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Chevalley groups over Dedekind domains and some problems for $K_2(2, \mathbb{Z}_S)$

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Abstract. We review Chevalley groups over Dedekind domains and associated K_2 groups. We also recall some old results on $K_2(2, \mathbb{Z}_S)$. Then, we show here several new examples and computations.

1. Introduction

Let $\mathbb{Z}$ be the ring of rational integers, and S a set of prime numbers. Put $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{q}]_{q \in S}$. We are interested in the following problems.

Problem 1. *Determine the group structure of $K_2(2, \mathbb{Z}_S)$ for a new set S of prime numbers.*

Problem 2. *Find a new set S of prime numbers such that*

$$K_2(2, \mathbb{Z}_S) \simeq \mathbb{Z} \oplus \amalg_{q \in S} (\mathbb{Z}/q\mathbb{Z})^\times \ .$$

Problem 3. *For a given prime number p, find a set $S = \{q_1, \cdots, q_r\}$ of prime numbers, satisfying $(*)$ below, as small as possible.*

$$(*) \quad \begin{cases} q_1 < q_2 < \cdots < q_{r-1} < q_r = p, \\ K_2(2, \mathbb{Z}_S) \simeq \mathbb{Z} \oplus \amalg_{q \in S} (\mathbb{Z}/q\mathbb{Z})^\times. \end{cases}$$

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Let p_i be the i-th prime number. That is,

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \cdots .$$

If we take $p = p_r$ and choose $S = \{p_1, p_2, p_3, \cdots, p_r\}$, then S is the largest set satisfying $(*)$ above (cf. Theorem 5.4 or [19]).

To discuss these problems, we will review Chevalley groups over Dedekind domains in Section 2, and their presentations in Section 3. In the same Section 3, we will also review $K_2(\Phi, \cdot)$, where Φ is the root system corresponding to a Chevalley group. We will specify $SL(2, \cdot)$ in Section 4, and we will recall several old results on $K_2(2, \mathbb{Z}_S)$ in Section 5. Then, we will introduce a criterion, in Section 6, to have new examples. In Section 7, we will mention several facts on Schur multipliers and loop groups, and so on.

2. Chevalley groups

Before explaining details, we just want to see what is interesting:

• A Chevalley group is a functor from the category of commutative rings to the category of groups. We can discuss rings and groups at the same time, which means that it is interesting to find a good relation between rings and groups.

• Tits systems, presentations and universal central extensions are very important in group theory. We discuss these topics for some algebraic groups over Dedekind domains. Then, we frequently obtain such interesting structures.

• Loop groups are Chevalley groups over Laurent polynomial rings, which are finite dimensional algebraic groups. Also, loop groups can be viewed as affine Kac-Moody groups, which are infinite dimensional algebraic groups. We can approach in two ways.

• It is interesting to find a certain property of each prime number if one takes $\mathbb{Z}_S$ as a Dedekind domain. Many relations in $K_2(2, \mathbb{Z}_S)$ are dependent on the choice of prime numbers in S. We can study prime numbers using Steinberg symbols.

In this sense, we can draw the following historical flow:

Eiichi Abe	Nagayoshi Iwahori	Robert V. Moody
Hopf Algebra	p-adic Algebraic Group	Kac-Moody Theory
Chevalley Group	Iwahori Subgroup	Infinite Root System
Algebraic K-theory	Iwahori-Hecke Algebra	Quasicrystal
$\searrow$	$\downarrow$	$\swarrow$

Chevalley Group over Dedekind Domain
- Loop Group (⇒ ∞ Dimension)
- Tits System (BN-Pair)
- Presentation ((A), (B), (B′), (C))
- Universal Central Extension
- K_2 group (Symbol $\{\,\cdot\,,\,\cdot\,\}$)
- Heredity Theorem ($D \Rightarrow D[1/p]$)

The author is deeply obliged to great professors, named E. Abe, N. Iwahori and R. V. Moody, to study Chevalley groups over Dedekind domains.

Let $\Phi \subset \mathcal{V}$ be a reduced irreducible root system in the sense of Bourbaki (cf. [3]), satisfying:

(RS1) $|\Phi| < \infty,\ \langle\Phi\rangle = \mathcal{V},\ 0 \notin \Phi.$

(RS2) $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ for all $\alpha \in \Phi$.

(RS3) $\sigma_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

(RS4) $2\dfrac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Here $\mathcal{V}$ is an n-dimensional Euclidean space, and $\sigma_\alpha(v) = v - 2\dfrac{(\alpha,v)}{(\alpha,\alpha)}\alpha$ for all $v \in \mathcal{V}$ ($\simeq \mathbb{R}^n$). We also assume that Φ is reduced and irreducible. Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra of type Φ (cf. [6]). For a finite dimensional faithful representation ρ of $\mathfrak{g}$ on V, we obtain an affine group scheme $G_\rho(\Phi,\ \cdot\ )$, that is a representable covariant functor from commutative rings to groups. This is simply called a Chevalley scheme and represented by a Hopf algebra $H_\mathbb{Z}$ over $\mathbb{Z}$, i.e. $G_\rho(\Phi,\ \cdot\ ) = \mathrm{Alg}_\mathbb{Z}(H_\mathbb{Z},\ \cdot\ )$, (cf. [1],[4]). A representation ρ is called simply connected if its weight lattice is full. We fix a fundamental system (or a base) Π of Φ (cf. [3],[6]).

For example, if Φ is of type A_n, then $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$. If ρ is simply connected (or ρ is a natural action on $\mathbb{C}^{n+1}$), then we obtain $G_\rho(\mathrm{A}_n,\ \cdot\ ) = SL(n+1,\ \cdot\ )$ and $H_{\mathbb{Z}} = \mathbb{Z}[\xi_{ij}]/(\det(\xi_{ij})-1)$, where $1 \le i,j \le n+1$ and $\Delta(\xi_{ij}) = \sum_{k=1}^{n+1} \xi_{ik} \otimes \xi_{kj}$.

For a commutative ring R, we call $G(\Phi,R) = G_\rho(\Phi,R)$ a Chevalley group over R of type Φ associated with ρ. For each $\alpha \in \Phi$, there is an "exponential" homomorphism x_α of the additive group R^+ to $G_\rho(\Phi,R)$. That is, $x_\alpha(t) = \exp \rho(te_\alpha) \in GL(V_R)$ for $t \in R$ and e_α is a part of a fixed Chevalley basis of $\mathfrak{g}$, where $V_R = R \otimes V_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$ is an addmissible lattice in V (cf. [6],[25]). Put $E(\Phi,R) = E_\rho(\Phi,R) = \langle x_\alpha(t) \mid \alpha \in \Phi, t \in R\rangle$, which is called the elementary subgroup of $G_\rho(\Phi,R)$. If ρ is simply connected, then we write $G_{sc}(\Phi,R)$ and $E_{sc}(\Phi,R)$ respectively. If R is a field, a local ring, or a Euclidean domain, then $G_{sc}(\Phi,R) = E_{sc}(\Phi,R)$, (cf. [1],[24],[25]).

In $E_\rho(\Phi,R) \subset G_\rho(\Phi,R)$, we obtain the following:

(A) $x_\alpha(s)x_\alpha(t) = x_\alpha(s+t)$,

(B) $[x_\beta(s), x_\gamma(t)] = \prod_{\substack{i\beta+j\gamma\in\Phi \\ i,j>0}} x_{i\beta+j\gamma}(N_{\beta,\gamma,i,j}s^i t^j)$,

(B′) $w_\alpha(u)x_\alpha(t)w_\alpha(-u) = x_{-\alpha}(-u^{-2}t)$,

(C) $h_\alpha(u)h_\alpha(v) = h_\alpha(uv)$

for all $\alpha,\beta,\gamma \in \Phi$ $(\beta \pm \gamma \neq 0)$, $s,t \in R$ and $u,v \in R^\times$, where $w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u)$, $h_\alpha(u) = w_\alpha(u)w_\alpha(-1)$, and $N_{\beta,\gamma,i,j} \in \mathbb{Z}$ (structure constants), and where $R^\times$ is the multiplicative group of R (cf. [1],[24],[25]).

Let D be a Dedekind domain, and we choose $p \in D$ as a nonzero prime element. Suppose that $D^\times \longrightarrow (D/pD)^\times$ is surjective. Then, we let

$$
\begin{aligned}
N &= \langle w_\alpha(u) \mid \alpha \in \Phi, u \in D[\tfrac{1}{p}]^\times\rangle,\\
T &= \langle h_\alpha(u) \mid \alpha \in \Phi, u \in D[\tfrac{1}{p}]^\times\rangle,\\
T_0 &= \langle h_\alpha(u) \mid \alpha \in \Phi, u \in D^\times\rangle,\\
W &= N/T \ : \ \text{“Weyl Group”},\\
W_a &= N/T_0 \simeq W \ltimes Q \ : \ \text{“Affine Weyl Group”},\\
Q &= T/T_0 \simeq \mathbb{Z}^n, \ \text{where } n = \dim \mathcal{V} = \mathrm{rank}(\Phi) = \mathrm{rank}(\mathfrak{g}),\\
S_a &= \{w_\alpha(1) \mid \alpha \in \Pi\} \cup \{w_{-\alpha_0}(p)\},
\end{aligned}
$$

where $\alpha_0 \in \Phi$ is the highest root. Set $Q_+ = \sum_{\alpha\in\Pi}\mathbb{Z}_{\geq 0}\alpha$ and $Q_- = -Q_+$, and put $\Phi^\pm = \Phi\cap Q_\pm$. Then, $\Phi = \Phi^+\cup\Phi^-$ and $\Phi^- = -\Phi^+$ (cf. [3],[6]).

Let B be the subgroup of $E_\rho(\Phi, D[\frac{1}{p}])$ generated by

$$x_\alpha(t), \quad h_\alpha(u), \quad x_{-\alpha}(pt)$$

for all $\alpha\in\Phi^+$, $t\in D$ and $u\in D^\times$. This B is called an Iwahori subgroup, which is usually denoted by B_I (cf. [7],[24]).

Theorem 2.1. (cf. [2]) *Notation is as above. Then,* $(E(\Phi, D[\frac{1}{p}]), B, N, S_a)$ *is a Tits system, and*

$$E(\Phi, D[\frac{1}{p}]) = \bigcup_{w\in W_a} BwB \quad \textit{(disjoint)}.$$

Remark 2.2. (cf. [7]) *If* $D=\mathbb{Z}_p$ *and* $D[\frac{1}{p}] = \mathbb{Z}_p[\frac{1}{p}] = \mathbb{Q}_p$*, then we obtain an original Iwahori-Matsumoto decomposition.*

3. Presentations and $K_2(\Phi,\ \cdot\ )$

Let $St(\Phi,R)$ be the group generated by $\hat{x}_\alpha(t)$ for all $\alpha\in\Phi$ and $t\in R$ with the defining relations corresponding to (A), (B), (B′). It is called a Steinberg group. There is a natural homomorphism ϕ of $St(\Phi,R)$ to $G_{sc}(\Phi,R)$ such that $\phi(\hat{x}_\alpha(t)) = x_\alpha(t)$, (cf. [1],[24],[25]).

Put $K_2(\Phi,R) = \mathrm{Ker}[St(\Phi,R)\xrightarrow{\phi} G_{sc}(\Phi,R)]$, and note $E_{sc}(\Phi,R) = \mathrm{Im}[St(\Phi,R)\xrightarrow{\phi} G_{sc}(\Phi,R)]$. For $\alpha\in\Phi$ and $u\in R^\times$, we put

$$\hat{w}_\alpha(u) = \hat{x}_\alpha(u)\hat{x}_{-\alpha}(-u^{-1})\hat{x}_\alpha(u),$$
$$\hat{h}_\alpha(u) = \hat{w}_\alpha(u)\hat{w}_\alpha(-1).$$

For a fixed long root $\alpha\in\Phi$, we define

$$\{u,v\} = \{u,v\}_\alpha = \hat{h}_\alpha(u)\hat{h}_\alpha(v)\hat{h}_\alpha(uv)^{-1},$$

called a Steinberg symbol. Let $C(R)$ be the subgroup of $St(\Phi,R)$ generated by $\{u,v\}$ for all $u,v\in R^\times$. Then, $C(R)$ is central, and $C(R)\subset K_2(\Phi,R)$.

Let $E_u(\Phi,R)$ be the group generated by $\tilde{x}_\alpha(t)$ for all $\alpha\in\Phi$ and $t\in R$ with the defining relations corresponding to (A), (B), (B′), (C). Then, the following two conditions are equivalent:

- $K_2(\Phi, R) = C(R)$;
- $E_u(\Phi, R) \xrightarrow{\sim} E_{sc}(\Phi, R)$.

A commutative ring R is called universal for Φ if $K_2(\Phi, R) = C(R)$.

Theorem 3.1. (cf. [2]) *Let D be a Dedekind domain, and $p \in D$ a nonzero prime element. Suppose that $D^\times \longrightarrow (D/pD)^\times$ is surjective. If D is universal for Φ, then $D[\frac{1}{p}]$ is also universal for Φ.*

Hence, we obtain the following two interesting topics:

- Surjectivity of $D^\times \to (D/pD)^\times$, and Tits System;
- Surjectivity of $D^\times \to (D/pD)^\times$, and Heredity of Universality.

Suppose that R is universal for Φ. If R satisfies some ideal or unit condition (for example, there exists a unit $u \in R^\times$ such that $u^2 - 1 \in R^\times$ for A_1 type), or if $\mathrm{rank}(\Phi) \gg 0$, then $St(\Phi, R)$ is a universal central extension of $E_{sc}(\Phi, R)$, and $K_2(\Phi, R)$ is the Schur multiplier of $E_{sc}(\Phi, R)$. Note that if $D^\times \to (D/pD)^\times$ is surjective, then $K_2(\Phi, D[\frac{1}{p}]) = K_2(\Phi, D)_p \cdot C(D[\frac{1}{p}])$, where $K_2(\Phi, D)_p = \mathrm{Im}[K_2(\Phi, D) \to K_2(\Phi, D[\frac{1}{p}])]$, (cf. [2],[19],[24],[25]).

Example 3.2. *We obtain the following picture starting from $\mathbb{Z}$.*

$$
\begin{array}{cccccccc}
 & & & & & \nearrow & \cdots & \\
 & & & & \mathbb{Z}[\frac{1}{2}, \frac{1}{q}] & \to & \cdots & \\
 & & & \nearrow & & \searrow & \cdots & \\
 & & \mathbb{Z}[\frac{1}{2}] & \to & \cdots & \cdots & \cdots & \rceil \\
 & \nearrow & & \searrow & & \nearrow & \cdots & | \\
\mathbb{Z} & & & & \mathbb{Z}[\frac{1}{2}, \frac{1}{3}] & \to & \cdots & \text{“perfect”} \\
 & \searrow & & \nearrow & & \searrow & \cdots & | \\
 & & \mathbb{Z}[\frac{1}{3}] & \to & \cdots & \cdots & \cdots & \rfloor \\
 & & & \searrow & & \nearrow & \cdots & \\
 & & & & \mathbb{Z}[\frac{1}{3}, \frac{1}{q'}] & \to & \cdots & \\
 & & & & & \searrow & \cdots &
\end{array}
$$

Namely, $D = \mathbb{Z}[\frac{1}{q_1}, \cdots, \frac{1}{q_n}]$ $(n \geq 0)$ appeared in this picture is universal for Φ, where we assume that $D^\times \twoheadrightarrow (D/pD)^\times$ is surjective in every step $D \to D[\frac{1}{p}]$. Here p is a prime number (see Theorem 3.1 above), *and the word "perfect" means that $SL(2,D)$ is perfect, i.e. $SL(2,D) = [SL(2,D), SL(2,D)]$. Note that $SL(2,D)$ is perfect if and only if $2,3 \in \{q_1,\cdots,q_n\}$ (cf. [25]). That is, we can see that $2^2-1=3 \in D^\times$ and $SL(2,D) = [SL(2,D), SL(2,D)]$ if $2,3 \in \{q_1,\cdots,q_n\}$, and we can also see that $D/p'D \simeq \mathbb{Z}/p'\mathbb{Z}$ and $SL(2,\mathbb{Z}/p'\mathbb{Z}) \neq [SL(2,\mathbb{Z}/p'\mathbb{Z}), SL(2,\mathbb{Z}/p'\mathbb{Z})]$, implying $SL(2,D) \neq [SL(2,D), SL(2,D)]$, if $p' = 2$ or 3 and if $p' \notin \{q_1,\cdots,q_n\}$. We also notice that $D=\mathbb{Z}$ and $p=2,3$ if $n=0$.*

4. $SL(2,\ \cdot\ )$ and $K_2(2,\ \cdot\ )$

The standard stable K_2 is given by

$$K_2(R) = \mathrm{Ker}[St(R) \longrightarrow SL(R)] = \lim_n K_2(n,R),$$

(cf. [25]), where

$St(R) = \lim_n St(n,R), \quad St(n,R) = St(\mathrm{A}_{n-1},R),$

$SL(R) = \lim_n SL(n,R), \quad SL(n,R) = G_{sc}(\mathrm{A}_{n-1},R),$

$K_2(n,R) = \mathrm{Ker}[St(n,R) \xrightarrow{\phi} SL(n,R)] = K_2(\mathrm{A}_{n-1},R),$

and the symbol $\lim_n$ means an inductive limit by inclusions $SL(n,R) \hookrightarrow SL(n+1,R)$ or homomorphisms $St(n,R) \to St(n+1,R)$. Note that $St(n,R)$ is not necessarily a subgroup of $St(n+1,R)$.

Let S be a set of (finite or infinite) prime numbers, and put

$$\mathbb{Z}_S = \mathbb{Z}[\frac{1}{q}]_{q\in S}.$$

Then, as is well-known,

$$K_2(\mathbb{Z}_S) = K_2(n,\mathbb{Z}_S) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \amalg_{q\in S}(\mathbb{Z}/q\mathbb{Z})^\times$$

for all $n \geq 3$ (cf. [24],[25]).

For a commutative ring R, we define the Steinberg group $St(2,R)$, as before, by the generators

$\hat{x}_{12}(t)$ and $\hat{x}_{21}(t)$ for all $t \in R$

and the following defining relations

(A) $\hat{x}_{ij}(s+t) = \hat{x}_{ij}(s)\hat{x}_{ij}(t),$

(B′) $\hat{w}_{ij}(u)\hat{x}_{ij}(t)\hat{w}_{ij}(-u) = \hat{x}_{ji}(-u^{-2}t)$

for all $s, t \in R$ and $u \in R^\times$, where $1 \leq i \neq j \leq 2$ and $\hat{w}_{ij}(u) = \hat{x}_{ij}(u)\hat{x}_{ji}(-u^{-1})\hat{x}_{ij}(u)$, and where $R^\times$ is the multiplicative group of R.

There is a canonical group homomorphism

$$\phi : St(2, R) \longrightarrow SL(2, R) \subset GL(2, R)$$

defined by

$$\phi(\hat{x}_{12}(t)) = x_{12}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \phi(\hat{x}_{21}(t)) = x_{21}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Put $K_2(2, R) = \mathrm{Ker}(\phi)$ and $E(2, R) = \mathrm{Im}(\phi)$. If R is a field, a local ring, or a Euclidean domain, then $E(2, R) = SL(2, R)$.

For $u, v \in R^\times$, put

$$\{u, v\} = \hat{h}_{12}(u)\hat{h}_{12}(v)\hat{h}_{12}(uv)^{-1},$$

which is called a Steinberg symbol, where $\hat{h}_{12}(u) = \hat{w}_{12}(u)\hat{w}_{12}(-1)$. Then, $\{u, v\} \in K_2(2, R)$, and $\{u, v\}$ is central. Set

$$C(R) = \langle \{u, v\} \mid u, v \in R^\times \rangle,$$

and $E_u(2, R) = St(2, R)/C(R)$. Then, we obtain the following Matsumoto relations (cf. [10], Section 7.3).

Theorem 4.1. (cf. [10]) *Notation is as above. Then:*

(M1) If $t, u, v \in R^\times$, then $\{t, u\}\{tu, v\} = \{t, uv\}\{u, v\}$.

(M2) If $t, u \in R^\times$, then $\{t, u\}\{t, -u^{-1}\} = \{t, -1\}$.

(M3) If $t, u \in R^\times$, then $\{t, u\} = \{u^{-1}, t\}$.

(M4) If $t, u \in R^\times$, then $\{t, u\} = \{t, -tu\}$.

(M5) If $t, u, v \in R^\times$, then $\{t, u^2v\} = \{t, u^2\}\{t, v\}$.

(M6) If $t, 1 - t, u \in R^\times$, then $\{t, u\} = \{t, (1 - t)u\}$.

As easy direct consequences, we obtain:

(M7) If $t \in R^\times$, then $\{t^2, -1\} = 1$.

(M8) If $t,u\in R^\times$ *and* $t+u=1$*, then* $\{t,u\}=1$.

(M9) If $t,u\in R^\times$ *and* $t+u=1$*, then* $\{t,u^k\}=1$ *for all* $k\in\mathbb{Z}$.

(M10) If $t,u,v\in R^\times$*, then* $\{t^2,u\}\{t^2,v\}=\{t^2,uv\}$.

By [2], we can reach the following two theorems.

Theorem 4.2. (cf. Theorem 2.1, [2]) *Let* D *be a Dedekind domain, and* $p\in D$ *a nonzero prime element. Suppose that the canonical homomorphism of* $D^\times$ *into* $(D/pD)^\times$ *is surjective. Then,*

$$E(2,D[\frac{1}{p}])=\bigcup_{m\in\mathbb{Z}} B\begin{pmatrix} p^m & 0\\ 0 & p^{-m}\end{pmatrix}B\ \cup\ B\begin{pmatrix} 0 & p^m\\ -p^{-m} & 0\end{pmatrix}B,$$

where B *is the Iwahori subgroup generated by*

$$\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix},\quad \begin{pmatrix} u & 0\\ 0 & u^{-1}\end{pmatrix},\quad \begin{pmatrix} 1 & 0\\ pt & 1\end{pmatrix}$$

for all $t\in D$ *and* $u\in D^\times$.

Theorem 4.3. (cf. [2]) *Under the same assumption, we have:*

$$St(2,D[\frac{1}{p}])=\bigcup_{m\in\mathbb{Z}}\hat{B}\hat{h}_{12}(p^m)\hat{B}\ \cup\ \hat{B}\hat{w}_{12}(p^m)\hat{B},$$

where $\hat{B}$ *is the Iwahori subgroup generated by*

$$C(D[\frac{1}{p}]),\quad \hat{x}_{12}(t),\quad \hat{h}_{12}(u),\quad \hat{x}_{21}(tp)$$

for all $t\in D$ *and* $u\in D^\times$.

Furthermore, we obtain the following as in Section 3.

Theorem 4.4. (cf. [2],[19]) *Under the same assumption, we have:*

$$K_2(2,D[\tfrac{1}{p}])\subset\hat{B},$$
$$K_2(2,D[\tfrac{1}{p}])=K_2(2,D)_p\cdot C(D[\tfrac{1}{p}]),$$
$$K_2(2,D[\tfrac{1}{p}])=K_2(2,D)_p\cdot\langle\{p,u\}\mid u\in D^\times\rangle,$$

where $K_2(2,D)_p=Im[K_2(2,D)\longrightarrow K_2(2,D[\frac{1}{p}])]$.

Proof. Put $\hat{B}' = \langle \hat{x}_{12}(t),\ \hat{h}_{12}(u),\ \hat{x}_{21}(tp) \mid t \in D,\ u \in D^\times \rangle \subset \hat{B}$, then $\hat{B} = \hat{B}' \cdot C(D[\frac{1}{p}])$. By Theorem 4.2 and Theorem 4.3, we obtain $K_2(2, D[\frac{1}{p}]) \subset \hat{B}$. Let $\hat{x} \in K_2(2, D[\frac{1}{p}])$. Then, we can write $\hat{x} = \hat{y}\hat{z}$ for some $\hat{y} \in \hat{B}'$ and $\hat{z} \in C(D[\frac{1}{p}])$, which implies $1 = \phi(\hat{x}) = \phi(\hat{y})\phi(\hat{z}) = \phi(\hat{y})$.

$$\begin{array}{ccccccc} K_2(2, D[\frac{1}{p}]) & \subset & \hat{B} = \hat{B}' \cdot C(D[\frac{1}{p}]) & \subset & St(2, D[\frac{1}{p}]) & \to & E(2, D[\frac{1}{p}]) \\ \uparrow & & & & \uparrow & & \bigcup \\ K_2(2, D) & & \subset & & St(2, D) & \to & E(2, D) \end{array}$$

Then, there is an element $\hat{y}' \in St(2, D)$ satisfying $\hat{\pi}(\hat{y}') = \hat{y}$ under the canonical map $\hat{\pi} : St(2, D) \to St(2, D[\frac{1}{p}])$. Hence, $\hat{y}' \in K_2(2, D)$ and $\hat{y} \in K_2(2, D)_p$, which implies $K_2(2, D[\frac{1}{p}]) = K_2(2, D)_p \cdot C(D[\frac{1}{p}])$. In particular, we have $K_2(2, D[\frac{1}{p}]) = K_2(2, D)_p \cdot \langle \{p, u\} \mid u \in D^\times \rangle$. $\square$

5. Old Results

By [19], we have the following additional relations (cf. Section 7.4, 7.5), which are essentially obtained from Matsumoto relations in Theorem 4.1.

Theorem 5.1. (cf. [19]) *Let R be a commutative ring, and let $R^\times$ be the multiplicative group of R. Then, in $K_2(2, R)$:*

(1) If $z, z', r, p \in R^\times$ and $z' = z + rp$, then, (cf. Section 7.3),
$$\{p, z'\} = \{p, z\}\{\frac{r}{z'}, \frac{1}{r}\}^{-1}\{\frac{r}{z'}, \frac{z}{rz'}\} = \{p, z\}\{z, \frac{r}{z'}\}^{-1}\{z', \frac{r}{z'}\}.$$

(2) If $t, u, 1-u \in R^\times$, then
$$\{t, u\}\{t, 1-u\} = \{t, u(1-u)\}.$$

(3) If $t, u, v, 1-u \in R^\times$, then
$$\{t, uv\}\{t, (1-u)v\} = \{t, v\}\{t, u(1-u)v\}.$$

(4) If $2, 3, t \in R^\times$, then
$$\{t, -3\}\{t, 2\} = \{t, -1\}\{t, 6\}.$$

(5) If $t, u, v, a \in R^\times$ and $u + v = a^2$, then
$$\{t, u\}\{t, v\} = \{t, uv\}.$$

(6) If $t, u \in R^\times$ and $t = u^2 + 1$, then
$$\{t, -1\}^2 = 1.$$

(7) If $2, t, u \in R^\times$*,* $k \in \mathbb{Z}$ *and* $2^k t = u^2 + 1$*, then*

$$\{t, -1\}^2 = 1.$$

(8) If $2, t, u \in R^\times$ *and* $t = 2u^2 + 1$*, then*

$$\{t, -1\}^2 = 1.$$

(9) If $t, u \in R^\times$*, then*

$$\langle \{t, \pm u^k\} \mid k \in \mathbb{Z} \rangle = \langle \{t, -1\},\ \{t, \pm u\},\ \{t, u^2\} \rangle.$$

(10) If $t, u \in R^\times$*, then the following two conditions (i) and (ii) are equivalent.*

(i) $\{t, u\}\{t, -1\} = \{t, -u\}$.
(ii) $\{t, u\}\{t, u^{-1}\} = 1$.

(11) If one of the two conditions in (10) holds, then

$$\{t, u^k\} = \{t, u\}^k$$

and

$$\{t, -u^k\} = \{t, -1\}\{t, u^k\} = \{t, -1\}\{t, u\}^k$$

for all $k \in \mathbb{Z}$*.*

(12) If $2, t \in R^\times$*, then*

$$\{t, 2\}\{t, -1\} = \{t, -2\}.$$

(13) If $2, t \in R^\times$*, then*

$$\{t, 2\}\{t, \tfrac{1}{2}\} = 1.$$

(14) If $2, t \in R^\times$*, then*

$$\{t, 2^k\} = \{t, 2\}^k \text{ for all } k \in \mathbb{Z}.$$

(15) If $2, t \in R^\times$*, then*

$$\{t, -2^k\} = \{t, -1\}\{t, 2^k\} = \{t, -1\}\{t, 2\}^k \text{ for all } k \in \mathbb{Z}.$$

(16) If $2, t \in R^\times$*, then*

$$\langle \{t, \pm 2^k\} \mid k \in \mathbb{Z} \rangle = \langle \{t, -1\}, \{t, 2\} \rangle.$$

On the other hand, we already know the following two facts.

Theorem 5.2. (cf. [26],[19]) *Let S be the set of all prime numbers. Then:*

$$K_2(2, \mathbb{Z}_S) = K_2(2, \mathbb{Q}) \simeq \mathbb{Z} \oplus \amalg_{q \in S} (\mathbb{Z}/q\mathbb{Z})^\times.$$

Theorem 5.3. (cf. [10],[12])

$$K_2(2,\mathbb{Z}) = \langle\{-1,-1\}\rangle \simeq \mathbb{Z}.$$

Furthermore, we obtain the following results using previous relations.

Theorem 5.4. (cf. [19]) *Let $S = \{p_1, p_2, p_3, \cdots, p_n\}$ be the set of the first n successive prime numbers, that is, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\cdots$. Put $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_n}]$. Then:*

$$K_2(2,\mathbb{Z}_S) \simeq \mathbb{Z} \oplus \amalg_{q\in S}(\mathbb{Z}/q\mathbb{Z})^\times.$$

Theorem 5.5. (cf. [19]) *Let $S = \{2,5\}$, $\{2,3,7\}$, $\{2,3,11\}$, $\{2,3,5,11\}$, $\{2,3,13\}$, $\{2,3,7,13\}$, $\{2,3,17\}$, $\{2,3,5,19\}$. Put $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{q}]_{q\in S}$. Then,*

$$K_2(2,\mathbb{Z}_S) \simeq \mathbb{Z} \oplus \amalg_{q\in S}(\mathbb{Z}/q\mathbb{Z})^\times.$$

Theorem 5.6. (Three Unit Formula, cf. [20]) *Let $t, u, v \in R^\times$ and put $a = u + v - t \in R$. Then :*

$$\hat{x}_{21}(a)\hat{x}_{12}(-\frac{1}{u})\hat{x}_{21}(\frac{au}{v})\hat{x}_{12}(\frac{v}{tu})\hat{x}_{21}(-\frac{at}{v})\hat{x}_{12}(\frac{1}{t}) = \{\frac{t}{v}, u\}\{t, -v\}^{-1}.$$

Theorem 5.7. (cf. [20]) $K_2(2,\mathbb{Z}[\frac{1}{3}]) \simeq \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^\times$.

Here, we notice the following fact.

Theorem 5.8. (cf. [2]) *Let p be a prime number. Then, the following four conditions are equivalent.*

(1) $\mathbb{Z}^\times \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^\times$*;*

(2) $p = 2, 3$*;*

(3) $K_2(2,\mathbb{Z}[\frac{1}{p}]) = C(\mathbb{Z}[\frac{1}{p}])$*;*

(4) $E_u(2,\mathbb{Z}[\frac{1}{p}]) \xrightarrow{\sim} SL(2,\mathbb{Z}[\frac{1}{p}])$.

Proof. We can easily find (1) $\Leftrightarrow$ (2) as well as (3) $\Leftrightarrow$ (4). By Theorem 5.1, we see (2) $\Rightarrow$ (3). To prove (3) $\Rightarrow$ (2), we suppose $K_2(2,\mathbb{Z}[\frac{1}{p}]) = C(\mathbb{Z}[\frac{1}{p}])$. Applying the surjection $K_2(2,\mathbb{Z}[\frac{1}{p}]) \to K_2(\mathbb{Z}[\frac{1}{p}])$, (cf. [5]), we obtain

$$K_2(\mathbb{Z}[\frac{1}{p}]) \simeq \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})^\times \simeq \langle\{-1,-1\},\ \{p,-1\}\rangle \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z}\ ; \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\ , \end{cases}$$

which shows $p = 2, 3$. $\square$

6. New Examples

We may simply use $a \equiv b$ instead of $a \equiv b \pmod{K'}$. In this section, we will have several new examples and computations.

Theorem 6.1. (Criteiron) *Let $q_1, \cdots, q_n, p$ be distinct prime numbers. Put $D = \mathbb{Z}[\frac{1}{q_1}, \cdots, \frac{1}{q_n}]$. Suppose that D is universal for $\Phi = \mathrm{A}_1$, and that $D^\times \to (D/pD)^\times$ is surjective. We also assume*

$$K_2(2, D) = C(D) \simeq \mathbb{Z} \oplus \amalg_{1 \le i \le n} (\mathbb{Z}/q_i\mathbb{Z})^\times,$$

which is naturally corresponding to $K_2(2, D) \to K_2(D)$. Put $K' = K_2(2, D)_p = Im[K_2(2, D) \to K_2(2, D[\frac{1}{p}])]$. If there exists $u \in D^\times$ such that

$$\{p, u\}^{p-1} \equiv 1 \ (mod\ K')$$

and

$$K_2(2, D[\frac{1}{p}]) = K' \cdot \langle \{p, u\} \rangle,$$

then we obtain:

$$K_2(2, D[\frac{1}{p}]) = C(D[\frac{1}{p}]) \simeq \mathbb{Z} \oplus \left(\amalg_{1 \le i \le n} (\mathbb{Z}/q_i\mathbb{Z})^\times \right) \oplus (\mathbb{Z}/p\mathbb{Z})^\times.$$

Proof. Put $G = K_2(2, D[\frac{1}{p}])$ and $G' = K_2(D[\frac{1}{p}])$, and let $\varphi : G \to G'$ be a natural homomorphism. Set $H = K' = K_2(2, D)_p = Im[K_2(2, D) \to K_2(D[\frac{1}{p}])]$, and $H' = \varphi(H)$. Then, $G/H = \langle \{p, u\} \bmod H \rangle$ and $\{p, u\}^{p-1} = 1$ in G/H. On the other hand, $G'/H' \simeq (\mathbb{Z}/p\mathbb{Z})^\times$. Since φ is surjective (cf. [5]) and induces a natural epimorphism

$$G/H \twoheadrightarrow G'/H',$$

we obtain $G/H \simeq G'/H'$, which means $\mathrm{Ker}\,\varphi \subset H$ and $\mathrm{Ker}\,\varphi = \langle \{-1, -1\}^2 \rangle$. Note that $K_2(2, D[\frac{1}{p}])$ is a finitely generated abelian group. We also note that there is a canonical homomorphism

$$\nu : K_2(2, D[\frac{1}{p}]) \xrightarrow{\mu} K_2(2, \mathbb{Q}) \overset{\lambda}{\twoheadrightarrow} \langle \{-1, -1\} \rangle \simeq \mathbb{Z},$$

where $\nu = \lambda \circ \mu$, and where λ is the projection (cf. Theorem 5.2) satisfying

$$\{v, w\} \overset{\lambda}{\mapsto} 1, \quad \{\pm v, \mp w\} \overset{\lambda}{\mapsto} 1, \quad \{-v, -w\} \overset{\lambda}{\mapsto} \{-1, -1\}$$

for all $v, w \in \mathbb{Q}_{>0}$ (cf. [10],[12],[25]) and μ is induced by the embedding $D[\frac{1}{p}] \to \mathbb{Q}$. Using ν and φ, we can show

$$\{p, u\}^{p-1} \in \amalg_{1\leq i\leq n}(\mathbb{Z}/q_i\mathbb{Z})^{\times} = \text{the torsion part of } K'.$$

Hence, $K_2(2, D[\frac{1}{p}])$ is a direct sum of a free abelian group $X = \langle\{-1, -1\}\rangle \simeq \mathbb{Z}$ and a finite abelian group Y. Using the fact $\mathrm{Ker}(\varphi) \subset X$, we have

$$Y \simeq \varphi(Y) \simeq \amalg_{1\leq i\leq n}(\mathbb{Z}/q_i\mathbb{Z})^{\times} \oplus (\mathbb{Z}/p\mathbb{Z})^{\times},$$

which implies

$$K_2(2, D[\frac{1}{p}]) = X \oplus Y \simeq \mathbb{Z} \oplus \left(\amalg_{1\leq i\leq n} (\mathbb{Z}/q_i\mathbb{Z})^{\times}\right) \oplus (\mathbb{Z}/p\mathbb{Z})^{\times}.$$

□

Theorem 6.2. *Let* $S = \{2, 3, 11, 23\}, \{2, 3, 5, 7, 29\}, \{2, 3, 5, 31\}, \{2, 3, 37\}, \{2, 3, 5, 7, 41\}, \{2, 3, 7, 43\}, \{2, 3, 7, 11, 43\}, \{2, 3, 11, 23, 47\}$. *Then,*

$$K_2(2, \mathbb{Z}_S) \simeq \mathbb{Z} \oplus \amalg_{q\in S}(\mathbb{Z}/q\mathbb{Z})^{\times}.$$

Proof 1. *Suppose* $S = \{2, 3, 11, 23\}$.

Let $K = K_2(2, D[\frac{1}{23}])$, $D = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{11}]$, $p = 23$ and put

$$K' = K_2(2, D)_{23} = \mathrm{Im}[K_2(2, D) \to K_2(2, D[\frac{1}{23}])].$$

Then,

$$\begin{aligned} K &= K' \cdot \langle\{-23, \pm 2^k 3^\ell 11^m\} \mid k, \ell, m \in \mathbb{Z}\rangle \\ &= K' \cdot \langle\{-23, \pm 2^k 11^\ell\} \mid k, \ell \in \mathbb{Z}\rangle \\ &= K' \cdot \langle\{23, \pm 2^k 11^\ell\} \mid k, \ell \in \mathbb{Z}\rangle \\ &= K' \cdot \langle\{23, \pm 2^k\} \mid k \in \mathbb{Z}\rangle \\ &= K' \cdot \langle\{23, -1\},\ \{23, 2\}\rangle. \end{aligned}$$

Note $11 + (-2) = 3^2$. Then

$$\begin{aligned} \{23, -1\}^2 &= \{23, -\tfrac{1}{2}\}\{23, 2\}\{23, -1\} \\ &= \{23, 11\}\{23, 2\}\{23, -1\} \\ &= \{23, 11\}\{23, -2\} \\ &= \{23, -22\} \\ &= 1. \end{aligned}$$

Note $\frac{11\cdot 23}{2^8} + \frac{3}{2^8} = 1$. Then, we obtain

$$\begin{aligned} &\{23, \tfrac{11\cdot 23}{2^8}\}\{23, \tfrac{3}{2^8}\} = \{23, \tfrac{11\cdot 23}{2^8} \cdot \tfrac{3}{2^8}\}, \\ &\{23, -\tfrac{11}{2^8}\}\{23, \tfrac{3}{2^8}\} = \{23, -\tfrac{11}{2^8} \cdot \tfrac{3}{2^8}\}, \\ &\{23, \tfrac{1}{2^9}\}\{23, \tfrac{3}{2^8}\} = \{23, \tfrac{1}{2^9} \cdot \tfrac{3}{2^8}\}, \\ &\{23, \tfrac{1}{2}\}\{23, 3\} = \{23, \tfrac{3}{2}\} \end{aligned}$$

and

$$\{23, 2\}\{23, 3\} = \{23, 6\}.$$

Hence,

$$\begin{aligned} \{23, 2\}^3\{23, 3\} &= \{23, 4\}\{23, 2\}\{23, 3\} \\ &= \{23, 4\}\{23, 6\} \\ &= \{23, 24\} \\ &= \{24, \tfrac{1}{23}\} \\ &= \{24, -1\} \\ &\equiv 1. \end{aligned}$$

Note $2^8 = 256 = 11 \cdot 23 + 3$. Then,

$$\begin{aligned} &\{23, 2^8\} \equiv \{23, 3\}, \\ &\{23, 2\}^8 = \{23, 2^8\} \equiv \{23, 3\} \equiv \{23, 2\}^{-3}, \\ &\{23, 2\}^{11} \equiv 1. \end{aligned}$$

Hence, we obtain $\{23, -2\}^{11} \equiv \{23, -1\}$ and $\{23, -2\}^{22} \equiv 1$. Then,

$$K = K' \cdot \langle \{23, -1\}, \{23, 2\} \rangle = K' \cdot \langle \{23, -2\} \rangle$$

and $\{23, -2\}^{22} \equiv 1$. Therefore,

$$K_2(2, \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{11}, \frac{1}{23}]) \simeq \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^{\times} \oplus (\mathbb{Z}/11\mathbb{Z})^{\times} \oplus (\mathbb{Z}/23\mathbb{Z})^{\times}.$$

Proof 2. *Suppose* $S = \{2, 3, 5, 7, 29\}$.

Let $K = K_2(2, D[\frac{1}{29}]),\ D = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}],\ p = 29$ and put

$$K' = K_2(2, D)_{29} = \mathrm{Im}[K_2(2, D) \to K_2(2, D[\frac{1}{29}])].$$

Then,

$$\begin{aligned} K &= K' \cdot \langle \{-29, \pm 2^k 3^\ell 5^m 7^n\} \mid k, \ell, m, n \in \mathbb{Z} \rangle \\ &= K' \cdot \langle \{-29, \pm 2^k 3^\ell 7^m\} \mid k, \ell, m \in \mathbb{Z} \rangle \\ &= K' \cdot \langle \{29, \pm 2^k 3^\ell 7^m\} \mid k, \ell, m \in \mathbb{Z} \rangle \\ &= K' \cdot \langle \{29, \pm 2^k 3^\ell\} \mid k, \ell \in \mathbb{Z} \rangle. \end{aligned}$$

Note $\frac{2}{29} + \frac{3^3}{29} = 1$. Then we obtain

$$\begin{aligned} &\{29, \tfrac{2}{29}\}\{29, \tfrac{3^3}{29}\} = \{29, \tfrac{2\cdot 3^3}{29^2}\}, \\ &\{29, -2\}\{29, -3^3\} = \{29, 2 \cdot 3^3\}, \\ &\{29, -2\}\{29, -3\}\{29, 3^2\} = \{29, 6\}\{29, 3^2\} \end{aligned}$$

and

$$\{29, -2\}\{29, -3\} = \{29, 6\}.$$

Here we notice $\{29, -2\}\{29, 3\} = \{29, -6\}$.

Note $\frac{2^5}{3} - \frac{29}{3} = 1$. Then,

$$\begin{aligned} &\{29, \tfrac{2^5}{3}\}\{29, -\tfrac{29}{3}\} = \{29, \tfrac{2^5\cdot(-29)}{3^2}\}, \\ &\{29, \tfrac{2^5}{3}\}\{29, \tfrac{1}{3}\} = \{29, \tfrac{2^5}{3^2}\}, \\ &\{29, 2^5 \cdot 3\}\{29, 3\} = \{29, 2^5\}\{29, 3^2\}, \\ &\{29, 6\}\{29, 3\} = \{29, 2\}\{29, 3^2\}, \\ &\{29, -2\}\{29, -3\}\{29, 3\} = \{29, 2\}\{29, 3^2\}, \\ &\{29, -1\}\{29, -3\}\{29, \tfrac{1}{3}\} = 1 \end{aligned}$$

and

$$\{29, -1\}^2 = 1.$$

Since $2^{11} = 2 \cdot 5 \cdot 7 \cdot 29 + 2 \cdot 3^2$, we obtain

$$\begin{aligned} &\{29, 2^{11}\} \equiv \{29, 2 \cdot 3^2\} = \{29, 2\}\{29, 3^2\}, \\ &\{29, 2^{10}\} \equiv \{29, 3^2\}. \end{aligned}$$

Note $2^5 = 29 + 3$. Then,

$$\{29, 2^5\} \equiv \{29, 3\}.$$

Since $-2^5 = (-1)\cdot 29 + (-3)$, we obtain

$$\begin{aligned}\{29,-3\} &\equiv \{29,-2^5\}\\ &= \{29,-1\}\{29,2^5\}\\ &\equiv \{29,-1\}\{29,3\}.\end{aligned}$$

Hence,

$$\begin{aligned}K &= K'\cdot\langle\{29,\pm 2^k 3^\ell\}\mid k,\ell\in\mathbb{Z}\rangle\\ &= K'\cdot\langle\{29,-1\},\{29,2\},\{29,\pm 3\},\{29,3^2\},\{29,\pm 6\}\rangle\\ &= K'\cdot\langle\{29,-1\},\{29,2\}\rangle.\end{aligned}$$

Note $3^3 = 29-2$. Then

$$\{29,3^3\}\equiv\{29,-2\}=\{29,-1\}\{29,2\},$$

which implies

$$\begin{aligned}\{29,2^{14}\} &= \{29,2^{10}\}\{29,2^4\}\\ &\equiv \{29,3^2\}\{29,2^4\}\\ &\equiv \{29,3\}^2\{29,2\}^4\\ &\equiv \{29,3\}^3\{29,2\}^{-1}\\ &\equiv \{29,-1\}.\end{aligned}$$

Therefore,

$$K = K'\cdot\langle\{29,2\}\rangle$$

and $\{29,2\}^{28}\equiv 1$. Hence,

$$K_2(2,\mathbb{Z}[\frac{1}{2},\frac{1}{3},\frac{1}{5},\frac{1}{7},\frac{1}{29}])$$

$$\simeq \mathbb{Z}\oplus(\mathbb{Z}/3\mathbb{Z})^\times\oplus(\mathbb{Z}/5\mathbb{Z})^\times\oplus(\mathbb{Z}/7\mathbb{Z})^\times\oplus(\mathbb{Z}/29\mathbb{Z})^\times.$$

Proof 3. *Suppose* $S=\{2,3,5,31\}$.

Let $K=K_2(2,D[\frac{1}{31}])$, $D=\mathbb{Z}[\frac{1}{2},\frac{1}{3},\frac{1}{5}]$, $p=31$ and put

$$K' = K_2(2,D)_{31} = \mathrm{Im}[K_2(2,D)\to K_2(2,D[\frac{1}{31}])].$$

Then,

$$\begin{aligned} K &= K' \cdot \langle \{31, \pm 2^k 3^\ell 5^m\} \mid k, \ell, m \in \mathbb{Z} \rangle \\ &= K' \cdot \langle \{31, \pm 2^k 3^\ell\} \mid k, \ell \in \mathbb{Z} \rangle \\ &= K' \cdot \langle \{31, -1\}, \{31, 2\}, \{31, \pm 3\}, \{31, 3^2\}, \{31, \pm 6\} \rangle \end{aligned}$$

Note $\frac{6}{5} - \frac{1}{5} = 1$. Then,

$$\begin{aligned} &\{31, \tfrac{6}{5}\}\{31, -\tfrac{1}{5}\} = \{31, \tfrac{6}{5} \cdot \tfrac{(-1)}{5}\}, \\ &\{31, -6^2\}\{31, 6\} = \{31, -6^3\}, \\ &\{31, 6^2\}\{31, -1\}\{31, 6\} = \{31, -6\}\{31, 6^2\}, \\ &\{31, -1\}\{31, 6\} = \{31, -6\} = \{31, -1\}\{31, 2\}\{31, 3\} \end{aligned}$$

and

$$\{31, 6\} = \{31, 2\}\{31, 3\}.$$

Since $\frac{5^2}{31} + \frac{6}{31} = 1$, we obtain

$$\begin{aligned} &\{31, \tfrac{5^2}{31}\}\{31, \tfrac{6}{31}\} = \{31, \tfrac{5^2}{31} \cdot \tfrac{6}{31}\}, \\ &\{31, -5^2\}\{31, -6\} = \{31, 5^2 \cdot 6\}, \\ &\{31, -1\}\{31, 5^2\}\{31, -6\} = \{31, 5^2\}\{31, 6\}, \\ &\{31, -1\}\{31, -1\}\{31, 2\}\{31, 3\} = \{31, 6\} \end{aligned}$$

and

$$\{31, -1\}^2 = 1.$$

Note $10 - 3^2 = 1$. Then,

$$\begin{aligned} &\{31, 10\}\{31, -3^2\} = \{31, -10 \cdot 3^2\}, \\ &\{31, -\tfrac{1}{3}\}\{31, -1\}\{31, 3^2\} = \{31, 3\}. \end{aligned}$$

Therefore, we have

$$\left[\begin{array}{l} \{31, -3\} = \{31, -1\}\{31, 3\}, \\ \{31, 3\}\{31, 3^{-1}\} = 1. \end{array} \right.$$

Since $3^3 = 31 - 2^2$, we obtain

$$\{31, 3\}^3 = \{31, 3^3\} \equiv \{31, -2^2\} = \{31 - 1\}\{31, 2\}^2.$$

Note $2^5 = 31+1$. Then,

$$\begin{aligned}\{31,2\}^5 &= \{31,2^5\}\\ &= \{2^5,\tfrac{1}{31}\}\\ &= \{32,\tfrac{1}{31}\}\\ &= \{32,-1\}\\ &= \{-1,2^5\}\\ &= \{-1,2\}^5 = 1.\end{aligned}$$

Therefore,

$$\begin{aligned}\{31,2\} &= \{31,2\}^{-4}\\ &\equiv \{31,-1\}^{-2}\{31,3\}^{-6}\\ &= \{31,3\}^{-6}\end{aligned}$$

and

$$\{31,3\}^{30} \equiv \{31,2\}^{-5} = 1.$$

Hence,

$$\begin{aligned}\{31,3\}^{15} &\equiv \{31-1\}^5\{31,2\}^{10}\\ &= \{31,-1\}.\end{aligned}$$

Thus, we obtain

$$K = K'\cdot\langle\{31,3\}\rangle$$

and $\{31,3\}^{30}\equiv 1$. Hence, we obtain

$$K_2(2,\mathbb{Z}[\frac{1}{2},\frac{1}{3},\frac{1}{5},\frac{1}{31}]) \simeq \mathbb{Z}\oplus(\mathbb{Z}/3\mathbb{Z})^\times\oplus(\mathbb{Z}/5\mathbb{Z})^\times\oplus(\mathbb{Z}/31\mathbb{Z})^\times.$$

Proof 4. *Suppose* $S=\{2,3,37\}$.

Let $K = K_2(2,D[\frac{1}{37}])$, $D=\mathbb{Z}[\frac{1}{2},\frac{1}{3}]$, $p=37$ and put

$$K' = K_2(2,D)_{37} = \mathrm{Im}[K_2(2,D)\to K_2(2,D[\frac{1}{37}])].$$

Then, we obtain

$$\begin{aligned}K &= K'\cdot\langle\{37,\pm 2^k3^\ell\}\mid k,\ell\in\mathbb{Z}\rangle\\ &= K'\cdot\langle\{37,-1\},\{37,2\},\{37,\pm3\},\{37,3^2\},\{37,\pm6\}\rangle.\end{aligned}$$

Note $37 - 1 = 36 = 6^2$. Then,

$$\{37, -1\}^2 = \{37, -1\}\{37, -1\} = \{37, 37\}\{37, -1\} = \{37, -37\} = 1.$$

On the other hand,

$$\{37, 2^2\}\{37, 3^2\} = \{37, 36\} = \{37, -1\}.$$

Since $2^6 = 64 = 37 + 3^3$, we obtain

$$\{37, 2^6\} \equiv \{37, 3^3\}.$$

Note $\frac{2^6}{37} - \frac{3^3}{37} = 1$. Then,

$$\begin{aligned} &\{37, \tfrac{2^6}{37}\}\{37, -\tfrac{3^3}{37}\} = \{37, \tfrac{2^6}{37} \cdot \tfrac{-3^3}{37}\}, \\ &\{37, -2^6\}\{37, 3^3\} = \{37, -2^6 \cdot 3^3\}, \\ &\{37, -1\}\{37, 3\} = \{37, -3\}. \end{aligned}$$

Since $\{37, 6\} = \{37, -\frac{1}{6}\}$, we obtain

$$\{37, 6\}^2 = \{37, -\tfrac{1}{6}\}^2 = \{37, 6\}\{37, -\tfrac{1}{6}\} = \{37, -1\}.$$

Therefore,

$$\begin{aligned} \{37, -1\} &= \{37, -6\}\{37, \tfrac{1}{6}\} \\ &= \{37, -1\}\{37, 2\}\{37, 3\}\{37, \tfrac{1}{6}\}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &\{37, 2\}^{-1}\{37, 3\}^{-1} = \{37, \tfrac{1}{6}\}, \\ &\{37, \tfrac{1}{2}\}\{37, \tfrac{1}{3}\} = \{37, \tfrac{1}{6}\} \end{aligned}$$

and

$$\{37, 2\}\{37, 3\} = \{37, \tfrac{1}{2}\}\{37, \tfrac{1}{3}\}\{37, 2^2\}\{37, 3^2\} = \{37, \tfrac{1}{6}\}\{37, 6^2\} = \{37, 6\}.$$

Hence,

$$\begin{aligned} &\langle \{37, \pm 2^k 3^\ell\} \mid k, \ell \in \mathbb{Z} \rangle \\ &= \langle \{37, -1\}, \{37, 2\}, \{37, \pm 3\}, \{37, 3^2\}, \{37, \pm 6\} \rangle \\ &= \langle \{37, -1\}, \{37, 2\}, \{37, 3\} \rangle. \end{aligned}$$

Since $\{37, 2\}^4\{37, 3\}^4 = \{37, -1\}^2 = 1$, we obtain

$$\left[\begin{array}{lcl} \{37, 2\}^6 & \equiv & \{37, 3\}^3, \\ \{37, 2\}^4 & = & \{37, 3\}^{-4}. \end{array} \right.$$

Hence,

$$\{37, 2\}^2 \equiv \{37, 3\}^7,$$

$$\{37,3\}^{14} \equiv \{37,2\}^4 = \{37,3\}^{-4}$$

and

$$\{37,3\}^{18} \equiv 1.$$

Thus,

$$\begin{aligned}
\{37,2\}^{18} &\equiv \{37,2\}^{18}\{37,3\}^{18} \\
&\equiv \{37,2^2\}^9\{37,3^2\}^9 \\
&= \{37,-1\}^9 \\
&= \{37,-1\}.
\end{aligned}$$

Since $\{37,3\}^9 \equiv \{37,2\}^{18} \equiv \{37,-1\}$, we obtain

$$\begin{aligned}
\{37,-1\}\{37,2\}^8 &= \{37,-1\}\{37,2\}^6\{37,2\}^2 \\
&\equiv \{37,-1\}\{37,3\}^3\{37,2\}^2 \\
&= \{37,-1\}\{37,3\}\{37,3\}^2\{37,2\}^2 \\
&= \{37,-1\}\{37,3\}\{37,-1\} = \{37,3\}.
\end{aligned}$$

Hence,

$$\{37,2\}^2\{37,3\}^2\{37,2\}^8 \equiv \{37,3\},$$
$$\{37,2\}^{10}\{37,3\}^2 \equiv \{37,3\}$$

and

$$\{37,3\} \equiv \{37,2\}^{-10}.$$

Since

$$K = K' \cdot \langle\{37,2\}\rangle$$

and $\{37,2\}^{36} \equiv 1$, we obtain

$$K_2(2,\mathbb{Z}[\frac{1}{2},\frac{1}{3},\frac{1}{37}]) \simeq \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^\times \oplus (\mathbb{Z}/37\mathbb{Z})^\times.$$

Proof 5. *Suppose* $S = \{2,3,5,7,41\}$.

Let $K = K_2(2, D[\frac{1}{41}])$, $D = \mathbb{Z}[\frac{1}{2},\frac{1}{3},\frac{1}{5},\frac{1}{7}]$, $p = 41$, and put

$$K' = K_2(2,D)_{41} = \mathrm{Im}[K_2(2,D) \to K_2(2,D[\frac{1}{41}])].$$

Then, we obtain

$$\begin{aligned} K &= K' \cdot \langle \{-41, \pm 2^k 3^\ell 5^m 7^n\} \mid k, \ell, m, n \in \mathbb{Z} \rangle \\ &= K' \cdot \langle \{-41, \pm 2^k 3^\ell 5^m\} \mid k, \ell, m \in \mathbb{Z} \rangle \\ &= K' \cdot \langle \{41, \pm 2^k 3^\ell 5^m\} \mid k, \ell, m \in \mathbb{Z} \rangle \\ &= K' \cdot \langle \{41, \pm 2^k 3^\ell\} \mid k, \ell \in \mathbb{Z} \rangle. \end{aligned}$$

Note $\frac{2^4}{41} + \frac{5^2}{41} = 1$. Then,

$$\{41, \tfrac{2^4}{41}\}\{41, \tfrac{5^2}{41}\} = \{41, \tfrac{2^4 \cdot 5^2}{41^2}\},$$
$$\{41, \tfrac{1}{41}\}\{41, \tfrac{1}{41}\} = \{41, \tfrac{1}{41^2}\}$$

and

$$\{41, -1\}^2 = 1.$$

Since

$$\{41, -1\}\{-41, u\} = \{41, -u\}\{-1, u\},$$

we obtain

$$\{-41, u\} \equiv \{41, -1\}\{41, -u\} \text{ for } u \in D^\times.$$

Note $3^4 = 81 = 2 \cdot 41 - 1$. This implies

$$\{41, 3^4\} \equiv \{41, -1\}$$

and

$$\{41, 3^8\} = \{41, 3^4\}^2 \equiv \{41, -1\}^2 = 1.$$

Since $2^5 = 32 = 41 - 3^2$, we obtain

$$\{41, 2^5\} \equiv \{41, -3^2\} = \{41, -1\}\{41, 3^2\},$$
$$\{41, 2^{10}\} \equiv \{41, -1\}^2\{41, 3^4\} \equiv \{41, -1\}$$

and

$$\{41, 2^{20}\} = \{41, 2^{10}\}^2 \equiv \{41, -1\}^2 = 1.$$

Note $-3^3 = -41 + 14$. Then,

$$\begin{aligned} \{41, 3^3\}\{41, -1\} &\equiv \{-41, -3^3\} \\ &\equiv \{-41, 14\} \\ &= \{-41, \tfrac{1}{3}\} \\ &= \{41, -\tfrac{1}{3}\}\{41, -1\} \end{aligned}$$

and

$$\{41, 3^3\} = \{41, 3\}\{41, 3^2\} \equiv \{41, -\frac{1}{3}\}.$$

Hence,

$$\begin{aligned}\{41,3\}^2\{41,3^2\} &\equiv \{41,-\tfrac{1}{3}\}\{41,3\}\\ &= \{41,-1\}\\ &\equiv \{41,3^4\}\\ &= \{41,3^2\}^2.\end{aligned}$$

Therefore,

$$\left[\begin{array}{l}\{41,3\}^2 \equiv \{41,3^2\},\\ \{41,3\}\{41,\tfrac{1}{3}\} \equiv 1,\\ \{41,-1\}\{41,3\} \equiv \{41,-3\}.\end{array}\right.$$

On the other hand,

$$\begin{aligned}\{41,2\}\{41,3\} &\equiv \{41,-1\}\{41,2\}\{41,-3\}\\ &\equiv \{-41,-2\}\{41,-1\}\{-41,3\}\\ &= \{41,-1\}\{-41,-6\}\\ &\equiv \{41,6\}.\end{aligned}$$

Note $2+7=3^2$. Then,

$$\begin{aligned}\{41,6\}\{41,7\} &\equiv \{41,2\}\{41,3\}\{41,7\}\\ &= \{41,3\}\{41,14\}\\ &\equiv \{41,3\}\{41,-1\}\{-41,-14\}\\ &= \{41,3\}\{41,-1\}\{-41,-\tfrac{1}{3}\}\\ &\equiv \{41,3\}\{41,\tfrac{1}{3}\}\\ &\equiv 1.\end{aligned}$$

Hence,

$$\begin{aligned}\{41,6\}^2 &\equiv \{41,2\}^2\{41,3\}^2\\ &\equiv \{41,2^2\}\{41,3^2\}\\ &\equiv \{41,2^2\}\{41,-2^5\}\\ &= \{41,-1\}\{41,2^7\}\\ &\equiv \{41,2^{10}\}\{41,2^7\}\\ &= \{41,2^{17}\}\\ &\equiv \{41,2^{-3}\}.\end{aligned}$$

Therefore,

$$\{41,6\}^{14} \equiv \{41,2^{-21}\} \equiv \{41,2^{-1}\},$$

$$\{41,7\}^{14} \equiv \{41,6\}^{-14} \equiv \{41,2\}$$

and

$$\begin{aligned}\{41,7\}^{-15} &\equiv \{41,7\}^{-1}\{41,7\}^{-14}\\ &\equiv \{41,6\}\{41,2\}^{-1}\\ &\equiv \{41,3\}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}K &= K' \cdot \langle \{41, \pm 2^k 3^\ell\} \mid k,\ell \in \mathbb{Z}\rangle\\ &= K' \cdot \langle \{41,-1\},\{41,2\},\{41,3\}\rangle\\ &= K' \cdot \langle \{41,7\}\rangle\end{aligned}$$

and $\{41,7\}^{40} \equiv \{41,6\}^{-40} \equiv \{41,2\}^{60} \equiv 1$, which implies

$$K_2(2, \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{41}])$$

$$\simeq \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^\times \oplus (\mathbb{Z}/5\mathbb{Z})^\times \oplus (\mathbb{Z}/7\mathbb{Z})^\times \oplus (\mathbb{Z}/41\mathbb{Z})^\times.$$

Proof 6. *Suppose* $S = \{2,3,7,43\}$.

Let $K = K_2(2, D[\frac{1}{43}]),\ D = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}],\ p = 43$, and put

$$K' = K_2(2,D)_{43} = \mathrm{Im}[K_2(2,D) \to K_2(2, D[\frac{1}{43}])].$$

Then, we obtain

$$\begin{aligned}K &= K' \cdot \langle \{43, \pm 2^k 3^\ell 7^m\} \mid k,\ell,m \in \mathbb{Z}\rangle\\ &= K' \cdot \langle \{43, \pm 2^k 3^\ell\} \mid k,\ell \in \mathbb{Z}\rangle.\end{aligned}$$

Note $\frac{2^4}{43} + \frac{3^3}{43} = 1$. Then,

$$\{43, \tfrac{2^4}{43}\}\{43, \tfrac{3^3}{43}\} = \{43, \tfrac{2^4 \cdot 3^3}{43^2}\},$$
$$\{43, -2^4\}\{43, -3^3\} = \{43, 2^4 \cdot 3^3\}$$

and

$$\{43,-1\}\{43,-3\} = \{43,3\}.$$

Since $-6^2 = (-1)\cdot 43 + 7$, we obtain

$$\{43, -6^2\} \equiv \{43, 7\}$$

and

$$\begin{aligned}\{43,6^2\}^{-1} &= \{43,\tfrac{1}{6^2}\}\\ &= \{43,-1\}\{43,-6^2\}^{-1}\\ &\equiv \{43,-1\}\{43,7\}^{-1}\\ &= \{43,-\tfrac{1}{7}\}\\ &= \{43,6\}.\end{aligned}$$

Therefore, we obtain

$$\{43,6^3\}\equiv 1.$$

Note $-6^3=(-6)\cdot 43+42$. Then, we have

$$\begin{aligned}\{43,-1\} &= \{43,42\}\\ &\equiv \{43,-6^3\}\\ &= \{43,-6\}\{43,6^2\}\\ &= \{43,-1\}\{43,2\}\{43,3\}\{43,6^2\},\end{aligned}$$

which implies

$$\{43,2\}\{43,3\}\{43,6^2\}\equiv 1,$$
$$\{43,2\}\{43,3\}\{43,6^3\}\equiv\{43,6\}$$

and

$$\{43,2\}\{43,3\}\equiv\{43,6\}.$$

In particular, we obtain

$$\{43,-1\}\{43,6\}\equiv\{43,-1\}\{43,2\}\{43,3\}=\{43,-6\}.$$

Since $2\cdot 6^2=72=2\cdot 43-14$, we calculate

$$\begin{aligned}\{43,\tfrac{1}{3}\} &= \{43,-14\}\\ &\equiv \{43,2\cdot 6^2\}\\ &= \{43,2\}\{43,6^2\}\\ &\equiv \{43,2\}\{43,6\}^{-1}\end{aligned}$$

and

$$\begin{aligned}\{43,2\}\{43,3^2\} &\equiv \{43,\tfrac{1}{3}\}\{43,6\}\{43,3^2\}\\ &= \{43,3\}\{43,6\}\\ &\equiv \{43,3\}^2\{43,2\}.\end{aligned}$$

Hence, we obtain

$$\{43,3\}^2\equiv\{43,3^2\},$$

$$\{43, 3\}\{43, \tfrac{1}{3}\} \equiv 1$$

and

$$\{43, -1\}\{43, 3\} \equiv \{43, -3\}.$$

Thus,

$$\begin{aligned} \{43, -1\}^2 &\equiv \{43, -1\}\{43, 7\}\{43, -\tfrac{1}{7}\} \\ &= \{43, -1\}\{43, 7\}\{43, 6\} \\ &\equiv \{43, 7\}\{43, -6\} \\ &\equiv \{43, -42\} \\ &= 1 \end{aligned}$$

and

$$\langle \{43, \pm 2^k 3^\ell\} \mid k, \ell \in \mathbb{Z} \rangle = \langle \{43, -1\}, \{43, 2\}, \{43, 3\} \rangle.$$

Note $2^6 = 64 = 43 + 3 \cdot 7$. Then,

$$\begin{aligned} &\{43, 2^6\} \equiv \{43, 21\} = \{43, -\tfrac{1}{2}\}, \\ &\{43, 2^7\} \equiv \{43, -1\} \end{aligned}$$

and

$$\{43, 2\}^{14} \equiv 1.$$

Therefore,

$$\begin{aligned} \{43, 2\}^3\{43, 3\}^3 &= \{43, 2^2\}\{43, 2\}\{43, 3^2\}\{43, 3\} \\ &= \{43, 6^2\}\{43, 6\} \\ &\equiv \{43, 6^3\} \\ &\equiv 1. \end{aligned}$$

Thus,

$$\begin{aligned} &\{43, 2^3\}^5\{43, 3^3\}^5 \equiv 1, \\ &\{43, 2\}^{14}\{43, 2\}\{43, 3\}^{15} \equiv 1 \end{aligned}$$

and

$$\{43, 2\} \equiv \{43, 3\}^{-15},$$

which shows

$$\{43, 3\}^{42} \equiv \{43, 3\}^{3 \cdot 14} \equiv \{43, 2\}^{-3 \cdot 14} \equiv 1.$$

Hence, we obtain

$$\begin{aligned} K &= K' \cdot \langle \{43, -1\}, \{43, 2\}, \{43, 3\} \rangle \\ &= K' \cdot \langle \{43, 3\} \rangle \end{aligned}$$

and $\{43,3\}^{42} \equiv 1$. Thus,

$$K_2(2,\mathbb{Z}[\frac{1}{2},\frac{1}{3},\frac{1}{7},\frac{1}{43}]) \simeq \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^\times \oplus (\mathbb{Z}/7\mathbb{Z})^\times \oplus (\mathbb{Z}/43\mathbb{Z})^\times.$$

Proof 7. *Suppose* $S = \{2,3,7,11,43\}$.

Let $D = \mathbb{Z}[\frac{1}{2},\frac{1}{3},\frac{1}{7},\frac{1}{11}] \ni p = 43$. Then, we already know by Proof 6,

$$\begin{aligned} K &= K' \cdot \langle \{-43, \pm 2^k 3^\ell 7^m 11^n\} \mid k,\ell,m,n \in \mathbb{Z}\rangle \\ &= K' \cdot \langle \{-43, \pm 2^k 3^\ell 7^m\} \mid k,\ell,m,n \in \mathbb{Z}\rangle \\ &= K' \cdot \langle \{43, \pm 2^k 3^\ell 7^m\} \mid k,\ell,m,n \in \mathbb{Z}\rangle \\ &= K' \cdot \langle \{43, \pm 2^k 3^\ell\} \mid k,\ell \in \mathbb{Z}\rangle \\ &= K' \cdot \langle \{43,-1\},\{43,2\},\{43,3\}\rangle \\ &= K' \cdot \langle \{43,3\}\rangle \end{aligned}$$

and $\{43,3\}^{42} \equiv 1$. Hence, we obtain

$$\begin{gathered} K_2(2,\mathbb{Z}[\frac{1}{2},\frac{1}{3},\frac{1}{7},\frac{1}{11},\frac{1}{43}]) \\ \simeq \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^\times \oplus (\mathbb{Z}/7\mathbb{Z})^\times \oplus (\mathbb{Z}/11\mathbb{Z})^\times \oplus (\mathbb{Z}/43\mathbb{Z})^\times. \end{gathered}$$

Proof 8. *Suppose* $S = \{2,3,11,23,47\}$.

Let $K = K_2(2,D[\frac{1}{47}])$, $D = \mathbb{Z}[\frac{1}{2},\frac{1}{3},\frac{1}{11},\frac{1}{23}]$, $p = 47$, and put

$$K' = K_2(2,D)_{47} = \mathrm{Im}[K_2(2,D) \to K_2(2,D[\frac{1}{47}])].$$

Then, we obtain

$$\begin{aligned} K &= K' \cdot \langle \{-47, \pm 2^k 3^\ell 11^m 23^n\} \mid k,\ell,m,n \in \mathbb{Z}\rangle \\ &= K' \cdot \langle \{-47, \pm 2^k 11^\ell 23^m\} \mid k,\ell,m \in \mathbb{Z}\rangle \\ &= K' \cdot \langle \{47, \pm 2^k 11^\ell 23^m\} \mid k,\ell,m \in \mathbb{Z}\rangle \\ &= K' \cdot \langle \{47, \pm 2^k 11^\ell\} \mid k,\ell \in \mathbb{Z}\rangle. \end{aligned}$$

Since

$$\pm 11 = \pm 47 \mp 6^2,\ \pm 22 = \pm 2\cdot 47 \mp 2\cdot 6^2,\ 11^2 = 11\cdot 47 - 6^2\cdot 11,$$

we obtain

$$\begin{aligned}&\{47,11\}\equiv\{47,-6^2\},\quad \{47,-11\}\equiv\{47,6^2\},\\ &\{47,22\}\equiv\{47,-2\}\{47,6^2\},\quad \{47,-22\}\equiv\{47,2\}\{47,6^2\},\\ &\{47,11^2\}\equiv\{47,-6^2\cdot 11\}\equiv\{47,-11\}\{47,6^2\}\equiv\{47,6^4\}.\end{aligned}$$

Note $\frac{24}{47}+\frac{23}{47}=1$. Then,

$$\begin{aligned}&\{47,\tfrac{24}{47}\}\{47,\tfrac{23}{47}\}=\{47,\tfrac{24\cdot 23}{47^2}\},\\ &\{47,-24\}\{47,-23\}=\{47,24\cdot 23\},\\ &\{47,-6\}\{47,2^2\}\{47,\tfrac{1}{2}\}=\{47,2^2\}\{47,6\cdot 23\},\\ &\{47,-6\}\{47,\tfrac{1}{2}\}=\{47,-3\},\\ &\{47,-1\}\{47,2\}\{47,3\}\{47,\tfrac{1}{2}\}=\{47,-3\}\end{aligned}$$

and

$$\{47,-1\}\{47,3\}=\{47,-3\}.$$

On the other hand,

$$\{47,-1\}\{47,-11\}\equiv\{47,-1\}\{47,6^2\}=\{47,-6^2\}\equiv\{47,11\}.$$

Note $23-22=1$. Then,

$$\{47,23\}\{47,-22\}=\{47,-23.\cdot 22\}$$

and

$$\{47,-\tfrac{1}{2}\}\{47,-22\}=\{47,11\}.$$

Therefore

$$\begin{aligned}\{47,-1\}\{47,-22\} &= \{47,-\tfrac{1}{2}\}\{47,2\}\{47,-22\}\\ &= \{46,11\}\{47,2\}\\ &\equiv \{47,-1\}\{47,-11\}\{47,2\}\end{aligned}$$

and

$$\{47,-22\}\equiv\{47,2\}\{47,-11\}.$$

Note $11-2=3^2$. Then,

$$\{47,11\}\{47,-2\}=\{47,-22\}\equiv\{47,2\}\{47,-11\}$$

and

$$\{47,-1\}\{47,11\}\equiv\{47,-11\}.$$

Hence, we obtain

$$\{47,-1\}^2\{47,11\} \equiv \{47,-1\}\{47,-11\} \equiv \{47,11\}$$

and

$$\{47,-1\}^2 \equiv 1.$$

Note $\pm 2\cdot 11 = \pm 2\cdot 47 \mp 2\cdot 6^2$ again. Then,

$$\{47,\pm 2\cdot 11\} \equiv \{47,\mp 2\cdot 6^2\},$$

and

$$\begin{aligned}\{47,22\} &\equiv \{47,-2\cdot 6^2\} = \{47,-1\}\{47,2\}\{47,6^2\}\\ &= \{47,2\}\{47,-6^2\} \equiv \{47,2\}\{47,11\}\end{aligned}$$

Recall

$$\{t,2\}\{t,-3\} = \{t,-1\}\{t,6\}.$$

Therefore, we obtain

$$\begin{aligned}&\langle\{47,\pm 2^k 11^\ell\} \mid k,\ell\in\mathbb{Z}\rangle\\ =\;&\langle\{47,-1\},\{47,2\},\{47,\pm 11\},\{47,11^2\},\{47,\pm 22\}\rangle\\ =\;&\langle\{47,-1\},\{47,2\},\{47,11\}\rangle\\ =\;&\langle\{47,-1\},\{47,2\},\{47,3\}\rangle\end{aligned}$$

Note $2^7 = 47 + 3^4$. Then,

$$\{47,2^7\} \equiv \{47,3^4\}.$$

Recall

$$\{47,u\} \equiv \{47,-1\}\{-47,-u\} \text{ for all } u\in D^\times.$$

Then,

$$\begin{aligned}\{47,3\} &\equiv \{47,-1\}\{-47,-3\}\\ &= \{47,-1\}\{-47,-\tfrac{1}{2^4}\}\\ &= \{47,2\}^{-4}.\end{aligned}$$

Therefore,

$$\{47,2\}^{23} \equiv \{47,2\}^{16}\{47,2\}^7 \equiv \{47,3\}^{-4}\{47,3\}^4 = 1$$

and

$$\{47,-2\}^{23} \equiv \{47,-1\},$$

which implies

$$\{47,-2\}^{46} \equiv 1.$$

Hence,

$$\begin{aligned} K &= K' \cdot \langle \{47,-1\}, \{47,2\}, \{47,3\} \rangle \\ K &= K' \cdot \langle \{47,-1\}, \{47,2\} \rangle \\ &= K' \cdot \langle \{47,-2\} \rangle \end{aligned}$$

and $\{47,-2\}^{46} \equiv 1$, which shows

$$K_2(2, \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{11}, \frac{1}{47}])$$
$$\simeq \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^\times \oplus (\mathbb{Z}/11\mathbb{Z})^\times \oplus (\mathbb{Z}/23\mathbb{Z})^\times \oplus (\mathbb{Z}/47\mathbb{Z})^\times .$$

□

7. Appendices

7.1. Schur Multipliers

A group epimorphism $\psi : G' \to G$ is called an extension of G. It is called a central extension if $\mathrm{Ker}(\psi)$ is central. If a central extension uniquely dominates all other central extensions, then it is called a universal central extension. For a universal central extension $\hat{\psi} : \hat{G} \to G$, the Schur multiplier of G is given by $M(G) = \mathrm{Ker}(\hat{\psi})$. For a group G, there exists a universal central extension of G if and only if $G = [G, G]$, i.e. G is perfect (cf. [25]).

Fact 1. (cf. Theorem 6.2 and [19]) *Let* $S = \{2,3,7\}$, $\{2,3,11\}$, $\{2,3,5,11\}$, $\{2,3,13\}$, $\{2,3,7,13\}$, $\{2,3,17\}$, $\{2,3,5,19\}$, $\{2,3,11,23\}$, $\{2,3,5,7,29\}$, $\{2,3,5,31\}$, $\{2,3,37\}$, $\{2,3,5,7,41\}$, $\{2,3,7,43\}$, $\{2,3,7,11,43\}$, $\{2,3,11,23,47\}$. *Then,* $St(2, \mathbb{Z}_S)$ *is a universal central extension of* $SL(2, \mathbb{Z}_S)$*, and the Schur multiplier of* $SL(2, \mathbb{Z}_S)$ *is as follows.*

$$M(SL(2, \mathbb{Z}_S)) \simeq \mathbb{Z} \oplus \amalg_{q \in S} (\mathbb{Z}/q\mathbb{Z})^\times$$

Fact 2. (cf. [19]) *Let* $S = \{p_1, p_2, \cdots, p_n\}$*, where* p_i *is the* i*-th prime number, i.e.* $p_1 = 2$*,* $p_2 = 3$*,* $p_3 = 5$*,* $\cdots$ *etc. Suppose* $n \geq 2$*. Then,* $St(2, \mathbb{Z}_S)$ *is a universal central extension of* $SL(2, \mathbb{Z}_S)$*, and the Schur multiplier of* $SL(2, \mathbb{Z}_S)$ *is as follows.*

$$M(SL(2, \mathbb{Z}_S)) \simeq \mathbb{Z} \oplus \amalg_{q \in S} (\mathbb{Z}/q\mathbb{Z})^\times$$

Fact 3. (cf. [25],[26],[19]) *Let S be the set of all prime numbers. That is, $\mathbb{Z}_S = \mathbb{Q}$. Then, $St(2, \mathbb{Q})$ is a universal central extension of $SL(2, \mathbb{Q})$, and the Schur multiplier of $SL(2, \mathbb{Q})$ is as follows.*

$$M(SL(2, \mathbb{Q})) \simeq \mathbb{Z} \oplus \amalg_{q \in S} (\mathbb{Z}/q\mathbb{Z})^{\times}$$

7.2. Loop Groups

Let F be a field, and $F[\xi, \xi^{-1}]$ the ring of Laurent polynomials in ξ over F. If $D = F[\xi]$ and $p = \xi \in D$, then D is a Dedekind domain and $p \in D$ is a nonzero prime element. Using this setting, $G_\rho(\Phi, D[\frac{1}{p}]) = G_\rho(\Phi, F[\xi, \xi^{-1}])$ is a loop group, which is a Kac-Moody group (cf. [2],[15],[21],[22],[23],[27]). We already know Iwahori-Matsumoto decompositions and universal central extensions of loop groups (cf. [16],[18]). Note that loop algebras are related to affine Kac-Moody Lie algebras (cf. [8],[14]). Recently we also discussed some word maps in Kac-Moody setting (cf. [9]).

Using Witt rings and Kac-Moody groups, we obtain the following two exact sequences.

Fact 4. (cf. [11],[26])

$$0 \to I^3(F) \to K_2(2, F) \to K_2(F) \to 0$$

Fact 5. (cf. [21],[22])

$$0 \to I^3(F) \oplus I^2(F) \to K_2(2, F[\xi, \xi^{-1}]) \to K_2(F[\xi, \xi^{-1}]) \to 0$$

Here $I(F)$ is the fundamental ideal of the Witt ring $W(F)$, (cf. [13]).

On the other hand, twisted loop groups are also important (cf. [17]), since they are affine Kac-Moody groups.

7.3. Matsumoto relations

We compute the following relations.
(M1):
$\{t, u\}\{tu, v\}$

$= \hat{h}_{12}(t)\hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1}\hat{h}_{12}(tu)\hat{h}_{12}(v)\hat{h}_{12}(tuv)^{-1}$
$= \hat{h}_{12}(t)\hat{h}_{12}(u)\hat{h}_{12}(v)\hat{h}_{12}(tuv)^{-1}$
$= \hat{h}_{12}(t)\hat{h}_{12}(u)\hat{h}_{12}(v) \cdot \hat{h}_{12}(uv)^{-1}\hat{h}_{12}(uv) \cdot \hat{h}_{12}(tuv)^{-1}$
$= \hat{h}_{12}(t) \cdot \{u, v\} \cdot \hat{h}_{12}(uv)\hat{h}_{12}(tuv)^{-1}$
$= \hat{h}_{12}(t)\hat{h}_{12}(uv)\hat{h}_{12}(tuv)^{-1} \cdot \{u, v\}$
$= \{t, uv\}\{u, v\}.$

(Remark):
$\hat{w}_{12}(t) = \hat{w}_{12}(t)\hat{w}_{12}(t)\hat{w}_{12}(-t) = \hat{w}_{21}(-t^{-1}).$

(Remark):
$\hat{w}_{12}(t)\hat{w}_{12}(u)\hat{w}_{12}(-t) = \hat{w}_{21}(-t^{-2}u) = \hat{w}_{12}(t^2u^{-1}).$

(Remark):
$\hat{x}_{12}(t)\hat{x}_{21}(-t^{-1})\hat{x}_{12}(t) = \hat{w}_{12}(t) = \hat{w}_{21}(-t^{-1}) = \hat{x}_{21}(-t^{-1})\hat{x}_{12}(t)\hat{x}_{21}(-t^{-1}).$

(Remark):
$\hat{w}_{12}(t)\hat{h}_{12}(u)\hat{w}_{12}(-t)$
$= \hat{w}_{12}(t)\hat{w}_{12}(u)\hat{w}_{12}(-1)\hat{w}_{12}(-t)$
$= \hat{w}_{12}(t^2u^{-1})\hat{w}_{12}(-t^2)$
$= \hat{w}_{12}(t^2u^{-1}) \cdot \hat{w}_{12}(-1)\hat{w}_{12}(1) \cdot \hat{w}_{12}(-t^2)$
$= \hat{h}_{12}(t^2u^{-1})\hat{h}_{12}(t^2)^{-1}.$

(Remark):
$\hat{h}_{12}(t)\hat{h}_{12}(u)\hat{h}_{12}(t)^{-1}$
$= \hat{w}_{12}(t)\hat{w}_{12}(-1)\hat{h}_{12}(u)\hat{w}_{12}(1)\hat{w}_{12}(-t)$
$= \hat{w}_{12}(t)\hat{h}_{12}(u^{-1})\hat{w}_{12}(-t)$
$= \hat{h}_{12}(t^2u)\hat{h}_{12}(t^2)^{-1}.$

(Remark):
$\{t, u\} = \hat{h}_{12}(t)\hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1} = \hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1}\hat{h}_{12}(t) = \hat{h}_{12}(tu)^{-1}\hat{h}_{12}(t)\hat{h}_{12}(u).$

(M3):
$\{t, u\}\{u^{-1}, t\}^{-1}$
$= \hat{h}_{12}(t)\hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1} \cdot \{u^{-1}, t\}^{-1}$
$= \hat{h}_{12}(t) \cdot \{u^{-1}, t\}^{-1} \cdot \hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1}$
$= \hat{h}_{12}(t) \cdot (\hat{h}_{12}(u^{-1})\hat{h}_{12}(t)\hat{h}_{12}(u^{-1}t)^{-1})^{-1} \cdot \hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1}$
$= \hat{h}_{12}(t) \cdot (\hat{h}_{12}(tu^{-1})^{-1}\hat{h}_{12}(u^{-1})\hat{h}_{12}(t))^{-1} \cdot \hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1}$
$= \hat{h}_{12}(t) \cdot \hat{h}_{12}(t)^{-1}\hat{h}_{12}(u^{-1})^{-1}\hat{h}_{12}(tu^{-1}) \cdot \hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1}$

$$
\begin{aligned}
&= \hat{h}_{12}(u^{-1})^{-1}\hat{h}_{12}(tu^{-1})\hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1} \\
&= \hat{w}_{12}(1)\hat{w}_{12}(-u^{-1})\hat{w}_{12}(tu^{-1})\hat{w}_{12}(-1)\cdot\hat{w}_{12}(u)\hat{w}_{12}(-tu) \\
&= \hat{w}_{12}(-u)\hat{w}_{12}(t^{-1}u)\hat{w}_{12}(u)\hat{w}_{12}(-tu) \\
&= \hat{w}_{12}(tu)\hat{w}_{12}(-tu) = 1.
\end{aligned}
$$

(M4):

$$
\begin{aligned}
&\{t,-tu\}^{-1}\{t,u\} \\
&= (\hat{h}_{12}(t)\hat{h}_{12}(-tu)\hat{h}_{12}(-t^2u)^{-1})^{-1}\cdot\hat{h}_{12}(t)\hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1} \\
&= \hat{h}_{12}(-t^2u)\hat{h}_{12}(-tu)^{-1}\hat{h}_{12}(t)^{-1}\cdot\hat{h}_{12}(t)\hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1} \\
&= \hat{h}_{12}(-t^2u)\hat{h}_{12}(-tu)^{-1}\hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1} \\
&= \hat{w}_{12}(-t^2u)\cdot\hat{w}_{12}(tu)\hat{w}_{12}(u)\hat{w}_{12}(-tu) \\
&= \hat{w}_{12}(-t^2u)\hat{w}_{12}(t^2u) = 1.
\end{aligned}
$$

(M2):

$$
\begin{aligned}
&\{t,u\}\{t,-u^{-1}\} = \{t,u\}\{-u,t\} = \{t,u\}\{-u,tu\} \\
&= \{t,u\}\{tu,-u^{-1}\} = \{t,u(-u^{-1})\}\{u,-u^{-1}\} \\
&= \{t,-1\}.
\end{aligned}
$$

(M5):

$$
\begin{aligned}
&\{t,u^2\}^{-1}\{t,u^2v\} \\
&= \hat{h}_{12}(tu^2)\hat{h}_{12}(u^2)^{-1}\hat{h}_{12}(t)^{-1}\cdot\hat{h}_{12}(t)\hat{h}_{12}(u^2v)\hat{h}_{12}(tu^2v)^{-1} \\
&= \hat{h}_{12}(tu^2)\hat{h}_{12}(u^2)^{-1}\cdot\hat{h}_{12}(u^2v)\hat{h}_{12}(u^2)^{-1}\cdot\hat{h}_{12}(u^2)\hat{h}_{12}(tu^2v)^{-1} \\
&= \hat{h}_{12}(u)\cdot\hat{h}_{12}(t)\hat{h}_{12}(v)\hat{h}_{12}(tv)^{-1}\cdot\hat{h}_{12}(u)^{-1} \\
&= \hat{h}_{12}(u)\cdot\{t,v\}\cdot\hat{h}_{12}(u)^{-1} = \{t,v\}.
\end{aligned}
$$

(M6):

$$
\begin{aligned}
&\{t,u\} = \hat{h}_{12}(t)\hat{h}_{12}(u)\hat{h}_{12}(tu)^{-1} \\
&= \hat{h}_{12}(t)\hat{w}_{12}(u)\hat{w}_{12}(-tu) \\
&= \hat{h}_{12}(t)\cdot\hat{x}_{12}(u)\hat{x}_{21}(-u^{-1})\hat{x}_{12}(u)\cdot\hat{w}_{12}(-tu) \\
&= \hat{h}_{12}(t)\hat{x}_{12}(u)\cdot\hat{w}_{12}(-tu)\hat{x}_{12}(t^2u)\hat{x}_{21}(-t^{-2}u^{-1}) \\
&= \hat{h}_{12}(t)\hat{x}_{12}(u)\hat{w}_{12}(-tu)\hat{x}_{12}(t^2u) \\
&\quad\times\hat{x}_{21}((1-t)^{-1}t^{-1}u^{-1})\hat{x}_{21}(-(1-t)^{-1}t^{-2}u^{-1}) \\
&= \hat{h}_{12}(t)\hat{x}_{12}(u)\hat{w}_{12}(-tu)\hat{x}_{12}(t^2u) \\
&\quad\times\hat{x}_{12}((1-t)tu)\hat{w}_{12}(-(1-t)tu)\hat{x}_{12}((1-t)tu)\cdot\hat{x}_{21}(-(1-t)^{-1}t^{-2}u^{-1}) \\
&= \hat{h}_{12}(t)\hat{x}_{12}(u)\hat{w}_{12}(-tu)\hat{x}_{12}(tu) \\
&\quad\times\hat{w}_{12}(-(1-t)tu)\hat{x}_{12}((1-t)tu)\cdot\hat{x}_{21}(-(1-t)^{-1}t^{-2}u^{-1}) \\
&= \hat{h}_{12}(t)\hat{x}_{12}(u)\hat{w}_{12}(-tu)\cdot\hat{x}_{12}(tu)
\end{aligned}
$$

$$
\begin{aligned}
&\times\hat{x}_{21}(-(1-t)^{-1}t^{-1}u^{-1})\hat{x}_{12}((1-t)u)\hat{w}_{12}(-(1-t)tu)\\
&=\hat{h}_{12}(t)\hat{x}_{12}(u)\cdot\hat{x}_{12}(-tu)\hat{x}_{21}(t^{-1}u^{-1})\hat{x}_{12}(-tu)\cdot\hat{x}_{12}(tu)\\
&\times\hat{x}_{21}(-(1-t)^{-1}t^{-1}u^{-1})\hat{x}_{12}((1-t)u)\hat{w}_{12}(-(1-t)tu)\\
&=\hat{h}_{12}(t)\hat{x}_{12}((1-t)u)\hat{x}_{21}(t^{-1}u^{-1})\\
&\times\hat{x}_{21}(-(1-t)^{-1}t^{-1}u^{-1})\hat{x}_{12}((1-t)u)\hat{w}_{12}(-(1-t)tu)\\
&=\hat{h}_{12}(t)\hat{x}_{12}((1-t)u)\hat{x}_{21}(-(1-t)^{-1}u^{-1})\hat{x}_{12}((1-t)u)\hat{w}_{12}(-(1-t)tu)\\
&=\hat{h}_{12}(t)\hat{w}_{12}((1-t)u)\hat{w}_{12}(-(1-t)tu)\\
&=\hat{h}_{12}(t)\hat{h}_{12}((1-t)u)\hat{h}_{12}((1-t)tu)^{-1}=\{t,(1-t)u\}.
\end{aligned}
$$

(M7):

$$\{t^2,-1\}=\{t,t\}^{-1}\{t,-t\}\{t,-1\}=\{t,-1\}^{-1}\{t,1\}\{t,-1\}=1.$$

(M8):

$$\{t,u\}=\{t,1-t\}=\{t,1\}=1.$$

(M9):

$$\{t,u^k\}=\{t,(1-t)^k\}=\cdots=\{t,1-t\}=\{t,u\}=1.$$

(M10):

$$\{t^2,uv\}=\{t^2,u\}\{t^2u,v\}\{u,v\}^{-1}=\{t,u^2\}\{t^2,v\}\{t,v\}^{-1}=\{t^2,u\}\{t^2,v\}.$$

7.4. Remark on Theorem 5.1(1)

For $\zeta\in R^\times$, we take $\sigma[\zeta]\in\mathrm{Aut}(St(2,R))$ defined by

$$\sigma[\zeta]:\begin{cases}\hat{x}_{12}(s)&\mapsto&\hat{x}_{12}(\zeta s),\\ \hat{x}_{21}(s)&\mapsto&\hat{x}_{21}(\zeta^{-1}s).\end{cases}$$

Then, (cf. [19],[20]),

$$
\begin{aligned}
&\hat{w}_{12}(u)\mapsto\hat{w}_{12}(\zeta u),\\
&\hat{w}_{21}(u)\mapsto\hat{w}_{21}(\zeta^{-1}u),\\
&\hat{h}_{12}(u)\mapsto\hat{h}_{12}(\zeta u)\hat{h}_{12}(\zeta)^{-1},\\
&\{u,v\}\mapsto\{u,\zeta\}^{-1}\{u,\zeta v\}.
\end{aligned}
$$

Let $z,z',r,p\in R^\times$ and suppose $z'=z+rp$.

First, note $p\dfrac{r}{z'}+\dfrac{z}{z'}=1$. Then,

$$\{p\frac{r}{z'},\frac{z}{z'}\}=1.$$

Therefore, we obtain

$$\{p,\frac{r}{z'}\}^{-1}\{p,\frac{rz}{z'^2}\}\{\frac{r}{z'},\frac{z}{z'}\}=1,$$

and

$$\{p,rz'\}=\{p,rz\}\{\frac{r}{z'},\frac{z}{z'}\}.$$

Applying $\sigma[\frac{1}{r}]$ to both sides, we have

$$\{p,\frac{1}{r}\}^{-1}\{p,z'\}=\{p,\frac{1}{r}\}^{-1}\{p,z\}\{\frac{r}{z'},\frac{1}{r}\}^{-1}\{\frac{r}{z'},\frac{z}{rz'}\},$$

which implies the following.

Fact 6. $\{p,z'\}=\{p,z\}\{\frac{r}{z'},\frac{1}{r}\}^{-1}\{\frac{r}{z'},\frac{z}{rz'}\}$

Second, notice again $p\frac{r}{z'}+\frac{z}{z'}=1$ and

$$\{p\frac{r}{z'},\frac{z}{z'}\}=1.$$

Applying $\sigma[z']$ to the left hand side, we obtain

$$\{p\frac{r}{z'},z'\}^{-1}\{p\frac{r}{z'},z\}=1.$$

Therefore,

$$\{p\frac{r}{z'},z\}=\{p\frac{r}{z'},z'\} \text{ and } \{z,\frac{z'}{pr}\}=\{z',\frac{z'}{pr}\}.$$

Applying $\sigma[\frac{r}{z'}]$ to both sides, we obtain

$$\{z,\frac{r}{z'}\}^{-1}\{z,\frac{1}{p}\}=\{z',\frac{r}{z'}\}^{-1}\{z',\frac{1}{p}\},$$

which implies the following.

Fact 7. $\{p,z'\}=\{p,z\}\{z,\frac{r}{z'}\}^{-1}\{z',\frac{r}{z'}\}$

Using Fact 6 and Fact 7, we obtain

$$\{\frac{r}{z'},\frac{1}{r}\}^{-1}\{\frac{r}{z'},\frac{z}{rz'}\}=\{z,\frac{r}{z'}\}^{-1}\{z',\frac{r}{z'}\}.$$

Hence, we have the following.

Fact 8. $\{z,\frac{r}{z'}\}\{\frac{rz'}{z},\frac{r}{z'}\}=\{z',\frac{r}{z'}\}\{r,\frac{r}{z'}\}$

In fact, there is a direct proof of Fact 8. Let us start from

$$\{\frac{r}{z'}, -1\} = \{\frac{r}{z'}, -r\}\{\frac{r}{z'}, \frac{1}{r}\} = \{\frac{r}{z'}, z'\}\{\frac{r}{z'}, \frac{1}{r}\}.$$

Then, we obtain the desired result as follows:

$$\{\frac{r}{z'}, \frac{1}{r}\}^{-1}\{\frac{r}{z'}, -1\} = \{\frac{r}{z'}, z'\},$$

$$\{\frac{r}{z'}, \frac{1}{r}\}^{-1}\{\frac{r}{z'}, -z\}\{\frac{r}{z'}, \frac{1}{z}\} = \{\frac{r}{z'}, z'\} = \{\frac{r}{z'}, \frac{1}{z'}\}\{\frac{r}{z'}, z'^2\},$$

$$\{\frac{r}{z'}, \frac{1}{r}\}^{-1}\{\frac{r}{z'}, -\frac{z}{z'^2}\} = \{\frac{r}{z'}, \frac{1}{z}\}^{-1}\{\frac{r}{z'}, \frac{1}{z'}\},$$

$$\{\frac{r}{z'}, \frac{1}{r}\}^{-1}\{\frac{r}{z'}, \frac{z}{rz'}\} = \{z, \frac{r}{z'}\}^{-1}\{z', \frac{r}{z'}, \}.$$

7.5. Remarks on Theorem 5.1 (2) - (16)

We will make the following remarks.

(2):

$\{t, u\} = \{u, t^{-1}\} = \{u, t^{-1}(1-u)^{-1}\} = \{t(1-u), u\}$
$= \{t, 1-u\}^{-1}\{t, (1-u)u\}\{1-u, u\} = \{t, 1-u\}^{-1}\{t, (1-u)u\}$ by (M1,3,6,8).

(3):

$\{t, v\}\{t, (1-u)uv\} = \{t, v\}\{v, (1-u)u\}^{-1}\{tv, (1-u)u\}\{t, v\}$
$= \{t, v\}\{v, 1-u\}^{-1}\{v, u\}^{-1}\{tv, 1-u\}\{tv, u\}\{t, v\}$
$= \{t, (1-u)v\}\{t, uv\}$ by (M1) and (2).

(4):

$\{t, -1\}\{t, 6\} = \{t, 2\}\{t, -3\}$ by $u = -2$ and $v = -1$ in (3).

(5):

$\{t, ua^{-2}\}\{t, va^{-2}\} = \{t, uva^{-4}\}$ implies $\{t, u\}\{t, v\} = \{t, uv\}$ by (2).

(6):

$\{t, -1\}^2 = \{t, -1\}\{t, t-1\} = \{t, -1\}\{t, u^2\} = \{t, -u^2\} = \{t, 1-t\} = 1$ by (M5,6,8).

(7):

$\{t, -1\} = \{-1, t\} = \{-1, 2^k t\} = \{2^k t, -1\}$ implies $\{t, -1\}^2 = \{2^k t, -1\}^2$ $= 1$ by (6).

(8):

$\{t, -1\}^2 = \{t, -1\}\{t, t-1\} = \{t, -1\}\{t, 2u^2\} = \{t, -1\}\{t, 2\}\{t, u^2\}$

$= \{t, -2\}\{t, u^2\} = \{t, -2u^2\} = \{t, 1-t\} = 1$ by (M5,6) and (2).

(9):

Note $\{t, \pm u^k\} = \{t, \pm u^{k-2}\}\{t, u^2\}$ by (M5).

(10):

This is true by $\{t, u\}\{t, -1\} = \{t, u\}\{t, u^{-1}\}\{t, -u\}$.

(11):

Note $\{t, u\}^2 = \{t, u\}\{t, u^{-1}\}\{t, u^2\} = \{t, u^2\}$ by (M5) and (10)(ii). Then, for $k = 2m$ and $k = 2m+1$, we see $\{t, u^k\} = \{t, u\}^k$ and $\{t, -u^k\} = \{t, -1\}\{t, u\}^k$ by (10).

(12):

Since $2 - 1 = 1$, we have $\{t, 2\}\{t, -1\} = \{t, -2\}$ by (2).

(13):

We have $\{t, 2\}\{t, \frac{1}{2}\} = 1$ by (11) and (12).

(14):

We have $\{t, 2^k\} = \{t, 2\}^k$ by (11).

(15):

We have $\{t, -2^k\} = \{t, -1\}\{t, 2\}^k$ by (10),(12) and (13).

(16):

We have $\langle \{t, \pm 2^k\} \mid k \in \mathbb{Z} \rangle = \langle \{t, -1\}, \{t, 2\} \rangle$ by (9),(14) and (15).

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