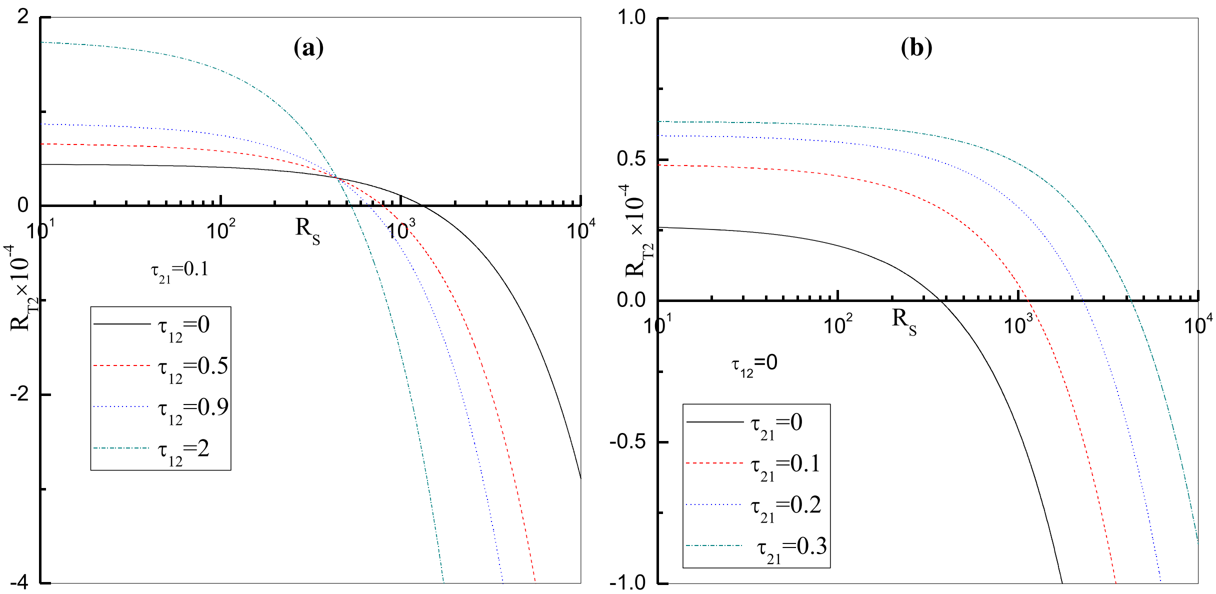

Fig. 14. Variation of R_{T2} with R_S for different values of **a** τ_{12}, and **b** τ_{21} when $Q = 100$, $\tau_m = 5$ and $\tau_{22} = 0.5$

wave number of stationary convection is independent of solute concentration and cross diffusion terms. The oscillatory instability is found to be possible even if the diffusivity ratios are greater than unity in contrast to double diffusive fluid systems. Although the cross diffusion effects show insignificant influence on the stationary onset, they affect the oscillatory onset significantly. More importantly, disconnected closed convex oscillatory neutral curves are identified under certain parametric conditions suggesting the requirement of three critical thermal Rayleigh numbers to specify the linear instability criteria instead of the usual single critical value. Small variations in the cross diffusion terms found to have profound effect on the nature of instability. Under different parametric spaces, a doubly cross diffusive fluid layer in the presence of magnetic field gets destabilized by the addition of a bottom-heavy solute gradient and also the magnetic field destabilizes a doubly cross diffusive conducting fluid layer. The range of solute Rayleigh number and the Chandrasekhar number up to which the system gets destabilized is influenced by Soret and/or Dufour coefficients. Both subcritical and supercritical bifurcations are possible depending on the magnitude of governing parameters. Increase in the value of Chandrasekhar number and in the presence of cross diffusion parameters as well as increase in the Soret number and decrease in the Dufour number is to increase the range of solute Rayleigh number beyond which the subcritical instability is possible. Thus, the foregoing study exemplifies the importance of considering cross-diffusion terms in the proper analysis of double/multidiffusive stability problems.

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