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Directed networks and self-similar systems

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Abstract. The formula $\frac{\partial L_r}{\partial r}\Big|_{r=\frac{1}{2}} = 2T$ in Hata and Yamaguti [1], where L_r is Salem's singular function and T is the Takagi function, was generalized to the formula $\frac{\partial^k L_r}{\partial r^k} = k! T_{r,k}$ in Sekiguchi and Shiota [17] by using the measure theoretic method, where $T_{r,k}$ is the k-th order Takagi function. In this paper we reconsider these functions from the viewpoint of de Rham's functional equation, and by investigating such functional equation on a directed network we expand the above formula without the measure theoretic method.

1. Introduction

Hata and Yamaguti [1] have obtained the formula $\frac{\partial L_r}{\partial r}\Big|_{r=\frac{1}{2}} = 2T$, which connects the Takagi function T with Salem's singular function L_r. The Takagi function takes the form $T(x) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n} f(\psi^n x)$ for $x \in [0,1]$, where $f(x) = ||2x-1|-1|$ and $\psi(x) = 2x(\bmod 1)$, and Salem's singular function L_r, which is called a "Lebesgue's singular function", is a unique continuous solution with $L_r(1) = 1$ of the following functional equation:

$$L_r(x) = rL_r(\psi(x))1_{J_{1,0}}(x) + \{r + (1-r)L_r(\psi(x))\}1_{J_{1,1}}(x) \quad (x \in [0,1]), \quad (1)$$

where r is a complex number with $\max\{|r|, |1-r|\} < 1$. This formula has been extended up to the k-th derivative of L_r in [17]. Namely, they took notice of that L_r is the distribution of the binomial probability measure, and by using the measure theoretical technique they proved the formula

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$\frac{\partial^k L_r}{\partial r^k} = k! T_{r,k}$, in which the higher-order Takagi function $T_{r,k}$ was defined by

$$T_{r,k}(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} r^{n-s(j)} (1-r)^{s(j)} B_{r,k} \circ \psi^n(x) 1_{J_{n,j}}(x) \tag{2}$$

where

$$B_{r,k}(x) = \begin{cases} L_r \circ \psi(x) 1_{J_{1,0}}(x) + (1 - L_r \circ \psi(x)) 1_{J_{1,1}}(x) & (k=1), \\ T_{r,k-1} \circ \psi(x) (1_{J_{1,0}}(x) - 1_{J_{1,1}}(x)) & (k \geq 2), \end{cases}$$

$s(n)$ is the sum of digits in the binary expansion of n, $J_{n,j} = [\frac{j}{2^n}, \frac{j+1}{2^n}[$ for $0 \leq j < 2^n - 2$ and $J_{n,2^n-1} = [\frac{2^n-1}{2^n}, 1]$. The higher-order Takagi function was used for the explicit representation of power sums of digital sums in [12]. Furthermore those results were extended for the digital sum problems on the different types of number system in [11], [3] and [2]. However it seems that the way to define the higher-order Takagi function is too technical, although their measure theoretic method is forceful.

In this paper we reconsider these functions from the viewpoint of some functional equation without the measure theoretic method, and investigate what causes the above definition of the higher-order Takagi function. We take notice of that the above functional equation (1) is a special case of de Rham's functional equation in [18]. By differentiating the equation (1) formally with respect to r, we get the functional equation

$$\begin{aligned} \frac{\partial L_r}{\partial r}(x) = \{ r \frac{\partial L_r}{\partial r}(\psi(x)) + L_r(\psi(x)) \} 1_{J_{1,0}}(x) \\ + \{ (1-r) \frac{\partial L_r}{\partial r}(\psi(x)) + 1 - L_r(\psi(x)) \} 1_{J_{1,1}}(x) \quad (x \in [0,1]), \end{aligned} \tag{3}$$

and the Takagi function appears in its solution for the case $r = \frac{1}{2}$. Furthermore, by setting that $\mathbf{r} = \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}$ and $\mathcal{L}_{\mathbf{r}}(x) = \begin{bmatrix} L_r(x) & \frac{\partial L_r}{\partial r}(x) \\ 0 & L_r(x) \end{bmatrix}$, we combine the above two functional equations (1) and (3) and then we get the following

$$\mathcal{L}_{\mathbf{r}}(x) = \mathbf{r} \mathcal{L}_{\mathbf{r}}(\psi(x)) 1_{J_{1,0}}(x) + \{ \mathbf{r} + (\mathfrak{e} - \mathbf{r}) \mathcal{L}_{\mathbf{r}}(\psi(x)) \} 1_{J_{1,1}}(x) \quad (x \in [0,1]), \tag{4}$$

where $\mathfrak{e}$ is a unit matrix. The functional equation (4) is the same as (1) except that those $\mathbf{r}$ and $\mathcal{L}_{\mathbf{r}}(x)$ are matrices, and so (4) is an extension of

(1). Moreover we reconsider (4) on $\prod_1^\infty\{0,1\}$ instead of $[0,1]$, because $[0,1]$ is considered to be $\prod_1^\infty\{0,1\}$ by the dyadic expansion, and then each of ψ, $J_{1,0}$ and $J_{1,1}$ in (4) are replaced by φ, $\{0\}\times\prod_1^\infty\{0,1\}$ and $\{1\}\times\prod_1^\infty\{0,1\}$, where φ is the shift on $\prod_1^\infty\{0,1\}$. As generalization of this functional equation, we define the system SRF($\mathbf{z}$) of functional equations on the directed network (G,m,τ) and show its fundamental properties in Section 2. This seems to be the first attempt to extending another aspect of Hata-Yamaguchi's formula. For more information on the relation between T, L_r and (G,m,τ), refer to examples in Section 5. In Section 3, by introducing the two kinds of transformations $\mathbf{D}_n$ and $\mathbf{U}_n$, we show the existence and uniqueness of solutions of SRF($\mathbf{z}$), and we also give some expansion to its solutions. In Section 4 we investigate SRF($\mathbf{z}$) of the type like (4), and by applying the result of Section 2 to it we get a general form of (2) for SRF($\mathbf{z}$). In Section 5 we shall define the mappings Ψ_g and Ψ_g^{-1} to translate the functional equations on Ω_g to the ones on $[0,1]$, and rewrite Theorem 4.1 in this case. In Section 6 we give glossary of symbols used in this paper.

We use the following notations. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural numbers(including 0), integers, real numbers and complex numbers respectively. Set $\mathbb{N}_+=\mathbb{N}\setminus\{0\}$. We denote the set of all mappings from a set X to a set Y by $\mathsf{Map}(X,Y)$, the set of all continuous mappings from a set X to a set Y by $\mathcal{C}(X,Y)$ if X and Y are topological spaces, the direct sum of a family of sets $\{B_a : a\in A\}$ with a parameter set A by $\coprod_{a\in A}B_a=\bigcup_{a\in A}\{a\}\times B_a$, the set of all mappings from A to $\bigcup_{a\in A}B_a$ such that the image f_a of a is in B_a for each a in A by $\Gamma(\coprod_{a\in A}B_a)$, and the number of all elements of a set C by $\sharp C$. Moreover we denote the set of $d\times d$ matrices with coefficients in $\mathbb{C}$ by $\mathsf{M}(d,\mathbb{C})$, and the set of upper triangle matrices in $\mathsf{M}(d,\mathbb{C})$ by $\Delta(d,\mathbb{C})$.

2. Functional equations on directed networks

We start introducing the directed network (G,m,τ).

Definition 2.1. *Let G be a non empty finite set, $S_0=\emptyset$ and $S_k=\{0,\dots,k-1\}$ for k in $\mathbb{N}_+$. Suppose that $m:G\longrightarrow\mathbb{N}_+$ and $\tau:G^{\triangleright}\longrightarrow G$ are mappings, where $G^{\triangleright}=\coprod_{g\in G}S_{m(g)}$. Then the triple (G,m,τ) is called the directed* *network. We also call an element of G a node, and an element (g,j) of $G^{\triangleright}$ a (communication) path of (G,m,τ) with the start point g and the end point $\tau(g,j)$.*

In the following we suppose that (G,m,τ) be a directed network.

Definition 2.2. *For each g in G and k in $\mathbb{N}$, we denote by S_g^{*k} the set of words with length k, which is defined by*

$$S_g^{*k} = \begin{cases} \{\epsilon\} & (k=0), \\ S_{m(g)} & (k=1), \\ \{i\boldsymbol{j} : i \in S_{m(g)}, \boldsymbol{j} = j_1 \dots j_{k-1} \in S_{\tau(g,i)}^{*(k-1)}\} & (k \geq 2), \end{cases}$$

where ϵ is the empty word and $i\boldsymbol{j}$ means $ij_1 \dots j_{k-1}$. We denote $\coprod_{g\in G} S_g^{*k}$ by $G^{\triangleright^k}$ and also use the notation $(\boldsymbol{h}, j_k)$ that means $(g, \boldsymbol{j})$ in $G^{\triangleright^k}$ such that $\boldsymbol{j} = j_1 \dots j_k$ in S_g^{*k} and $\boldsymbol{h} = (g, j_1 \dots j_{k-1})$ in $G^{\triangleright^{k-1}}$. Next we define the mapping $\tilde{\tau} : \bigcup_{n\in\mathbb{N}} G^{\triangleright^n} \longrightarrow G$ by

$$\tilde{\tau}(\boldsymbol{g}) = \begin{cases} g & \text{if } \boldsymbol{g} = (g,\epsilon) \in G^{\triangleright^0}, \\ \tau(g,j) & \text{if } \boldsymbol{g} = (g,j) \in G^{\triangleright^1}, \\ \tau(\tilde{\tau}(\boldsymbol{h}), j) & \text{if } \boldsymbol{g} = (\boldsymbol{h}, j) \in G^{\triangleright^k}, \boldsymbol{h} \in G^{\triangleright^{k-1}}, j \in S_{\tilde{\tau}(\boldsymbol{h})}^{*1} \quad (k \geq 2). \end{cases}$$

The mapping $\tilde{\tau}$ is the extension of τ, and so we use the same notation τ instead of $\tilde{\tau}$, and we call an element in $G^{\triangleright^k}$ a path with length k, because $(g, j_1 \dots j_k)$ is a connection of paths (g, j_1), $(\tau(g,j_1), j_2)$, $\dots$ and $(\tau(g, j_1 \dots j_{k-1}), j_k)$ sequentially. We note that $G^{\triangleright^0}$ is identified with G and $G^{\triangleright^1}$ is $G^{\triangleright}$.

Definition 2.3. *Let $\Omega = \prod_1^\infty G^{\triangleright}$. We define the mapping $\varphi : \Omega \longrightarrow \Omega$ as*

$$\varphi(\omega) = (\omega_2, \dots, \omega_n, \dots) \quad \text{for } \omega = (\omega_1, \omega_2, \dots, \omega_n, \dots) \text{ in } \Omega,$$

and, for each $\boldsymbol{g}$ in $\bigcup_{n\in\mathbb{N}} G^{\triangleright^n}$, we define the mapping $\sigma_{\boldsymbol{g}} : \Omega \longrightarrow \Omega$ by

$$\sigma_{\boldsymbol{g}}(\omega) = \begin{cases} \omega & \text{if } \boldsymbol{g} = (g,\epsilon) \in G^{\triangleright^0}, \\ (\boldsymbol{g}, \omega_1, \dots, \omega_n, \dots) & \text{if } \boldsymbol{g} = (g,j) \in G^{\triangleright^1}, \\ \sigma_{\boldsymbol{h}}(\sigma_{(\tau(\boldsymbol{h}),j)}(\omega)) & \text{if } \boldsymbol{g} = (\boldsymbol{h}, j) \in G^{\triangleright^k}, \\ & \qquad \boldsymbol{h} \in G^{\triangleright^{k-1}}, j \in S_{\tau(\boldsymbol{h})}^{*1} \ (k \geq 2), \end{cases}$$

where $\omega = (\omega_1, \dots, \omega_n, \dots)$ in Ω.

Definition 2.4. *Let $G^{\triangleright}$ have the discrete topology and Ω have the product topology. We define Ω_g for each g in G by*

$$\Omega_g = \left\{ ((g_1, j_1), \dots, (g_n, j_n), \dots) : \begin{array}{l} g_1 = g, j_1 \in S_g^{*1} \text{ and } g_{n+1} = \tau(g_n, j_n), \\ j_{n+1} \in S_{g_{n+1}}^{*1} \text{ for } n \in \mathbb{N}_+ \end{array} \right\},$$

and we denote $\sigma_{\boldsymbol{g}}(\Omega_{\tau(\boldsymbol{g})})$ by $I_{\boldsymbol{g}}$ for each $\boldsymbol{g}$ in $\bigcup_{n\in\mathbb{N}} G^{\triangleright^n}$. Then it is clear that each $I_{\boldsymbol{g}}$ is open, closed and compact, and that $I_g = \Omega_g = \bigcup_{\boldsymbol{j}\in S_g^{*n}} I_{(g,\boldsymbol{j})}$ for each g in G and n in $\mathbb{N}$.

Definition 2.5. *For each g in G and n in $\mathbb{N}_+$, the mappings $\boldsymbol{\pi}_{g,n} : \Omega_g \longrightarrow S_g^{*n}$ and $\pi_{g,n} : \Omega_g \longrightarrow \bigcup_{\boldsymbol{i}\in S_g^{*(n-1)}} S_{\tau(g,\boldsymbol{i})}^{*1}$ are defined by $\boldsymbol{\pi}_{g,n}(\omega) = j_1 \dots j_n$ and $\pi_{g,n}(\omega) = j_n$, where $\omega = ((g_1, j_1), \dots, (g_n, j_n), \dots)$ in Ω_g. We use $\boldsymbol{\pi}_{g,0}$ and $\pi_{g,0}$ as the mappings form Ω_g to S_g^{*0} defined by $\boldsymbol{\pi}_{g,0}(\omega) = \pi_{g,0}(\omega) = \epsilon$.*

We remark that $\boldsymbol{\pi}_{g,n}(\omega) = \pi_{g,0}(\omega) \dots \pi_{g,n}(\omega)$ and $\omega = \sigma_{(g,\boldsymbol{\pi}_{g,n}(\omega))} \circ \varphi^n(\omega)$ for ω in $\Omega_g$and n in $\mathbb{N}$.

Definition 2.6. *The mappings $\mathfrak{t}, \mathfrak{t}^+, \mathfrak{t}^- : \Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ are defined by*

$$\begin{aligned} \mathfrak{t}(\omega) &= \min\{k \in \mathbb{N} : \sup_{n>k} m(g_n) = 1\}, \\ \mathfrak{t}^+(\omega) &= \min\{k \in \mathbb{N} : \sup_{n>k} j_n = 0\}, \\ \mathfrak{t}^-(\omega) &= \min\{k \in \mathbb{N} : \sup_{n>k} (m(g_n) - j_n) = 1\}, \end{aligned}$$

and the mapping $\mathfrak{s} : \mathbb{N} \times \Omega \longrightarrow \mathbb{N}$ is defined by

$$\mathfrak{s}(n, \omega) = \sharp\{k \in \mathbb{N}_+ : m(g_k) > 1, k \leq n\},$$

where $\min \emptyset = \infty$ and $\omega = ((g_1, j_1), \dots, (g_n, j_n), \dots)$ in Ω. Moreover we denote by $\overline{0}_g$ and $\overline{m}_g-$ the elements in Ω_g satisfying $\mathfrak{t}^+(\omega) = 0$ and $\mathfrak{t}^-(\omega) = 0$ respectively, and we use the following notations $\omega-$ and $\omega+$ for ω as follows:

$$\begin{aligned} \omega- &= \sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-) && \text{if } 0 < \mathfrak{t}^+(\omega) < \infty, \\ \omega+ &= \sigma_{(g,\boldsymbol{j})}(\overline{0}_{\tau(g,\boldsymbol{j})}) && \text{if } 0 < \mathfrak{t}^-(\omega) < \infty, \end{aligned}$$

where $\boldsymbol{i} = \boldsymbol{\pi}_{g,\mathfrak{t}^+(\omega)-1}(\omega)(\pi_{g,\mathfrak{t}^+(\omega)}(\omega) - 1)$ and $\boldsymbol{j} = \boldsymbol{\pi}_{g,\mathfrak{t}^-(\omega)-1}(\omega)(\pi_{g,\mathfrak{t}^-(\omega)}(\omega) + 1)$. We also use $\overline{0}_g-$ as an imaginary point, which is not in Ω_g.

The next proposition is easily checked.

Proposition 2.1. *The mappings* $\mathfrak{s}$, $\mathfrak{t}$, $\mathfrak{t}^+$, $\mathfrak{t}^-$ *have the following properties:*
1) $\mathfrak{t}^+ \le \mathfrak{t}$, $\mathfrak{t}^- \le \mathfrak{t}$ *and* $\mathfrak{t}^- \vee \mathfrak{t}^+ = \mathfrak{t}$.
2) *if* $n \le \mathfrak{t}(\omega)$, $\omega \in \Omega_g$, *then* $\mathfrak{s}(n,\omega) \ge \lfloor \frac{n}{\sharp G} \rfloor$.

Definition 2.7. *Let the mapping* $\mathbf{z}: G^{\rhd} \longrightarrow \mathsf{M}(d,\mathbb{C})$ *satisfy* $\sum_{k\in S_g^{*1}} \mathbf{z}_g(k) = \mathfrak{e}$ *for* g *in* G, *where* $\mathfrak{e}$ *is the unit matrix and* $\mathbf{z}_g(k)$ *is the image of* (g,k) *in* $G^{\rhd}$. *We define the system of de Rham functional equations with the weighted parameter* $\mathbf{z}$ *associated with* (G,m,τ), *that is abbreviated to be " SRF*$(\mathbf{z})$ *on* (G,m,τ) *", by the following equations:*

$$\begin{cases} \mathcal{L}_g(\overline{m}_g -) = \mathfrak{e}, \\ \mathcal{L}_g(\omega) = \sum_{0\le k<j} \mathbf{z}_g(k) + \mathbf{z}_g(j)\mathcal{L}_{\tau(g,j)}(\varphi(\omega)) \quad (\omega \in I_{(g,j)}, j \in S_g^{*1}) \end{cases} \tag{5}$$

for g *in* G. *We only deal with continuous solutions* $\mathcal{L}_{\mathbf{z}}$ *of* (5). *Strictly speaking, we denote a continuous solution by* $\mathcal{L}_{\mathbf{z},g}$, *which belongs to* $\mathcal{C}(\Omega_g, \mathsf{M}(d,\mathbb{C}))$ *and which index* $\mathbf{z}$ *means its weighted parameter for each* g *in* G.

In the subsequent, we assume that the mapping $\mathbf{z}: G^{\rhd} \longrightarrow \mathsf{M}(d,\mathbb{C})$ satisfies $\sum_{k\in S_g^{*1}} \mathbf{z}_g(k) = \mathfrak{e}$ for each g in G, unless otherwise stated.

Lemma 2.1. *Let* $\mathcal{L}_{\mathbf{z}}$ *be a solution of* SRF$(\mathbf{z})$. *Then we have*

$$\begin{aligned} \mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i}j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-)) &= \mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})-) \textstyle\sum_{j<k<m(\tau(g,\boldsymbol{i}))} \mathbf{z}_{\tau(g,\boldsymbol{i})}(k) \\ &\quad + \mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-)) \textstyle\sum_{0\le k\le j} \mathbf{z}_{\tau(g,\boldsymbol{i})}(k) \end{aligned} \tag{6}$$

for $\boldsymbol{i}$ *in* S_g^{*n}, j *in* $S_{\tau(g,\boldsymbol{i})}^{*1}$, n *in* $\mathbb{N}$ *and* g *in* G, *where we set* $\mathcal{L}_{\mathbf{z},g}(\overline{0}_g -) = \mathbf{0}$.

Proof. We prove this lemma by the induction. Let g in G and j in S_g^{*1}. By substituting $\sigma_{(g,j)}(\overline{m}_{\tau(g,j)}-)$ for ω in (5), we have

$$\begin{aligned} \mathcal{L}_{\mathbf{z},g}(\sigma_{(g,j)}(\overline{m}_{\tau(g,j)}-)) &= \textstyle\sum_{0\le k<j} \mathbf{z}_g(k) + \mathbf{z}_g(j)\mathcal{L}_{\mathbf{z},\tau(g,j)}(\varphi(\sigma_{(g,j)}(\overline{m}_{\tau(g,j)}-))) \\ &= \textstyle\sum_{0\le k\le j} \mathbf{z}_g(k) \\ &= \mathcal{L}_{\mathbf{z},g}(\overline{0}_g-) \textstyle\sum_{j<k<m(g)} \mathbf{z}_g(k) + \mathcal{L}_{\mathbf{z},g}(\overline{m}_g-) \textstyle\sum_{0\le k\le j} \mathbf{z}_g(k), \end{aligned}$$

which is (6) in the case $n=0$. Next let h in S_g^{*1}, $\boldsymbol{i}$ in $S_{\tau(g,h)}^{*n}$ and j in $S_{\tau(g,h\boldsymbol{i})}^{*1}$. By using (5) again, we have the following three equations:

$$\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,h\boldsymbol{i}j)}(\overline{m}_{\tau(g,h\boldsymbol{i}j)}-))$$

$$= \textstyle\sum_{0\le k<h} \mathbf{z}_g(k) + \mathbf{z}_g(h)\mathcal{L}_{\mathbf{z},\tau(g,h)}(\sigma_{(\tau(g,h),\boldsymbol{i}j)}(\overline{m}_{\tau(g,h\boldsymbol{i}j)}-)) \quad (7)$$

$$\mathbf{z}_g(h)\mathcal{L}_{\mathbf{z},\tau(g,h)}(\sigma_{(\tau(g,h),\boldsymbol{i})}(\overline{m}_{\tau(g,h\boldsymbol{i})}-)) = \mathcal{L}_{\mathbf{z},g}(\sigma_{(g,h\boldsymbol{i})}(\overline{m}_{\tau(g,h\boldsymbol{i})}-)) - \textstyle\sum_{0\le k<h} \mathbf{z}_g(k) \quad (8)$$

$$\mathbf{z}_g(h)\mathcal{L}_{\mathbf{z},\tau(g,h)}(\sigma_{(\tau(g,h),\boldsymbol{i})}(\overline{0}_{\tau(g,h\boldsymbol{i})})-) = \mathcal{L}_{\mathbf{z},g}(\sigma_{(g,h\boldsymbol{i})}(\overline{0}_{\tau(g,h\boldsymbol{i})})-) - \textstyle\sum_{0\le k<h} \mathbf{z}_g(k), \quad (9)$$

and, by the induction assumption, we have

$$\begin{aligned} &\mathcal{L}_{\mathbf{z},\tau(g,h)}(\sigma_{(\tau(g,h),\boldsymbol{i}j)}(\overline{m}_{\tau(g,h\boldsymbol{i}j)}-)) \\ &\quad= \mathcal{L}_{\mathbf{z},\tau(g,h)}(\sigma_{(\tau(g,h),\boldsymbol{i})}(\overline{0}_{\tau(g,h\boldsymbol{i})})-)\textstyle\sum_{j<k<m(\tau(g,h\boldsymbol{i}))} \mathbf{z}_{\tau(g,h\boldsymbol{i})}(k) \\ &\qquad+ \mathcal{L}_{\mathbf{z},\tau(g,h)}(\sigma_{(\tau(g,h),\boldsymbol{i})}(\overline{m}_{\tau(g,h\boldsymbol{i})}-))\textstyle\sum_{0\le k\le j} \mathbf{z}_{\tau(g,h\boldsymbol{i})}(k). \end{aligned} \quad (10)$$

Then we substitute (10), (9) and (8) in (7) sequentially, and we have

$$\begin{aligned} &\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,h\boldsymbol{i}j)}(\overline{m}_{\tau(g,h\boldsymbol{i}j)}-)) \\ &= \textstyle\sum_{0\le k<h} \mathbf{z}_g(k) \\ &\quad+ \mathbf{z}_g(h)\{\mathcal{L}_{\mathbf{z},\tau(g,h)}(\sigma_{(\tau(g,h),\boldsymbol{i})}(\overline{0}_{\tau(g,h\boldsymbol{i})})-)\textstyle\sum_{j<k<m(\tau(g,h\boldsymbol{i}))} \mathbf{z}_{\tau(g,h\boldsymbol{i})}(k) \\ &\quad+ \mathcal{L}_{\mathbf{z},\tau(g,h)}(\sigma_{(\tau(g,h),\boldsymbol{i})}(\overline{m}_{\tau(g,h\boldsymbol{i})}-))\textstyle\sum_{0\le k\le j} \mathbf{z}_{\tau(g,h\boldsymbol{i})}(k)\} \\ &= \textstyle\sum_{0\le k<h} \mathbf{z}_g(k) \\ &\quad+ \{\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,h\boldsymbol{i})}(\overline{0}_{\tau(g,h\boldsymbol{i})})-) - \textstyle\sum_{0\le k<h} \mathbf{z}_g(k)\}\textstyle\sum_{j<k<m(\tau(g,h\boldsymbol{i}))} \mathbf{z}_{\tau(g,h\boldsymbol{i})}(k) \\ &\quad+ \{\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,h\boldsymbol{i})}(\overline{m}_{\tau(g,h\boldsymbol{i})}-)) - \textstyle\sum_{0\le k<h} \mathbf{z}_g(k)\}\textstyle\sum_{0\le k\le j} \mathbf{z}_{\tau(g,h\boldsymbol{i})}(k) \\ &= \mathcal{L}_{\mathbf{z},g}(\sigma_{(g,h\boldsymbol{i})}(\overline{0}_{\tau(g,h\boldsymbol{i})})-)\textstyle\sum_{j<k<m(\tau(g,h\boldsymbol{i}))} \mathbf{z}_{\tau(g,h\boldsymbol{i})}(k) \\ &\quad+ \mathcal{L}_{\mathbf{z},g}(\sigma_{(g,h\boldsymbol{i})}(\overline{m}_{\tau(g,h\boldsymbol{i})}-))\textstyle\sum_{0\le k\le j} \mathbf{z}_{\tau(g,h\boldsymbol{i})}(k). \end{aligned}$$

Namely we get (6) for the next n. □

The above Lemma 2.1 means that any solution of SRF($\mathbf{z}$) is determined only by the parameter $\mathbf{z}$ on the set $\{\sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-) : \boldsymbol{i} \in S_g^{*n}, n \in \mathbb{N}\}$, which is dense in Ω_g, for each g in G. That implies the following proposition.

Proposition 2.2. *The continuous solution $\mathcal{L}_{\mathbf{z}}$ of* SRF($\mathbf{z}$) *is unique, and each $\mathcal{L}_{\mathbf{z},g}$ belongs to $\mathcal{C}(\Omega_g, \Delta(d,\mathbb{C}))$ for g in G if $\mathbf{z}: G^{\triangleright} \longrightarrow \Delta(d,\mathbb{C})$.*

Definition 2.8. *For the mapping* $\mathbf{z}:G^{\rhd}\longrightarrow \mathsf{M}(d,\mathbb{C})$ *we define the mapping* $\tilde{\mathbf{z}}:\bigcup_{n\in\mathbb{N}} G^{\rhd^n}\longrightarrow \mathsf{M}(d,\mathbb{C})$ *by*

$$\tilde{\mathbf{z}}(\boldsymbol{g}) = \begin{cases} \mathfrak{e} & \text{if } \boldsymbol{g}=(g,\epsilon)\in G^{\rhd^0},\\ \mathbf{z}_g(j) & \text{if } \boldsymbol{g}=(g,j)\in G^{\rhd^1},\\ \tilde{\mathbf{z}}(\boldsymbol{h})\mathbf{z}_{\tau(\boldsymbol{h})}(j) & \text{if } \boldsymbol{g}=(\boldsymbol{h},j)\in G^{\rhd^k}, \boldsymbol{h}\in G^{\rhd^{k-1}},\\ & \qquad\qquad j\in S^{*1}_{\tau(\boldsymbol{h})} \quad (k\geq 2). \end{cases}$$

The mapping $\tilde{\mathbf{z}}$ *is the extension of* $\mathbf{z}$, *and so we use the same notation* $\mathbf{z}$ *instead of* $\tilde{\mathbf{z}}$. *We also use the notation* $\mathbf{z}_g(\boldsymbol{j})$ *as* $\mathbf{z}(\boldsymbol{g})$ *if* $\boldsymbol{g}=(g,\boldsymbol{j})$ *for* g *in* G *and* $\boldsymbol{j}$ *in* S^{*k}_g.

Lemma 2.2. *Let* $\mathcal{L}_{\mathbf{z}}$ *be a solution of* $\mathrm{SRF}(\mathbf{z})$. *If* ω *in* Ω_g *and* $\mathsf{t}^+(\omega)<\infty$, *then we have*

$$\mathcal{L}_{\mathbf{z},g}(\omega)-\mathcal{L}_{\mathbf{z},g}(\omega-) = \mathbf{z}_g(\boldsymbol{i})\mathcal{L}_{\mathbf{z},\tau(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})}) \tag{11}$$

for g *in* G, *where* $\boldsymbol{i}$ *in* $S^{*\mathsf{t}^+(\omega)}_g$ *and* $\omega=\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})$.

Proof. If $\mathsf{t}^+(\omega)=0$ then the left-hand side of (11) equals $\mathbf{z}_g(\epsilon)\mathcal{L}_{\mathbf{z},\tau(g,\epsilon)}(\overline{0}_{\tau(g,\epsilon)})$ because of $\omega=\overline{0}_g$ and $\boldsymbol{i}=\epsilon$. If $\mathsf{t}^+(\omega)=1$ then there exists j in S^{*1}_g such that $j\geq 1$ and $\omega=\sigma_{(g,j)}(\overline{0}_{\tau(g,j)})$, and so $\omega-=\sigma_{(g,j-1)}(\overline{m}_{\tau(g,j-1)}-)$. Hence

$$\mathcal{L}_{\mathbf{z},g}(\omega) = \textstyle\sum_{0\leq k<j}\mathbf{z}_g(k)+\mathbf{z}_g(j)\mathcal{L}_{\mathbf{z},\tau(g,j)}(\overline{0}_{\tau(g,j)})$$

and

$$\mathcal{L}_{\mathbf{z},g}(\omega-) = \textstyle\sum_{0\leq k<j-1}\mathbf{z}_g(k)+\mathbf{z}_g(j-1)\mathcal{L}_{\mathbf{z},\tau(g,j-1)}(\overline{m}_{\tau(g,j-1)}-)$$

by (5), and then the left-hand side of (11) equals $\mathbf{z}_g(j)\mathcal{L}_{\mathbf{z},\tau(g,j)}(\overline{0}_{\tau(g,j)})$.

Next suppose that $\mathsf{t}^+(\omega)=n+2$ and $n\in\mathbb{N}$, that is, $\omega=\sigma_{(g,h\boldsymbol{i}j)}(\overline{0}_{\tau(g,h\boldsymbol{i}j)})$ where $h\in S^{*1}_g$, $\boldsymbol{i}\in S^{*n}_{\tau(g,h)}$ and $0<j\in S^{*1}_{\tau(g,h\boldsymbol{i})}$. Then we have $\omega-=\sigma_{(g,h\boldsymbol{i}(j-1))}(\overline{m}_{\tau(g,h\boldsymbol{i}(j-1))}-)$ and $\varphi(\omega)=\sigma_{(\tau(g,h),\boldsymbol{i}j)}(\overline{0}_{\tau(g,h\boldsymbol{i}j)})$. Moreover we have $\varphi(\omega-)=\sigma_{(\tau(g,h),\boldsymbol{i}(j-1))}(\overline{m}_{\tau(g,h\boldsymbol{i}(j-1))}-)=\varphi(\omega)-$ and $\mathsf{t}^+(\varphi(\omega))=n+1$. Hence

$$\mathcal{L}_{\mathbf{z},g}(\omega)-\mathcal{L}_{\mathbf{z},g}(\omega-) = \mathbf{z}_g(h)\{\mathcal{L}_{\mathbf{z},\tau(g,h)}(\varphi(\omega))-\mathcal{L}_{\mathbf{z},\tau(g,h)}(\varphi(\omega)-)\}$$

by (5). Therefore (11) is obtained. □

3. The existence and some expansion formulas of solutions of functional equations systems

We start by defining transformations $\mathbf{D}_n$ and $\mathbf{U}_n$.

Definition 3.1. *For each n in $\mathbb{N}$, we define the transformations $\mathbf{D}_n$ and $\mathbf{U}_n$ on $\Gamma(\coprod_{g\in G}\mathsf{Map}(\Omega_g,\mathsf{M}(d,\mathbb{C})))$, as follows:*

$$(\mathbf{D}_n\mathcal{F})_g(\omega)=\textstyle\sum_{\boldsymbol{j}\in S_g^{*n}}\{\mathcal{F}_g(\sigma_{(g,\boldsymbol{j})}(\overline{m}_{\tau(g,\boldsymbol{j})}-))-\mathcal{F}_g(\sigma_{(g,\boldsymbol{j})}(\overline{0}_{\tau(g,\boldsymbol{j})})-)\}1_{I_{(g,\boldsymbol{j})}}(\omega),$$
$$(\mathbf{U}_n\mathcal{F})_g(\omega)=\textstyle\sum_{\boldsymbol{j}\in S_g^{*n}}\mathcal{F}_{\tau(g,\boldsymbol{j})}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{j})}}(\omega),$$

where $\mathcal{F}$ in $\Gamma(\coprod_{g\in G}\mathsf{Map}(\Omega_g,\mathsf{M}(d,\mathbb{C})))$, ω in Ω_g, and $\mathcal{F}_g(\overline{0}_g-)=\mathbf{0}$ by using the imaginary point $\overline{0}_g-$.

Definition 3.2. *We define the mapping $s:G^{\triangleright}\times\bigcup_{n\in\mathbb{N}}G^{\triangleright^n}\longrightarrow\mathbb{N}$ by*

$$s((g,j);(h,\boldsymbol{i}))=\sharp\{k:(\tau(h,i_1\ldots i_{k-1}),i_k)=(g,j),1\le k\le n\},$$

where $(g,j)\in G^{\triangleright}$, $\boldsymbol{i}=i_1\ldots i_n\in S_h^{*n}$ and $h\in G$. (Do not confuse with $\mathfrak{s}(n,\omega)$ in Definition 2.6.*)*

Definition 3.3. *We define $\rho(\mathbf{z})$ and $\eta(\mathbf{z})$ by*

$$\rho(\mathbf{z})=\max_{g\in G,m(g)>1}\max_{j\in S_g^{*1}}\max_{1\le i\le d}|(\mathbf{z}_g(j))_{i,i}|,$$
$$\eta(\mathbf{z})=\max_{h\in G}\max_{k\in S_h^{*1}}\max_{1\le i<j\le d}|(\mathbf{z}_h(k))_{i,j}|$$

for $\mathbf{z}:G^{\triangleright}\longrightarrow\Delta(d,\mathbb{C})$, and use the notation $\|\mathcal{F}\|$ by

$$\|\mathcal{F}\|=\max_{g\in G}\sup_{\omega\in\Omega}\|\mathcal{F}(\omega)\|\qquad\text{for }\mathcal{F}\in\Gamma(\coprod_{g\in G}\mathsf{Map}(\Omega_g,\mathsf{M}(d,\mathbb{C}))).$$

We next show fundamental properties of $\mathbf{D}_n$ and $\mathbf{U}_n$.

Proposition 3.1. 1) *For n in $\mathbb{N}$, $\mathbf{D}_n$ and $\mathbf{U}_n$ are $\mathsf{M}(d,\mathbb{C})$-linear transformations on $\Gamma(\coprod_{g\in G}\mathsf{Map}(\Omega_g,\mathsf{M}(d,\mathbb{C})))$, that is, these transformations satisfy*

$$\mathbf{D}_n(\mathbf{a}\mathcal{F}+\mathbf{b}\mathcal{G})=\mathbf{a}\mathbf{D}_n(\mathcal{F})+\mathbf{b}\mathbf{D}_n(\mathcal{G}),\quad \mathbf{D}_n(\mathcal{F}\mathbf{a}+\mathcal{G}\mathbf{b})=\mathbf{D}_n(\mathcal{F})\mathbf{a}+\mathbf{D}_n(\mathcal{G})\mathbf{b},$$
$$\mathbf{U}_n(\mathbf{a}\mathcal{F}+\mathbf{b}\mathcal{G})=\mathbf{a}\mathbf{U}_n(\mathcal{F})+\mathbf{b}\mathbf{U}_n(\mathcal{G}),\quad \mathbf{U}_n(\mathcal{F}\mathbf{a}+\mathcal{G}\mathbf{b})=\mathbf{U}_n(\mathcal{F})\mathbf{a}+\mathbf{U}_n(\mathcal{G})\mathbf{b}$$

for $\mathcal{F}$ *and* $\mathcal{G}$ *in* $\Gamma(\coprod_{g\in G}\mathsf{Map}(\Omega_g,\mathsf{M}(d,\mathbb{C}))$ *and* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathsf{M}(d,\mathbb{C})$*, where* $(\mathbf{a}\mathcal{F})_g(\omega)=\mathbf{a}(\mathcal{F}_g(\omega))$ *and* $(\mathcal{F}\mathbf{a})_g(\omega)=(\mathcal{F}_g(\omega))\mathbf{a}$ *for* ω *in* Ω_g *and* g *in* G*.*

2) *If* $\mathcal{F}$ *is in* $\Gamma(\coprod_{g\in G}\mathcal{C}(\Omega_g,\mathsf{M}(d,\mathbb{C})))$ *then both* $\mathbf{U}_n\mathcal{F}$ *and* $\mathbf{D}_n\mathcal{F}$ *are so.*

3) *For* n,n_1 *and* n_2 *in* $\mathbb{N}$*,* $\mathbf{U}_n$ *satisfies that* $\mathbf{U}_n(\mathcal{F}\mathcal{G})=(\mathbf{U}_n\mathcal{F})(\mathbf{U}_n\mathcal{G})$ *and* $\mathbf{U}_{n_1}\mathbf{U}_{n_2}\mathcal{F}=\mathbf{U}_{n_1+n_2}\mathcal{F}$ *for* $\mathcal{F}$ *and* $\mathcal{G}$ *in* $\Gamma(\coprod_{g\in G}\mathsf{Map}(\Omega_g,\mathsf{M}(d,\mathbb{C})))$ *, where* $\mathcal{F}\mathcal{G}$ *is defined by* $(\mathcal{F}\mathcal{G})_g(\omega)=\mathcal{F}_g(\omega)\mathcal{G}_g(\omega)$ *for* g *in* G *and* ω *in* Ω_g*.*

4) *Let* $\mathcal{L}_{\mathbf{z}}$ *be a solution of* $\mathrm{SRF}(\mathbf{z})$*. Then we have the followings:*

$$(\mathbf{D}_n\mathcal{L}_{\mathbf{z}})_g(\omega)=\textstyle\sum_{\boldsymbol{j}\in S_g^{*n}}\mathbf{z}_g(\boldsymbol{j})1_{I_{(g,\boldsymbol{j})}}(\omega)\qquad(\omega\in\Omega_g,g\in G,n\in\mathbb{N}),\tag{12}$$

$$\mathbf{D}_{n_1+n_2}\mathcal{L}_{\mathbf{z}}=(\mathbf{D}_{n_1}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1}\mathbf{D}_{n_2}\mathcal{L}_{\mathbf{z}})\qquad(n_1,n_2\in\mathbb{N}).\tag{13}$$

Proof. We directly get 1), 2) and 3) from Definition 3.1. Since $\mathcal{L}_{\mathbf{z}}$ is a solution of (5), (12) with $n=0,1$ and (13) with $n_1=n_2=0$ are clear. By Definition 3.1, we have

$$(\mathbf{D}_{n+1}\mathcal{L}_{\mathbf{z}})_g(\omega)=\textstyle\sum_{\boldsymbol{i}\in S_g^{*n}}\sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}}\{\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i}j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-))\\ -\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i}j)}(\overline{0}_{\tau(g,\boldsymbol{i}j)})-)\}1_{I_{(g,\boldsymbol{i}j)}}(\omega),$$

and by Lemma 2.1 we get

$$\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i}j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-))=\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})-)\textstyle\sum_{j<k<m(\tau(g,\boldsymbol{i}))}\mathbf{z}_{\tau(g,\boldsymbol{i})}(k)\\ +\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-))\textstyle\sum_{0\le k\le j}\mathbf{z}_{\tau(g,\boldsymbol{i})}(k)\tag{14}$$

and

$$\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i}j)}(\overline{0}_{\tau(g,\boldsymbol{i}j)})-)=\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})-)\textstyle\sum_{j\le k<m(\tau(g,\boldsymbol{i}))}\mathbf{z}_{\tau(g,\boldsymbol{i})}(k)\\ +\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-))\textstyle\sum_{0\le k<j}\mathbf{z}_{\tau(g,\boldsymbol{i})}(k)\tag{15}$$

for $\boldsymbol{i}$ in S_g^{*n}, j in $S^{*1}_{\tau(g,\boldsymbol{i})}$, n in $\mathbb{N}$ and g in G. Therefore we have

$$(\mathbf{D}_{n+1}\mathcal{L}_{\mathbf{z}})_g(\omega)\\ =\textstyle\sum_{\boldsymbol{i}\in S_g^{*n}}\sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}}\{\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-))-\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})-)\}\\ \times\mathbf{z}_{\tau(g,\boldsymbol{i})}(j)1_{I_{(g,\boldsymbol{i}j)}}(\omega)\\ =\textstyle\sum_{\boldsymbol{i}\in S_g^{*n}}\{\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-))-\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})-)\}\\ \times\{\textstyle\sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}}\mathbf{z}_{\tau(g,\boldsymbol{i})}(j)1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega))\}1_{I_{(g,\boldsymbol{i})}}(\omega)$$

$$
\begin{aligned}
&= \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \{\mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-)) - \mathcal{L}_{\mathbf{z},g}(\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})-)\} \\
&\qquad\qquad\qquad\qquad \times (\mathbf{D}_1\mathcal{L}_{\mathbf{z}})_{\tau(g,\boldsymbol{i})}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{i})}}(\omega) \\
&= (\mathbf{D}_n\mathcal{L}_{\mathbf{z}})_g(\omega)(\mathbf{U}_n\mathbf{D}_1\mathcal{L}_{\mathbf{z}})_g(\omega),
\end{aligned}
$$

that is, we get (13) with $n_1 = n$ and $n_2 = 1$. Next we shall prove (12) and (13) with the remaining n, n_1 and n_2, by the induction as follows:

$$
\begin{aligned}
(\mathbf{D}_{n+1}\mathcal{L}_{\mathbf{z}})_g(\omega) &= (\mathbf{D}_n\mathcal{L}_{\mathbf{z}})_g(\omega)(\mathbf{U}_n\mathbf{D}_1\mathcal{L}_{\mathbf{z}})_g(\omega) \\
&= \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \mathbf{z}_g(\boldsymbol{i})1_{I_{(g,\boldsymbol{i})}}(\omega)\sum_{\boldsymbol{k}\in S_g^{*n}}(\mathbf{D}_1\mathcal{L}_{\mathbf{z}})_{\tau(g,\boldsymbol{k})}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{k})}}(\omega) \\
&= \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \mathbf{z}_g(\boldsymbol{i})(\mathbf{D}_1\mathcal{L}_{\mathbf{z}})_{\tau(g,\boldsymbol{i})}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{i})}}(\omega) \\
&= \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \mathbf{z}_g(\boldsymbol{i})\sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}} \mathbf{z}_{\tau(g,\boldsymbol{i})}(j)1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{i})}}(\omega) \\
&= \textstyle\sum_{\boldsymbol{j}\in S_g^{*(n+1)}} \mathbf{z}_g(\boldsymbol{j})1_{I_{(g,\boldsymbol{j})}}(\omega),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{D}_{n_1+n_2+1}\mathcal{L}_{\mathbf{z}} &= (\mathbf{D}_{n_1+n_2}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1+n_2}\mathbf{D}_1\mathcal{L}_{\mathbf{z}}) \\
&= (\mathbf{D}_{n_1}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1}\mathbf{D}_{n_2}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1}\mathbf{U}_{n_2}\mathbf{D}_1\mathcal{L}_{\mathbf{z}}) \\
&= (\mathbf{D}_{n_1}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1}((\mathbf{D}_{n_2}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_2}\mathbf{D}_1\mathcal{L}_{\mathbf{z}}))) \\
&= (\mathbf{D}_{n_1}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1}\mathbf{D}_{n_2+1}\mathcal{L}_{\mathbf{z}})
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathbf{D}_{n_1}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1}\mathbf{D}_{n_2+1}\mathcal{L}_{\mathbf{z}}) &= (\mathbf{D}_{n_1}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1}((\mathbf{D}_1\mathcal{L}_{\mathbf{z}})(\mathbf{U}_1\mathbf{D}_{n_2}\mathcal{L}_{\mathbf{z}}))) \\
&= (\mathbf{D}_{n_1}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1}\mathbf{D}_1\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1+1}\mathbf{D}_{n_2}\mathcal{L}_{\mathbf{z}}) \\
&= (\mathbf{D}_{n_1+1}\mathcal{L}_{\mathbf{z}})(\mathbf{U}_{n_1+1}\mathbf{D}_{n_2}\mathcal{L}_{\mathbf{z}}).
\end{aligned}
$$

□

We must estimate the product of upper triangle matrices before describing the main result in this section.

Lemma 3.1. *Let* $\mathbf{w}_k \in \Delta(d,\mathbb{C})$ $(k = 1,\ldots,n)$ *and set*

$$\alpha = \max\nolimits_{1\le k\le n, 1\le i\le d}|(\mathbf{w}_k)_{i,i}| \text{ and } \beta = \max\nolimits_{1\le k\le n, 1\le i<j\le d}|(\mathbf{w}_k)_{i,j}|,$$

where $(\mathbf{w}_k)_{i,j}$ $(1\le i,j\le d)$ *are components of the matrix* $\mathbf{w}_k$*. Then we have*

$$\|\mathbf{w}_1\mathbf{w}_2\cdots\mathbf{w}_n\| \le C(n,\alpha,\beta,d),$$

where

$$C(n,\alpha,\beta,d)=\alpha^n\prod_{0\le j<d\wedge n}(n-j)\exp(\sqrt{\tfrac{d(d-1)}{2}\tfrac{\beta}{\alpha}})+\alpha^n(\sqrt{d}-1).$$

We remark that $\sum_{n\in\mathbb{N}}C(n,\alpha,\beta,d)<\infty$ *if* $\alpha<1$.

Proof. Let $\mathbf{a}=(a_{i,j})\in\Delta(d,\mathbb{C})$ such that $a_{i,i}=\alpha$ $(i=1,\dots,d)$ and $a_{i,j}=\beta$ $(1\le i<j\le d)$. We also set $\mathbf{w}_1\mathbf{w}_2\cdots\mathbf{w}_n=(b_{i,j})$ and $\mathbf{a}^n=(a^{(n)}_{i,j})$, then $|b_{i,j}|\le a^{(n)}_{i,j}$ $(1\le i,j\le d)$ and so $\|\mathbf{w}_1\mathbf{w}_2\cdots\mathbf{w}_n\|\le\|\mathbf{a}^n\|$. On the other hand, by letting $\mathbf{c}=\mathbf{a}-\alpha\mathfrak{e}$, we have $\mathbf{a}^n=\sum_{0\le j<d\wedge n}\binom{n}{j}\alpha^{n-j}\mathbf{c}^j$ since $\mathbf{c}^k=\mathbf{0}$ $(k\ge d)$. Therefore we have

$$\begin{aligned}\|\mathbf{a}^n\|&\le\sum_{0\le j<d}\tbinom{n}{j}\alpha^{n-j}\|\mathbf{c}\|^j+\alpha^n(\sqrt{d}-1)\\&\le\alpha^n\prod_{0\le k<d\wedge n}(n-k)\sum_{0\le j<d}\tfrac{1}{j!}(\tfrac{\|\mathbf{c}\|}{\alpha})^j+\alpha^n(\sqrt{d}-1)\\&\le\alpha^n\prod_{0\le k<d\wedge n}(n-k)\exp(\tfrac{\|\mathbf{c}\|}{\alpha})+\alpha^n(\sqrt{d}-1)\\&\le C(n,\alpha,\beta,d).\end{aligned}$$

□

Theorem 3.1. *Let* $\mathbf{z}:G^{\triangleright}\longrightarrow\Delta(d,\mathbb{C})$ *with* $\sum_{k\in S^{*1}_g}\mathbf{z}_g(k)=\mathfrak{e}$ *for* g *in* G *and* $\rho(\mathbf{z})<1$. *Then we have the followings:*

1) *There exists a unique continuous solution* $\mathcal{L}_{\mathbf{z}}$ *of* $\mathrm{SRF}(\mathbf{z})$.
2) *The mapping* $\mathbf{z}\longmapsto\mathcal{L}_{\mathbf{z},g}$ *is* $\mathcal{C}(\Omega_g,\mathsf{M}(d,\mathbb{C}))$*-valued analytic.*
3) *By using the notations* $\mathbf{v}$, $\mathbf{w}$, $\mathcal{V}$ *and* $\mathcal{W}$ *defined by*

$$(\mathbf{v}_g(j))_{k,l}=\begin{cases}(\mathbf{z}_g(j))_{k,l} & (k=l)\\ 0 & (k\ne l)\end{cases}\qquad(1\le k,l\le d,\ \ j\in S^{*1}_g),$$

$$\begin{aligned}\mathbf{z}&=\mathbf{v}+\mathbf{w},\\ \mathcal{V}_g(\omega)&=\sum_{j\in S^{*1}_g}\{\sum_{0\le k<j}\mathbf{v}_g(k)\}1_{I_{(g,j)}}(\omega),\\ \mathcal{W}_g(\omega)&=\sum_{j\in S^{*1}_g}\{\sum_{0\le k<j}\mathbf{w}_g(k)+\mathbf{w}_g(j)\mathcal{L}_{\mathbf{z},\tau(g,j)}(\varphi(\omega))\}I_{(g,j)}(\omega)\end{aligned}$$

for each ω *in* Ω_g *and* g *in* G, *the solution* $\mathcal{L}_{\mathbf{z}}$ *is represented as*

$$\mathcal{L}_{\mathbf{z},g}=\sum_{n\in\mathbb{N}}\{(\mathbf{D}_n\mathcal{L}_{\mathbf{v}})_g(\mathbf{U}_n\mathcal{V})_g+(\mathbf{D}_n\mathcal{L}_{\mathbf{v}})_g(\mathbf{U}_n\mathcal{W})_g\}\tag{16}$$

uniformly on $\{\omega\in\Omega_g:\mathfrak{t}(\omega)=\infty\}$ *and*

$$\mathcal{L}_{\mathbf{z},g}(\omega)=\sum_{0\le n\le\mathfrak{t}(\omega)}\{(\mathbf{D}_n\mathcal{L}_{\mathbf{v}})_g(\omega)(\mathbf{U}_n\mathcal{V})_g(\omega)$$

$$+ (\mathbf{D}_n\mathcal{L}_{\mathbf{v}})_g(\omega)(\mathbf{U}_n\mathcal{W})_g(\omega)\} + (\mathbf{D}_{\mathfrak{t}(\omega)}\mathcal{L}_{\mathbf{v}})_g(\omega) \tag{17}$$

for ω in Ω_g with $\mathfrak{t}(\omega) < \infty$, where $\mathcal{L}_{\mathbf{v}}$ appearing in the above expansions is the continuous solution of SRF($\mathbf{v}$).

Proof. 1) The uniqueness comes from Proposition 2.2 and so we show the existence by the mathematical inductive as the parameter d.

(i) We first show it in the case $d = 1$, that is, $\mathbf{z}$ is a mapping from $G^{\triangleright}$ to $\mathbb{C}$. For n in $\mathbb{N}$, g in G, $\boldsymbol{i}$ in S_g^{*n} and ω in Ω_g, we set

$$\mathcal{J}_{n,g,\boldsymbol{i}}(\omega) = \mathbf{z}_g(\boldsymbol{i})\textstyle\sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}}\sum_{0\le k<j}\mathbf{z}_{\tau(g,\boldsymbol{i})}(k)1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega)).$$

Then we have that $\mathcal{J}_{n,g,\boldsymbol{i}}(\omega) = 0$ if $n > \mathfrak{t}(\omega)$ and

$$\|\mathcal{J}_{n,g,\boldsymbol{i}}(\omega)\| \le C(\mathfrak{s}(n,\omega), \rho(\mathbf{z}), 0, 1) \cdot \max_{h\in G} m(h)$$

by Lemma 3.1, and so we can define the continuous mapping $\mathcal{K}_g : \Omega_g \longrightarrow \mathbb{C}$ by $\sum_{n\in\mathbb{N}}\sum_{\boldsymbol{i}\in S_g^{*n}}\mathcal{J}_{n,g,\boldsymbol{i}}(\omega)1_{I_{(g,\boldsymbol{i})}}(\omega)$ for g in G, because the summation converges uniformly by Proposition 2.1. By noticing that $\mathcal{J}_{n+1,g,h\boldsymbol{i}}(\omega) = \mathbf{z}_g(h)\mathcal{J}_{n,\tau(g,h),\boldsymbol{i}}(\varphi(\omega))$ and $\mathcal{J}_{n,g,\boldsymbol{i}_n}(\overline{m}_g-) = \mathbf{z}_g(\boldsymbol{i}_n) - \mathbf{z}_g(\boldsymbol{i}_{n+1})$, where h in S_g^{*1}, $\boldsymbol{i}$ in $S^{*n}_{\tau(g,h)}$ and $\boldsymbol{i}_k = \boldsymbol{\pi}_{g,k}(\overline{m}_g-)$, we have

$$\begin{aligned}\mathcal{K}_g(\omega) =&\mathcal{J}_{0,g,\epsilon}(\omega) + \textstyle\sum_{n\in\mathbb{N}}\sum_{h\in S_g^{*1}} \\ &\qquad\textstyle\sum_{\boldsymbol{i}\in S^{*n}_{\tau(g,h)}}\mathcal{J}_{n+1,g,h\boldsymbol{i}}(\omega)1_{I_{(\tau(g,h),\boldsymbol{i})}}(\varphi(\omega))1_{I_{(g,h)}}(\omega) \\ =&\textstyle\sum_{j\in S_g^{*1}}\{\sum_{0\le k<j}\mathbf{z}_g(k) + \mathbf{z}_g(j)\mathcal{K}_{\tau(g,j)}(\varphi(\omega))\}1_{I_{(g,j)}}(\omega)\end{aligned}$$

and $\mathcal{K}_g(\overline{m}_g-) = \sum_{n\in\mathbb{N}}(\mathbf{z}_g(\boldsymbol{i}_n) - \mathbf{z}_g(\boldsymbol{i}_{n+1})) = 1$. Namely $\mathcal{K}$ is a continuous solution of SRF($\mathbf{z}$).

(ii) Let $\mathbf{z}: G^{\triangleright} \longrightarrow \Delta(d{+}1, \mathbb{C})$ with $\sum_{k\in S_g^{*1}}\mathbf{z}_g(k) = \mathfrak{e}$ for g in G and $\rho(\mathbf{z}) < 1$, and let the mappings $\mathbf{x} : G^{\triangleright} \longrightarrow \mathbb{C}$, $\mathbf{y} : G^{\triangleright} \longrightarrow \Delta(d, \mathbb{C})$ and $\mathbf{r} : G^{\triangleright} \longrightarrow \mathbb{C}^d$ defined by $\mathbf{z} = \begin{bmatrix}\mathbf{x} & \mathbf{r}\\ \mathbf{0} & \mathbf{y}\end{bmatrix}$. We assume that both $\mathcal{X}$ and $\mathcal{Y}$ are continuous solutions of SRF($\mathbf{x}$) and SRF($\mathbf{y}$) respectively. We denote the continuous mapping $\mathcal{Q}_{n,g,\boldsymbol{i}} : \Omega_g \longrightarrow \mathbb{C}^d$ by

$$\mathbf{x}_g(\boldsymbol{i})\textstyle\sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}}\{\sum_{0\le k<j}\mathbf{r}_{\tau(g,\boldsymbol{i})}(k)$$

$$+\mathbf{r}_{\tau(g,\boldsymbol{i})}(j)\mathcal{Y}_{\tau(g,\boldsymbol{i}j)}(\varphi^{n+1}(\omega))\}1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega))$$

for n in $\mathbb{N}$, g in G and $\boldsymbol{i}$ in S_g^{*n}. Then we have $\mathcal{Q}_{n,g,\boldsymbol{i}}(\omega)=0$ if $n>\mathfrak{t}(\omega)$ and

$$\|\mathcal{Q}_{n,g,\boldsymbol{i}}(\omega)\|\leq|\mathbf{x}_g(\boldsymbol{i})|\eta(\mathbf{z})\{\max_{h\in G}m(h)+\|\mathcal{Y}\|\}$$

by the same estimation of $\mathcal{J}_{n,g,\boldsymbol{i}}(\omega)$, and so we can define the continuous mapping $\mathcal{R}_g:\Omega_g\longrightarrow\mathbb{C}^d$ by $\mathcal{R}_g(\omega)=\sum_{n\in\mathbb{N}}\sum_{\boldsymbol{i}\in S_g^{*n}}\mathcal{Q}_{n,g,\boldsymbol{i}}(\omega)1_{I_{(g,\boldsymbol{i})}}(\omega)$ for g in G. By noticing that

$$\mathcal{Q}_{0,g,\epsilon}(\omega)=\textstyle\sum_{0\leq k<j}\mathbf{r}_g(k)+\mathbf{r}_g(j)\mathcal{Y}_{\tau(g,j)}(\varphi(\omega)),$$
$$\mathcal{Q}_{n+1,g,h\boldsymbol{i}}(\omega)=\mathbf{x}_g(h)\mathcal{Q}_{n,\tau(g,h),\boldsymbol{i}}(\varphi(\omega))\ \text{ and }\ \mathcal{Q}_{n,g,\boldsymbol{i}}(\overline{m}_g-)=\mathbf{0},$$

we have

$$\mathcal{R}_g(\omega)=\textstyle\sum_{0\leq k<j}\mathbf{r}_g(k)+\mathbf{r}_g(j)\mathcal{Y}_{\tau(g,j)}(\varphi(\omega))+\mathbf{x}_g(j)\mathcal{R}_{\tau(g,j)}(\varphi(\omega))$$

for ω in $I_{(g,j)}$ and $\mathcal{R}_g(\overline{m}_g-)=\mathbf{0}$. Hence $\mathcal{L}$, which is defined by $\mathcal{L}=\begin{bmatrix}\mathcal{X} & \mathcal{R}\\ \mathbf{0} & \mathcal{Y}\end{bmatrix}$, is a continuous solution of SRF($\mathbf{z}$).

2) The above $\mathcal{K}_g$ is analytic, because it is a limit of polynomials of $\mathbf{z}$. Hence, by using the same inductive method in 1), the above $\mathcal{L}$ is also analytic.

Next before proving 3) we describe the terms in the right-hand sides of (16) and (17) in detail as the next lemma, which is easy to prove by Proposition 3.1 and Lemma 3.1.

Lemma 3.2. *For ω in Ω_g, g in G and n in $\mathbb{N}$ we have*

$$\begin{aligned}(\mathbf{D}_n\mathcal{L}_{\mathbf{v}})_g(\omega)&=\textstyle\sum_{\boldsymbol{i}\in S_g^{*n}}\mathbf{v}_g(\boldsymbol{i})1_{I_{(g,\boldsymbol{i})}}(\omega)\\&=\textstyle\sum_{\boldsymbol{i}\in S_g^{*n}}\prod_{(h,j)\in G^{\triangleright}}\mathbf{v}_h(j)^{s((h,j);(g,\boldsymbol{i}))}1_{I_{(g,\boldsymbol{i})}}(\omega),\\(\mathbf{U}_n\mathcal{V})_g(\omega)&=\textstyle\sum_{\boldsymbol{i}\in S_g^{*n}}\sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}}\sum_{0\leq k<j}\mathbf{v}_{\tau(g,\boldsymbol{i})}(k)1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{i})}}(\omega),\\(\mathbf{U}_n\mathcal{W})_g(\omega)&=\textstyle\sum_{\boldsymbol{i}\in S_g^{*n}}\sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}}\{\sum_{0\leq k<j}\mathbf{w}_{\tau(g,\boldsymbol{i})}(k)\\&\quad+\mathbf{w}_{\tau(g,\boldsymbol{i})}(j)\mathcal{L}_{\mathbf{z},\tau(g,\boldsymbol{i}j)}(\varphi^{n+1}(\omega))\}1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{i})}}(\omega),\end{aligned}$$

and also if $n > \mathfrak{t}(\omega)$ *then we have*

$$\begin{aligned}(\mathbf{U}_n\mathcal{V})_g(\omega) &= (\mathbf{U}_n\mathcal{W})_g(\omega) = \mathbf{0},\\ (\mathbf{U}_n\mathcal{L}_{\mathbf{z}})_g(\omega) &= \mathfrak{e},\\ (\mathbf{D}_n\mathcal{L}_{\mathbf{v}})_g(\omega) &= (\mathbf{D}_{\mathfrak{t}(\omega)}\mathcal{L}_{\mathbf{v}})_g(\omega).\end{aligned}$$

Moreover, we get the inequalities:

$$\begin{aligned}\|(\mathbf{D}_n\mathcal{L}_{\mathbf{v}})_g(\omega)\| &\le C(\mathfrak{s}(n,\omega),\rho(\mathbf{z}),0,d),\\ \|(\mathbf{U}_n\mathcal{V})_g(\omega)\| &\le C(\mathfrak{s}(n,\omega),\rho(\mathbf{z}),0,d)\cdot \max_{h\in G} m(h),\\ \|(\mathbf{U}_n\mathcal{W})_g(\omega)\| &\le d\cdot\eta(\mathbf{z})\cdot(\max_{h\in G} m(h) + \|\mathcal{L}_{\mathbf{z}}\|).\end{aligned}$$

3) Since $\mathcal{L}_{\mathbf{z}}$ is a continuous solution of SRF($\mathbf{z}$), we get

$$\begin{aligned}\mathcal{L}_{\mathbf{z},g}(\omega) &= \textstyle\sum_{j\in S_g^{*1}}\{\sum_{0\le k<j}\mathbf{z}_g(k) + \mathbf{z}_g(j)\mathcal{L}_{\mathbf{z},\tau(g,j)}(\varphi(\omega))\}1_{I_{(g,j)}}(\omega)\\ &= \mathcal{V}_g(\omega) + \mathcal{W}_g(\omega) + \textstyle\sum_{j\in S_g^{*1}}\mathbf{v}_g(j)\mathcal{L}_{\mathbf{z},\tau(g,j)}(\varphi(\omega))1_{I_{(g,j)}}(\omega)\\ &= \mathcal{V}_g(\omega) + \mathcal{W}_g(\omega) + (\mathbf{D}_1\mathcal{L}_{\mathbf{v}})_g(\omega)(\mathbf{U}_1\mathcal{L}_{\mathbf{z}})_g(\omega)\end{aligned}$$

for ω in Ω_g and g in G, that means

$$\begin{aligned}\mathcal{L}_{\mathbf{z}} &= \mathcal{V} + \mathcal{W} + (\mathbf{D}_1\mathcal{L}_{\mathbf{v}})(\mathbf{U}_1\mathcal{L}_{\mathbf{z}})\\ &= (\mathbf{D}_0\mathcal{L}_{\mathbf{v}})(\mathbf{U}_0\mathcal{V}) + (\mathbf{D}_0\mathcal{L}_{\mathbf{v}})(\mathbf{U}_0\mathcal{W}) + (\mathbf{D}_1\mathcal{L}_{\mathbf{v}})(\mathbf{U}_1\mathcal{L}_{\mathbf{z}}).\end{aligned}$$

By using Proposition 3.1 we get

$$\mathbf{U}_1\mathcal{L}_{\mathbf{z}} = (\mathbf{U}_1\mathbf{D}_0\mathcal{L}_{\mathbf{v}})(\mathbf{U}_1\mathcal{V}) + (\mathbf{U}_1\mathbf{D}_0\mathcal{L}_{\mathbf{v}})(\mathbf{U}_1\mathcal{W}) + (\mathbf{U}_1\mathbf{D}_1\mathcal{L}_{\mathbf{v}})(\mathbf{U}_2\mathcal{L}_{\mathbf{z}}),$$

and so we get

$$(\mathbf{D}_1\mathcal{L}_{\mathbf{v}})(\mathbf{U}_1\mathcal{L}_{\mathbf{z}}) = (\mathbf{D}_1\mathcal{L}_{\mathbf{v}})(\mathbf{U}_1\mathcal{V}) + (\mathbf{D}_1\mathcal{L}_{\mathbf{v}})(\mathbf{U}_1\mathcal{W}) + (\mathbf{D}_2\mathcal{L}_{\mathbf{v}})(\mathbf{U}_2\mathcal{L}_{\mathbf{z}}).$$

By repeating the above we get

$$\mathcal{L}_{\mathbf{z}} = \textstyle\sum_{0\le k\le n}\{(\mathbf{D}_k\mathcal{L}_{\mathbf{v}})(\mathbf{U}_k\mathcal{V}) + (\mathbf{D}_k\mathcal{L}_{\mathbf{v}})(\mathbf{U}_k\mathcal{W})\} + (\mathbf{D}_{n+1}\mathcal{L}_{\mathbf{v}})(\mathbf{U}_{n+1}\mathcal{L}_{\mathbf{z}}).$$

Hence (16) and (17) are obtained by Lemma 3.2. □

Finally we shall add a few properties on SRF($\mathbf{z}$).

Theorem 3.2. *Let* $\mathbf{z} : G^{\triangleright} \longrightarrow \mathsf{M}(d,\mathbb{C})$ *with* $\sum_{k\in S_g^{*1}} \mathbf{z}_g(k) = \mathfrak{e}$ *for* g *in* G*, and* $\mathcal{L}$ *be in* $\Gamma(\coprod_{g\in G} \mathcal{C}(\Omega_g, \mathsf{M}(d,\mathbb{C})))$*. Then the followings* 1), 2) *and* 3) *are equivalent.*

1) $\mathcal{L}$ *is a solution of* $\mathrm{SRF}(\mathbf{z})$*.*

2) $\mathcal{L}$ *satisfies the following system of difference equations:*

$$\begin{cases} \mathcal{L}_g(\overline{m}_g-) = \mathfrak{e}, \\ \mathcal{L}_g(\sigma_{(g,\boldsymbol{i}j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-)) = \mathcal{L}_g(\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})-)\sum_{j<k<m(\tau(g,\boldsymbol{i}))} \mathbf{z}_{\tau(g,\boldsymbol{i})}(k) \\ \qquad + \mathcal{L}_g(\sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-))\sum_{0\le k\le j} \mathbf{z}_{\tau(g,\boldsymbol{i})}(k) \\ \qquad (\boldsymbol{i}\in S_g^{*n}, j \in S^{*1}_{\tau(g,\boldsymbol{i})}, n\in\mathbb{N}) \end{cases} \tag{18}$$

for g *in* G*.*

3) $\mathcal{L}$ *satisfies the following equations:*

$$\begin{cases} (\mathbf{D}_0\mathcal{L})_g(\omega) = \mathfrak{e} & (\omega\in\Omega_g, g\in G), \\ (\mathbf{D}_1\mathcal{L})_g(\omega) = \sum_{j\in S_g^{*1}} \mathbf{z}_g(j) 1_{I_{(g,j)}}(\omega) & (\omega\in\Omega_g, g\in G), \\ \mathbf{D}_{n+1}\mathcal{L} = (\mathbf{D}_n\mathcal{L})(\mathbf{U}_n\mathbf{D}_1\mathcal{L}) & (n\in\mathbb{N}). \end{cases} \tag{19}$$

Proof. In Proposition 3.1,4) we already proved that 1) implies 3).

By calculating according to the definitions of $\mathbf{D}_n$ and $\mathbf{U}_n$ we get

$$(\mathbf{D}_0\mathcal{L})_g(\omega) = \mathcal{L}_g(\overline{m}_g-),$$
$$(\mathbf{D}_{n+1}\mathcal{L})_g(\sigma_{(g,\boldsymbol{i}j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-)) = \mathcal{L}_g(\sigma_{(g,\boldsymbol{i}j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-)) - \mathcal{L}_g(\sigma_{(g,\boldsymbol{i}j)}(\overline{0}_{\tau(g,\boldsymbol{i}j)})-),$$
$$(\mathbf{D}_n\mathcal{L})_g(\sigma_{(g,\boldsymbol{i}j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-)) = \mathcal{L}_g(\sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-)) - \mathcal{L}_g(\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})-)$$

and

$$(\mathbf{U}_n\mathbf{D}_1\mathcal{L})_g(\sigma_{(g,\boldsymbol{i}j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-)) = (\mathbf{D}_1\mathcal{L})_{\tau(g,\boldsymbol{i})}(\sigma_{(\tau(g,\boldsymbol{i}),j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-))$$

for g in G, $\boldsymbol{i}$ in S_g^{*n}, j in $S^{*1}_{\tau(g,\boldsymbol{i})}$ and n in $\mathbb{N}$. Then we assume 3) and we get $\mathcal{L}_g(\overline{m}_g-) = \mathfrak{e}$ and

$$\begin{aligned} &\mathcal{L}_g(\sigma_{(g,\boldsymbol{i}j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-)) - \mathcal{L}_g(\sigma_{(g,\boldsymbol{i}j)}(\overline{0}_{\tau(g,\boldsymbol{i}j)})-) \\ &\qquad = \{\mathcal{L}_g(\sigma_{(g,\boldsymbol{i})}(\overline{m}_{\tau(g,\boldsymbol{i})}-)) - \mathcal{L}_g(\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})-)\}\mathbf{z}_{\tau(g,\boldsymbol{i})}(j) \end{aligned}$$

because of $(\mathbf{D}_1\mathcal{L})_{\tau(g,\boldsymbol{i})}(\sigma_{(\tau(g,\boldsymbol{i}),j)}(\overline{m}_{\tau(g,\boldsymbol{i}j)}-)) = \mathbf{z}_{\tau(g,\boldsymbol{i})}(j)$. We then notice that $\sigma_{(g,\boldsymbol{i}j)}(\overline{0}_{\tau(g,\boldsymbol{i}j)})- = \sigma_{(g,\boldsymbol{i}j)}(\overline{m}_{\tau(g,\boldsymbol{i}(j-1))}-)$ for $j>0$ and $\sigma_{(g,\boldsymbol{i}0)}(\overline{0}_{\tau(g,\boldsymbol{i}0)}) = \sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})$, and so we get 2).

Finally we shall show that 2) implies 1). It is sufficient to prove the equation (5) for ω in Ω_g such that $\mathfrak{t}^-(\omega) < \infty$ and g in G. It is clear for ω with $\mathfrak{t}^-(\omega) \leq 1$ and so we assume the induction assumption, that is, we assume (5) for ω with $\mathfrak{t}^-(\omega) \leq n$ and g in G. Let ω in Ω_g, g in G and $\mathfrak{t}^-(\omega) = n+1$, and set $j = \pi_{g,1}(\omega)$, $\boldsymbol{i} = \boldsymbol{\pi}_{\tau(g,j),n-1}(\varphi(\omega))$ and $\ell = \pi_{g,n+1}(\omega)$. Since $\omega = \sigma_{(g,j\boldsymbol{i}\ell)}(\overline{m}_{\tau(g,j\boldsymbol{i}\ell)}-)$ and $\varphi(\omega) = \sigma_{(\tau(g,j),\boldsymbol{i}\ell)}(\overline{m}_{\tau(g,j\boldsymbol{i}\ell)}-)$, the equation (18) implies

$$\begin{aligned}\mathcal{L}_g(\omega) = \mathcal{L}_g(\sigma_{(g,j\boldsymbol{i})}(\overline{0}_{\tau(g,j\boldsymbol{i})})-)\textstyle\sum_{\ell<k<m(\tau(g,j\boldsymbol{i}))}\mathbf{z}_{\tau(g,j\boldsymbol{i})}(k) \\ + \mathcal{L}_g(\sigma_{(g,j\boldsymbol{i})}(\overline{m}_{\tau(g,j\boldsymbol{i})}-))\textstyle\sum_{0\leq k\leq \ell}\mathbf{z}_{\tau(g,j\boldsymbol{i})}(k)\end{aligned} \tag{20}$$

and

$$\begin{aligned}\mathcal{L}_{\tau(g,j)}(\varphi(\omega)) = \mathcal{L}_{\tau(g,j)}(\sigma_{(\tau(g,j),\boldsymbol{i})}(\overline{0}_{\tau(g,j\boldsymbol{i})})-)\textstyle\sum_{\ell<k<m(\tau(g,j\boldsymbol{i}))}\mathbf{z}_{\tau(g,j\boldsymbol{i})}(k) \\ + \mathcal{L}_{\tau(g,j)}(\sigma_{(\tau(g,j),\boldsymbol{i})}(\overline{m}_{\tau(g,j\boldsymbol{i})}-))\textstyle\sum_{0\leq k\leq \ell}\mathbf{z}_{\tau(g,j\boldsymbol{i})}(k).\end{aligned} \tag{21}$$

The induction assumption also implies

$$\begin{aligned}&\mathcal{L}_g(\sigma_{(g,j\boldsymbol{i})}(\overline{m}_{\tau(g,j\boldsymbol{i})}-)) \\ &\quad = \textstyle\sum_{0\leq k<j}\mathbf{z}_g(k) + \mathbf{z}_g(j)\mathcal{L}_{\tau(g,j)}(\varphi(\sigma_{(g,j\boldsymbol{i})}(\overline{m}_{\tau(g,j\boldsymbol{i})}-))),\end{aligned} \tag{22}$$

because of $\sigma_{(g,j\boldsymbol{i})}(\overline{m}_{\tau(g,j\boldsymbol{i})}-)$ in $I_{(g,j)}$. Now we substitute (22) for (20) and, by combining with (21), we have

$$\begin{aligned}&\mathcal{L}_g(\omega) - \textstyle\sum_{0\leq k<j}\mathbf{z}_g(k) - \mathbf{z}_g(j)\mathcal{L}_{\tau(g,j)}(\varphi(\omega)) \\ &= \{\mathcal{L}_g(\sigma_{(g,j\boldsymbol{i})}(\overline{0}_{\tau(g,j\boldsymbol{i})})-) - \textstyle\sum_{0\leq k<j}\mathbf{z}_g(k) \\ &\qquad - \mathbf{z}_g(j)\mathcal{L}_{\tau(g,j)}(\sigma_{(\tau(g,j),\boldsymbol{i})}(\overline{0}_{\tau(g,j\boldsymbol{i})})-)\}\textstyle\sum_{\ell<k<m(\tau(g,j\boldsymbol{i}))}\mathbf{z}_{\tau(g,j\boldsymbol{i})}(k).\end{aligned} \tag{23}$$

Since $\varphi(\sigma_{(g,j\boldsymbol{i})}(\overline{0}_{\tau(g,j\boldsymbol{i})})-) = \sigma_{(\tau(g,j),\boldsymbol{i})}(\overline{0}_{\tau(g,j\boldsymbol{i})})-$ and $\mathfrak{t}^-(\sigma_{(g,j\boldsymbol{i})}(\overline{0}_{\tau(g,j\boldsymbol{i})})-) \leq n$ if $\mathfrak{t}^+(\sigma_{(g,j\boldsymbol{i})}(\overline{0}_{\tau(g,j\boldsymbol{i})})-) \geq 2$, the first factor of the right-hand side of (23) is $\mathbf{0}$ and so we complete the proof. □

Proposition 3.2. *Suppose the same assumption in* Theorem 3.1. *Then we have the following.*

1) *If g in G and $\mathfrak{t}^-(\overline{0}_g) = \infty$, then $\mathcal{L}_{\mathbf{z},g}(\overline{0}_g) = \mathbf{0}$.*
2) *If ω in Ω_g, $\mathfrak{t}^+(\omega) < \infty$ and $\mathfrak{t}^-(\omega) = \infty$, then $\mathcal{L}_{\mathbf{z},g}(\omega) = \mathcal{L}_{\mathbf{z},g}(\omega-)$.*

Proof. By using n times (5), we get

$$\mathcal{L}_{\mathbf{z},g}(\overline{0}_g) = \mathbf{z}_g(0\dots0)\mathcal{L}_{\mathbf{z},\tau(g,0\dots0)}(\overline{0}_{\tau(g,0\dots0)}),$$

where $0\dots0 \in S_g^{*n}$. By Lemma 3.1

$$\begin{aligned}\|\mathcal{L}_{\mathbf{z},g}(\overline{0}_g)\| &\leq \|\mathbf{z}_g(0\dots0)\|\|\mathcal{L}_{\mathbf{z},\tau(g,0\dots0)}(\overline{0}_{\tau(g,0\dots0)})\| \\ &\leq C(\mathfrak{s}(n,\overline{0}_g),\rho(\mathbf{z}),\eta(\mathbf{z}),d)\|\mathcal{L}_{\mathbf{z}}\|,\end{aligned}$$

and so if $\mathfrak{t}^-(\overline{0}_g)=\infty$ then $\|\mathcal{L}_{\mathbf{z},g}(\overline{0}_g)\|=0$ by Proposition 2.1, that is , 1) is obtained.

Suppose that ω in Ω_g, $\mathfrak{t}^+(\omega)<\infty$ and $\mathfrak{t}^-(\omega)=\infty$. By Lemma 2.2 $\mathcal{L}_{\mathbf{z},g}(\omega)-\mathcal{L}_{\mathbf{z},g}(\omega-) = \mathbf{z}_g(\boldsymbol{i})\mathcal{L}_{\mathbf{z},\tau(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})$ when $\boldsymbol{i}$ in $S_g^{*\mathfrak{t}^+(\omega)}$, and by 1) $\mathcal{L}_{\mathbf{z},\tau(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})=\mathbf{0}$ because of $\mathfrak{t}^-(\overline{0}_{\tau(g,\boldsymbol{i})})=\infty$. Hence 2) is obtained. □

Proposition 3.3. *Let $\mathcal{L}_{\mathbf{z}}$ be a solution of* SRF($\mathbf{z}$) *and $X_g=\mathcal{L}_{\mathbf{z},g}(\Omega_g)$, and define the mappings $\Phi_{(g,j)}:\mathsf{M}(d,\mathbb{C})\longrightarrow\mathsf{M}(d,\mathbb{C})$ by*

$$\Phi_{(g,j)}(\mathbf{w}) = \textstyle\sum_{0\leq k<j}\mathbf{z}_g(k)+\mathbf{z}_g(j)\mathbf{w} \qquad (\mathbf{w}\in\mathsf{M}(d,\mathbb{C}))$$

for (g,j) in $G^{\triangleright}$. Then each X_g is compact and satisfies

$$X_g = \textstyle\bigcup_{j\in S_g^{*1}}\Phi_{(g,j)}(X_{\tau(g,j)}) \tag{24}$$

for g in G.

Proof. Since $\mathcal{L}_{\mathbf{z}}$ is a solution of SRF($\mathbf{z}$), we have

$$\begin{aligned}\mathcal{L}_{\mathbf{z},g}(I_{(g,j)}) &= \textstyle\sum_{0\leq k<j}\mathbf{z}_g(k)+\mathbf{z}_g(j)\mathcal{L}_{\mathbf{z},\tau(g,j)}(\varphi(I_{(g,j)})) \\ &= \Phi_{(g,j)}(X_{\tau(g,j)})\end{aligned}$$

for j in S_g^{*1} and g in G, and so we obtain (24) because of $\mathcal{L}_{\mathbf{z},g}(\Omega_g) = \bigcup_{j\in S_g^{*1}}\mathcal{L}_{\mathbf{z},g}(I_{(g,j)})$. □

4. Representations of differentials of L_{z} with respect to the parameter z

In this section, we suppose that $\mathsf{z}:G^{\triangleright}\longrightarrow\mathbb{C}$ satisfy $\sum_{j\in S_g^{*1}}\mathsf{z}_g(j)=1$ and $\rho(\mathsf{z})<1$. Then there exists a unique continuous solution of SRF(z), that is denoted by L_{z}, because of Theorem 3.1,1). We use the notations:

$$\partial_g(j) = \frac{\partial}{\partial\mathsf{z}_g(j)} \qquad ((g,j)\in G^{\triangleright}),$$

$$\partial^{\mathsf{p}} = \prod_{(g,j)\in G^{\rhd}, j<m(g)-1} \partial_g(j)^{\mathsf{p}_g(j)} \qquad (\mathsf{p} \in \mathsf{Map}(G^{\rhd}, \mathbb{N})),$$
$$\mathsf{a}\cdot\partial = \sum_{(g,j)\in G^{\rhd}, j<m(g)-1} \mathsf{a}_g(j)\partial_g(j) \qquad (\mathsf{a} \in \mathsf{Map}(G^{\rhd}, \mathbb{C})),$$
$$\mathsf{L}^{(\mathsf{p})}_{\mathsf{z},g}(\omega) = \partial^{\mathsf{p}}\mathsf{L}_{\mathsf{z},g}(\omega) \qquad (\mathsf{p} \in \mathsf{Map}(G^{\rhd}, \mathbb{N}),\ \omega\in\Omega_g,\ g\in G),$$
$$\mathsf{p}! = \prod_{(g,j)\in G^{\rhd}, j<m(g)-1} \mathsf{p}_g(j)! \qquad (\mathsf{p} \in \mathsf{Map}(G^{\rhd}, \mathbb{N})),$$
$$|\mathsf{p}| = \sum_{(g,j)\in G^{\rhd}, j<m(g)-1} \mathsf{p}_g(j),$$

where we do not need the notations $\partial_g(m(g)-1)$, because $\mathsf{z}_g(m(g)-1) = 1-\sum_{0\le j<m(g)-1}\mathsf{z}_g(j)$. By using the special upper triangle matrices as the weighted parameter, we shall show that Theorem 3.1 implies the representations of differentiation of L_{z} with respect to the parameter z.

Theorem 4.1. *Let* $\mathsf{a}:G^{\rhd}\longrightarrow\mathbb{C}$ *satisfy* $\sum_{j\in S^{*1}_g}\mathsf{a}_g(j)=0$, *and let* $\mathbf{z}:G^{\rhd}\longrightarrow \mathsf{M}(d,\mathbb{C})$ *be defined by* $\mathbf{z}_g(j)=\mathsf{z}_g(j)\mathfrak{e}+\mathsf{a}_g(j)\mathfrak{n}$ *for* (g,j) *in* $G^{\rhd}$, *where* $\mathfrak{n}$ *is the* $d\times d$*-matrix* $(n_{k,l})$ *such that* $n_{k,l}=0$ *except* $n_{i,i+1}=1$ $(i=1,\dots,d-1)$. *Then we have*

1) *The continuous solution* $\mathcal{L}_{\mathbf{z}}$ *of* $\mathrm{SRF}(\mathbf{z})$ *is given by*

$$\mathcal{L}_{\mathbf{z},g} = \sum_{0\le q<d}\frac{(\mathsf{a}\cdot\partial)^q}{q!}\mathsf{L}_{\mathsf{z},g}\mathfrak{n}^q \qquad (g\in G). \tag{25}$$

2) *The differentials of* L_{z} *are expanded as follows:*

$$(\mathsf{a}\cdot\partial)^q\mathsf{L}_{\mathsf{z},g}(\omega) = q!\sum_{n\in\mathbb{N}}\sum_{\boldsymbol{i}\in S^{*n}_g}\prod_{(h,l)\in G^{\rhd}}\mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))} \times\mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),q}(\mathsf{a})\circ\varphi^n(\omega)1_{I_{(g,\boldsymbol{i})}}(\omega) \qquad (q\in\mathbb{N}), \tag{26}$$
$$\frac{1}{\mathsf{p}!}\mathsf{L}^{(\mathsf{p})}_{\mathsf{z},g}(\omega) = \sum_{n\in\mathbb{N}}\sum_{\boldsymbol{i}\in S^{*n}_g}\prod_{(h,l)\in G^{\rhd}}\mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))} \times\mathsf{B}_{\mathsf{z},\tau(g,\boldsymbol{i}),\mathsf{p}}\circ\varphi^n(\omega)1_{I_{(g,\boldsymbol{i})}}(\omega) \quad (\mathsf{p}\in\mathsf{Map}(G^{\rhd},\mathbb{N})) \tag{27}$$

uniformly on $\{\omega\in\Omega_g : \mathfrak{t}(\omega)=\infty\}$ *for* g *in* G, *and*

$$\mathsf{L}_{\mathsf{z},g}(\omega) = \sum_{0\le n\le\mathfrak{t}(\omega)}\sum_{\boldsymbol{i}\in S^{*n}_g}\prod_{(h,l)\in G^{\rhd}}\mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))} \times\mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),0}(\mathsf{a})\circ\varphi^n(\omega)1_{I_{(g,\boldsymbol{i})}}(\omega) + \sum_{\boldsymbol{i}\in S^{*\mathfrak{t}(\omega)}_g}\prod_{(h,l)\in G^{\rhd}}\mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))}, \tag{28}$$
$$(\mathsf{a}\cdot\partial)^q\mathsf{L}_{\mathsf{z},g}(\omega) = q!\sum_{0\le n\le\mathfrak{t}(\omega)}\sum_{\boldsymbol{i}\in S^{*n}_g}\prod_{(h,l)\in G^{\rhd}}\mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))} \times\mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),q}(\mathsf{a})\circ\varphi^n(\omega)1_{I_{(g,\boldsymbol{i})}}(\omega) \qquad (q\in\mathbb{N}^+), \tag{29}$$

$$\tfrac{1}{\mathsf{p}!}\mathsf{L}^{(\mathsf{p})}_{\mathsf{z},g}(\omega)=\sum_{0\le n\le \mathsf{t}(\omega)}\sum_{\boldsymbol{i}\in S_g^{*n}}\prod_{(h,l)\in G^{\triangleright}}\mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))}$$
$$\times \mathsf{B}_{\mathsf{z},\tau(g,\boldsymbol{i}),\mathsf{p}}\circ\varphi^n(\omega)1_{I_{(g,\boldsymbol{i})}}(\omega)\quad (\mathsf{p}\in\mathsf{Map}(G^{\triangleright},\mathbb{N}),|\mathsf{p}|>0)\quad (30)$$

for ω in Ω_g with $\mathsf{t}(\omega)<\infty$ and g in G, where

$$\mathsf{A}_{\mathsf{z},g,q}(\mathsf{a})(\omega)=\begin{cases}\sum_{j\in S_g^{*1}}\sum_{0\le k<j}\mathsf{z}_g(k)1_{I_{(g,j)}}(\omega) & (q=0),\\ \sum_{j\in S_g^{*1}}\Big(\sum_{0\le k<j}\mathsf{a}_g(k) & \\ \qquad +\mathsf{a}_g(j)\mathsf{L}_{\mathsf{z},\tau(g,j)}\circ\varphi(\omega)\Big)1_{I_{(g,j)}}(\omega) & (q=1),\\ \sum_{j\in S_g^{*1}}\mathsf{a}_g(j)\frac{(\mathsf{a}\cdot\partial)^{q-1}}{(q-1)!}\mathsf{L}_{\mathsf{z},\tau(g,j)}\circ\varphi(\omega)1_{I_{(g,j)}}(\omega) & (q\ge 2)\end{cases}$$

and

$$\mathsf{B}_{\mathsf{z},g,\mathsf{p}}(\omega)$$
$$=\begin{cases}\sum_{j\in S_g^{*1}}\sum_{0\le k<j}\mathsf{z}_g(k)1_{I_{(g,j)}}(\omega) & (|\mathsf{p}|=0),\\ \sum_{0\le j<m(g)-1}\mathsf{p}_g(j)\big\{\mathsf{L}_{\mathsf{z},\tau(g,j)}\circ\varphi(\omega)1_{I_{(g,j)}}(\omega) & \\ \qquad +\sum_{j<k<m(g)-1}1_{I_{(g,k)}}(\omega) & \\ \qquad +(1-\mathsf{L}_{\mathsf{z},\tau(g,m(g)-1)}\circ\varphi(\omega))1_{I_{(g,m(g)-1)}}(\omega)\big\} & (|\mathsf{p}|=1),\\ \sum_{0\le j<m(g)-1}\mathsf{p}_g(j)\frac{\partial^{\mathsf{p}-\delta(g,j)}}{(\mathsf{p}-\delta(g,j))!}\big\{\mathsf{L}_{\mathsf{z},\tau(g,j)}\circ\varphi(\omega)1_{I_{(g,j)}}(\omega) & \\ \qquad -\mathsf{L}_{\mathsf{z},\tau(g,m(g)-1)}\circ\varphi(\omega)1_{I_{(g,m(g)-1)}}(\omega)\big\} & (|\mathsf{p}|\ge 2)\end{cases}$$

for ω in Ω_g, where $\delta(g,j)$ is in $\mathsf{Map}(G^{\triangleright},\mathbb{N})$ such that $\delta(g,j)_h(l)=0$ for $(h,l)\in G^{\triangleright}$ and $0\le l<m(h)-1$ without $\delta(g,j)_g(j)=1$.

Proof. 1) We apply the Leibniz formula to the equation

$$\mathsf{L}_{\mathsf{z},g}(\omega)=\sum_{0\le k<j}\mathsf{z}_g(k)+\mathsf{z}_g(j)\mathsf{L}_{\mathsf{z},\tau(g,j)}(\varphi(\omega))$$

by using $\sum_{i\in S_g^1}\mathsf{z}_g(i)=1$ and $\sum_{i\in S_g^1}\mathsf{a}_g(i)=0$ for g in G, and get

$$(\mathsf{a}\cdot\partial)\mathsf{L}_{\mathsf{z},g}(\omega)=\sum_{0\le k<j}\mathsf{a}_g(k)+\mathsf{z}_g(j)(\mathsf{a}\cdot\partial)\mathsf{L}_{\mathsf{z},\tau(g,j)}(\varphi(\omega))+\mathsf{a}_g(j)\mathsf{L}_{\mathsf{z},\tau(g,j)}(\varphi(\omega)),$$

and

$$\tfrac{(\mathsf{a}\cdot\partial)^n}{n!}\mathsf{L}_{\mathsf{z},g}(\omega)=\mathsf{z}_g(j)\tfrac{(\mathsf{a}\cdot\partial)^n}{n!}\mathsf{L}_{\mathsf{z},\tau(g,j)}(\varphi(\omega))+\mathsf{a}_g(j)\tfrac{(\mathsf{a}\cdot\partial)^{n-1}}{(n-1)!}\mathsf{L}_{\mathsf{z},\tau(g,j)}(\varphi(\omega)),$$

where $n \geq 2$ and $\omega \in I_{(g,j)}$. Hence we can verify that $\mathcal{L}_{\mathbf{z}}$ defined by (25) is a solution of SRF($\mathbf{z}$) as follows:

$$\begin{aligned}
&\textstyle\sum_{0\le k<j} \mathbf{z}_g(k) + \mathbf{z}_g(j)\mathcal{L}_{\mathbf{z},\tau(g,j)}(\varphi(\omega)) \\
&= \textstyle\sum_{0\le k<j} \big(\mathsf{z}_g(k)\mathfrak{e} + \mathsf{a}_g(k)\mathfrak{n}\big) + \big(\mathsf{z}_g(j)\mathfrak{e} + \mathsf{a}_g(j)\mathfrak{n}\big) \sum_{0\le q<d} \frac{(\mathsf{a}\cdot\partial)^q}{q!} \mathsf{L}_{\mathsf{z},\tau(g,j)}\big(\varphi(\omega)\big)\mathfrak{n}^q \\
&= \big\{\textstyle\sum_{0\le k<j} \mathsf{z}_g(k) + \mathsf{z}_g(j)\mathsf{L}_{\mathsf{z},\tau(g,j)}\big(\varphi(\omega)\big)\big\}\mathfrak{e} \\
&\qquad + \big\{\textstyle\sum_{0\le k<j} \mathsf{a}_g(k) + \mathsf{z}_g(j)(\mathsf{a}\cdot\partial)\mathsf{L}_{\mathsf{z},\tau(g,j)}\big(\varphi(\omega)\big) + \mathsf{a}_g(j)\mathsf{L}_{\mathsf{z},\tau(g,j)}\big(\varphi(\omega)\big)\big\}\mathfrak{n} \\
&\qquad + \textstyle\sum_{2\le q<d} \big\{\mathsf{z}_g(j)\frac{(\mathsf{a}\cdot\partial)^q}{q!}\mathsf{L}_{\mathsf{z},\tau(g,j)}\big(\varphi(\omega)\big) + \mathsf{a}_g(j)\frac{(\mathsf{a}\cdot\partial)^{q-1}}{(q-1)!}\mathsf{L}_{\mathsf{z},\tau(g,j)}\big(\varphi(\omega)\big)\big\}\mathfrak{n}^q \\
&= \textstyle\sum_{0\le q<d} \frac{(\mathsf{a}\cdot\partial)^q}{q!}\mathsf{L}_{\mathsf{z},g}(\omega)\mathfrak{n}^q \\
&= \mathcal{L}_{\mathbf{z},g}(\omega)
\end{aligned}$$

for g in G and ω in Ω_g.

2) The notations $\mathbf{v}$ and $\mathbf{w}$ in Theorem 3.1 are given by

$$\mathbf{v}_g(j) = \mathsf{z}_g(j)\mathfrak{e}, \quad \mathbf{w}_g(j) = \mathsf{a}_g(j)\mathfrak{n} \qquad ((g,j) \in G^{\triangleright})$$

for $\mathbf{z}$ in this case, and by Lemma 3.2 we get

$$\begin{aligned}
&(\mathbf{D}_n\mathcal{L}_{\mathbf{v}})_g(\omega) = \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \prod_{(h,l)\in G^{\triangleright}} \mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))}\mathfrak{e}1_{I_{(g,\boldsymbol{i})}}(\omega), \\
&(\mathbf{U}_n\mathcal{V})_g(\omega) \\
&= \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}} \sum_{0\le k<j} \mathsf{z}_{\tau(g,\boldsymbol{i})}(k)\mathfrak{e}1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{i})}}(\omega) \\
&= \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),0}(\mathsf{a})(\varphi^n(\omega))\mathfrak{e}1_{I_{(g,\boldsymbol{i})}}(\omega), \\
&(\mathbf{U}_n\mathcal{W})_g(\omega) \\
&= \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}} \{\sum_{0\le k<j} \mathsf{a}_{\tau(g,\boldsymbol{i})}(k)\mathfrak{n} \\
&\qquad\qquad + \mathsf{a}_{\tau(g,\boldsymbol{i})}(j)\mathfrak{n}\mathcal{L}_{\mathbf{z},\tau(g,\boldsymbol{i}j)}(\varphi^{n+1}(\omega))\}1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{i})}}(\omega) \\
&= \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}} \{\sum_{0\le k<j} \mathsf{a}_{\tau(g,\boldsymbol{i})}(k)\mathfrak{n} \\
&\qquad + \mathsf{a}_{\tau(g,\boldsymbol{i})}(j)\mathfrak{n}\textstyle\sum_{0\le q<d} \frac{(\mathsf{a}\cdot\partial)^q}{q!}\mathsf{L}_{\mathsf{z},\tau(g,\boldsymbol{i}j)}(\varphi^{n+1}(\omega))\mathfrak{n}^q\}1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{i})}}(\omega) \\
&= \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}} \{\sum_{0\le k<j} \mathsf{a}_{\tau(g,\boldsymbol{i})}(k)\mathfrak{n} \\
&\qquad + \mathsf{a}_{\tau(g,\boldsymbol{i})}(j)\textstyle\sum_{1\le q<d} \frac{(\mathsf{a}\cdot\partial)^{q-1}}{(q-1)!}\mathsf{L}_{\mathsf{z},\tau(g,\boldsymbol{i}j)}(\varphi^{n+1}(\omega))\mathfrak{n}^q\}1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{i})}}(\omega) \\
&= \textstyle\sum_{\boldsymbol{i}\in S_g^{*n}} \sum_{j\in S^{*1}_{\tau(g,\boldsymbol{i})}} \big\{\big(\sum_{0\le k<j} \mathsf{a}_{\tau(g,\boldsymbol{i})}(k) + \mathsf{a}_{\tau(g,\boldsymbol{i})}(j)\mathsf{L}_{\mathsf{z},\tau(g,\boldsymbol{i}j)}(\varphi^{n+1}(\omega))\big)\mathfrak{n} \\
&\qquad + \mathsf{a}_{\tau(g,\boldsymbol{i})}(j)\textstyle\sum_{2\le q<d} \frac{(\mathsf{a}\cdot\partial)^{q-1}}{(q-1)!}\mathsf{L}_{\mathsf{z},\tau(g,\boldsymbol{i}j)}(\varphi^{n+1}(\omega))\mathfrak{n}^q\big\}1_{I_{(\tau(g,\boldsymbol{i}),j)}}(\varphi^n(\omega))1_{I_{(g,\boldsymbol{i})}}(\omega)
\end{aligned}$$

$= \sum_{\boldsymbol{i}\in S_g^{*n}} \{\mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),1}(\mathsf{a})(\varphi^n(\omega))\mathfrak{n} + \sum_{2\le q<d}\mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),q}(\mathsf{a})(\varphi^n(\omega))\mathfrak{n}^q\} 1_{I_{(g,\boldsymbol{i})}}(\omega).$

Moreover, by substituting the above $\mathbf{D}_n\mathcal{L}_{\mathbf{v}}$, $\mathbf{U}_n\mathcal{V}$ and $\mathbf{U}_n\mathcal{W}$ for the ones in the equations (16) and (17), we get

$$\mathcal{L}_{\mathbf{z},g}(\omega) = \sum_{n\in\mathbb{N}}\sum_{\boldsymbol{i}\in S_g^{*n}} \prod_{(h,l)\in G^{\triangleright}} \mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))} \{\mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),0}(\mathsf{a})(\varphi^n(\omega))\mathfrak{e} + \mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),1}(\mathsf{a})(\varphi^n(\omega))\mathfrak{n} + \sum_{2\le q<d}\mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),q}(\mathsf{a})(\varphi^n(\omega))\mathfrak{n}^q\} 1_{I_{(g,\boldsymbol{i})}}(\omega)$$

uniformly on $\{\omega\in\Omega_g : \mathfrak{t}(\omega)=\infty\}$, and

$$\mathcal{L}_{\mathbf{z},g}(\omega) = \sum_{0\le n\le \mathfrak{t}(\omega)}\sum_{\boldsymbol{i}\in S_g^{*n}} \prod_{(h,l)\in G^{\triangleright}} \mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))} \{\mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),0}(\mathsf{a})(\varphi^n(\omega))\mathfrak{e} + \mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),1}(\mathsf{a})(\varphi^n(\omega))\mathfrak{n} + \sum_{2\le q<d}\mathsf{A}_{\mathsf{z},\tau(g,\boldsymbol{i}),q}(\mathsf{a})(\varphi^n(\omega))\mathfrak{n}^q\} 1_{I_{(g,\boldsymbol{i})}}(\omega) + \sum_{\boldsymbol{i}\in S_g^{*\mathfrak{t}(\omega)}} \prod_{(h,l)\in G^{\triangleright}} \mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))} \mathfrak{e} 1_{I_{(g,\boldsymbol{i})}}(\omega)$$

for ω in Ω_g with $\mathfrak{t}(\omega) < \infty$. Hence we write out the components of $\mathcal{L}_{\mathbf{z}}$ and get (26), (28) and (29). Finally, by removing the parameter a from (26) and (29) we obtain (27) and (30). □

5. Fractal functions with the parameter space $[0,1]$ on directed networks

We start this section to define the mappings Ψ_g and Ψ_g^{-1+} for translating the functional equations on Ω_g to the ones on $[0,1]$.

Definition 5.1. *Let us set* $\mathbf{z}_g(j) = \frac{1}{m(g)}$ *for* j *in* S_g^{*1} *and* g *in* G. *Since the mapping* $\mathbf{z} : G^{\triangleright} \longrightarrow \mathbb{R}$ *satisfies the assumption of* Theorem 3.1*, the unique continuous solution of* SRF($\mathbf{z}$) *exist, and we denote it by* Ψ. *It is clear that* $\Psi_g(\overline{0}_g-) = 0$, $\Psi_g(\overline{m}_g-) = 1$, $0 \le \Psi_g(\omega) \le 1$ *and*

$$\Psi_g(\omega) = \sum_{n\in\mathbb{N}_+} \frac{j_n}{\prod_{k=1}^n m(g_k)} + \lim_{n\to\infty} \frac{1}{\prod_{k=1}^n m(g_k)}, \tag{31}$$

for g *in* G *and* ω *in* Ω_g, *where* $g_1 = g$, $j_n = \pi_{g,n}(\omega)$ *and* $g_{n+1} = \tau(g_n, j_n)$ *for* n *in* $\mathbb{N}_+$.

Then it is easy to check the following properties of Ψ_g, and so we can define Ψ_g^{-1+}.

Lemma 5.1. *Let x in $[0,1]$ and g in G. Then the set $\Psi_g^{-1}(\{x\})$ has at most two elements, and $\Psi_g^{-1}(\{x\})$ is empty if and only if there exists ω in Ω_g such that $\mathfrak{t}(\omega) < \infty$ and $\Psi_g(\omega-) < x < \Psi_g(\omega)$. In more detail, let ω in Ω_g and $\Psi_g(\omega) = x$, we have*

1) $\Psi_g^{-1}(\{x\}) = \{\omega\}$ *if* $\mathfrak{t}^-(\omega) = \mathfrak{t}^+(\omega) = \infty$,
2) $\Psi_g^{-1}(\{x\}) = \{\omega-, \omega\}$ *if* $\mathfrak{t}^-(\omega) = \infty$ *and* $\mathfrak{t}^+(\omega) < \infty$,
3) $\Psi_g^{-1}(\{x\}) = \{\omega, \omega+\}$ *if* $0 < \mathfrak{t}^-(\omega) < \infty$ *and* $\mathfrak{t}^-(\omega+) = \infty$,
4) $\Psi_g(\omega) = x < \Psi_g(\omega+)$ *if* $0 < \mathfrak{t}^-(\omega) < \infty$ *and* $\mathfrak{t}^-(\omega+) < \infty$.
5) $\omega = \overline{m}_g-$ *and* $\Psi_g(\omega) = 1$ *if* $\mathfrak{t}^-(\omega) = 0$.

Definition 5.2. *Define the mapping $\Psi_g^{-1+} : [0,1] \longrightarrow \Omega_g$ for g in G as follows:*

$$\Psi_g^{-1+}(x) = \begin{cases} \omega & \text{if } \mathfrak{t}(\omega) < \infty \text{ and } \Psi_g(\omega-) \le x < \Psi_g(\omega), \\ \omega & \text{if } \mathfrak{t}^-(\omega) = \infty \text{ and } \Psi_g(\omega) = x, \\ \overline{m}_g- & \text{if } x = 1, \end{cases}$$

where x in $[0,1]$ and ω in Ω_g.

Definition 5.3. *Let g in G, $\boldsymbol{j}$ in S_g^{*n} and n in $\mathbb{N}$. We denote by the set $I_{g,\boldsymbol{j}}^+$ $\{x \in [0,1]; \Psi_g^{-1+}(x) \in I_{(g,\boldsymbol{j})}\} \setminus \{\Psi_g(\sigma_{(g,\boldsymbol{j})}(\overline{m}_{\tau(g,\boldsymbol{j})}-))\}$ for $\sigma_{(g,\boldsymbol{j})}(\overline{m}_{\tau(g,\boldsymbol{j})}-) \neq \overline{m}_g-$, and $\{x \in [0,1]; \Psi_g^{-1+}(x) \in I_{(g,\boldsymbol{j})}\}$ for $\sigma_{(g,\boldsymbol{j})}(\overline{m}_{\tau(g,\boldsymbol{j})}-) = \overline{m}_g-$. Define $\phi_{g,n} : [0,1] \longrightarrow [0,1]$ as follows:*

$$\phi_{g,n}(x) = \begin{cases} \prod_{k=1}^n m(g_k)\big(x - \sum_{1\le k\le n} \frac{j_k}{\prod_{l=1}^k m(g_l)}\big) & \text{for } x \in I_{g,\boldsymbol{j}}^+ \text{ if } n \ge 1, \\ x & \text{for } x \in I_{g,\epsilon}^+ \text{ if } n = 0, \end{cases}$$

where $\boldsymbol{j} = j_1 \dots j_n$ and $g_k = \tau(g, j_1 \dots j_{k-1})$ for $k = 1, \dots, n$. For simplicity we denote $\phi_{g,1}$ by ϕ_g.

Lemma 5.2. *Let g in G and j in S_g^{*1}. We have*

$$I_{g,\epsilon}^+ = [0,1], \tag{32}$$

$$I_{g,j}^+ = \begin{cases} [\frac{j}{m(g)}, \frac{j+1}{m(g)}[& if \ \ 0 \le j < m(g) - 1, \\ [\frac{j}{m(g)}, \frac{j+1}{m(g)}] & if \ \ j = m(g) - 1, \end{cases} \tag{33}$$

$$\Psi_{\tau(g,j)}(\varphi(\omega)) = m(g)(\Psi_g(\omega) - \tfrac{j}{m(g)}) \qquad \text{for } \omega \in I_{(g,j)}, \tag{34}$$

$$\varphi \circ \Psi_g^{-1+}(x) = \Psi_{\tau(g,j)}^{-1+} \circ \phi_g(x) \qquad \text{for } x \in I_{g,j}^+. \tag{35}$$

Proof. (32), (33) and (34) are clear from Definition 5.2 and 5.3. Next we show (35). In the case $x=1$, $j=m(g)-1$ and so

$$\begin{aligned}\varphi\circ\Psi_g^{-1+}(1) &= \varphi(\overline{m}_g-) = \overline{m}_{\tau(g,m(g)-1)}- \\ &= \Psi^{-1+}_{\tau(g,m(g)-1)}(1) = \Psi^{-1+}_{\tau(g,m(g)-1)}\circ\phi_g(1).\end{aligned}$$

If $\omega\in\Omega_g$, $\mathfrak{t}(\omega)<\infty$ and $\Psi_g(\omega-)\le x<\Psi_g(\omega)$, then $\varphi(\omega)\in\Omega_{\tau(g,j)}$, $\mathfrak{t}(\varphi(\omega))<\infty$ and $\Psi_{\tau(g,j)}(\varphi(\omega)-)\le\phi_g(x)<\Psi_{\tau(g,j)}(\varphi(\omega))$ by (34). Hence

$$\varphi\circ\Psi_g^{-1+}(x)=\varphi(\omega)=\Psi^{-1+}_{\tau(g,j)}\circ\phi_g(x).$$

We can check other cases by similar method mentioned above. □

Proposition 5.1. *Let g in G, $\boldsymbol{j}$ in S_g^{*n} and n in $\mathbb{N}_+$. In addition, let $g_1=g$, $\boldsymbol{j}=j_1\dots j_n$ and $g_{k+1}=\tau(g,j_1\dots j_k)$ for $k=1,\dots,n-1$. Suppose x in $I^+_{g,\boldsymbol{j}}$. Then x in $I^+_{g_1,j_1}$ and $\phi_{g_k}\circ\cdots\circ\phi_{g_1}(x)$ in $I^+_{g_{k+1},j_{k+1}}$ $(k=1,\dots,n-1)$. Moreover*

$$\phi_{g,n}=\phi_{g_n}\circ\cdots\circ\phi_{g_1} \qquad \text{on } I^+_{g,\boldsymbol{j}}, \tag{36}$$

$$\varphi^n\circ\Psi_g^{-1+}=\Psi^{-1+}_{\tau(g,\boldsymbol{j})}\circ\phi_{g,n} \qquad \text{on } I^+_{g,\boldsymbol{j}}. \tag{37}$$

Proof. Let x in $I^+_{g,\boldsymbol{j}}$. By using (35), we get $x\in I^+_{g,\boldsymbol{j}}\subset I^+_{g_1,j_1\cdots j_{n-1}}\subset\cdots\subset I^+_{g_1,j_1j_2}\subset I^+_{g_1,j_1}$, and $\phi_{g_1}(x)\in I^+_{g_2,j_2\cdots j_n}\subset\cdots\subset I^+_{g_2,j_2j_3}\subset I^+_{g_2,j_2}$. By repeating the process, $\phi_{g_k}\circ\cdots\circ\phi_{g_1}(x)\in I^+_{g_{k+1},j_{k+1}\cdots j_n}\subset\cdots\subset I^+_{g_{k+1},j_{k+1}}$ for $k=1,\dots,n-1$. Since $\phi_{g_k}(y)=m(g_k)(y-\frac{j_k}{m(g_k)})$ on $I^+_{g_k,j_k}$ for $k=1,\dots,n$, $\phi_{g_n}\circ\cdots\circ\phi_{g_1}(x)=\prod_{k=1}^n m(g_k)(x-\sum_{1\le k\le n}\frac{j_k}{\prod_{l=1}^k m(g_l)})$, and so (36) is obtained.

(37) is also obtained by (35) and (36) as follows:

$$\begin{aligned}\varphi^n\circ\Psi_g^{-1+}(x) &= \varphi^{n-1}\circ\varphi\circ\Psi_g^{-1+}(x)=\varphi^{n-1}\circ\Psi_{g_2}^{-1+}\circ\phi_{g_1}(x)\\ &= \Psi^{-1+}_{\tau(g,\boldsymbol{j})}\circ\phi_{g_n}\circ\cdots\circ\phi_{g_1}\\ &= \Psi^{-1+}_{\tau(g,\boldsymbol{j})}\circ\phi_{g,n}.\end{aligned}$$

□

Proposition 5.2. *Ψ_g^{-1+} has the following properties for each g in G.*

1) *The mapping $\Psi_g^{-1+}:[0,1]\longrightarrow\Omega_g$ is right continuous with left-hand limits.*

2) Ψ_g^{-1+} *is discontinuous at* x_0 *if and only if there exist* $\omega \in \Omega_g$ *such that* (i) $\mathfrak{t}(\omega) < \infty$ *or* (ii) $\mathfrak{t}^-(\omega) = \infty$ *and* $\mathfrak{t}^+(\omega) < \infty$, *and* $\Psi_g(\omega-) = x_0$. *In this case,* $\lim_{x \uparrow x_0} \Psi_g^{-1+}(x) = \omega-$.

Proof. By noticing that $\{I_{(g,j_1 \cdots j_n)} : n \in \mathbb{N}_+\}$ is a basis of neighbourhoods at $\omega = ((g_1, j_1), \dots, (g_n, j_n), \dots)$ in Ω_g, these properties are immediately obtained from Definition 5.2. □

Proposition 5.3. *Let* $\mathcal{L}_{\mathbf{z}}$ *be the continuous solution of* $\mathrm{SRF}(\mathbf{z})$. *Define* $L_{\mathbf{z}}$ *by*

$$L_{\mathbf{z},g}(x) = \mathcal{L}_{\mathbf{z},g} \circ \Psi_g^{-1+}(x) \qquad \textit{for } x \in [0,1] \textit{ and } g \in G. \tag{38}$$

Then $L_{\mathbf{z}}$ *has the following properties.*

1) *The mapping* $x \longmapsto L_{\mathbf{z},g}(x)$ *is* $\mathsf{M}(d, \mathbb{C})$*-valued right continuous with left-hand limits on* $[0,1]$ *for each* g *in* G.

2) *If there is* ω *in* Ω_g *with* $\Psi_g(\omega-) = x_0$ *such that* (i) $\mathfrak{t}(\omega) < \infty$ *or* (ii) $\mathfrak{t}^-(\omega) = \infty$ *and* $\mathfrak{t}^+(\omega) < \infty$, *then* $L_{\mathbf{z},g}(x_0) - L_{\mathbf{z},g}(x_0-) = \mathcal{L}_{\mathbf{z},g}(\omega) - \mathcal{L}_{\mathbf{z},g}(\omega-)$, *where* $L_{\mathbf{z},g}(x_0-) = \lim_{y \nearrow x_0} L_{\mathbf{z},g}(y)$ *for* $x_0 > 0$ *and* $L_{\mathbf{z},g}(0-) = 0$.

3) $L_{\mathbf{z}}$ *satisfies the following system of functional equations with the weighted parameter* $\mathbf{z}$*:*

$$\begin{cases} L_{\mathbf{z},g}(1) = \mathfrak{e}, \\ L_{\mathbf{z},g}(x) = \sum_{0 \le k < j} \mathbf{z}_g(k) + \mathbf{z}_g(j) L_{\mathbf{z},\tau(g,j)} \circ \phi_g(x) \quad (x \in I^+_{g,j}, j \in S_g^{*1}), \\ L_{\mathbf{z},g}(x) = \mathfrak{e} \qquad (x \in [0,1]) \qquad \textit{if} \quad \mathfrak{t}(\overline{0}_g) = 0 \end{cases} \tag{39}$$

for each g *in* G.

We call the above system $\mathrm{SRF}(\mathbf{z})$ *with the parameter space* $[0,1]$ *on* (G, m, τ), *that is abbreviated to be "* $\mathrm{SRF}(\mathbf{z})$ with $[0,1]$ on (G, m, τ)" *or "* $\mathrm{SRF}(\mathbf{z})$ with $[0,1]$ ".

Proof. 1) and 2) follow directly from Proposition 5.2, and also 3) is easily checked by using (5) and (37). □

Theorem 5.1. *Let* $\mathbf{z} : G^{\triangleright} \longrightarrow \Delta(d, \mathbb{C})$ *with* $\sum_{k \in S_g^{*1}} \mathbf{z}_g(k) = \mathfrak{e}$ *for* g *in* G *and* $\rho(\mathbf{z}) < 1$. *Then we have*

1) *There exists a unique right continuous solution* $L_{\mathbf{z}}$ *with left-hand limits of* $\mathrm{SRF}(\mathbf{z})$ *with* $[0,1]$.

2) *The mapping* $\mathbf{z}\longmapsto L_{\mathbf{z},g}$ *is* $\mathcal{B}([0,1],\mathbb{C})$*-valued analytic for each* g *in* G*, where* $\mathcal{B}([0,1],\mathbb{C})$ *is the set of all bounded mappings from* $[0,1]$ *to* $\mathbb{C}$ *with the uniform norm.*

3) *If the mapping* $x\longmapsto L_{\mathbf{z},g}(x)$ *is discontinuous at* x_0*, then there exists* ω *in* Ω_g *such that* $\mathfrak{t}(\omega)<\infty$*,* $\Psi_g(\omega-)=x_0$ *and* $L_{\mathbf{z},g}(x_0)-L_{\mathbf{z},g}(x_0-)=\mathbf{z}_g(\boldsymbol{i})\mathcal{L}_{\mathbf{z},\tau(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})$*, where* g *in* G*,* $\boldsymbol{i}$ *in* $S_g^{*\mathfrak{t}(\omega)}$ *and* $\omega=\sigma_{(g,\boldsymbol{i})}(\overline{0}_{\tau(g,\boldsymbol{i})})$*.*

Proof. It is sufficient to prove the uniqueness of the solution of SRF($\mathbf{z}$) with $[0,1]$, because of Theorem 3.1, Proposition 3.2 and Proposition 5.3 . Let $L_{\mathbf{z}}$ and $L'_{\mathbf{z}}$ be two solutions of SRF($\mathbf{z}$) with $[0,1]$. For $x\in I^+_{g,j_1j_2}$, where $j_1j_2\in S_g^{*2}$,

$$L_{\mathbf{z},g}(x)-L'_{\mathbf{z},g}(x)=\mathbf{z}_g(j_1)\{L_{\mathbf{z},\tau(g,j_1)}\circ\phi_g(x)-L'_{\mathbf{z},\tau(g,j_1)}\circ\phi_g(x)\}$$
$$=\mathbf{z}_g(j_1j_2)\{L_{\mathbf{z},\tau(g,j_1j_2)}\circ\phi_{g,2}(x)-L'_{\mathbf{z},\tau(g,j_1j_2)}\circ\phi_{g,2}(x)\},$$

because of $\phi_g(x)\in I^+_{\tau(g,j_1),j_2}$ and $\phi_{g,2}(x)=\phi_{\tau(g,j_1)}\circ\phi_g(x)$ by Proposition 5.1. By repeating this operation, for x in $I^+_{g,\boldsymbol{j}}$, where $\boldsymbol{j}$ in S_g^{*n} and n in $\mathbb{N}$,

$$L_{\mathbf{z},g}(x)-L'_{\mathbf{z},g}(x)=\mathbf{z}_g(\boldsymbol{j})\{L_{\mathbf{z},\tau(g,\boldsymbol{j})}\circ\phi_{g,n}(x)-L'_{\mathbf{z},\tau(g,\boldsymbol{j})}\circ\phi_{g,n}(x)\}.$$

Since $\Psi_g^{-1+}(x)$ in $I_{(g,\boldsymbol{j})}$, if $\mathfrak{t}^-(\Psi_g^{-1+}(x))=\infty$ then $\mathbf{z}_g(\boldsymbol{j})\to\mathbf{0}$ as $n\to\infty$ by Lemma 3.1, and if $\mathfrak{t}(\Psi_g^{-1+}(x))<\infty$ then there exists n such that $\mathfrak{t}(\Psi^{-1+}_{\tau(g,\boldsymbol{j})}(x))=0$ and $\Psi_g^{-1+}(x)=\sigma_{(g,\boldsymbol{j})}(\Psi^{-1+}_{\tau(g,\boldsymbol{j})}(x))$. Hence $L_{\mathbf{z}}=L'_{\mathbf{z}}$. □

Theorem 5.2. *Let* $\mathsf{z}:G^{\triangleright}\longrightarrow\mathbb{C}$ *and* $\mathsf{a}:G^{\triangleright}\longrightarrow\mathbb{C}$ *be the mappings such that* $\sum_{j\in S_{m(g)}}\mathsf{z}_g(j)=1$ *and* $\sum_{j\in S_{m(g)}}\mathsf{a}_g(j)=0$ *for* g *in* G*. Suppose that* $\rho(\mathsf{z})<1$*, and let* L_{z} *be a unique right continuous solution with left-hand limits of* SRF(z) *with* $[0,1]$*. Then the mapping* $\mathsf{z}\longmapsto L_{\mathsf{z},g}$ *is* $\mathcal{B}([0,1],\mathbb{C})$*-valued analytic, and is expanded as follows:*

$$(\mathsf{a}\cdot\partial)^qL_{\mathsf{z},g}(x)=q!\textstyle\sum_{n\in\mathbb{N}}\sum_{\boldsymbol{i}\in S_g^{*n}}\prod_{(h,l)\in G^{\triangleright}}\mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))}$$
$$\times A_{\mathsf{z},\tau(g,\boldsymbol{i}),q}(\mathsf{a})\circ\phi_{g,n}(x)1_{I^+_{g,\boldsymbol{i}}}(x)\qquad(q\in\mathbb{N}),\quad(40)$$
$$\tfrac{1}{\mathsf{p}!}L^{(\mathsf{p})}_{\mathsf{z},g}(x)=\textstyle\sum_{n\in\mathbb{N}}\sum_{\boldsymbol{i}\in S_g^{*n}}\prod_{(h,l)\in G^{\triangleright}}\mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))}$$
$$\times B_{\mathsf{z},\tau(g,\boldsymbol{i}),\mathsf{p}}\circ\phi_{g,n}(x)1_{I^+_{g,\boldsymbol{i}}}(x)\quad(\mathsf{p}\in\mathsf{Map}(G^{\triangleright},\mathbb{N}))\ (41)$$

uniformly on $\{x \in [0,1]; t(\Psi_g^{-1+}(x)) = \infty\}$, *and*

$$L_{\mathsf{z},g}(x) = \sum_{0\le n\le \mathsf{t}(\Psi_g^{-1+}(x))} \sum_{\boldsymbol{i}\in S_g^{*n}} \prod_{(h,l)\in G^{\triangleright}} \mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))} \times A_{\mathsf{z},\tau(g,\boldsymbol{i}),0}(\mathsf{a}) \circ \phi_{g,n}(x) 1_{I^+_{g,\boldsymbol{i}}}(x) + \sum_{\boldsymbol{i}\in S_g^{*\mathsf{t}(\Psi_g^{-1+}(x))}} \prod_{(h,l)\in G^{\triangleright}} \mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))}, \tag{42}$$

$$(\mathsf{a}\cdot\partial)^q L_{\mathsf{z},g}(x) = q! \sum_{0\le n\le \mathsf{t}(\Psi_g^{-1+}(x))} \sum_{\boldsymbol{i}\in S_g^{*n}} \prod_{(h,l)\in G^{\triangleright}} \mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))} \times A_{\mathsf{z},\tau(g,\boldsymbol{i}),q}(\mathsf{a}) \circ \phi_{g,n}(x) 1_{I^+_{g,\boldsymbol{i}}}(x) \qquad (q\in\mathbb{N}^+), \tag{43}$$

$$\frac{1}{\mathsf{p}!} L^{(\mathsf{p})}_{\mathsf{z},g}(x) = \sum_{0\le n\le \mathsf{t}(\Psi_g^{-1+}(x))} \sum_{\boldsymbol{i}\in S_g^{*n}} \prod_{(h,l)\in G^{\triangleright}} \mathsf{z}_h(l)^{s((h,l);(g,\boldsymbol{i}))} \times B_{\mathsf{z},\tau(g,\boldsymbol{i}),\mathsf{p}} \circ \phi_{g,n}(x) 1_{I^+_{g,\boldsymbol{i}}}(x) \quad (\mathsf{p}\in \mathsf{Map}(G^{\triangleright},\mathbb{N}), |\mathsf{p}|>0) \tag{44}$$

for x *in* $[0,1]$ *and* g *in* G *with* $t(\Psi_g^{-1+}(x)) < \infty$, *where*

$$A_{\mathsf{z},g,q}(\mathsf{a})(x) = \begin{cases} \sum_{j\in S_g^{*1}} \sum_{0\le k<j} \mathsf{z}_g(k) 1_{I^+_{g,j}}(x) & (q=0), \\ \sum_{j\in S_g^{*1}} \Big(\sum_{0\le k<j} \mathsf{a}_g(k) + \mathsf{a}_g(j) L_{\mathsf{z},\tau(g,j)} \circ \phi_g(x) \Big) 1_{I^+_{g,j}}(x) & (q=1), \\ \sum_{j\in S_g^{*1}} \mathsf{a}_g(j) \frac{(\mathsf{a}\cdot\partial)^{q-1}}{(q-1)!} L_{\mathsf{z},\tau(g,j)} \circ \phi_g(x) 1_{I^+_{g,j}}(x) & (q\ge 2) \end{cases}$$

and

$$B_{\mathsf{z},g,\mathsf{p}}(x) = \begin{cases} \sum_{j\in S_g^{*1}} \sum_{0\le k<j} \mathsf{z}_g(k) 1_{I^+_{g,j}}(x) & (|\mathsf{p}|=0), \\ \sum_{0\le j<m(g)-1} \mathsf{p}_g(j) \big\{ L_{\mathsf{z},\tau(g,j)} \circ \phi_g(x) 1_{I^+_{g,j}}(x) + \sum_{j<k<m(g)-1} 1_{I^+_{g,k}}(x) + (1 - L_{\mathsf{z},\tau(g,m(g)-1)} \circ \phi_g(x)) 1_{I^+_{g,m(g)-1}}(x) \big\} & (|\mathsf{p}|=1), \\ \sum_{0\le j<m(g)-1} \mathsf{p}_g(j) \frac{\partial^{\mathsf{p}-\delta(g,j)}}{(\mathsf{p}-\delta(g,j))!} \big\{ L_{\mathsf{z},\tau(g,j)} \circ \phi_g(x) 1_{I^+_{g,j}}(x) - L_{\mathsf{z},\tau(g,m(g)-1)} \circ \phi_g(x) 1_{I^+_{g,m(g)-1}}(x) \big\} & (|\mathsf{p}|\ge 2) \end{cases}$$

for x *in* $[0,1]$ *and* g *in* G, *where* $\delta(g,j)$ *is in* $\mathsf{Map}(G^{\triangleright},\mathbb{N})$ *such that* $\delta(g,j)_h(l) = 0$ *for* $(h,l)\in G^{\triangleright}$ *and* $0 \le l < m(h)-1$ *without* $\delta(g,j)_g(j) = 1$.

Proof. By using $\mathsf{L}_{\mathsf{z},g}$, $\mathsf{A}_{\mathsf{z},g,q}(\mathsf{a})$ and $\mathsf{B}_{\mathsf{z},g,q}$ in Theorem 4.1, we define L_{z}, $A_{\mathsf{z},g,q}$ and $B_{\mathsf{z},g,q}$ by

$$L_{\mathsf{z},g}(x) = \mathsf{L}_{\mathsf{z},g} \circ \Psi_g^{-1+}(x),$$

$$A_{\mathsf{z},g,q}(\mathsf{a})(x) = \mathsf{A}_{\mathsf{z},g,q}(\mathsf{a}) \circ \Psi_g^{-1+}(x),$$
$$B_{\mathsf{z},g,q}(x) = \mathsf{B}_{\mathsf{z},g,q} \circ \Psi_g^{-1+}(x),$$

where x in $[0,1]$, g in G and q in $\mathbb{N}$. Then Theorem 4.1 implies this theorem. □

Finally we give two examples of (G,m,τ), which have a continuous solution of SRF(z) with $[0,1]$ and include the Takagi function and the Gray Takagi function.

Example 5.1. *Let* $G=\{0\}$ *and* $m = constant(\geq 2)$.

In this case we may use S_m, Ω, S^{*n}, $I_{\boldsymbol{j}}^+$, ϕ_n, ϕ, $\mathsf{z}(j)$, $\mathsf{a}(j)$, $\mathsf{p}(j)$, $\delta(j)$, $s(j;\boldsymbol{i})$, …instead of $G^{\triangleright}$, Ω_g, S_g^{*n}, $I_{g,\boldsymbol{j}}^+$, $\phi_{g,n}$, ϕ_g, $\mathsf{z}_g(j)$, $\mathsf{a}_g(j)$, $\mathsf{p}_g(j)$, $\delta(g,j)$, $s((g,j);(h,\boldsymbol{i}))$, …, because G is one point set. Then τ is identity, $\phi(x) = mx \,(\bmod 1)$ and the notations $I_{n,j}$ and $s(j,k)$ in [11] are expressed by $I_{\boldsymbol{i}}^+$ and $s(k;\boldsymbol{i})$ as follows:

$I_{n,j} = I_{\boldsymbol{i}}^+$ and $\quad s(j,k) = s(k;\boldsymbol{i})$,

where $j = i_1 m^{n-1} + \ldots + i_{n-1}m + i_n$, $\boldsymbol{i} = i_1 \ldots i_n$ in S^{*n} and $k = 1, \ldots, m-1$. According to (36), (39), (40) and (41), the SRF(z) with $[0,1]$ is

$$\begin{cases} L_{\mathsf{z}}(1) = 1, \\ L_{\mathsf{z}}(x) = \sum_{0\leq k<j} \mathsf{z}(k) + \mathsf{z}(j) L_{\mathsf{z}} \circ \phi(x) \quad (x \in I_{1,j}, 0 \leq j < m) \end{cases} \tag{45}$$

and its solution L satisfies that

$$\begin{aligned}&(\mathsf{a}\cdot\partial)^q L_{\mathsf{z}}(x) \\ &= q! \sum_{0\leq n<\infty} \sum_{0\leq j<m^n} \mathsf{z}(0)^{n-\sum_{1\leq k<m} s(j,k)} \prod_{1\leq k<m} \mathsf{z}(k)^{s(j,k)} \\ &\qquad \times A_{\mathsf{z},q}(\mathsf{a}) \circ \phi^n(x) 1_{I_{n,j}}(x) \qquad (q \in \mathbb{N})\end{aligned} \tag{46}$$

and

$$\begin{aligned}&\tfrac{1}{\mathsf{p}!} L_{\mathsf{z}}^{(\mathsf{p})}(x) \\ &= \sum_{0\leq n<\infty} \sum_{0\leq j<m^n} \mathsf{z}(0)^{n-\sum_{1\leq k<m} s(j,k)} \prod_{1\leq k<m} \mathsf{z}(k)^{s(j,k)} \\ &\qquad \times B_{\mathsf{z},\mathsf{p}} \circ \phi^n(x) 1_{I_{n,j}}(x) \qquad (\mathsf{p} \in \mathsf{Map}(S_m, \mathbb{N}))\end{aligned} \tag{47}$$

for $x \in [0,1]$, where

$$A_{\mathsf{z},q}(\mathsf{a})(x) = \begin{cases} \sum_{0\le i<m}\sum_{0\le k<i} \mathsf{z}(k)1_{I_{1,i}}(x) & (q=0),\\ \sum_{0\le i<m}\Big(\sum_{0\le k<i} \mathsf{a}(k)+\mathsf{a}(i)L_{\mathsf{z}}\circ\phi(x)\Big)1_{I_{1,i}}(x) & (q=1),\\ \sum_{0\le i<m}\mathsf{a}(i)\frac{(\mathsf{a}\cdot\partial)^{q-1}}{(q-1)!}L_{\mathsf{z}}\circ\phi(x)1_{I_{1,i}}(x) & (q\ge 2)\end{cases}$$

and

$$B_{\mathsf{z},\mathsf{p}}(x)=\begin{cases} \sum_{0\le i<m}\sum_{0\le k<i} \mathsf{z}(k)1_{I_{1,i}}(x) & (|\mathsf{p}|=0),\\ \sum_{0\le i<m-1} \mathsf{p}(i)\big\{L_{\mathsf{z}}\circ\phi(x)1_{I_{1,i}}(x)+\sum_{i<k<m-1} 1_{I_{1,k}}(x) & \\ \qquad\qquad +(1-L_{\mathsf{z}}\circ\phi(x))1_{I_{1,m-1}}(x)\big\} & (|\mathsf{p}|=1),\\ \sum_{0\le i<m-1} \mathsf{p}(i)\frac{\partial^{\mathsf{p}-\delta(i)}}{(\mathsf{p}-\delta(i))!}\big\{L_{\mathsf{z}}\circ\phi(x)1_{I_{1,i}}(x) & \\ \qquad\qquad -L_{\mathsf{z}}\circ\phi(x)1_{I_{1,m-1}}(x)\big\} & (|\mathsf{p}|\ge 2).\end{cases}$$

Moreover according to (12) we get

$$L_{\mathsf{z}}(\frac{K}{m^N}) =\sum_{0\le j<K} \mathsf{z}(0)^{N-\sum_{1\le k<m} s(j,k)} \prod_{1\le k<m} \mathsf{z}(k)^{s(j,k)} \tag{48}$$

for integers K and N such that $0\le K<m^N$, which implies Theorem 2.1 in [11], which is a key lemma to connect Takagi function and the digital sum problem.

Especially in the case of $m=2$, $\mathsf{z}(0)=r$, $\mathsf{z}(1)=1-r$ and $\mathsf{a}(0)=1$, $\mathsf{a}(1)=-1$ (45) coincides with (1) and (46) implies (2), that is, denoting L_r by L_{z} we get $T_{r,k}=\frac{1}{k!}\frac{\partial^k L_r}{\partial r^k}$.

Example 5.2. *Let* $\sharp G=d$, $G=\{0,1,\dots,d-1\}$, $m=constant(\ge\max\{d,2\})$ *and* $\tau(g,j)=j\ (\mathrm{mod}\ d)$ *for* $0\le g<d$ *and* $0\le j<m$.

In this case we may use S^{*n}, ϕ_n, ϕ and $I^+_{\boldsymbol{j}}$ instead of S^{*n}_g, $\phi_{g,n}$, ϕ_g and $I^+_{g,\boldsymbol{j}}$, because these notations do not depend on g. Moreover $\phi(x)$ and $I_{n,j}$ are the same ones in Example 5.1. We can give such expression as (45) – (48) in Example 5.1, but we consider more special case: $m=d=2$, $\mathsf{z}_g(j)=r$ if $g\oplus j=0$, $\mathsf{z}_g(j)=1-r$ if $g\oplus j=1$ and $\mathsf{a}_g(j)=(-1)^{g+j}$, where $\oplus$ is the addition modulo 2. Then, by setting $\tilde{L}_{\mathsf{r},g}(x)=L_{\mathsf{z},g}(x)$ and $\tilde{\phi}(x)=\phi(x)1_{I_{1,0}}(x)+(1-\phi(x))1_{I_{1,1}}(x)$ for x in $[0,1]$ and $g\in G$, we get the followings:

$$\begin{cases}\tilde{L}_{\mathsf{r},g}(1)=1,\\ \tilde{L}_{\mathsf{r},g}(x)=\mathsf{z}_g(0)\tilde{L}_{\mathsf{r},0}\circ\phi(x)1_{I_{1,0}}(x)+\{\mathsf{z}_g(0)+\mathsf{z}_g(1)\tilde{L}_{\mathsf{r},1}\circ\phi(x)\}1_{I_{1,1}}(x),\end{cases}$$

$$\frac{\partial^q}{\partial r^q}\tilde{L}_{\mathsf{r},g}(x) = q!\sum_{0\le n<\infty}\sum_{0\le j<2^n} r^{n-\tilde{s}(g,j,n)}(1-r)^{\tilde{s}(g,j,n)} \times \tilde{A}_{r,j(\bmod 2),q}\circ\phi_n(x)1_{I_{n,j}}(x) \qquad (q\in\mathbb{N}),$$

$$\tilde{L}_{\mathsf{r},g}(\frac{K}{2^N}) = \sum_{0\le j<K} r^{N-\tilde{s}(g,j,N)}(1-r)^{\tilde{s}(g,j,N)} \qquad (K,N\in\mathbb{N}, 0\le K<2^N),$$

where $x\in[0,1]$,

$$\tilde{A}_{r,g,q}(x) = \begin{cases} \{\frac{1}{2}+(-1)^g\frac{2r-1}{2}\}1_{I_{1,1}}(x) & (q=0), \\ \frac{(-1)^g}{(q-1)!}\frac{\partial^{q-1}}{\partial r^{q-1}}\tilde{L}_{r,0}\circ\tilde{\phi}(x) & (q\ge 1), \end{cases}$$

and $\tilde{s}(g,j,n) = s((0,1);(g,\boldsymbol{i})) + s((1,0);(g,\boldsymbol{i}))$ with $\boldsymbol{i} = i_1\dots i_n$ for $j = i_1 2^{n-1}+\dots+i_{n-1}2+i_n$ with $i_k\in\{0,1\}$ $(k=1,\dots,n)$, because of $\tilde{L}_{\mathsf{r},g}(x) = 1-\tilde{L}_{r,1\oplus g}(1-x)$ and $\frac{\partial}{\partial r} = \frac{\partial}{\partial \mathsf{z}_0(0)} - \frac{\partial}{\partial \mathsf{z}_1(0)}$. Moreover $\tilde{L}_{\mathsf{r},0}$ and $\frac{1}{k!}\frac{\partial^k \tilde{L}_{\mathsf{r},0}}{\partial r^k}$ equal to $\tilde{L}_{\mathsf{r}}$ and $\tilde{T}_{\mathsf{r},k}$ in [3] respectively, by noticing $\tilde{s}(g,j,n) = g\oplus i_1 + \sum_{k=2}^n i_{k-1}\oplus i_k$.

6. Glossary of symbols

(G,m,τ) $G^{\triangleright}$ S_k	Definition 2.1
S_g^{*k} $\tilde{\tau}(\boldsymbol{g})$ $G^{\triangleright^k}$	Definition 2.2
Ω $\varphi(\omega)$ $\sigma_{\boldsymbol{g}}$	Definition 2.3
Ω_g $I_{\boldsymbol{g}}$	Definition 2.4
$\boldsymbol{\pi}_{g,n}$ $\pi_{g,n}$	Definition 2.5
$\mathfrak{t}$ $\mathfrak{t}^+$ $\mathfrak{t}^-$ $\mathfrak{s}(n,\omega)$ $\overline{0}_g$ $\overline{m}_g-$ $\omega-$ $\omega+$	Definition 2.6
$\mathbf{z}_g$ $\mathrm{SRF}(\mathbf{z})$	Definition 2.7
$\tilde{\mathbf{z}}(\boldsymbol{g})$	Definition 2.8
$\mathbf{D}_n$ $\mathbf{U}_n$	Definition 3.1
$s((g,j);(h,\boldsymbol{i}))$	Definition 3.2
$\rho(\mathbf{z})$ $\eta(\mathbf{z})$ $\Vert\mathcal{F}\Vert$	Definition 3.3
$\partial_g(j)$ ∂^{p} $\mathsf{a}\cdot\partial$ $\mathsf{L}_{\mathsf{z},g}^{(\mathsf{p})}$ $\mathsf{p}!$ $\lvert\mathsf{p}\rvert$	Section 4
Ψ_g	Definition 5.1
Ψ_g^{-1+}	Definition 5.2
$I_{g,\boldsymbol{j}}^+$ $\phi_{g,n}$ ϕ_g	Definition 5.3

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