

On first order rigidity for linear groups

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To Prof. Jun Morita with great respect and admiration

Abstract. The paper is a short survey of recent developments in the area of first order descriptions of linear groups. It is aimed to illuminate the known results and to pose the new problems relevant to logical characterizations of Chevalley groups and Kac–Moody groups.

1. Introduction

Questions we are going to illuminate in this paper are concentrated around the interaction between algebra, logic, model theory and geometry.

The main question behind further considerations is as follows. Suppose we have two algebras equipped with a sort of logical description.

Problem 1. When the coincidence of logical descriptions provides an isomorphism between algebras in question?

With this aim we consider different kinds of logical equivalences between algebras. Some of the notions we are dealing with are not formally defined in the text. For precise definitions and references use [9], [16], [23], [24], [26], [30], [31].

First, we make emphasis on elementary equivalence of groups. Importance of the elementary classification of algebraic structures goes back to

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the famous works of A. Tarski and A. Malcev. The main problem is to figure out *what are the algebras elementarily equivalent to a given one*. We will describe the current state of art of the problem: *when elementary equivalence of groups implies their isomorphism*. Situation of such kind will be called, for short, *elementary rigidity*.

Our second aim is to describe the notions of isotypicity of algebras and logical equivalence of algebras. These notions are much less known than elementary equivalence. However, they can logically characterize algebras in a very rigid way and one can expect affirmative answers to most of the problems formulated.

We discuss these notions from the perspectives of Chevalley groups and some other linear groups, and Kac-Moody groups.

2. Elementary equivalence of algebras

2.1. Definitions

Given an algebra H , its *elementary theory* $Th(H)$ is the set of all sentences (closed formulas) valid on H .

Definition 2. Two algebras H_1 and H_2 are said to be *elementarily equivalent* if their elementary theories coincide.

Very often we fix a class of algebras \mathcal{C} and ask what are the algebras elementarily equivalent to a given algebra inside the class \mathcal{C} . So, the rigidity question with respect to elementary equivalence looks as follows.

Problem 3. Let a class of algebras \mathcal{C} and an algebra $H \in \mathcal{C}$ be given. Suppose that the elementary theories of algebras H and $A \in \mathcal{C}$ coincide. Are they *elementarily rigid*, that is, are H and A isomorphic?

For example, \mathcal{C} can be the class of all groups, the class of finitely generated groups, the class of profinite groups, etc.

Remark 4. What we call elementary rigidity has different names. This notion appeared in the papers by A. Nies [18] under the name of quasi definability of groups. The corresponding name used in [1] with respect to the class of finitely generated groups is first order rigidity. For some reasons which will be clear later on we use another term.

In other words we ask for which algebras their logical characterization by means of the elementary theory is strong enough and defines the algebra in the unique, up to an isomorphism, way?

We restrict our attention to the case of groups. Elementary rigidity of groups occurs not very often. Usually various extra conditions are needed. Consider examples of elementary rigidity for linear groups. First of all, a group which is elementarily equivalent to a finitely generated linear group is a residually finite linear group [12]. The incomplete list of known rigidity cases is given in the following theorem.

2.2. Chevalley groups

Theorem 5. *Historically, the first result was obtained by A. Malcev:*

- *If two linear groups $GL_n(K)$ and $GL_m(F)$, where K and F are fields, are elementarily equivalent, then $n = m$ and the fields K and F are elementarily equivalent, see [13].*
- *This result was generalized to the wide class of Chevalley groups. Let $G_1 = G_\pi(\Phi, R)$ and $G_2 = G_\mu(\Psi, S)$ be two elementarily equivalent Chevalley groups. Here Φ, Ψ denote the root systems of rank > 1 , R and S are commutative rings, and π, μ are weight lattices. Then root systems of G_1 and G_2 coincide, while the rings are elementarily equivalent. In other words Chevalley groups over commutative rings are elementarily rigid in the class of such groups modulo elementary equivalence of ground rings [4].*
- *Let $G_\pi(\Phi, K)$ be a simple Chevalley group over the algebraically closed field K . Then $G_\pi(\Phi, K)$ is elementarily rigid in the class of all groups (cardinality is fixed). This result can be deduced from [42]. In fact, this is true for a much wider class of algebraic groups over algebraically closed fields and, modulo elementary equivalence of fields, over arbitrary fields [42].*
- *Any irreducible non-uniform higher-rank characteristic zero arithmetic lattice is elementarily rigid in the class of all finitely generated groups, see [1]. In particular, $SL_n(\mathbb{Z})$, $n > 2$ is elementarily rigid.*

- *Recently, the results of [1] have been extended to a much wider class of lattices, see [2].*
- *Let \mathcal{O} be the ring of integers of a number field, and let $n \geq 3$. Then every group G which is elementarily equivalent to $SL_n(\mathcal{O})$ is isomorphic to $SL_n(\mathcal{R})$, where the rings \mathcal{O} and \mathcal{R} are elementarily equivalent. In other words $SL_n(\mathcal{O})$ is elementarily rigid in the class of all groups modulo elementary equivalence of rings. The similar results are valid with respect to $GL_n(\mathcal{O})$ and to the triangular group $T_n(\mathcal{O})$ [39]. These results intersect in part with the previous items, since the ring $R = \mathbb{Z}$ is elementarily rigid in the class of all finitely generated rings [18], and thus $SL_n(\mathbb{Z})$ is elementarily rigid in the class of all finitely generated groups.*
- *For the case of arbitrary Chevalley groups the results similar to above cited are obtained in [35] by different machinery for a wide class of ground rings. Suppose the Chevalley group $G = G(\Phi, R)$ of rank ≥ 2 over the ring R is given. Suppose that the ring R is elementarily rigid in the class \mathcal{C} of rings. Then $G = G(\Phi, R)$ is elementarily rigid in the corresponding class \mathcal{C}_1 of groups if R is a field, R is a local ring and G is simply connected, R is a Dedekind ring of arithmetic type, that is the ring of S -integers of a number field, R is Dedekind ring with at least 4 units and G is adjoint. In particular, if a ring of such kind is finitely generated then it gives rise to elementary rigidity of $G = G(\Phi, R)$ in the class of all finitely generated groups (see [3]). If R of such kind is not elementarily rigid then $G = G(\Phi, R)$ is elementarily rigid in the class of all groups modulo elementary equivalence of rings.*
- *The Chevalley group $G = G(\Phi, R)$ of rank ≥ 2 is elementarily rigid in the class of all finitely generated groups, if R is a ring of one-variable polynomials over the finite field, i.e., $R = F_q[x]$, $\text{char} F_q \neq 2$, see [35], [3] and [5], where the bounded generation of such $G(\Phi, R)$ in elementary generators is proven.*
- *The Chevalley group $G = G(\Phi, R)$ is elementarily rigid in the class of all finitely generated groups, if R is a ring of Laurent polynomials*

over the finite field, i.e., $R = F_q[x, x^{-1}]$, $\text{char} F_q \neq 2$. The proof also relies on [35], [3] and [5].

- The Chevalley group $G = G(\Phi, R)$ is elementarily rigid in the class of all finitely generated groups, if R is a finitely generated ring of S -integers in a global function field of positive characteristic which has infinitely many units and satisfy some additional condition on S , see [35], [3], [5] and [34].

For rank 1 simple linear groups the situation is quite different. For $R = \mathbb{Z}$ and $G = PSL_2(\mathbb{Z})$ there is the following result of Z. Sela, cf. [37], see [10].

Theorem 6. *A finitely generated group G is elementarily equivalent to $PSL(2, \mathbb{Z})$ if and only if G is a hyperbolic tower (over $PSL(2, \mathbb{Z})$).*

Let us make some comments regarding Theorem 6. It was A. Tarski who asked whether one can distinguish between finitely generated free groups by means of their elementary theories. This formidable problem has been solved in affirmative, that is all free groups have one and the same elementary theory. Moreover, all finitely generated groups elementarily equivalent to a given non-abelian free group have been explicitly described, see [11], [36]. Paper [37] extends this line.

The result for $SL_2(\mathbb{Z})$ can be deduced from Theorem 6, see [10].

Theorem 7. ^{*1} *A finitely generated group G is elementarily equivalent to $SL_2(\mathbb{Z})$ if and only if G is the central extension of a hyperbolic tower over $PSL_2(\mathbb{Z})$ by \mathbb{Z}_2 with the cocycle $f : PSL_2(\mathbb{Z}) \times PSL_2(\mathbb{Z}) \rightarrow \mathbb{Z}_2$, where $f(x, x) = 1$ for all $x \in PSL_2(\mathbb{Z})$ of order 2, and $f(x, y) = 0$, otherwise.*

However, the situation in rank one Chevalley groups over the rings with infinitely many invertible elements looks different. Bounded generation of $SL_2(R)$ in elementary generators, where R is a ring of S -integers of a number field with infinitely many units is proved in [14]. The same fact is true if R is a ring of S -integers in a global function field of positive characteristic which has infinitely many units and satisfies some additional condition on S , see [34]. Hence, in these cases one can think about elementary rigidity.

^{*1}The proof of Theorem 7 contains a gap (I.Kazachkov, private correspondence).

Before going over the situation for Kac-Moody groups we shall cite an important model-theoretic fact noticed by F. Point on collaboration between Chevalley-Demazure functor and the operation of taking ultraproducts. Let F be a non-principal ultrafilter on the set I and let $G_\Phi(\cdot)$ be a simple Chevalley-Demazure group scheme over \mathbb{Z} defined by a root system Φ . Let $K_i, i \in I$ be a collection of fields and $G_i = G_\Phi(K_i)$ be the corresponding set of Chevalley groups. Denote by $\prod_F G_i$ the ultraproduct of the groups G_i with respect to ultrafilter F . Theorem of F. Point (see [33]) basically says that the functors $G_\Phi(\cdot)$ and \prod_F commute. Namely,

Theorem 8. $G_\Phi(\prod_F K_i) = \prod_F G_\Phi(K_i)$. Moreover the ultraproducts of the unipotent, diagonal and monomial subgroups of $G_\Phi(K_i)$ are isomorphic to the corresponding subgroups of $G_\Phi(\prod_F K_i)$.

Note that Theorem 8 remains true (modulo minor conditions on the fields) also in the case of twisted Chevalley groups. From the perspectives of Kac-Moody groups and elementary rigidity of Chevalley groups, the following question is of interest.

Question 9. ^{*2} Does the statement of Theorem 8 remain true for arbitrary rings?

2.3. Kac–Moody groups

Given a generalized Cartan matrix A and a field k (or a ring R), the value $G_A(k)$ of the Tits functor $G_A: \mathbb{Z}\text{-Alg} \rightarrow \text{Grp}$ defines a minimal Kac–Moody group over k , see [40] (cf. [7], [16]). One can view this functor as a generalization of the Chevalley–Demazure group scheme. We assume that A is indecomposable. As a rule, we assume that the functor G_A is simply connected. However, speaking about isomorphisms of affine Kac-Moody groups over fields and Chevalley groups over rings, we will freely, often without special notice, use the common language abuse, assuming that we go over to its subquotient, taking the derived subgroup (resp. subalgebra) and factoring out the centre, if necessary.

^{*2}This question is answered in positive in [4].

If A is a definite matrix, the group $G_A(k)$ is a Chevalley group $G_\Phi(k)$ where Φ is the root system corresponding to A . These groups were considered in the previous section. If A is of affine type, $G_A(k)$ is isomorphic to the Chevalley group $G_\Phi(k[t, t^{-1}])$ where $k[t, t^{-1}]$ is the ring of Laurent polynomials. The general case of Kac-Moody groups is covered by A of indefinite type. The first question is related to Theorem 8.

Let F be a non-principal ultrafilter on the set I , let K_i , $i \in I$ be a collection of fields and let the Tits functor $G_A: \mathbb{Z}\text{-Alg} \rightarrow \text{Grp}$ defines a minimal Kac-Moody group over K_i . Denote by $\prod_F G_i$ the ultraproduct of the groups G_i with respect to ultrafilter F .

Question 10. Let A be of affine type. Is it true that the Tits functor and ultraproducts commute, that is the formula $G_A(\prod_F K_i) = \prod_F G_A(K_i)$ holds? Is it true that the same property is satisfied for arbitrary A ? Consider separately the hyperbolic case.

The penultimate item of Theorem 5 implies that

Theorem 11. *Let $G_A(k)$ be an affine Kac-Moody group over a finite field k . Then $G_A(k)$ is elementarily rigid in the class of all finitely generated groups.*

Let now $G_A(k)$ be a Kac-Moody group of indefinite type. B. Rémy [21] and P.-E. Caprace-B. Rémy [8] showed that the minimal indefinite adjoint Kac-Moody groups $G_A(\mathbb{F}_q)$ are simple provided $q > n > 2$ where n is the size of A . These groups are also simple for some matrices A if $n = 2$ and $q > 3$. J. Morita and B. Rémy [15] proved that in the case where k is the algebraic closure of \mathbb{F}_q the groups $G_A(k)$ are simple. P.-E. Caprace and K. Fujiwara [6] showed that over finite fields these (infinite) simple groups have infinite commutator width. It seems extremely unlikely that these groups are elementarily rigid.

Let $G_A(k)$ be an incomplete Kac-Moody group. There are several ways to complete this group with respect to an appropriate topology.

Let A be of affine type, that is, $\overline{G}_A(k)$ is a complete affine Kac-Moody group. Then $\overline{G}_A(k)$ is isomorphic to a Chevalley group of the form $G_\Phi(k((t)))$ where $k((t))$ is the field of formal Laurent series over k . By Theorem 5 this

group is elementarily rigid modulo elementary equivalence of the ground field. If the field k is finite then this group is elementarily rigid in the class of finitely generated groups.

3. Isotypic equivalence of algebras

The aim of this section is to introduce another logical invariant which describes algebras more rigidly than elementary equivalence. Elementary equivalence of algebras H_1 and H_2 assumes coincidence of all first order sentences valid on H_1 and H_2 . What we are going to introduce requires coincidence of all types valid on H_1 and H_2 . We call such a situation *isotypicity* of algebras. Before going over results in this direction, we need to make some preparations.

3.1. Basics of universal algebraic geometry

Fix a variety of algebras Θ . Let $W(X)$, $X = \{x_1, \dots, x_n\}$ denote the finitely generated free algebra in Θ . By equations in Θ we mean expressions of the form $w \equiv w'$, where w, w' are words in $W(X)$ for some X . This is our first syntactic object. Next, let $\tilde{\Phi} = (\Phi(X), X \in \Gamma)$ be the multi-sorted Halmos algebra of first order logical formulas based on atoms $w \equiv w'$, w, w' in $W(X)$, see [26], [27], [31]. There is a special procedure to construct such an algebraic object which plays the same role with respect to First Order Logic as Boolean algebras do with respect to Propositional calculus. One can view elements of $\tilde{\Phi} = (\Phi(X), X \in \Gamma)$ just as first order formulas over $w \equiv w'$.

Let $X = \{x_1, \dots, x_n\}$ and let H be an algebra in the variety Θ . We have an affine space $H^n = H^X$ of points $\mu : X \rightarrow H$. For every μ we have also the n -tuple $(a_1, \dots, a_n) = \bar{a} \in H^n$ with $a_i = \mu(x_i)$. For the given Θ we have the homomorphism $\mu : W(X) \rightarrow H$ and, hence, the affine space H^n is viewed as the set of homomorphisms $Hom(W(X), H)$. The classical kernel $Ker(\mu)$ corresponds to each point $\mu : W(X) \rightarrow H$. This is exactly the set of equations for which the point μ is a solution. Every point μ has also the logical kernel $LKer(\mu)$, see [30], [25], [27]. Logical kernel $LKer(\mu)$ consists of all formulas $u \in \Phi(X)$ valid on the point μ . This is always an ultrafilter

in $\Phi(X)$.

So we define syntactic and semantic areas where logic and geometry operate, respectively. Connect them by a sort of Galois correspondence.

Let T be a system of equations in $W(X)$. The set $A = T'$ in the affine space $Hom(W(X), H)$ consisting of all solutions of the system T corresponds to T . Sets of such kind are called *algebraic sets*. Vice versa, given a set A of points in the affine space consider all equations $T = A'$ having A as the set of solutions. Sets T of such kind are called *closed congruences* over W .

We can do the same correspondence with respect to arbitrary sets of formulas. Given a set T of formulas in algebra of formulas (set of elements) $\Phi(X)$, consider the set $A = T^L$ in the affine space, such that every point of A satisfies every formula of Φ . Sets of such kind are called *definable sets*. Points of A are called solutions of the set of formulas T . Conversely, given a set A of points in the affine space consider all formulas (elements) $T = A^L$ having A as the set of solutions. Sets T of such kind are *closed filters* in $\Phi(X)$. Given arbitrary T and A we can make their Galois closures T'' and A'' , and T^{LL} and A^{LL} .

3.2. Logical equivalence of algebras

All algebraic sets constitute a category with special rational maps as morphisms [31]. The same is true with respect to definable sets [31]. So, we can formulate logical closeness of algebras geometrically.

Definition 12. We call algebras H_1 and H_2 *logically similar*, if the categories of definable sets $LG_{\Theta}(H_1)$ and $LG_{\Theta}(H_2)$ are isomorphic.

Definition 13. Algebras are called logically equivalent, if for every X and every set of formulas T in $\Phi(X)$ the equality $T_{H_1}^{LL} = T_{H_2}^{LL}$ holds .

The set T_H^{LL} is called the *logical radical* of T with respect to H . So algebras H_1 and H_2 are *logically equivalent* if for every set of formulas T the logical radicals with respect to H_1 and H_2 coincide.

First of all, it is easy to see that if algebras H_1 and H_2 are logically equivalent then they are logically similar. Now we want to understand what is the meaning of logical equivalence.

Definition 14. Two algebras H_1 and H_2 are called LG-isotypic if for every point $\mu : W(X) \rightarrow H_1$ there exists a point $\nu : W(X) \rightarrow H_2$ such that $LKer(\mu) = LKer(\nu)$ and, conversely, for every point $\nu : W(X) \rightarrow H_2$ there exists a point $\mu : W(X) \rightarrow H_1$ such that $LKer(\nu) = LKer(\mu)$.

We can reformulate isotypicity of algebras in more standard logical notations.

Definition 15. Let \mathcal{L} be a first-order language, H and G be \mathcal{L} -algebras. Then H and G are isotypic, if for any finite tuple \bar{a} in H^n , there exists a tuple \bar{b} in G^n such that $tp^H(\bar{a}) = tp^G(\bar{b})$ and vice versa.

The meaning of the Definition 14 is the following. Two algebras are isotypic if the sets of realizable types over H_1 and H_2 coincide. So, by some abuse of language these algebras have the same logic of types. Some references for the notion of isotypic algebras are contained in [29], [28], [30], [31], [32], [41]. Note that the notion was introduced in [32], [29] while [31] gives the most updated survey.

The principal property is as follows, see [41].

Theorem 16. *Algebras H_1 and H_2 are logically equivalent if and only if they are isotypic.*

Definition 17. We call the condition \mathcal{A} rigid (or \mathcal{A} -rigid) in the class of algebras \mathcal{C} if two algebras H_1 and H_2 from \mathcal{C} subject to \mathcal{A} are isomorphic.

Now we are in a position to study rigidity of algebras with respect to isotypicity property. It is easy to see that

Proposition 18. *If algebras H_1 and H_2 are logically equivalent then they are elementarily equivalent.*

3.3. Isotypic algebras

It is clear, that since isotypicity is stronger than elementary equivalence, this phenomenon can occur quite often. Let us state this problem explicitly.

Problem 19. Let a class of algebras \mathcal{C} and an algebra $H \in \mathcal{C}$ be given. Suppose that algebras $H \in \mathcal{C}$ and $A \in \mathcal{C}$ are isotypic. Are they *isotypically rigid*, that is are H and A isomorphic?

Remark 20. In many papers isotypically rigid algebras are called *logically separable* [31], [28], or *type definable* [17].

The following principal problem was stated in [31], see also [17] and is widely open.

Problem 21 (Rigidity problem, B.Plotkin). Is it true that every two isotypic finitely generated groups are isomorphic?

Meantime, Problem 21 is answered in affirmative for many groups. Some of the cases are collected in Theorem 22 and in the consequent Corollary.

Theorem 22. *The following cases of isotypically rigid groups are known:*

- *Every finitely generated co-Hopfian group is isotypically rigid in the class of all groups, see [41], [38].*
- *Every finitely presented Hopfian group is isotypically rigid in the class of all groups, see [38].*
- *Let Θ be a variety of groups. If a finitely generated free group in Θ is Hopfian then it is isotypically rigid in the class of all groups, see [41].*
- *Finitely generated metabelian groups are isotypically rigid in the class of all groups [17].*
- *Finitely generated virtually polycyclic groups are isotypically rigid in the class of all groups [17].*
- *Finitely generated torsion free hyperbolic groups are isotypically rigid in the class of all groups [38].*
- *All surface groups, which are not non-orientable surface groups of genus 1,2 or 3 are isotypically rigid in the class of all groups [17].*

Corollary 23. *Finitely generated absolutely free, free abelian, free nilpotent, free solvable groups are isotypically rigid.*

Conjecture 24. *Every finitely generated linear group is isotypically rigid.*

Conjecture 25. *Let a Chevalley group $G_\Phi(R)$ over a ring R be isotypic to a group H . Then H is isomorphic to $G_\Phi(S)$ such that R and S are isotypic rings.*

Problem 26. What are the isotypicity classes of fields? When two isotypic fields are isomorphic?

Remark 27. In fact, using either logical equivalence of algebras, or what is the same, the isotypicity of algebras, we compare the possibilities of individual points in the affine space to define the sets of formulas (in fact ultrafilters in $\Phi(X)$) which are valid in these points. Given a point μ in the affine space, the collection of formulas valid in the point μ is a *type* of μ . If these individual types are, roughly speaking, the same for both algebras, then these algebras are declared isotypic. Thus, for isotypic algebras we compare types of formulas realizable on these algebras. Of course, this is significantly stronger than elementary equivalence, where the individuality of points disappeared and we compare only formulas valid in all points of the affine space.

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