UNIVERSIDADE DE LISBOA

INSTITUTO SUPERIOR DE ECONOMIA E GESTÃO

# Real World Economic Scenario Generator

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Tese especialmente elaborada para obtenção do grau de Doutora em Matemática Aplicada à Economia e Gestão

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"*Man has always learned from the past. After all, you can’t learn history in reverse!*"

***The Sword in the Stone***

# Resumo

Neste trabalho apresentamos uma metodologia para simular a evolução das taxas de juros sob medida de probabilidade real. Mais precisamente, usando o modelo de mercado *Shifted Lognormal LIBOR* multidimensional e uma especificação do vetor do preço de mercado do risco, explicamos como realizar simulações das taxas de juro futuras, usando o método de Euler-Maruyama com preditor-corretor. A metodologia proposta permite acomodar a presença de taxas de juro negativas, tal como é observado atualmente em vários mercados.

Após definir a estrutura livre de default, generalizamos os resultados para incorporar a existência de risco de crédito nos mercados financeiros e desenvolvemos um modelo LIBOR para obrigações com risco de crédito classificadas por ratings. Neste trabalho modelamos diretamente os spreads entre as classificações de ratings de acordo com uma dinâmica estocástica que garante a monotonicidade dos preços dos títulos relativamente às classificações por ratings.

**Palavras-Chave:** Modelo sob Medida Real, Simulação de Cenários, Solvência II, Taxas de juro, Taxas Forward Lognormais Ajustadas, Preço de Risco, Risco de Crédito, *Spreads*, Notações de Risco

# Abstract

In this work, we present a methodology to simulate the evolution of interest rates under real world probability measure. More precisely, using the multidimensional Shifted Lognormal LIBOR market model and a specification of the market price of risk vector process, we explain how to perform simulations of the real world forward rates in the future, using the Euler-Maruyama scheme with a predictor-corrector strategy. The proposed methodology allows for the presence of negative interest rates as currently observed in many markets.

After setting the default-free framework we generalize the results to incorporate the existence of credit risk to our model and develop a LIBOR model for defaultable bonds with credit ratings. We model directly the inter-rating spreads according to a stochastic dynamic that guarantees the monotonicity of bond prices with respect to the credit ratings.

**Key Words:** Real World Model, Scenario Simulation, Solvency II, Interest Rate, Shifted Lognormal Forward Rates, Market Price of Risk, Credit Risk, Spreads, Credit Ratings

# Resumo alargado


O aumento sucessivo das exigências regulatórias e internas relacionadas com a correta avaliação e gestão de riscos no sector bancário e segurador tornou imprescindível o melhor entendimento da incerteza existente no mercado. Esta incerteza pode ser parcialmente mitigada com a utilização de cenários que permitem avaliar como os balanços económicos das empresas reagiriam em situações de mercado adversas. Neste trabalho pretende-se construir um gerador de cenários económicos sob a medida de probabilidade real de alguns dos fatores que mais afectam as empresas de seguros: taxas de juro e risco de crédito.

Para definição do preço de produtos financeiros, cujo valor depende de realizações futuras de um determinado fator de risco, utilizam-se modelos formulados sob a medida neutra ao risco, desta forma os preços são obtidos como o valor descontado dos retornos futuros esperados sob a hipótese habitual de não arbitragem no mercado. No entanto, quando o objetivo é a simulação de valores futuros desses fatores, como necessário para avaliação de estratégias de investimento em carteiras sensíveis às taxas de juros em estudos de Gestão de Ativos e Passivos e cálculos de requisitios de capital, as probabilidades neutras ao risco não representam probabilidades reais, uma vez que sob esta medida assume-se que a os preços das ações crescem à taxa de juro sem risco e as taxas *forward* são estimativas não enviesadas das taxas futuras, o que não é realista, uma vez que isso implicaria que os investidores não requerem compensação pelo risco de mudanças imprevisíveis no futuro.

Para simular trajetórias futuras de fatores financeiros, isto é, para simular como o mundo será no futuro, deve-se usar uma medida de probabilidade que reflita o facto de que os investidores exigem um prémio de risco para manter nas suas carteiras os ativos com risco.

Sob este paradigma, foram apresentadas várias possibilidades para geração de cenários de taxas de juros. Por exemplo, Rebonato em [60] apresenta um método semiparamétrico para simulação das trajectórias da estrutura temporal de taxas de juro, Hull e White [38] propuseram uma abordagem simples para construir um modelo de *short rate* com um fator sob a medida neutra ao risco e sob a medida real. Utilizando modelos de mercado (*market models*), Norman em [58] introduziu uma forma paramétrica para o vector do preço de mercado do risco (*market price of risk*) e Takashi em [72] apresenta o modelo de mercado *forwad* LIBOR sob a medida do mundo real de uma forma rigorosa, obtendo o preço de mercado do risco, resolvendo um problema mínimos quadrados para um modelo de regressão com atraso.

Uma das principais dificuldades na modelização das taxas de juros é a escolha do processo apropriado que se pretende estudar. Muitos dos modelos de taxa de juros são baseados na evolução estocástica de uma determinada taxa de juro, geralmente a taxa de curto prazo (*short rate*), uma vez que através destes modelos se conseguem obter fórmulas fechadas para o preço de derivados financeiros. Uma desvantagem apresentada por esta classe de modelos é que apresenta um número bastante limitado de graus de liberdade para permitir o ajuste correto entre a estrutura de taxas de juro observada e a teórica.

Uma alternativa aos modelos de taxa de curto prazo é a especificação da dinâmica de toda a curva de taxas de juro usando o modelo HJM [34], no qual a dinâmica das taxas *forward* instantâneas é totalmente caracterizada pelas volatilidades instantâneas. Porém, a principal desvantagem dos modelos do tipo HJM, advém do fato de as taxas instantâneas não serem observáveis diretamente nos mercados, tornando por isso difícil a calibração aos preços de mercado. Tendo em vista essa desvantagem, os modelos de mercado foram introduzidos por Brace, Gararek e Musiela [13] e, desde então, tornaram-se muito populares principalmente devido ao acordo existente entre esses modelos e fórmulas utilizadas no mercado para a formação de preço de dois produtos derivados *standard*: *caps* e *swaptions*. Mais precisamente, o modelo Lognormal *Forward* LIBOR (LFM) permite obter o preço de *caps* com a fórmula de *Black-76* usualmente utilizada pelos *traders*.

Sob a abordagem LFM, as taxas forward LIBOR seguem uma distribuição *lognormal* e, consequentemente, pressupõe-se que as taxas forward são positivas, o que era, até recentemente, uma propriedade desejável. No entanto, taxas de juros negativas estão presentes na economia atual. Por exemplo, o Banco Central Europeu e os bancos centrais da Suíça, Dinamarca, Suécia e Japão estabeleceram taxas de juros negativas nas reservas como mecanismo de política monetária com o objectivo de criar um estimulo à taxa de crescimento economico, reduzindo a poupança e incentivando empréstimos a custos mais baixos.

Como consequência, essa mudança do limite inferior das taxas de juros trouxe alguns problemas aos modelos adoptados até agora, uma vez que muitas das metodologias adotadas não conseguem lidar com taxas negativas.

Dadas as condições atuais do mercado, é possível obter uma solução alterando a condição de fronteira das taxas de juros de zero para um valor negativo adequado e isso pode ser alcançado usando o modelo de mercado *Shifted Lognormal Forward LIBOR* (SLFM). Sob esta abordagem, optamos por modelar um conjunto de taxas forward que podem ser observadas diretamente no mercado. Assim, usando o modelo SLFM, podemos garantir a não existência de oportunidades de arbitragem nos mercados mantendo a flexibilidade do modelo para capturar todos os movimentos possíveis de curvas sob uma estrutura coerente para taxas de juros negativas.

Após a definição do modelo adoptado para a estrutura temporal de taxas de juro sem risco, generalizamos os resultados para incorporar a existência de risco de crédito no mercado e desenvolvemos um modelo LIBOR para títulos com risco de *default* classificados através de um sistema de *ratings*. Modelamos diretamente os *spreads* entre os *ratings* de acordo com uma dinâmica estocástica que garante a monotonicidade dos preços dos títulos em relação às classificações de crédito.

Este trabalho tem a seguinte estrutura: no Capítulo 1, apresentamos a estrutura regulatória e como os cálculos de requisitos de capital podem ser realizados no âmbito da Solvência II. O Capítulo 2 analisa os resultados relevantes na literatura sobre modelos de taxa de juro sob a medida neutra ao risco e apresenta as técnicas para passar da medida neutra ao risco para a medida real. No Capítulo 3, apresentamos o modelo proposto para o gerador de cenários sob a medida real para taxas de juro e explicamos em detalhes como estimamos os parâmetros e discutimos os resultados obtidos. O Capítulo 4 analisa os resultados relevantes na literatura sobre modelos de risco de crédito e, no capítulo 5, apresentamos a abordagem proposta para gerar cenários para spreads de crédito na medida real.

Os Capítulos 3 e 5 são as principais contribuições desta tese onde são apresentados novos desenvolvimentos. Por fim, algumas conclusões e comentários sobre novos temas de investigação são apresentadas no Capítulo 6.

# Long abstract

The increase of regulatory and internal demands on risk assessment and management of assets and liabilities within banks and insurance companies led to the need of a better understanding of the uncertainty in market risk factors, particularly in future scenarios of interest rates. In this project we aim to construct a real world Economic Scenario Generator (ESG) modeling some of the main financial risk factors that affect insurance companies: interest rates, and credit spreads.

For pricing financial products, the value of which depends on future realizations of certain risk factors, one can rely on risk neutral models so that prices are obtained as the discounted value of expected future payoffs under the standard hypotheses on frictionless and complete markets.

However, in the assessment of investment strategies in interest-rate sensitive portfolios for Asset-Liability Management studies and calculations of Economic Capital for Solvency II, the objective is the simulation of future values of these underlying factors and products. In these cases, the risk neutral probabilities do not represent real probabilities, as the drift of the stock prices is assumed to be the risk free rate and the forward rates are unbiased predictors of future rates, which is not realistic because that would imply that investors require no compensation for the risk of unpredictable changes in the future.

In order to simulate future trajectories of financial factors, i.e. to simulate how the world will look like in the future, one should use a probability measure that reflects the fact

that investors demand a risk premium to hold risky assets. Under this paradigm many approaches have been presented for modelling interest rates. For instance, Rebonato in [60] presents a semiparametric method to simulate yield curve paths using random sampling of historical changes in the yield curve with a spring mechanism to guarantee correct shapes of the generated curves. Hull and White [38] proposed a simple approach to construct a one factor short rate model for both the risk neutral measure and the real world measure. By using the LIBOR market model, Norman [58], introduced a parametric form for the market price of risk vector and Takashi [72] presents the LIBOR market model under the real world measure in a rigorous manner, thus obtaining the market price of risk by solving a least square problem for a lag regression model.

One of the main problems in interest rate modeling is the choice of the appropriate process to model. Many interest rate models focus on the stochastic evolution of a given interest rate, usually the short rate, because they can provide closed pricing formulas due to their analytical tractability. However, this class of models has a rather limited number of degrees of freedom to allow for a correct match between the observed term structure and the theoretical one. An alternative to short rate models is to specify the dynamics of the entire yield curve using the arbitrage-free HJM [34] framework where the instantaneous forward rates dynamics are fully specified through their instantaneous volatility structure. However, the main disadvantage of HJM type models comes from the fact that the instantaneous rates are not directly observed in the markets, so that the calibration to current market prices turns out to be difficult. Having in view this drawback, LIBOR market models were introduced by Brace, Gatarek and Musiela [13] and have since become very popular mainly due to the agreement between such models and market formulas for pricing two basic derivative products: caps and swaptions. More precisely, the Lognormal Forward LIBOR model (LFM) allows to price caps with Black's cap formula used in the markets.

Under the LFM approach, the implicit forward LIBOR rates follow a lognormal distribution and, consequently, forward rates are guaranteed to be positive, which was, until recently, a desirable property. However, negative interest rates are present in many current economies. For example, the European Central Bank and central banks of Switzerland, Denmark, Sweden and Japan have set negative interest rates on reserves with the argument of economic growth rate estimulation by reducing savings and encouraging borrowing at lower costs. Of course, this change of the lower bound for interest rates involves many consequences in the models used until now, since many adopted methodologies cannot cope with negative rates.

Given the current market conditions, a solution can be obtained by shifting the boundary condition of interest rates from zero to an adequate negative value and this can be achieved using the Shifted Lognormal Forward LIBOR market model (SLFM). Under this approach, we choose to model a set of key forward LIBOR rates which can be directly observed in the market. Then, using the SLFM model, we can guarantee no arbitrage opportunities in interest rate markets and provide more flexibility to capture all possible curve movements under a coherent framework for negative interest rates.

After setting the default-free framework, we generalize the results to incorporate the existence of credit risk to our model and develop a LIBOR model for defaultable bonds with credit ratings. For this purpose, we directly model the inter-rating spreads according to a stochastic dynamics that guarantees the monotonicity of bond prices with respect to the credit ratings.

The outline of this work is as follows. In Chapter 1, we present the regulatory framework and how capital requirement calculations can be accomplished under Solvency II. Chapter 2 briefly reviews the relevant results in the literature on interest rate models under the risk neutral measure and presents the techniques to move from risk neutral to real world measure. In Chapter 3, we present the proposed modeling approach for the real world scenario generator for interest rates and explain in detail how we estimate the parameters and present the obtained results. Chapter 4 reviews the relevant results in the literature on credit risk models. In Chapter 5, we present the proposed approach to generate scenarios for credit spreads under the real world measure.

Chapters 3 and 5 are the main original contributions of this thesis where new developments are presented. Finally, some conclusions and comments on further research are set out in Chapter 6.

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# Abbreviations

| | |
|---|---|
| ESG | Economic Scenario Generator(s) |
| EIOPA | European Insurance and Occupational Pensions Authority |
| SCR | Solvency Capital Requirement |
| MCR | Minimum Capital Requirement |
| VaR | Value at Risk |
| ES | Expected Shortfall |
| $SCR_{mkt}$ | Solvency Capital Requirement for Market Risk |
| $Mkt_{interest}$ | Solvency Capital Requirement for Interest Rate Risk |
| $Mkt_{spread}$ | Solvency Capital Requirement for Spread Risk |
| PCA | Principal Component Analysis |
| PCs | Principal Components |
| HJM | Heath-Jarrow-Morton model |
| LFM | Lognormal Forward LIBOR Model |
| LSM | Lognormal Swap Model |
| FRA | Forward Rate Agreement |
| IRS | Interest Rate Swap |
| ECB | European Central Bank |
| ALM | Asset - Liability Management |
| SDE | Stochastic Differential Equation |

# Notation

## Default-Free Term Structure of Interest Rates

| | |
|---|---|
| $F_j(t) = F(t, T_j, T_{j+1})$ | Simply compounded default-free forward rate over $[T_j, T_{j+1}]$ |
| $B(t, T_j) = B_j(t)$ | Price at time $t$ of a zero-coupon bond with maturity date $T_j$ |
| $F_j^D(t) = F_j(t) + \alpha_j$ | Simply compounded default-free shifted forward rate over $[T_j, T_{j+1}]$ |
| $Cpl(T_j, T_{j+1}, \delta_j, K)$ | $T_j - caplet$ Price at time $t = 0$ with strike $K$ |
| $BL^{call}(K, F, V)$ | Black price of a call option |
| $B_d(t)$ | Value at time $t$ of a discretely balanced bank account |
| $\theta(t)$ | $r$-dimensional market price of risk process |
| $L_j^D(t)$ | Constant maturity shifted forward rate |

## Defaultable Term Structure of Interest Rates

---

| | |
|---|---|
| $\overline{B}_j(t)$ | Price at $t$ of a defaultable zero-coupon bond with maturity date $T_j$ |
| $F_j^i(t)$ | Forward LIBOR rate for credit rating $i$ the period $[T_j, T_{j+1}]$ |
| $F_j^{i,D}(t)$ | Shifted forward LIBOR rate for credit rating $i$ the period $[T_j, T_{j+1}]$ |
| $S_j^i(t)$ | Forward inter-rating LIBOR spread between credit rating $i$ and $i-1$ |
| $D_j^i(t)$ | Default-risk factor for credit rating $i$ at time $t$ for maturity $T_j$ |
| $H_j^i(t)$ | Discrete-tenor forward default intensity for credit rating $i$ over the period $[T_j, T_{j+1}]$ |
| $\theta^S(t)$ | $m$-dimensional market price of risk process |
| $\mathbf{P}$ | One-year transition matrix |

# Chapter 1

# Motivation and financial framework

Solvency II directive [24] was approved in 2009. It aimed to enhance policyholders' protection, to improve stability of the financial system, and to unify the insurance market in the European Union as a whole, by establishing harmonized solvency requirements across all member states. Solvency II has been in preparation since 2007 and came into effect on 1 January 2016.

Along with many important new guidelines for risk management of insurance companies, Solvency II directive mandates that the valuation of assets and liabilities should be done using market consistent techniques. This means that the value of an asset or liability is its market value, if it is readily traded on a deep, liquid and transparent market at the point in time. Otherwise, its value would be given by a reasonable best estimate of what its market value would have when readily traded at the relevant valuation date.

A market is defined as liquid when an individual or firm can quickly purchase or sell an asset without causing a drastic adjustment in the asset price. A market is defined as deep when a large number of assets can be bought and sold without significantly affecting the price. A market is transparent when information about supply and prices is readily available to the public.

In order to obtain the best estimate, insurance companies can use analytical techniques, deterministic techniques or a scenario approach. Nevertheless, using an analytical approach would imply that the insurance company is able to find closed-form solutions to value guarantees which is a very difficult task because some of the financial products sold by insurance companies have embedded path-dependent options and high number of risk factors. On the other hand, deterministic approaches imply simplified assumptions of the market behaviour. Therefore, for many insurance companies Economic Scenario Generators (ESG) are the only practical and robust way to determine the market consistent present value of the liabilities. An accurate and robust valuation can result in competitive premiums for policyholders and allow for an optimal amount of reserving for the insurance company, while maintaining the risk management thresholds.

Another very important regulatory change introduced by the Solvency II directive concerns to the way risks should be accounted for and the introduction of a solvency margin entirely based on risk sources and risk mitigation techniques, namely, Solvency Capital Requirement (SCR).

In order to calculate SCR, insurance companies can choose the standard formula proposed by EIOPA, where the SCR is decomposed in simpler terms divided by the risk they refer to and it relies on some assumptions that are still under debate [27]. For instance, using the standard formula, it is assumed that: correlations between risk factors can be fully captured by using a linear correlation coefficient approach. Moreover, not all quantifiable risks are explicitly formulated and, consequently, some risks, whose nature and calibration depend on the single undertaking specificity, may not be covered by the standard formula. In particular, the simplifications considered for the market risk module include: the assumption that only changes in the level of the market risk factors have impact on the solvency level of an insurer and, in the specific case of the interest rate risk sub-module, the underlying assumption that in times of lower interest rates the absolute shocks are lower implies that the risk of deflation is not entirely captured.

Alternatively, SCR can be estimated using internal models, which would have to be approved by the supervisory authorities. It is in the latter case that a real world ESG becomes relevant.

The position of an insurance portfolio is influenced by a large number of macroeconomic risk factors, such as inflation, stock prices, real estate prices, and correlations between all the different factors. Market risk accounts for 64% of the net SCR before diversification benefits for standard formula users [28]. However, among all the market risks, the interest rate risk represents the main contribution to the the market risk component of SCR [25]. Actually, the changes in the interest rate curves affect both sides of the insurer financial books. On the other hand, the asset side is affected because insurers invest a significant amount in government bonds usually with longer maturities (see Fig: 1.1). On the liabilities side, the today's value of future cash flows are directly influenced by the discount rates. Typically, the effect of interest rate movements in the liabilities side has a more material impact than in the assets side.

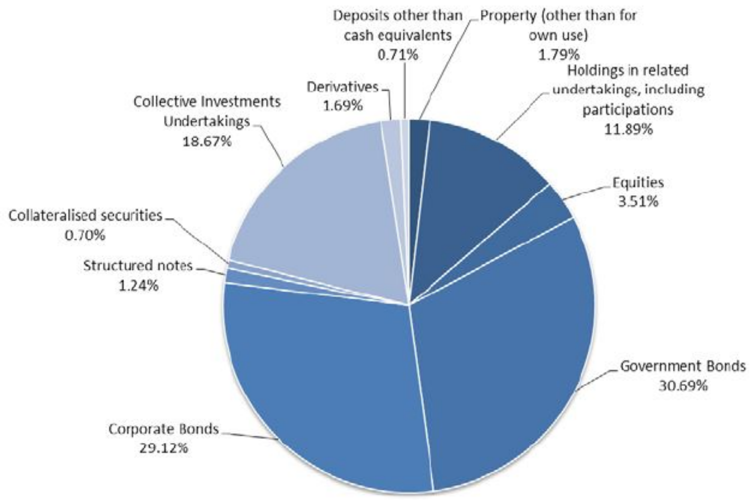


Figure 1.1: Investment mix by insurers in EEA in Q42018
Source: EIOPA Statistics - accompanying note

The second most relevant asset class of insurance companies investments is the class of corporate bonds. This implies that the risk related to credit quality changes, and consequently, spread changes, should be accurately estimated and the corresponding adequately capital reserved.

In this project, we aim to construct a real world ESG modeling two of the main financial risk factors that affect insurance companies: interest rates and credit spreads.

## 1.1 Solvency II

The economic conditions faced by insurance companies during the last two decades and the shortfalls in the previous regulatory framework, Solvency I, led European authorities to rethink and reformulate the way insurance companies should calculate their solvency positions.

Under Solvency I, the solvency requirements - the funds' amount that insurance and reinsurance companies in the European Union are required to hold - were calculated as a percentage of the technical provisions. This simplified method had some shortcomings, such as penalizing insurance companies with high technical provisions even if the value was determined by prudence and risk averse managing actions. Moreover, the ratio focuses mainly on the liability side of the balance sheet and ignores risks occurring in the assets side [69]. In order to increase internal regulations, local authorities across the European Union, acting individually and independently, put in place several actions. These specific actions led to significant differences in the criteria applied by each of the member states.

Solvency II Framework Directive 2009/138/EC [24] was the response from the European Commission to unify the regulatory structure in the EU insurance market. It is inspired by the Basel II accord, [7], for the banking industry introduced in 2006. This new solvency regime points to the risk profile of insurance and reinsurance undertakings, and it thrives on creating better conditions to protect policyholders.

The proposed Solvency II framework has three main pillars defined by the European Insurance and Occupational Pensions Authority (EIOPA[1]):

- **Pillar 1** covers all the quantitative requirements that insurers must fulfill to demonstrate that they have sufficient capital resources. It covers all components of the economic balance sheet and defines two capital requirements: the Minimal Capital Requirement (MCR) and the Solvency Capital Requirement (SCR);

- **Pillar 2** sets out requirements for the governance and risk management of insurers, as well as for the supervisory activities and powers of regulators;

- **Pillar 3** focuses on disclosure and transparency requirements through public disclosures in the form of narrative and quantitative reports encouraging early warning systems.

As a risk-based system, Solvency II focuses on risk identification and the accurate allocation of capital to the identified risks. It is expected that undertakings with more risk exposures will now have higher capital requirements, thus punishing risk-seeking, or at least imprudent behaviors and reward risk mitigation actions.

In order to enter in more detail in SCR calculations, it is important to define some main concepts of the Economic Balance Sheet as pictured in Fig 1.2

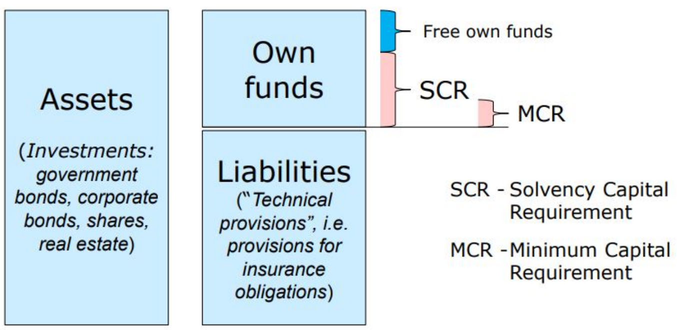


Figure 1.2: Balance sheet under Solvency II
Source: EIOPA Presentations - Understanding the Solvency II Balance sheet 2013

[1]Former Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS)

Technical provisions are the amount that an insurance company must hold to ensure that it can meet its expected future obligations on insurance contracts. They are obtained by summing the best estimate of the expected liabilities - in the form of a probability-weighted average - plus a risk margin that takes into account the cost of capital that would be required to sell the liabilities to a new knowledgeable undertaking.

Basic own funds are the value of the subordinated liabilities and the excess of assets over liabilities, valued accordingly to the market consistent valuation principle, reduced by the amount of own shares held by the insurance or reinsurance undertaking. They are classified into tiers that represent how well and how fast they can absorb losses.

The Minimum Capital Requirement (MCR) represents the threshold below which the insurance undertaking is exposed to an unacceptable level of risk leading to a necessary intervention from the national regulatory agency .

Finally, the SCR is the total amount of funds that insurance and reinsurance companies in the European Union are required to hold to ensure that their obligations to policyholders over the following 12 months can be met with a 99.5% probability.

### 1.1.1 Capital requirements

The Minimum Capital Requirement (MCR) is the solvency threshold and, it is set to represent a 12 months Value-at-Risk (VaR) calibrated to an 85% confidence level. If the amount of eligible basic own funds of an insurance undertaking falls below this threshold, then regulatory authorities will act in order to transfer the insurer's liabilities to another company and withdrawn the license of the undertaking.

The Solvency Capital Requirement (SCR) is defined as the VaR of the own funds of an insurer set at a level of 99.5% over a one year period. SCR is usually interpreted as the value that guarantees that only once in 200 years, the funds held are not enough to meet the insurer's obligations.

According to Solvency II directive, SCR should be calibrated in order to ensure that all quantifiable risks to which an insurance or reinsurance undertaking is exposed are taken into account. It should cover existing business, as well as the new business expected to be written over the following 12 months. Furthermore, when calculating the SCR, insurance and reinsurance undertakings must take into account the effect of risk-mitigation techniques, provided that credit risk and other risks arising from the use of such techniques are properly reflected in the SCR.

Both MCR and SCR are based on the concept of VaR, as it is easy to understand and implement, although it has been noted that it is not a coherent risk measure, see [4] e.g., as it does not fulfill the required property of subadditivity. An alternative risk measure has been proposed in the literature, see [1] and [41]: the Expected Shortfall (ES). ES is defined as the expected value of the losses which are greater or equal than the VaR. As a result, ES takes more into consideration the shape of the loss distribution in the tail of the distribution. ES answers the question "*if things go bad, how much do we expect to loose?*" where VaR answers the question "*how bad can things get within a certain probability?*" [36].

### 1.1.2 European standard formula

The standard formula provided in the EIOPA Techical Specifications [26] is a simplified calculation to obtain the SCR of a particular undertaking. Thus, the overall SCR for an undertaking is defined as:

$$SCR = BSCR + SCR_{op} + Adj \ , \tag{1.1}$$

where $SCR_{op}$ is the capital requirement for operational risk, $Adj$ is the sum of the adjustment for the risk absorbing effect of technical provisions and deferred taxes, and $BSCR$ is the Basic Solvency Capital Requirement that combines capital requirements for six major risk categories, as shown in Fig 1.3.

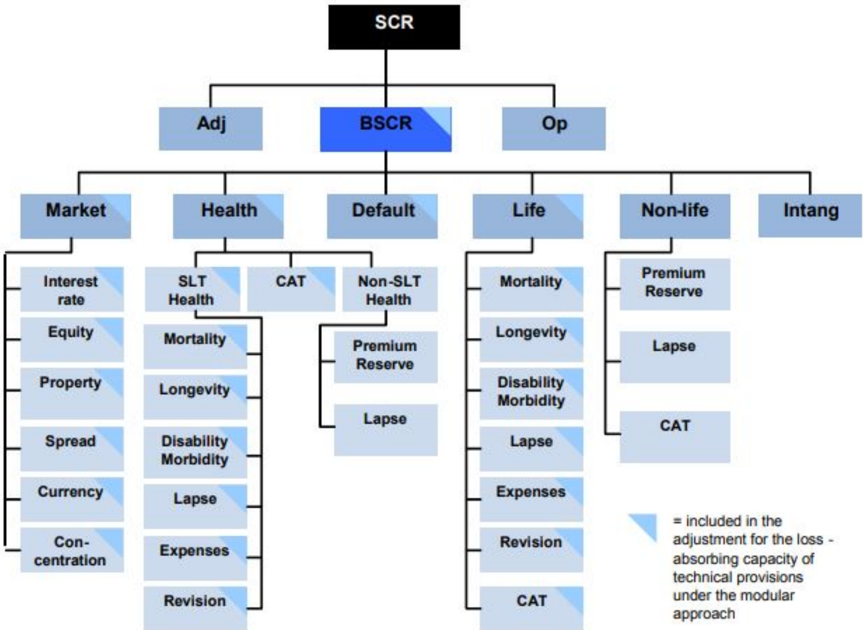


Figure 1.3: Solvency Capital Requirement (SCR) according to the standard formula
Source: EIOPA Technical Specification for the Preparatory Phase - 2014

The BSCR is given by:

$$BSCR = \sqrt{\sum_{ij} Corr_{ij} SCR_i SCR_j} + SCR_{intangible} \tag{1.2}$$

where $Corr_{ij}$ are the coefficients of a pre-determined correlation matrix between the individual SCR modules. The SCR modules are denoted by $SCR_i$ and represent the capital requirements for the individual SCR risks $SCR_{Mkt}$, $SCR_{Health}$, $SCR_{Default}$, $SCR_{Life}$, $SCR_{NonLife}$. Moreover, $SCR_{intangible}$ denotes the capital requirement for intangible asset risk.

In what follows we will discuss in more detail the modules inside the market risk module $SCR_{mkt}$. Specifically the modules of interest rate risk, $SCR_{Int}$, and credit spread risk, $SCR_{Spread}$, as those are the main focus of our work. For detail about the calculations of other modules we refer to the EIOPA Techical Specifications [26].

**Capital requirement for market risk**

Under Solvency II, the market risk module reflects the risk arising from the level of market prices of financial instruments, which have an impact upon the value of the assets and liabilities of the undertaking, and it should reflect the structural mismatch between assets and liabilities. These are scenario-based calculations and are based on the impact of instantaneous shocks of the risk factors.

The standard formula of the market risk module aggregates equity risk, interest rate risk, property risk, currency risk, spread risk and concentration risk. For each type of risk , the formula assesses the required capital to overcome a set of specified scenarios. Then, the individual capital requirements are aggregated, taking into account correlations between risk factors, providing the market risk solvency capital requirement, $SCR_{Mkt}$.

The $SCR_{Mkt}$ is given by :

$$SCR_{mkt} = \sqrt{\sum_{ij} Corr_{ij} Mkt_i Mkt_j} \tag{1.3}$$

where again $Corr_{ij}$ are the coefficients of a pre-determined correlation matrix between the individual $SCR_{Mkt}$ components. These components, $Mkt_i$, are the capital requirements for the individual market risk modules, $Mkt_{equity}$, $Mkt_{interest}$, $Mkt_{property}$, $Mkt_{currency}$, $Mkt_{spread}$ and $Mkt_{concentration}$.

**Capital requirement for interest rate risk**

The capital requirement for interest rate risk is determined as the maximum change in the net value of assets and liabilities due to the revaluing of all interest rate sensitive items under two pre-defined scenarios: an instantaneous upward movement of interest rates and a downward movement.

These two scenarios for the term structures are obtained by multiplying the current interest rate curve by $(1 + s^{up})$ and $(1 + s^{down})$, where the upward stress $s^{up}(t)$ and the downward stress $s^{down}(t)$ for individual maturities are specified as in **Table 1.1**:

| Maturity | $s^{up}(t)$ | $s^{down}(t)$ | Maturity | $s^{up}(t)$ | $s^{down}(t)$ |
|---|---|---|---|---|---|
| 1 or shorter | 70% | -75% | 12 | 37% | -28% |
| 2 | 70% | -65% | 13 | 35% | -28% |
| 3 | 64% | -56% | 14 | 34% | -28% |
| 4 | 59% | -50% | 15 | 33% | -27% |
| 5 | 55% | -46% | 16 | 31% | -28% |
| 6 | 52% | -42% | 17 | 30% | -28% |
| 7 | 49% | -39% | 18 | 29% | -28% |
| 8 | 47% | -36% | 19 | 27% | -29% |
| 9 | 44% | -33% | 20 | 26% | -29% |
| 10 | 42% | -31% | 90 or longer | 20% | 30% |
| 11 | 39% | -30% | | | |

Table 1.1: Interest rate curve shocks by maturity

Moreover, for maturities not specified above, the value of the shock is obtained by linear interpolation. Also, irrespective of the above stress scenarios, the absolute increase of interest rates in the upward scenario at any maturity should be at least one percentage point. When, for a given maturity, the initial value of the interest rate is negative, the undertaking should calculate the increase or decrease of the interest rate as the product between the $s^{up}$ and $s^{down}$ shock and the absolute value of the initial interest rate.

**Capital requirement for credit spread risk**

The spread risk is the risk of changes in the market value assets, liabilities, and financial instruments caused by changes in credit spreads. It reflects the change in the market value due to a movement in the yield curve relative to the risk-free interest rate term structure.

The spread risk module $Mkt_{Spread}$, applies to bonds, in particular, corporate bonds, securitization positions, and credit derivatives.

For simplicity, we will focus the exposure in the capital requirement for the spread risk of bonds and loans. In this case, the capital requirement is the immediate effect on the net value of asset and liabilities expected in the event of an instantaneous decrease of values in bonds and loans due to the widening of their credit spreads. This capital requirement is given by the formula:

$$\sum_i MV_i F^{up}(rating_i, duration_i) \tag{1.4}$$

where $MV_i$ is the market value of the position and $F^{up}(rating_i, duration_i)$ is a function of the rating and the duration of the exposure, which is calibrated to deliver a shock consistent with VaR 99.5% following a widening of credit spreads. The spread risk factor is capped at a level of 100%.

## 1.2 Economic scenario generators in Solvency II

Even though it is not required by the authorities, insurance companies are encouraged to implement their own internal model for the calculation of the SCR instead of using the simplifications implied by the standard formula. The implementation of an internal model has many advantages. First, it gives the undertakings a better understanding of the risks they are exposed to, which leads to a better risk assessment. Secondly, it allows to develop and obtain a tailor-made solution that represents all the specific business lines and strategies, instead of using a formula that only takes into account the risks to which an average undertaking is exposed.

In the case the undertaking chooses to use a full or partial internal model for capital requirements calculation, one of the main required tools is an ESG that allows the computation of VaR using simulation methods.

An ESG generates future scenarios for different risk factors. Moreover, it allows for the possibility of generating full distributions of capital, rather than just point estimates at given percentiles, which give a deeper understanding of the market risk.

In the insurance industry, there are two types of ESG with two different applications. On the one hand, there are market consistent ESG that are used in the calculation of technical provisions for insurance contracts with financial options and guarantees. On the other hand, there are real world ESG which generate scenarios that reflect the expected future evolution of the economy to support the calculation of the SCR. In both kinds of ESG, the underlying models for the risk factors can be very similar. However, the parameters of the models will change when we move from risk free to real world modeling, since real world scenarios account for the risk premium so that calibration is done using historical values instead of market prices, as it is the case in market consistent models. In this work, we focus on the second type of ESG.

There are several ways to generate future scenarios. The simplest approach to scenario generation is to use historical data (observations) as scenarios. This technique is known as bootstrapping. It involves sampling, with replacement, from historical observations. Even though it is a simple and intuitive approach, it has some disadvantages as it only allows observed events to be simulated and assumes that the structure and conditions of the market do not change. Also, this approach does not model the existing relationships between macroeconomic variables and does not allow for expert intervention. Another simplified methodology is to draw future observation of the risk factors from a standard normal distribution as proposed by [42]. However this method is not able to capture the long term dynamics of the risk factors and only linear correlations are modeled.

A popular approach for scenario generation under the real world measure is the use of Principal Component Analysis (PCA) to reduce the dimensionality of the market risks into a smaller number of factors and then to model these factors using possible multidimensional models. The applications of PCA in scenario simulation can be found in multiple studies, in particular [30] apply this methodology for specifying stress scenarios for interest rates and Value at Risk (VaR) calculations and [47] propose the use of principal component analysis for projection of macroeconomic variables related to stress-testing exercises in banking.

Another possibility is the use of vector autoregressive models (VAR) that allow the estimation of the relationships between different risk factors and are able to capture long term dynamics between the risk factors since factors are modeled as a system of auto-regressive equations with explicit dependencies between equations. The continuous-time generalization of this methodology is the use of stochastic differential equations, and this is the object of this thesis.

The outline of the rest of the document is as follows. Chapter 2 reviews the relevant results in the literature on interest rate models under the risk neutral measure and presents the techniques to move from risk neutral to real world measure. In Chapter 3 we present the proposed modeling approach for the real world scenario generator for interest rates and explain in detail how we estimated the parameters and present the obtained results. In Chapter 4 we discuss the main setting and techniques for modeling credit risk and in Chapter 5 we present the proposed approach to generate scenarios for credit spreads and defaultable bond prices under the real world measure.

Chapters 3 and 5 are the main contributions of this thesis where new developments are presented. At last, some conclusions and comments on further research are set out in Chapter 6.

# Chapter 2

# Interest rate models

In this chapter, we set out the main characterization of the interest rate models and some tools we use later in this work. In Section 2.1, we provide an overview of the existing literature in interest rate modeling. In Section 2.2, we explain how the models have been modified to accommodate negative interest rates observed in current markets. Section 2.3 is devoted to the introduction of the risk neutral version of the Shifted Forward LIBOR market model. Finally, in Section 2.4 we introduce the fundamental tools to rewrite the model under the real world measure and we conclude with a discussion on previous studies on the market price of risk process.

## 2.1 Overview of interest rate models

The term structure of interest rates is an essential element in finance. It is one of the most important factors for pricing contingent claims, determining the cost of capital and managing financial risk.

Some of the desired properties and objectives of interest rate models are its adherence to data. More precisely, the ability to calibrate to market prices or historical data, the time or cost needed to calibrate and simulate with the model. Also, it is desirable that the model is intuitive enough and easy to understand for decision-makers.

We can distinguish two major classes of interest rate models that have been proposed in the literature: deterministic and stochastic models.

In deterministic models, the spot or instantaneous forward rate is modelled by means of a deterministic function of time and the maturity of the rate. Some of the most important deterministic models used in the market are the Nelson-Siegel model [57], the Svensson model [67] and the Bjork-Christensen model [8]. One key advantage of these models is that they are parsimonious, which in turn leads to a lack in flexibility since they are not able to account for all possible shapes of the interest rate term structure we see in practice. Also, when we are interested in pricing fixed-income securities that pay uncertain cash-flows and where the potential correlations between interest rates and future cash-flows play an important role, the deterministic models fail to provide this information. Even with all these drawbacks, many actuaries continue to use deterministic scenarios for modelling interest rates in performing asset adequacy analysis [3].

Another deterministic model worth referencing in view of the relevant role it plays in the Solvency II framework is the Smith-Wilson method [66] for the projection of the risk-free rates on a span of 135 years. In this extrapolation method, bond prices are modelled directly and are defined as linear combinations of kernel functions depending on a set of parameters defined by the supervision. The advantages of this approach are that it is a simple, linear and a mechanized approach. Moreover, it provides a perfect fit for the liquid zero-coupon bonds used in the calibration step. However, Lageras and Lindholm [48] show that there are a number of problems with the Smith-Wilson method. In particular, they show that discount factors extrapolated by the Smith-Wilson method may become negative when the market curve exhibits a steep slope for high tenors and that hedging strategies present oscillating behaviour. Moreover, Gourieroux and Monfort in [33] show that the Smith-Wilson model is not consistent with the absence of arbitrage required by Solvency II Directive.

In the group of stochastic models for the interest rate, we distinguish between short rate, the HJB framework for the instantaneous forward rates and the market models approach.

Short rate models describe the spot interest rate evolution via a possible multi-dimensional driving diffusion process in terms of some parameters. These parameters depend only in the spot rate in endogenous models, such as Vasicek [70] and Cox, Ingersoll and Ross [18] models. In the case of exogenous models such as Hull-White [37] and Black-Karasinski [11] these parameters depend initially on time. These models are characterized by their analytical tractability and consequent ease of use. The main disadvantages of short rate models are that they focus on unobservable instantaneous interest rates, they rely on unrealistic correlation patterns between points of the curve with different maturities and they have poor calibration capabilities. Furthermore, in order to obtain realistic volatility structures, the analytical tractability feature can be lost as we need to add more complexity and stochastic factors to the model. Given the drawback of one-factor short-rate models in assuming a perfect correlation between rates with different maturities whenever the correlation plays a more relevant role, we need to move to models allowing for more realistic correlation patterns [15]. This can be achieved with multi-factor models, in particular with two-factor short rate models such as two factor Hull and White [39] and two factor CIR models [18]. However, even with a multi-factor model, the term structure of interest rate exhibits a limited number of degrees of freedom.

An alternative to short rate models relies on the specification of the dynamics of the entire yield curve. One of the most significant contributions for this type of models was presented by Heath, Jarrow and Morton [34], who extended the discrete binomial model of forward rates by Ho and Lee [35], to continuous time. In the arbitrage-free HJM framework, the instantaneous forward rates dynamics are fully specified through their instantaneous volatility structure. However, the main disadvantage of HJM type models comes from the fact that the instantaneous rates are not directly observed in the markets, so that the calibration to current market prices is turn out to be difficult. Having in view this drawback, LIBOR market models were introduced by Brace, Gatarek and Musiela [13] and have since become very popular mainly due to the agreement between such models and market formulas for pricing two basic derivative products: caps and swaptions. In the Lognormal Forward LIBOR Model (LFM), in the terminology of Brigo and Mercurio [15], forward LIBOR rates are assumed to follow a lognormal distribution. From this hypothesis, traders in the markets can use a Black/Scholes–like formula to price caps. Analogously, assuming that swap rates follow a lognormal distribution, they can price swaptions using the Lognormal Swap Model (LSM). Some advantages of these models are that they model rates that are observable in the market (the forward rates and the swap rates), avoid arbitrage among bonds and allow calibration to market data. However, LFM and LSM are not compatible with each other, this meaning that if forward LIBOR rates are lognormal under the associated forward measure, as assumed by the LFM, then forward swap rates cannot be lognormal under the same measure, as assumed by the LSM. Brace, Dun and Barton [12] suggest the adoption the LFM as the central model for the two markets, mainly for its mathematical tractability. Moreover, they argue that LFM can be considered for swaption pricing by using approximate equations which closely match market prices.

More recently, Eberlein and Ozkan [23] introduced an extension of the LIBOR market models based on Lévy processes. The consideration of jump processes has several advantages since their distributional flexibility allows to better capture the empirical distributions of logarithmic returns. Moreover, they allow for the introduction of infinitely many sources of risk by the use of a one-dimensional Lévy process with an infinite jump activity [32]. However, option pricing and calibration are significantly less tractable in this setting compared with the LFM.

## 2.2 Models under negative interest rates

Negative interest rates are present in current economies. For example, the European Central Bank and central banks of Switzerland, Denmark, Sweden and Japan have set negative interest rates on reserves with the argument of economic growth rate stimulation by reducing savings and encouraging borrowing at lower costs. Of course, this change of the lower bound for interest rates involves many economic consequences. However, besides this, they also involve a certain technical impact as the previously discussed models cannot cope with negative rates.

Given this change of range in the interest rates, two options are available: remove the boundary condition and allow for interest rates to assume any negative value or change the boundary condition, such that a new floor less than zero is admissible. In the first approach, examples of modelling include the short rate one-factor Hull-White model [37] and the forward rate Bachelier model [6], under which forward rate dynamics is described as a Brownian motion. The main disadvantage of both these models is that they allow the occurrence of large negative rates.

The other possibility is the class of shifted or displayed models. Brigo and Mercurio [14] propose the shifted Cox-Ingersoll-Ross (CIR++), as it is analytically tractable and can reproduce volatility smiles. In the class of market models, the Shifted LIBOR market model (SLFM) [46] provides the possibility for modeling negative interest rates while maintaining the desirable characteristics of market models. The displaced models provide a good interpretation. Moreover, if analytically formulas for pricing instruments exist in the non-shifted version of the model, they will still be attainable under the displaced version. One drawback of this methodology is that an additional shift parameter needs to defined a priori. However, since historical data offer little guidance to the lower limit that interest rates can take, the estimation of this shift parameter becomes a difficult task.

In current market practice, either the implied shifted lognormal volatility is quoted together with the shift parameter or the implied normal volatility from Bachelier model is quoted.

In this thesis, we choose to model a set of key forward rates, which can be easily obtained through prices of zero coupon bond prices observed in the market. Also, using the SLFM model, we can guarantee no arbitrage opportunities in interest rate markets and provide more flexibility to capture all possible curve movements under a coherent framework for negative interest rates.

## 2.3 Risk neutral shifted LIBOR market model

### 2.3.1 Some definitions and notations

**Definition 2.1.** A $T$**-maturity zero-coupon bond** is a contract that guarantees its holder the payment of one unit of currency at time $T$, with no intermediate payments. The contract value at time $t < T$ is denoted by $B(t, T)$.

**Definition 2.2.** The **simply compounded spot interest rate** prevailing at time $t$ for the maturity $T$ is denoted by $F(t, T)$ and it is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity starting from $B(t, T)$ units of currency at time $t$.

We can obtain $F(t, T)$ in terms of $B(t, T)$ as follows :

$$F(t, T) = \frac{1 - B(t, T)}{\delta(t, T)B(t, T)}$$

where we denote the time measure between $t$ and $T$ by $\delta(t, T)$.

**Definition 2.3.** A **forward rate agreement (FRA)** is a contract involving three time instants: the current time $t$, the expiry time $T > t$, and the maturity time $S > T$. The contract gives its holder an interest rate payment for the period from $T$ to $S$ with fixed rate $K$ at maturity $S$ against an interest payment over the same period with rate $F(T, S)$.

The value of the FRA is denoted by $FRA(t, T, S, K)$, and is given by:

$$FRA(t, T, S, K) = B(t, S)\delta(T, S)K - B(t, T) + B(t, S) \; .$$

**Definition 2.4.** The value of $K$ which makes the contract fair is the **forward LIBOR interest rate** prevailing at time $t$ for the expiry $T$ and maturity $S$. Thus, the forward LIBOR interest rate is given by

$$F(t, T, S) = \frac{1}{\delta(S - T)} \left( \frac{B(t, T)}{B(t, S)} - 1 \right) \; .$$

**Definition 2.5.** An **interest rate swap (IRS)** is a contract that exchanges payments between two differently indexed legs, starting from a future time-instant. More precisely, at every instant $T_i$ in a prespecified set of dates $T_\alpha, T_{\alpha+1}, \ldots, T_\beta$, the fixed leg pays out the amount $NK\delta_i$ corresponding to a fixed interest rate $K$, a nominal value $N$ and a year fraction $\delta_i$ between $T_{i-1}$ and $T_i$. The floating leg pays the amount $N\delta_i F(T_{i-1}, T_i)$ corresponding to the floating interest rate $F(T_{i-1}, T_i)$ resetting at the previous instant $T_{i-1}$ for the maturity given by the current payment instant $T_i$.

When the fixed leg is paid and the floating leg is received the IRS is termed Payer IRS (PFS), conversely in the other case we have a Receiver IRS (RFS). The value at time $t \leq T_\alpha$ of a RFS is given by

$$RFS(t, T, \delta, N, K) = -NB(t, T_\alpha) + NB(t, T_\beta) + N \sum_{i=\alpha+1}^{\beta} \delta_i K B(t, T_i) \; .$$

**Definition 2.6.** The value of $K$ such that the RFS contract value equals zero at time $t$

defines the **swap LIBOR rate**, $S_{\alpha,\beta}$. Thus, we have

$$S_{\alpha,\beta} = \frac{B(t,T_\alpha) - B(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \delta_i B(t,T_i)} \quad .$$

**Definition 2.7.** A **cap** is a contract that can be viewed as a payer IRS where each exchange payment is executed only if it has positive value. At every instant $T_i$ in a prespecified set of dates $T_\alpha, T_{\alpha+1}, \ldots, T_\beta$, the cap holder receives $(F(T_{i-1}, T_i) - K)^+$ for a predefined strike/cap value $K$.

When the cap has only one payment date it is called a **caplet**. A cap contract can be additively decomposed as a collection of caplets. This is exactly the market practice to price a cap, as a sum of caplet prices. Each caplet price is the price of a call on a lognormally distributed interest rate, so that the Black-formula can be applied.

**Definition 2.8.** A **floor** is a contract that can be viewed as a receiver IRS, where each exchange payment is executed only if it has positive value. At every instant $T_i$ in a prespecified set of dates $T_\alpha, T_{\alpha+1}, \ldots, T_\beta$ the floor holder receives $(K - F(T_{i-1}, T_i))^+$ for a predefined strike/floor value $K$.

Similarly to caps, the contract type where the floor has only one payment date is called a **floorlet**, and the price is obtained as the price each floorlet is obtained as the price of a put option on the interest rate.

### 2.3.2 The LIBOR market model

We consider the tenor structure $\mathcal{T} = \{T_0, T_1, \ldots, T_{N+1}\}$, with $T_0 = 0$ and where $T_j < T_k$ for $0 \leq j < k \leq N$. We define the corresponding accruals as $\delta_j = T_{j+1} - T_j$, $0 \leq j \leq N$. For $j = 0, 1, \ldots, N+1$, let us denote by $B_j(t)$ the price at time $t$ of a zero-coupon bond that matures at the tenor date $T_j$ with $T_j \geq t$. Moreover, for $j = 1, \ldots, N$, let us define by $F_j(t) = F(t, T_j, T_{j+1})$ the value at time $t \leq T_j$ of the forward LIBOR rate for the period

$[T_j, T_{j+1}]$.

The forward LIBOR rates can be obtained in terms of the bond prices by using the following relation:

$$1 + \delta_j F_j(t) = \frac{B_j(t)}{B_{j+1}(t)} \ , \ j = 1, \ldots, N. \tag{2.1}$$

In the setting of possible negative forward rates, we assume the that the diffusion coefficient for $F_j$ is given by:

$$\epsilon_j(t) \left[F_j(t) + \alpha_j\right], \tag{2.2}$$

where the shift parameter $\alpha_j$ is a constant and $\epsilon_j$ is a deterministic function of time. In this modified LIBOR market model, the dynamics of the forward rates in the terminal measure $Q^{j+1}$ is given by:

$$dF_j(t) = \left[F_j(t) + \alpha_j\right] \epsilon_j(t) \cdot dW^{j+1}, \tag{2.3}$$

where $\epsilon_j(t) = \{\epsilon_j^1(t), \ldots, \epsilon_j^r(t)\}$ is the vector of volatility functions, and $W^{j+1}(t) = \{W_1^{j+1}(t), \ldots, W_r^{j+1}(t)\}$ denotes a multidimensional Brownian motion. We note that the measure $Q^{j+1}$ is associated with the numeraire $B_{j+1}$.

We also define the process of Forward Shifted rates, as they will be useful in future results:

$$F_j^D(t) = F_j(t) + \alpha_j, \tag{2.4}$$

where $\alpha_j$ is such that $F_j^D(t) > 0$, for all $t > 0$ and $0 < j \leq N$. Note that the dynamics of $F_j^D$ under the terminal measure $Q^{j+1}$ is given by:

$$dF_j^D(t) = F_j^D(t)\epsilon_j(t) \cdot dW^{j+1}. \tag{2.5}$$

In order to express the dynamics of all different forward LIBOR rates using a common numeraire we choose a specific bond with fixed tenor, say $B_k$. However, when we discretize the drift terms that appear in the dynamics of all forward rates with tenor different from $k-1$, some rates will become more biased than others [15]. Another alternative could be the use of the standard continuously compounded bank account. In this case, the drift term of all the rates would depend on the instantaneous forward rate volatility, which cannot be deduced from discrete forward rates. The most obvious alternative that yields a more well-behaved dynamics for all the tenors comes from the consideration of a discretely balanced bank account whose value at time $t$ is given by

$$B_d(t) = B(t, T_{m(t)}) \prod_{j=0}^{m(t)-1} (1 + \delta_j F_j(T_j)) \ ,$$

where $m(t)$ is the notation for the next tenor date after time $t$, i.e., $m(t) = T_j$ if $T_{j-1} \leq t < T_j$.

In this setting $B_d$ can be understood as the value of a portfolio that starts with one unit of currency at time 0 and this unit currency is invested in $T_1$ zero-coupon bonds. Next, for each tenor, the current value is reinvested in zero-coupon bonds for the next tenor. Therefore, it can be thought as the discrete version of the continuously compounded bank account. The measure associated with the numeraire $B_d$ is called the spot measure.

**Proposition 2.1.** Under the spot measure, $Q^d$, associated with the numeraire $B_d$, the dynamics of $F_j^D(t)$, for $j = 1, \dots N$ and $t < T_j$, is the given by

$$\frac{dF_j^D(t)}{F_j^D(t)} = \epsilon_j(t) \cdot \sum_{i=m(t)}^{j} \beta_i^D(t) dt + \epsilon_j(t) \cdot dW^d(t) \ , \tag{2.6}$$

where:

$$\beta_i^D(t) = \frac{\epsilon_i(t)\delta_i F_i^D(t)}{1 + \delta_i(F_i^D(t) - \alpha_i)} \ ,$$

and $W^d$ is a multidimensional Brownian motion.

*Proof.* First, we consider the relative prices of the bonds $B(t, T_i)$ with respect to the numeraire $B_d(t)$,

$$\frac{B(t,T_i)}{B_d(t)} = \frac{B(t,T_i)}{B(t,T_{m(t)})}\left(\prod_{j=0}^{m(t)-1}(1+\delta_j F_j(T_j))\right)^{-1}$$
$$= \prod_{j=m(t)}^{i-1}(1+\delta_j F_j(t))^{-1}\left(\prod_{j=0}^{m(t)-1}(1+\delta_j F_j(T_j))\right)^{-1} . \tag{2.7}$$

Since $\frac{B(t,T_i)}{B_d(t)}$ are martingales under the spot measure, then we have

$$\text{drift}\left(\frac{B(t,T_i)}{B_d(t)}\right) = 0 . \tag{2.8}$$

By using the identity (2.7) we obtain

$$d\left(\frac{B(t,T_i)}{B_d(t)}\right) = d\left(\prod_{j=m(t)}^{i-1}(1+\delta_j F_j(t))^{-1}\left(\prod_{j=0}^{m(t)-1}(1+\delta_j F_j(T_j))\right)^{-1}\right) \tag{2.9}$$
$$= \left(\prod_{j=0}^{m(t)-1}(1+\delta_j F_j(T_j))\right)^{-1} d\left(\prod_{j=m(t)}^{i-1}(1+\delta_j F_j(t))^{-1}\right) ,$$

Moreover, the following computations in the second term can be done,

$$
\begin{aligned}
d\left(\prod_{j=m(t)}^{i-1}(1+\delta_j F_j(t))^{-1}\right) &= \sum_{j=m(t)}^{i-1}\prod_{\substack{k=m(t)\\k\neq j}}^{i-1}\frac{1}{1+\delta_k F_k(t)}d\left(\frac{1}{1+\delta_j F_j(t)}\right)\\
&+\sum_{\substack{j,l=m(t)\\j>l}}^{i-1}\prod_{\substack{k=m(t)\\k\neq j,l}}^{i-1}\frac{1}{1+\delta_k F_k(t)}d\left(\frac{1}{1+\delta_j F_j(t)}\right)d\left(\frac{1}{1+\delta_l F_l(t)}\right)\\
&=\sum_{j=m(t)}^{i-1}\prod_{k=m(t)}^{i-1}\frac{1}{1+\delta_k F_k(t)}\left(\frac{-\delta_j dF_j(t)}{(1+\delta_j F_j(t))^2}+\frac{\delta_j^2(dF_j(t))^2}{(1+\delta_j F_j(t))^3}\right)\\
&+\sum_{\substack{j,l=m(t)\\j>l}}^{i-1}\prod_{\substack{k=m(t)\\k\neq l,j}}^{i-1}\frac{1}{1+\delta_k F_k(t)}\left(\frac{-\delta_j dF_j(t)}{(1+\delta_j F_j(t))^2}+\frac{\delta_j^2(dF_j(t))^2}{(1+\delta_j F_j(t))^3}\right)\left(\frac{-\delta_l dF_l(t)}{(1+\delta_l F_l(t))^2}+\frac{\delta_l^2(dF_l(t))^2}{(1+\delta_l F_l(t))^3}\right)\\
&=\prod_{k=m(t)}^{i-1}\frac{1}{1+\delta_k F_k(t)}\left(\sum_{j=m(t)}^{i-1}\frac{-\delta_j dF_j(t)}{1+\delta_j F_j(t)}+\frac{\delta_j^2(dF_j(t))^2}{(1+\delta_j F_j(t))^2}+\right.\\
&+\left.\sum_{\substack{j,l=m(t)\\j>l}}^{i-1}\left(\frac{-\delta_l dF_l(t)}{1+\delta_l F_l(t)}+\frac{\delta_l^2(dF_l(t))^2}{(1+\delta_l F_l(t))^2}\right)\left(\frac{-\delta_j dF_j(t)}{1+\delta_j F_j(t)}+\frac{\delta_j^2(dF_j(t))^2}{(1+\delta_j F_j(t))^2}\right)\right)\\
&=\prod_{k=m(t)}^{i-1}\frac{1}{1+\delta_k F_k(t)}\sum_{j=m(t)}^{i-1}\left(\frac{-\delta_j dF_j(t)}{1+\delta_j F_j(t)}+\sum_{l=m(t)}^{j}\frac{\delta_j dF_j(t)}{(1+\delta_j F_j(t))}\frac{\delta_l dF_l(t)}{(1+\delta_l F_l(t))}\right)
\end{aligned}
\tag{2.10}
$$

Therefore, combining equations (2.7), (2.9) and (2.10) , for $i = 0, \ldots, N+1$, we obtain:

$$
\sum_{j=m(t)}^{i-1}\text{drift}\left(-\frac{\delta_j dF_j(t)}{1+\delta_j F_j(t)}+\sum_{l=m(t)}^{j}\frac{\delta_j dF_j(t)}{1+\delta_j F_j(t)}\frac{\delta_l dF_l(t)}{1+\delta_l F_l(t)})\right)=0\ .
$$

If we now consider

$$
dF_j(t)=dF_j^D(t)=F_j^D(t)\mu_j^D dt+F_j^D(t)\epsilon_j\cdot dW^d(t)\ ,
$$

and

$$
dF_j(t)dF_i(t)=F_j^D F_i^D(t)\epsilon_j\cdot\epsilon_i dt\ ,
$$

we can obtain

$$
-\frac{\mu_j^D\delta_j F_j^D(t)dt}{1+\delta_j F_j(t)}+\sum_{l=m(t)}^{j}\frac{F_j^D(t)F_l^D(t)\epsilon_j\cdot\epsilon_l\delta_j\delta_l dt}{(1+\delta_l F_l(t))(1+\delta_j F_j(t))}=0\ .
$$

Finally, we deduce that

$$\mu_j^D(t) = \sum_{l=m(t)}^{j} \frac{F_l^D(t)\epsilon_j \cdot \epsilon_l \delta_l}{1+\delta_l F_l(t)} = \sum_{l=m(t)}^{j} \frac{(F_l(t)-\alpha_l)\epsilon_j \cdot \epsilon_l \delta_l}{1+\delta_l F_l(t)} \; . \tag{2.11}$$

□

Furthermore, from (2.6) we can deduce immediately that:

$$dF_j(t) = (F_j(t)+\alpha_j) \sum_{i=m(t)}^{j} \frac{\epsilon_j(t)\cdot\epsilon_i(t)\;\delta_i\;(F_i(t)+\alpha_i)}{1+\delta_i F_i(t)} dt + (F_j(t)+\alpha_j)\;\epsilon_j(t)\cdot dW^d(t)\; . \tag{2.12}$$

One of the advantages of the SLFM framework is that it preserves the analytical tractability of the LFM model. In particular, if we consider a $T_j - caplet$, i.e., a call-option on the future LIBOR rate, set at time $T_j$ and with the payoff at time $T_{j+1}$ given by :

$$\delta_j\,[F_j(T_j) - K]^+ \; .$$

The price of the caplet at time $t = 0$ can be obtained as:

$$\begin{aligned} Cpl(T_j, T_{j+1}, \delta_j, K) &= \delta_j B_{j+1}(0)\mathbb{E}^{j+1}\left[(F_j(T_j) - K)^+\right] \\ &= \delta_j B_{j+1}(0)\mathbb{E}^{j+1}\left[(F_j^D(T_j) - (K+\alpha_j))^+\right] \; . \end{aligned}$$

Next, since $F_j^D$ follows a lognormal distribution, we can apply Black's formula [9] for pricing call-options and obtain:

$$Cpl(T_j, T_{j+1}, \delta_j, K) = \delta_j B_{j+1}(0) BL^{call}(K+\alpha_j, F_j(0)+\alpha_j, v_j)\; , \tag{2.13}$$

where $v_j = \sqrt{\int_0^{T_j} |\epsilon_j(t)|^2}$ and

$$BL^{call}(K, F, V) = F\Phi\left(\frac{\ln(F/K) + V^2/2}{V}\right) - K\left(\frac{\ln(F/K) - V^2/2}{V}\right)\; ,$$

with $\Phi$ denoting the standard normal distribution. A similar result can be obtained for pricing floorlets using Black's formula for put options.

## 2.4 Real world and risk neutral models

For pricing financial products, the value of which depends on future realizations of a certain risk factor, one can rely on risk neutral models. In this case, prices are obtained as the discounted value of expected future payoffs under the standard hypotheses on frictionless and complete markets. However, when the objective is the simulation of real future values of these underlying factors and products, as it is the case in the assessment of investment strategies in interest-rate sensitive portfolios for Asset-Liability Management studies and calculations of Economic Capital for Solvency II, the risk neutral probabilities do not represent real probabilities, as the drift of the stock prices is assumed to be the risk free rate and the forward rates are unbiased predictors of future rates. This is not realistic because that would imply that investors require no compensation for the risk of unpredictable changes in the future.

An important remark is the fact that when we generate a set of scenarios in a risk-neutral way, each individual scenario can be considered a real world scenario. The difference between risk-neutral scenarios and real world scenarios is not the paths themselves. The difference is in the probability of these scenarios occurring or, more correctly, the distribution of the scenarios. Both probability measures are equivalent, so if a path is possible in a real world setting, it is also possible in a risk-neutral setting and vice-versa.

The main mathematical tool to change from the risk neutral measure to the real world one is the Girsanov's Theorem [31], which will be the topic of the next section.

### 2.4.1 Change of measure

Girsanov's Theorem describes how the dynamics of stochastic processes change when the original measure is changed into an equivalent probability measure. In mathematical finance, it is usually used to move from the real world measure to the risk-neutral measure as a tool for pricing derivatives. Here the idea is the opposite, as we have presented the dynamics of the shifted forward LIBOR rates in the spot measure, and now we are interested in obtaining the dynamics of the process under the (physical) real world measure.

As in Equation 2.6 we consider a multidimensional Brownian motion, we will consider the multidimensional version of Girsanov's Theorem.

We start introducing important definitions in order to present the main results. For a detailed proof of the results presented, we refer to [65].

**Definition 2.9.** For any probability measure $\mathbf{Q}$ defined on the filtered space $(\Omega, \mathcal{F})$, we define the **Q-null-set** as follows:

$$\mathcal{N}_{\mathbf{Q}} := \{A \in \mathcal{F} : \mathbf{Q}(A) = 0\} \ .$$

**Definition 2.10.** Let $\mathbf{Q}$ and $\mathbf{P}$ be two measures defined on the filtered space $(\Omega, \mathcal{F})$. $\mathbf{Q}$ and are $\mathbf{P}$ equivalent when $\mathcal{N}_{\mathbf{Q}} = \mathcal{N}_{\mathbf{P}}$. In this case, we use the notation $\mathbf{Q} \sim \mathbf{P}$.

**Theorem 2.1.** Let $(\Omega, \mathcal{F}, \mathbf{Q})$ be a probability space and let $Z$ be a nonnegative random variable satisfying $E^Q[Z] = 1$. Defining $\mathbf{P}$ as

$$\mathbf{P}(A) = \int_A Z(w) d\mathbf{Q}(w) \ , \ \forall A \in \mathcal{F} \ , \tag{2.14}$$

then, $\mathbf{P}$ is a probability measure and $\mathbf{P} \sim \mathbf{Q}$. Furthermore, if $X$ is a nonnegative random variable, then $E^P[X] = E^Q[XZ]$.

**Theorem 2.2.** Let $\mathbf{Q}$ and $\mathbf{P}$ be equivalent measures defined on $(\Omega, \mathcal{F})$ . Then, there exists an almost surely positive random variable $Z$, such that, $E[Z] = 1$ and $\mathbf{P}$ satisfies (2.14).

**Definition 2.11.** Let $(\Omega, \mathcal{F}, \mathbf{Q})$ be a probability space and let $\mathbf{P}$ be another probability measure on $(\Omega, \mathcal{F})$ that is equivalent to $\mathbf{Q}$. Moreover, let Z be an almost surely positive random variable that relates $\mathbf{Q}$ and $\mathbf{P}$ through (2.14). Then $Z$ is called the **Radon-Nikodym derivative of $\mathbf{P}$ with respect to $\mathbf{Q}$** and we write $Z = \frac{d\mathbf{P}}{d\mathbf{Q}}$.

Girsanov's theorem shows the change in the dynamics of a process when we change the underlying probability measure.

**Theorem 2.3.** ***(Girsanov's Theorem)*** Let $W(t),\ 0 \le t \le T$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$, and let $\mathcal{F}_t,\ 0 \le t \le T$, be a filtration for this Brownian motion. Let $\theta(t) = \{\theta_1(t), \ldots, \theta_d(t)\}$ be a $d$-dimensional adapted process.

Define:

$$Z(t) = \exp\left\{-\int_0^t \theta(u) \cdot dW(u) - \frac{1}{2}\int_0^t \|\theta(u)\|^2 du\right\}$$

and

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(u) du$$

and assume that

$$E\left[\|\theta(u)\|^2 Z^2(u) du\right] < \infty .$$

Then, $E[Z(T)] = 1$ and, under the probability measure $\mathbf{P}$ given by (2.14), the process $\widetilde{W}(t)$ is a $d$-dimensional Brownian motion.

## 2.4.2 Market price of risk

In the setting of Theorem 2.3, we denote by $\mathbf{P}$ the real-world probability measure and by $\mathbf{Q}$ the risk-neutral measure. The problem of finding the equivalent real-world probability measure is equivalent to the problem of defining the process $\theta$ known as the market price of risk process.

By using US data, Willmot and Ahmad [2] examined the statistical properties of the spot interest rate and the yield curve to identify the behavior of the market price of interest rate risk. In their study, they concluded that the market price of interest rate risk is not constant nor even a deterministic function of the short rate as it varies wildly from day to day, and is not always negative. Hull and White [38] proposed a simple approach to construct a one-factor short rate model for both the risk neutral measure and the real world measure in which historical data are used jointly with market prices to create a joint measure model for the short rate and estimate the real world drift. Their study shows that the market price of interest rate risk depends on the maturity of interest rates used for its estimation.

In the LIBOR market model framework Norman [58] introduces a parametric form for the market price of risk vector that leads to reasonable levels and shapes for real world evolution of the yield curve. Takashi [72] presents the LIBOR market model under the real world measure in a rigorous manner and obtains the market price of risk by solving a least square problem for a lag regression model. Finally, he concludes that the market price of risk can be mostly explained by changes in the historical forward LIBOR curve.

# Chapter 3

# LIBOR Market Model under the real world measure

In this section we move from the risk neutral measure to the real world measure.

In Section 3.1, we describe the interest rates dynamics under the real world measure, prove that the SLFM is arbitrage-free under the real world measure, and we present the market price of risk process adopted for the real world setting. In Section 3.2, we explain how to estimate the parameters based on historical information. The simulation methodologies we follow are presented in Section 3.3. In the last section, we present the results of the projected rates one year ahead from the last data point by using historical data of AAA-rated European bonds.

The main results in this chapter have been published in [52].

## 3.1 The model

From the adopted risk neutral model and using Girsanov's Theorem, we can write the dynamics of the LIBOR rates under the real world measure using the following relation between the $r$-dimensional Brownian motion under the real world measure, $W^P$, and the $r$-dimensional Brownian motion under the spot measure $W^d$:

$$dW^P(t) = dW^d(t) - \theta(t)dt \, , \tag{3.1}$$

where $\theta(t)$ is the $r$-dimensional market price of risk process.

Let $T > 0$ be the time horizon and let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$, $t \in [0, T]$ be the probability space, where $\mathcal{F}_t$ is the augmented filtration, and $\mathbf{P}$ is the real world measure.
Let $\mathcal{E}$ be a set of continuous semi-martingales on $[0, T]$ , $\mathcal{E}_+ = \{X \in \mathcal{E} | X > 0\}$ and $\mathcal{E}^n = \{X | X = (X_1, \dots, X_n), X_i \in \mathcal{E}\}$.

In the following, we omit the time dependency of the variables to enhance readability and write, for instance, $F_j^D(t)$ as $F_j^D$ and $B_j(t)$ as $B_j$.

**Definition 3.1.** The price system $B \in \mathcal{E}^n$ is said to be arbitrage-free if there exists $\xi \in \mathcal{E}_+$ with $\xi(0) = 1$, such that $\xi B_j$ are $\mathbf{P}$-martingales for all $j$. $\xi$ is called the state price deflator.

**Proposition 3.1.** Let $\theta : \Omega \times [0, T] \to \mathbb{R}^d$ be an arbitrary predictable process with $\int_0^T |\theta|^2 ds < \infty$ and let $\epsilon_j : [0, T] \to \mathbb{R}^d$ be an arbitrary deterministic process for all $j$ and $\alpha_j \in \mathbb{R}_0^+$. Let $F_j^D$ be the solution to the following stochastic differential equation (SDE) with initial condition $F_j^D(0) > 0$,

$$\frac{dF_j^D}{F_j^D} = \left\{ \epsilon_j \cdot \sum_{i=m(t)}^{j} \beta_i^D + \epsilon_j \cdot \theta \right\} dt + \epsilon_j \cdot dW^P \tag{3.2}$$

where:

$$\beta_i^D = \frac{\epsilon_i \delta_i F_i^D}{1 + \delta_i (F_i^D - \alpha_i)} \, ,$$

Moreover, assume that $B_j$ has the following dynamics under the real world measure:

$$\frac{dB_j}{B_j} = \left\{ \overline{\mu} - \sum_{i=m(t)}^{j-1} \beta_i^D \cdot \theta \right\} dt - \sum_{i=m(t)}^{j-1} \beta_i^D \cdot dW^P \tag{3.3}$$

where $\overline{\mu} = \log\{1 + \delta_{m(t)-1} F_{m(t)-1}\}/\delta_{m(t)-1}$, with initial values

$B_j(0) = \prod_{i=0}^{j-1}(1 + \delta_i F_i(0))^{-1}$. Then, B is arbitrage-free and $\theta$ is a market price of risk.

*Proof.* At $t = 0$ we assume that

$$1 + \delta_j F_j(0) = B_j(0)/B_{j+1}(0),\ j = 1, \ldots, n-1\ . \tag{3.4}$$

We will prove that

$$1 + \delta_j F_j = B_j / B_{j+1}\ . \tag{3.5}$$

Since $F_j = F_j^D - \alpha_j$, we have that $dF_j = dF_j^D$ and using (3.2) we obtain:

$$\begin{aligned} \frac{d(\delta_j F_j)}{1 + \delta_j F_j} &= \frac{\delta_j F_j^D \epsilon_j}{1 + \delta_j F_j} \cdot \sum_{i=m(t)}^{j} \beta_i^D dt + \frac{\delta_j F_j^D \epsilon_j \cdot \theta}{1 + \delta_j F_j} dt + \frac{\delta_j F_j^D \epsilon_j}{1 + \delta_j F_j} \cdot dW^P \\ &= \sum_{i=m(t)}^{j} \beta_j^D \beta_i^D dt + \beta_j^D \cdot \theta dt + \beta_j^D \cdot dW^P \end{aligned} \tag{3.6}$$

The unique solution of (3.6) is given by:

$$1 + \delta_j F_j(t) = (1 + \delta_j F_j(0)) \exp\left\{ \int_0^t \left( \beta_j^D \cdot \theta + \sum_{i=m(t)}^{j} \beta_j^D \beta_i^D - \frac{|\beta_j^D|^2}{2} \right) ds + \int_0^t \beta_j^D \cdot dW^P(s) \right\} \tag{3.7}$$

Moreover, equation (3.3) has the following solution:

$$B_j(t) = B_j(0) \exp\left\{ \int_0^t \left( \overline{\mu} - \sum_{i=m(t)}^{j-1} \beta_i^D \cdot \theta - \frac{1}{2} \left| \sum_{i=m(t)}^{j-1} \beta_i^D \right|^2 \right) ds - \int_0^t \sum_{i=m(t)}^{j-1} \beta_i^D \cdot dW^P(s) \right\} \tag{3.8}$$

and it follows that:

$$\frac{B_j(t)}{B_{j+1}(t)} = \frac{B_j(0)}{B_{j+1}(0)} \exp\left\{ \int_0^t \left( \beta_j^D \cdot \theta + \sum_{i=m(t)}^{j} \beta_i^D \beta_j^D - \frac{|\beta_j^D|^2}{2} \right) ds + \int_0^t \beta_j^D \cdot dW^P(s) \right\}. \tag{3.9}$$

Thus, using the initial assumption (3.4) we can conclude that the left hand side of equation (3.7) and (3.9) are equal.

In order to prove that $B$ is arbitrage-free we define the process $\xi : \Omega \times [0,T] \to \mathbb{R}$, satisfying the SDE:

$$\frac{d\xi}{\xi} = -\overline{\mu} dt - \theta \cdot dW^P, \tag{3.10}$$

with $\xi(0) = 1$ and obtain the dynamics of the process $\xi B_j$:

$$\begin{aligned} d(\xi B_j) &= B_j d\xi + \xi dB_j + dB_j d\xi = \\ &= B_j \xi \left\{ -\overline{\mu} dt - \theta \cdot dW^P \right\} + \xi B_j \left\{ \left\{ \overline{\mu} - \sum_{i=m(t)}^{j-1} \beta_i^D \cdot \theta \right\} dt - \sum_{i=m(t)}^{j-1} \beta_i^D \cdot dW^P \right\} \\ &+ B_j \xi \left\{ -\overline{\mu} dt - \theta \cdot dW^P \right\} \left\{ \left\{ \overline{\mu} - \sum_{i=m(t)}^{j-1} \beta_i^D \cdot \theta \right\} dt - \sum_{i=m(t)}^{j-1} \beta_i^D \cdot dW^P \right\} = \\ &= -\left( B_j \xi \right) \left\{ \theta + \sum_{i=m(t)}^{j-1} \beta_i^D \right\} \cdot dW^P. \end{aligned}$$

Hence $\xi B_j$ is a $\mathbf{P}$-martingale for all $j$ , so we can conclude that $B$ is arbitrage-free. □

Next, we assume that $F_j$ has a piecewise constant instantaneous volatility, and therefore also $F_j^D$. In this case, the volatilities depend only on the time-to-maturity (whole reset periods between time $t$ and the maturity of the rate), i.e:

$$\epsilon_j^k(t) = \lambda_{j-m(t)}^k \ ,$$

and we can write the volatility functions in terms of an orthonormal basis[1] $\Lambda_i^k$ :

$$\lambda_i^k = \sigma_k \Lambda_i^k \ .$$

Let us consider the process of the market price of risk $\theta(t)$ as the excess return for taking a unit amount of interest rate risk. [2] argue that since investors are not always rational, this process should not be modeled as a constant (nor even as a piecewise constant function). Also, for a fixed time, the market price of risk must be common to every interest rate derivative with the same set of interest rates as underlying.

Since the specification of the market price of risk process adopted in [58] takes into account these two properties, we will consider an appropriate variation. In this case, the $k$ component of the market price of risk is assumed to have the following structure:

$$\theta^k(t) = \frac{a^k}{\sigma^k}\Big(b^k - \sum_{i=m(t)}^{N+m(t)-1} \Lambda_{i-m(t)}^k \ln(F_i^D(t))\Big) \ , \tag{3.11}$$

where $a^k$, $\sigma^k$ and $b^k$ are parameters to be estimated from historical data and the vectors $\Lambda_i$ are the coefficients obtained from the principal component analysis of the historical covariance matrix of the logarithm of shifted rates.

Under this specification of the market price of risk process, we assume that the market price of risk is a linear function of the observed forward rates and introduce a mean reversion dynamics in the forward rates, where the parameters $b^k$ and $a^k$ represent the long run mean and speed of reversion of the $k$th factor, respectively.

Since we only have a finite number of observed rates, we need to extrapolate the interest rates curve. For this purpose, we use the following recursive formula for future and

[1] $\sum_{i=0}^{N-1} \Lambda_i^k \Lambda_i^l = \delta^{kl}$ where $N$ can be interpreted to be the number of maturities observable on the forward rate curve.

unobserved rates:

$$F_{i+N}(t) = \alpha\Big((1+\beta)F_{i+N-1}(t) - \beta F_{i+N-2}(t)\Big) + (1-\alpha)c + \sigma\epsilon\,, \quad i = 1, 2, \dots, \tag{3.12}$$

where the parameters $\alpha$ and $\beta$ reflect, respectively, the dependence on the long term interest rate $c$ and how much dependence on the slope of the yield curve the new rate has. Moreover, $\epsilon$ is a normal random variate and $\sigma$ its standard deviation. With this specification, $\alpha$, $\beta$ and $\sigma$ can be chosen by using the risk manager's view of the future and $c$ can reflect the market expectations on the long term evolution of the interest rates.

## 3.2 Parameter estimation

In this section, the method for calibrating the real world model to historical data is detailed. As already introduced, the real world dynamics of interest rates will be useful to predict future evolutions in the markets. Moreover, in order to consistently perform the simulations, we need to understand the past and incorporate that information in the model in which the decision making process will be based. So, instead of calibrating the model to current prices, we use historical estimation of the parameters of the model.

Adopting the methodology introduced in [58] and generalizing for a setting with negative rates. We first consider a common constant accrual, i.e. $\delta = \delta_j = T_{j+1} - T_j$, and introduce the following notation for the constant maturity default-free forward shifted rates:

$$L_j^D(t) = F^D(t, t + j\delta, t + (j+1)\delta)\ , \ \ j = 1, \dots, N. \tag{3.13}$$

More precisely, $L_j^D(t)$ denotes the observed value at time $t$ of the forward shifted rates for the period $[t + \delta j, t + \delta(j+1)]$.

Under this notation, the dynamics of $L_j^D$ satisfies:

$$dL_j^D(t) = L_j^D(t)\{\lambda_j \cdot \theta(t) + \lambda_j \cdot \sum_{i=1}^{j} \beta_i^D(t)\}dt + L_j^D(t)\lambda_j \cdot dW^P(t). \tag{3.14}$$

For convenience, we consider the stochastic differential equation for the dynamics of the logarithm of the forward shifted rates:

$$d\ln L_j^D(t) = \left(\lambda_j \cdot \theta(t) + \lambda_j \cdot \sum_{i=1}^{j} \beta_i^D(t) - \frac{1}{2}\|\lambda_j\|^2\right) dt + \lambda_j \cdot dW^P(t)\ . \tag{3.15}$$

Next, we consider the covariance matrix $\Sigma$ of the historical changes in the logarithms of forward shifted rates and, using Principal Components Analysis, we recover the $\Lambda_i^k$'s and $\sigma_k$'s as the eigenvectors and eigenvalues of $\Sigma$, respectively. Also in this step, we define the dimension of the volatility functions and the number of components of the market price of risk process. Empirically, changes in interest rates can be largely explained by a small number of factors: level, slope, bow, and higher order perturbations.

By considering a small time step, $\Delta t$, between observations and using the Euler-Maruyama scheme, we approximate the evolution of the logarithm of forward shifted rates as follows:

$$\ln L_j^D(t+\Delta t) \approx \ln L_j^D(t) + \left(\lambda_j \cdot \theta(t) + \overline{\mu}_j^D(t) - \frac{1}{2}\|\lambda_j\|^2\right) \Delta t + \lambda_j \cdot \Delta W^P(t)\ , \tag{3.16}$$

where

$$\overline{\mu}_j^D(t) = \lambda_j \cdot \sum_{i=1}^{j} \beta_i^D(t) = \sum_{i=1}^{j} \frac{\delta\lambda_j\lambda_i L_i^D(t)}{1+\delta L_i(t)}\ . \tag{3.17}$$

If we define

$$\ln \widetilde{L}_j^D(t) = \ln L_j^D(t) + \{\overline{\mu}_j^D(t) - \frac{1}{2}\|\lambda_j\|^2\}\Delta t \tag{3.18}$$

we have that

$$\ln L_j^D(t+\Delta t) = \ln \widetilde{L}_j^D(t) + \lambda_j \cdot \theta(t)\Delta t + \lambda_j \cdot \Delta W^P \ . \tag{3.19}$$

Thus, by multiplying (3.19) by $\Lambda_j^k$ and summing over $j$, we obtain

$$f^k(t+\Delta t) = \widetilde{f}^k(t) + a^k(b^k - f^k(t))\Delta t + \sigma^k \Delta W^k \text{ for } k = 1, 2, \dots r, \tag{3.20}$$

where we have introduced

$$f^k(t) = \sum_{j=1}^{N} \Lambda_j^k \ln L_j^D(t) \ , \tag{3.21}$$

and

$$\widetilde{f}^k(t) = \sum_{j=1}^{N} \Lambda_j^k \ln \widetilde{L}_j^D(t) \ . \tag{3.22}$$

In order to estimate the parameters $a^k$ and $b^k$, first from the historical data we define the following series of observations:

$$Y^k(t) = f^k(t+\Delta t) - \widetilde{f}^k(t)$$

and

$$X^k(t) = f^k(t) \ .$$

Next, we estimate the $k$ regression models obtained from equation (3.20):

$$Y^k(t) = c^k + m^k X^k(t) + e^k(t) \ , \tag{3.23}$$

where

$$e^k(t) = \sigma^k \Delta W^k \ , \quad m^k = -a^k \Delta t \ , \quad c^k = a^k b^k \Delta t \ , \tag{3.24}$$

so that we can recover $a^k$ and $b^k$ from $m^k$ and $c^k$.

## 3.3 Real world scenarios simulation

After obtaining all the market price of risk coefficients, we take the discretized model and the predictor-corrector method for our simulations. In this case, the SDE describing the evolution of the shifted forward rates (2.6) involves a state-dependent drift, which implies that there is no analytic solution to the SDE so that it must be numerically approximated.

Many authors proposed different approximations schemes to simulate the risk neutral dynamics of forward LIBOR rates as the predictor-corrector approximation in [40], the drift-free simulation method in [73] and also the parameterized drift-free simulation in [29].

Here we use the predictor-corrector method. The idea is first to evolve forward rates assuming that all state variables in the drift are constant (frozen at the previous time step), recompute the drift at the evolved time, and average the two drifts. Next, we recompute the forward rates using this averaged drift and the same random numbers.

At step one, we compute drifts by using the observed rates:

$$\widetilde{\mu}_j(t) = \sum_{i=1}^{j} \frac{\delta \lambda_j \lambda_i (F_i(t) - \alpha_i)}{1 + \delta F_i(t)}, \tag{3.25}$$

thus obtaining the first approximation of the rates as :

$$\begin{aligned} \overline{F}_j(t+\Delta t) + \alpha_j =& (\overline{F}_j(t) + \alpha_j) \\ & \times \left\{ \left( \lambda_j \cdot \theta(t) + \widetilde{\mu}_j(t) - \frac{1}{2}\|\lambda_j\|^2 \right) \Delta t + \sqrt{\Delta t} \lambda_j \cdot \Delta W \right\}, \end{aligned} \tag{3.26}$$

where

$$\theta^q(t) = \frac{a^q}{\sigma^q}\left( b^q - \sum_{j=1}^{N} \Lambda_j^q \ln(F_j(t) + \alpha_j) \right) \text{ for } q = 1, \ldots, d \tag{3.27}$$

and $\Delta W \sim N(0,1)$ .

Next, we repeat the previous step using the $\overline{F}_j$'s instead of the $F_j$'s to compute $\widetilde{\mu^*}_j(t)$ and $\theta^{*q}(t)$, thus obtaining the final approximation of the rates as:

$$F_j(t + \Delta t) + \alpha_j = (F_j(t) + \alpha_j) \times \exp\left\{\left(\lambda_j \cdot \frac{\theta(t) + \theta^*(t)}{2} + \frac{\widetilde{\mu}_j(t) + \widetilde{\mu^*}_j(t)}{2} - \frac{1}{2}\|\lambda_j\|^2\right)\Delta t + \sqrt{\Delta t}\lambda_j \cdot \Delta W\right\}, \tag{3.28}$$

using the same values for $\Delta W$.

## 3.4 Results

In this section, we present the estimation and simulation results of the real world LIBOR market model. We use a three-factor version of the model.

The parameter estimates are obtained from the monthly historical observations of European AAA-government yield curves, ranging from 31 January 2015 up to 30 June 2016, which are available in the European Central Bank database. **Table 3.1** presents the forward rate curve observed on 30 June 2016. At that time, negative forward rates of one year tenor for the maturities of one to four years were prevailing.

| Maturity | Forward Rate | Maturity | Forward Rate |
|---|---|---|---|
| 1 | -0.671% | 16 | 0,760% |
| 2 | -0.611% | 17 | 0,763% |
| 3 | -0.446% | 18 | 0,766% |
| 4 | -0.229% | 19 | 0,767% |
| 5 | -0.008% | 20 | 0,768% |
| 6 | 0.189% | 21 | 0,769% |
| 7 | 0.350% | 22 | 0,769% |
| 8 | 0.474% | 23 | 0.770% |
| 9 | 0.566% | 24 | 0.770% |
| 10 | 0.632% | 25 | 0.770% |
| 11 | 0.678% | 26 | 0.770% |
| 12 | 0.709% | 27 | 0.770% |
| 13 | 0.730% | 28 | 0.770% |
| 14 | 0.744% | 29 | 0.770% |
| 15 | 0.754% | 30[a] | 0.770% |

[a] Computed using extrapolation.

Table 3.1: One year forward rates observed on 30 June 2016

By using the historical forward rates, we compute the covariance matrix $\Sigma$ between monthly changes in the logarithm of one year shifted forward rates of term $i$ and term $j$ $(i, j = 1, \dots, 29)$. Since the lowest observed value of a forward rate for the period in study was -0.67% (one-year forward rate with maturity on 28 of February of 2016), we defined $\alpha_k = \alpha = 0.7\%$ as the shift parameter for the following results to be presented.

The first three principal components corresponding to the decomposition of the covariance matrix are shown in **Figure 3.1**. In this figure, we can identify the level, slope, and bow factors, as usually in yield curve studies. The first three principal components explain 98.84% of the covariance of the data.

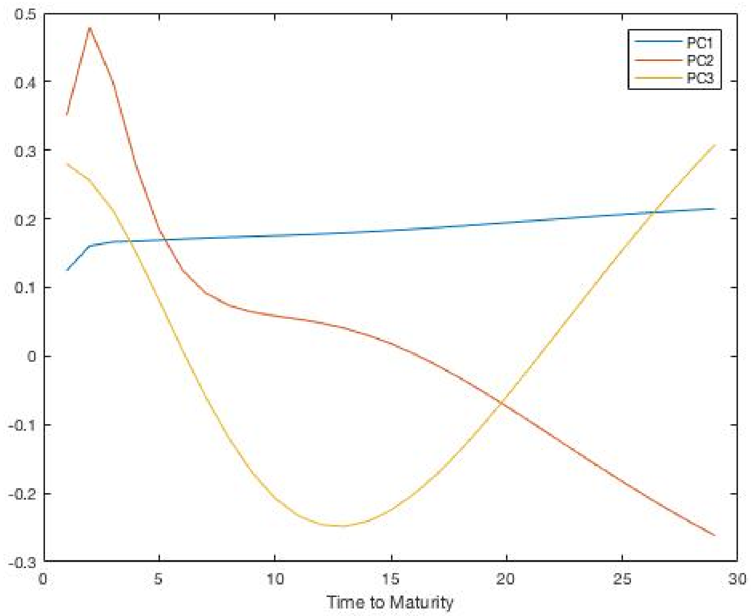


Figure 3.1: The first three principal components obtained from a principal components analysis of monthly observed European shifted forward rates.

The parameters of the market price of risk process are shown in **Table 3.2**. The results show low mean reversion rate parameters, meaning that any disturbance on the factors has a long term effect on the future rates. This can be explained by the strong influence that European Central Bank has had on the current levels of the European bond interest rates, specifically with the Expanded Asset Purchase Programme and Covered Bond Purchase Programmes.

| Factor p | $a^p$ | $b^p$ | $\sigma^p$ |
|---|---|---|---|
| 1 | 0.0267 | -20.3854 | 0.6033 |
| 2 | 0.0112 | -2.9118 | 0.0565 |
| 3 | 0.0248 | -0.9319 | 0.0230 |

Table 3.2: Estimated market price of risk Parameters

In **Figure 3.2** we exhibit ten thousand paths of the forward rate curve in the 1st year of simulations and we can compare it with the observed forward curve in June 2017. In **Figure 3.3** we present the corresponding simulation results for the zero coupon bond prices.

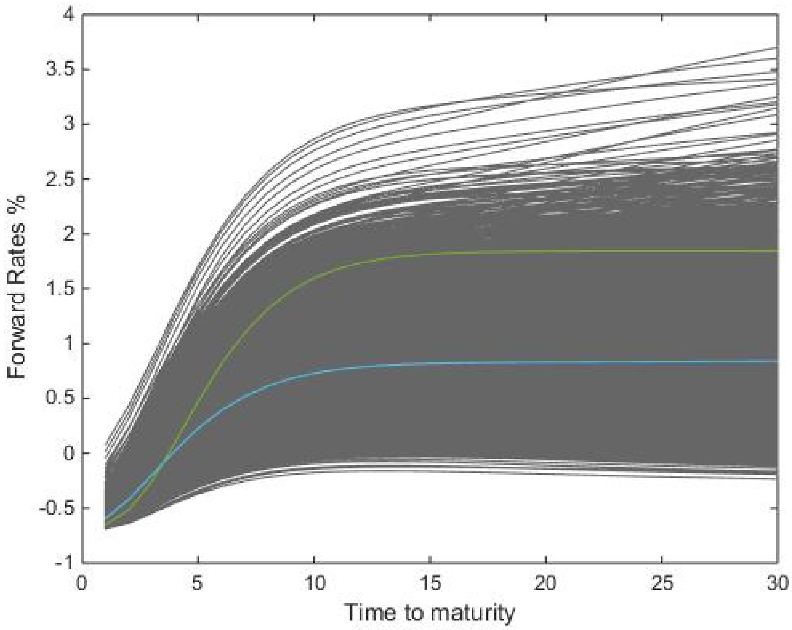


Figure 3.2: Ten thousand simulations of 1 year-ahead forward rates (grey), mean of the simulatons (blue) and observed forward curve in 30 of June of 2017 (green). For simulation, data from December 2015 to June 2016 have been used.

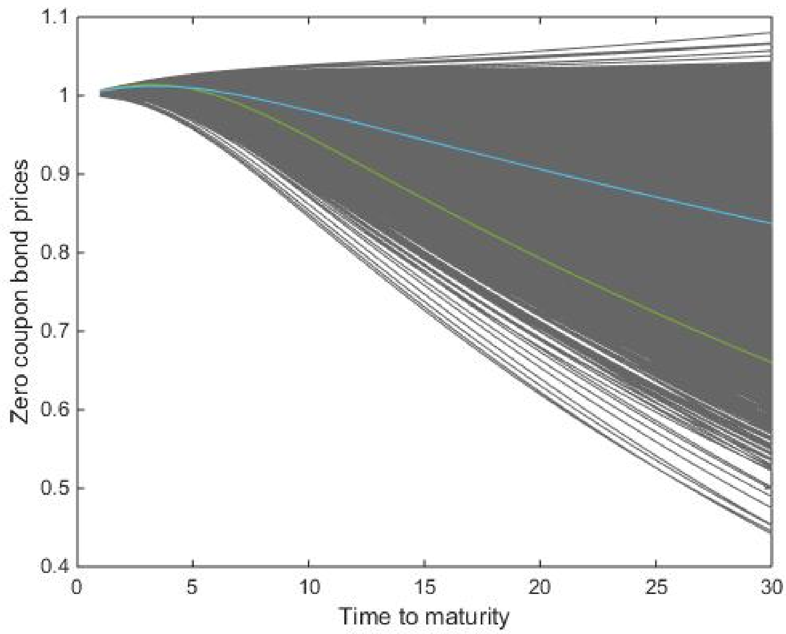


Figure 3.3: Ten thousand simulations of the zero coupon bond prices at the end of the first projection year (grey), mean of simulations (blue) and observed zero coupon bond prices at 30 of June of 2017 (green). For simulation, data from December 2015 to June 2016 have been used.

The expected value and 97% confidence interval for each zero-coupon bond price with maturities 1 to 30 years were computed using the ten thousand simulations, and the results are shown in **Table 3.3**. By comparing the observed values for the bond prices in June of 2017 with the one year ahead predictions, we can report that for all maturities, the market prices of bonds fall inside the confidence interval.

| Maturity | Mean | Conf Interval | Obs June 2017 |
|---|---|---|---|
| 1 | 1.0060 | [1.0046,1.0073] | 1.0065 |
| 2 | 1.0102 | [1.0065,1.0139] | 1.0117 |
| 3 | 1.0122 | [1.0052,1.0192] | 1.0143 |
| 4 | 1.0120 | [1.0006,1.0233] | 1.0133 |
| 5 | 1.0097 | [0.9932,1.0263] | 1.0086 |
| 6 | 1.0058 | [0.9833,1.0283] | 1.0004 |
| 7 | 1.0007 | [0.9717,1.0296] | 0.9896 |
| 8 | 0.9946 | [0.9588,1.0304] | 0.9767 |
| 9 | 0.9880 | [0.9451,1.0308] | 0.9624 |
| 10 | 0.9809 | [0.9308,1.0310] | 0.9473 |
| 11 | 0.9736 | [0.9162,1.0309] | 0.9316 |
| 12 | 0.9661 | [0.9014,1.0308] | 0.9157 |
| 13 | 0.9585 | [0.8865,1.0306] | 0.8998 |
| 14 | 0.9510 | [0.8717,1.0303] | 0.8838 |
| 15 | 0.9434 | [0.8569,1.0300] | 0.8681 |
| 16 | 0.9359 | [0.8422,1.0296] | 0.8525 |
| 17 | 0.9284 | [0.8276,1.0293] | 0.8372 |
| 18 | 0.9210 | [0.8131,1.0289] | 0.8221 |
| 19 | 0.9136 | [0.7987,1.0286] | 0.8072 |
| 20 | 0.9063 | [0.7844,1.0283] | 0.7926 |
| 21 | 0.8991 | [0.7703,1.0279] | 0.7783 |
| 22 | 0.8920 | [0.7563,1.0276] | 0.7642 |
| 23 | 0.8849 | [0.7424,1.0273] | 0.7503 |
| 24 | 0.8778 | [0.7287,1.0270] | 0.7368 |
| 25 | 0.8709 | [0.7151,1.0267] | 0.7234 |
| 26 | 0.8640 | [0.7016,1.0264] | 0.7103 |
| 27 | 0.8572 | [0.6883,1.0261 | 0.6975 |
| 28 | 0.8504 | [0.6751,1.0258] | 0.6848 |
| 29 | 0.8437 | [0.6620,1.0255] | 0.6724 |
| 30[a] | 0.8371 | [0.6491,1.0251] | 0.6602 |

[a] Computed using extrapolation.

Table 3.3: Expected value, 97% confidence interval of zero coupon bond prices and 30 June 2017 observations

In **Figure 3.4** we present ten thousand paths for the forward rate curve for June 2018, using the historical observations from January 2015 through June 2017. For this period in study, the lowest observed rate was -0.91% (one-year forward rate with maturity one year, observed on February 28th 2017) so we defined $\alpha_k = \alpha = 1\%$ as the shift parameter for the next results presented. **Table 3.4** summarizes the results of the mean and standard deviations of simulations for the zero-coupon bond prices.

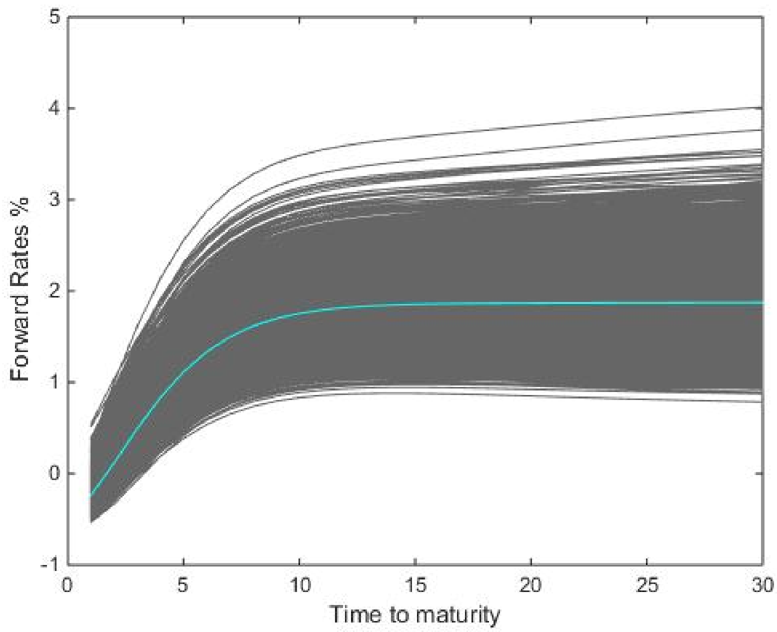


Figure 3.4: Ten thousand simulations(grey) of one year-ahead forward rates and mean curve (blue). For simulation, data from December 2015 to June 2017 have been used

| Maturity | Mean | Standard Deviation |
|---|---|---|
| 1 | 1.0023 | 0.0020 |
| 2 | 1.0011 | 0.0023 |
| 3 | 0.9962 | 0.0025 |
| 4 | 0.9880 | 0.0027 |
| 5 | 0,9772 | 0.0029 |
| 6 | 0.9643 | 0.0030 |
| 7 | 0.9501 | 0.0031 |
| 8 | 0.9351 | 0.0032 |
| 9 | 0.9195 | 0.0033 |
| 10 | 0.9037 | 0.0034 |
| 11 | 0.8878 | 0.0034 |
| 12 | 0.8721 | 0,0035 |
| 13 | 0.8565 | 0,0035 |
| 14 | 0.8410 | 0,0036 |
| 15 | 0.8258 | 0,0036 |
| 16 | 0.8109 | 0,0036 |
| 17 | 0.7962 | 0,0037 |
| 18 | 0.7818 | 0,0037 |
| 19 | 0.7676 | 0,0037 |
| 20 | 0.7537 | 0,0038 |
| 21 | 0.7401 | 0,0039 |
| 22 | 0.7267 | 0,0039 |
| 23 | 0.7135 | 0,0039 |
| 24 | 0.7006 | 0,0040 |
| 25 | 0.6880 | 0,0040 |
| 26 | 0.6755 | 0,0040 |
| 27 | 0.6633 | 0,0041 |
| 28 | 0.6514 | 0,0041 |
| 29 | 0.6396 | 0.0041 |
| 30 | 0.6281 | 0.0042 |

Table 3.4: Expected value and standard deviation of one year-ahead zero-coupon bond prices

# Chapter 4

# Credit risk models

In this chapter, we provide an overview of credit risk models and introduce some definitions and tools we use later in this work. In Section 4.1, we give an overview of the existing literature in credit risk modeling. In Section 4.2, we present the main definitions used to set the model approach under the risk neutral measure.

## 4.1 Overview of credit risk models

Credit risk represents an important subject in mathematical finance and contributes significantly to the solvency capital requirement of insurance undertakings.

According to [61] credit risk refers to the risk associated with any kind of credit-linked events, such as: changes in the credit quality (including downgrades or upgrades in credit ratings), variations of credit spreads, and the default event. Default occurs when a counterparty fails to meet its obligations in accordance with agreed terms in a contract. The simplest example of such a contract is a bond, where the issuer may default (with a certain probability) before or at the maturity date. In case that the issuer defaults, the bondholder receives only a partial amount of the promised payment or nothing at all. Therefore, investors of corporate and sovereign bonds must assume not only interest rate risk but also credit risk.

Many issuers of defaultable debt, including sovereigns, are assigned a publicly available credit rating by rating agencies, such as Standard and Poor's, Moody's, or Fitch. A credit rating represents a quantified evaluation of the creditworthiness of a borrower, in general terms or with respect to a particular debt or financial obligation. It signals to market investors the level of default risk associated with certain company or instrument. In some cases, due to regulation guidelines of risk mitigation, investors are restricted to investments in certain rating classes. This classification into rating classes allows the derivation of benchmark credit curves for each rating class, which leads to complexity reduction. Since both the current credit rating of a bond and the changes in credit ratings have a significant effect on how the price is formed in the markets, when modeling credit spreads, one should take into account the dynamics of the spreads for each credit rating and also the transition probabilities between these ratings.

Modelling default is a difficult task given that default occurrences are rare. Sometimes they occur unexpectedly, as firms try to avoid public dissemination of information when faced with financial problems. Moreover, default implies losses that are unknown before default occurs. Schönbucher in [64] decomposes credit risk in four components. First component is arrival risk, a term for the uncertainty whether a default will occur or not measured by the probability of default. Second component is timing risk, which refers to the uncertainty about the precise time of default. Third components is recovery risk that describes the uncertainty about the severity of the losses if a default has happened. The fourth component is market risk, which describes the risk of changes in the market price of a defaultable asset, even if no default occurs. A credit risk model should aim to capture all the different components. However, this implies the model can be extremely complex, which can make it computational unfeasible.

We refer to two different approaches to credit risk modelling: structural models and reduced-form models. The first type was introduced by Merton [55] and Black and Cox [10]. In structural models, credit risk of a corporate bond is measured by relating the firm value of the issuing company to its liabilities since the value of the firms assets is modeled exogenously, and default takes place as the first time the value process of the firm hits a certain barrier. The foundation on fundamentals allows us to obtain correct relationships between different securities from the same entity, although it has the drawback that it turns out to be very hard to define a meaningful process for the firm's value, and data are rarely available. Another problem faced with this modeling approach comes from the unrealistic short-term spreads it implies. Spreads result very low and tend towards zero as the maturity of the debt decreases, which is contrary to empirical observations where short maturity spreads are not negligible because even close to maturity the bondholder is uncertain whether the full amount of money will be paid back or not.

In reduced-form models, the time of default is modeled directly as a stopping time with an intensity process to be specified. Usually a Poisson process with random intensity (a Cox process) is considered. This second approach is followed by Jarrow and Turnbull in [44], where they consider that default is driven by a Poisson process with constant intensity and known payoff at default. In Madan and Unal [54], the constant intensity is extended to the case of random intensity. In Singleton and Duffie [20], the recovery rate is modeled as a fraction of the value of the security immediately before default occurs.

The idea to make the probability of default of a corporate bond depend on the rating of the issuer was proposed later with Lando [49]. Jarrow, Lando and Turnbull in [43] presented a model for valuing risk debt that explicitly incorporates a firm's rating as an indicator of the likelihood of default using a deterministic Markov chain. Arvantis et al. [5] follow the same idea while proposing a stochastic generator for a continuous-time Markov chain. These credit models can be combined with any desired term-structure model for a default-free debt and take into account the historical transition probabilities for the various credit classes to determine the risk-adjusted probabilities used in valuation.

The inclusion of credit risk in interest rate market models was a recent development introduced by Schönbucher in [63], who extended the forward LIBOR market model by adding the so called defaultable forward LIBOR rates to the model. Moreover, he introduced the survival-based pricing measures as a tool for the pricing of defaultable payoffs, which allows the derivation of the no-arbitrage dynamics of defaultable forward rates and forward credit spreads.

Even more recently, Eberlein, Kluge, and Schönbucher [22] constructed the Lévy LIBOR model with default risk driven by a time-inhomogeneous Lévy process. Later Eberlein and Grbac [21] extended the classical definition of the default-free forward Libor rate and developed a rating based LIBOR market model to cover defaultable bonds with credit ratings. Moreover, they modeled credit migration by a conditional Markov chain.

## 4.2 Notation and risk neutral model setup

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{Q})$ be the filtered probability space under the spot risk-neutral probability measure, $\mathbf{Q}$.

Credit ratings is a relevant concept in credit risk models. We consider the case where the credit ratings are identified as elements of a finite set $\mathcal{K} = \{1, 2, \dots, K\}$. The rating classes are ordered by their credit risk. This means that rating class 1 represents the best possible credit quality while class $K-1$ is the worst non-defaulted credit quality, and the last class $K$ represents the default state.

**Definition 4.1.** The **rating of a bond** is a stochastic process, denoted by $C$, so that for each time $t$ the rating of the bond is a random variable $C(t) : \Omega \to \mathcal{K}$.

Moreover, we consider the case where the migration between credit ratings is modeled by a continuous-time conditional Markov chain $C$ with state space $\mathcal{K}$.

Under the previous assumption, the rating transition process satisfies the Markov property, thus, at any time $t$, the probability of transition to another rating until time $T > t$ only depends on the current rating $C(t)$ of the process:

$$P[C(T) = r|\mathcal{F}_t] = P[C(T) = r|C(t)] \ , \ r = 1, \ldots K \ .$$

A stochastic process satisfying the Markov property is defined a Markov process. Additionally, if the state space of this Markov process is countable, the process is defined a Markov chain.

**Definition 4.2.** The **transition probability matrix** $P(t, T)$ for the time interval $[t, T]$ is defined as:

$$P(t,T) = \begin{pmatrix} p_{1,1}(t,T) & p_{1,2}(t,T) & \ldots & p_{1,K}(t,T) \\ \vdots & \vdots & & \vdots \\ p_{K-1,1}(t,T) & p_{K-1,2}(t,T) & \ldots & p_{K-1,K}(t,T) \\ p_{K,1}(t,T) & p_{K,2}(t,T) & \ldots & p_{K,K}(t,T) \end{pmatrix}$$

where, for all $i, j \in \mathcal{K}$, the component $p_{i,j}(t, T)$ represents the probability that the rating process changes to rating $j$ at time $T$, given that it was in rating $i$ at time $t$; that is

$$p_{i,j}(t,T) = P[C(T) = j|C(t) = i] \ , \ t \leq T.$$

In particular, the last element in each row of the matrix represents the default probability of each credit rating, i.e. $p_{1,K}(t, T), \ldots, p_{K-1,K}(t, T), p_{K,K}(t, T)$.

Note that the sum of the coefficients of each row equals one, since we always consider the case where the entity can only transition to one of the credit ratings.[1] That is,

$$\sum_{j=1}^{K} p_{i,j}(t,T) = 1 \ , \ \forall i \in \mathcal{K}.$$

Moreover, we assume that if an entity enters default it can not recover, this meaning that the default state $K$ is absorbing. So, it holds that:

$$p_{K,1}(t,T) = \ldots = p_{K,K-1}(t,T) = 0 \quad \text{and} \quad p_{K,K}(t,T) = 1 \quad \text{with} \quad t \leq T \ .$$

Moreover, we assume the transition probabilities only depend on the time interval over which the transitions take place (time-homogeneity), that is

$$P(t,T) = P(T-t) \ , \ \forall \, t \leq T.$$

The generator matrix is another useful concept when modeling using Markov chains, as it allows to obtain transition matrices for different time intervals.

**Definition 4.3.** A matrix $\mathbf{G} = (g_{i,j})_{i,j=1,\ldots,K}$ is called a **generator matrix** if it has the following three properties:

- All diagonal entries are not positive, i.e. $g_{i,i} \leq 0, \ i = 1, \ldots, K$
- All non-diagonal entries are not negative. i.e. $g_{i,j} \geq 0, \ i,j = 1, \ldots, K$ and $i \neq j$
- The sum of each row is zero, i.e., $\sum_{j=1}^{K} g_{i,j} = 0, \ i = 1, \ldots, K$

**Definition 4.4.** Let $P(0,1)$ be the one-period transition probability matrix of a time-homogeneous Markov chain. If there exists a generator matrix with the properties stated in Definition 4.3 and such that $P(0,1) = e^{\mathbf{G}}$ then the matrix $\mathbf{G}$ is the **generator matrix of** $P$.

[1]in practice, a company can leave the rating system and get an unrated status.

**Definition 4.5.** For $A \in \mathcal{F}$ we denote by $\mathbf{1}_A$ the **indicator function** of $A$, i.e

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

**Definition 4.6.** The **default time** is defined as the first time the Markov chain reaches the state $K$: $\tau = \inf\{t > 0,\ C(t) = K\}$. The **survival indicator function** $I$ is defined as $I(t) = \mathbf{1}_{\{\tau > t\}}$. Thus, the survival indicator function $I$ is one before default and jumps to zero at the time of default $\tau$.

**Definition 4.7.** A $T$**-maturity defaultable zero-coupon bond** is a contract that pays its holder one unit of currency at time $T$, with no intermediate payments if no default occurs before $T$. If default occurs before the maturity the bond's holder receives a reduced payment, called the recovery payment. The contract value at time $t < T$ is denoted by $\overline{B}(t,T)$.

**Definition 4.8.** The **recovery rate** is the amount, expressed as a percentage, recovered from a loan that is unable to settle the full outstanding amount.

In the literature, the most common assumptions on the recovery rate are: the zero-recovery, recovery of market value, recovery of par value and recovery of treasury. In the case of zero-recovery assumption, if default occurs the bondholder receives nothing. When recovery of face value is assumed, a fixed fraction of the face value of the bond is paid to the investors at time of default, whereas, under the recovery of treasury assumption, a fixed fraction of the face value of the bond is paid to the investors at time of maturity. Finally, if recovery of treasury is assumed, the recovery value is expressed in terms of the market value of equivalent default-free assets. For a detailed comparison of traditional recovery models see [50] and [64].

We assume that recovery rates depend on the rating class from which the bond has defaulted. We represent the recovery rate by a vector $q = (q_1, q_2, \ldots, q_{K-1})$. Moreover, we assume the recovery of treasury value hypothesis.

In this setup, the payoff at maturity of a defaultable zero-coupon bond with maturity $T_j$ and migration process $C$ is given by

$$\overline{B}_j(T_j) = \mathbf{1}_{\{\tau > T_j\}} + \mathbf{1}_{\{\tau \leq T_j\}} q_{C_{\tau-}} = \sum_{i=1}^{K-1} \mathbf{1}_{\{C(T_j)=i\}} + \mathbf{1}_{\{C(T_j)=K\}} q_{C_{\tau-}} \ , \tag{4.1}$$

where $C_{\tau-} := C(\tau-)$ denotes the last rating prior to default (i.e. the rating class from which the bond has defaulted). Hence,we postulate that the defaultable bond price process is given by

$$\overline{B}_j(t) = \sum_{i=1}^{K-1} B_j^i(t)\mathbf{1}_{\{C(t)=i\}} + B_j(t) q_{C_{\tau-}} \mathbf{1}_{\{C(t)=K\}} \ , \tag{4.2}$$

where $B_j^i(t)$ denotes the price at time $t$ of a defaultable zero-coupon bond that matures at $T_j$ with $T_j \geq t$, provided that the bond is in rating $i$ during the time interval $[0, t]$.

**Definition 4.9.** We define by $F_j^i(t) = F^i(t, T_j, T_{j+1})$ the value at time $t \leq T_j$ of the **forward LIBOR rate associated to the bond with credit rating i** in the period $[T_j, T_{j+1}]$. This forward rate can be obtained as

$$F_j^i(t) = \frac{B_j^i(t) - B_{j+1}^i(t)}{\delta_j B_{j+1}^i(t)}, \ j = 1, \ldots, N, i = 1, \ldots, K-1. \tag{4.3}$$

For simplicity of notation, we assume that $F_j^0(t) := F_j(t)$ and that $F_j^{0,D}(t) := F_j^D(t)$.

Next, we also define the forward shifted LIBOR rates for credit rating class $i$ as

$$F_j^{i,D}(t) = F_j^i(t) + \alpha_j, \tag{4.4}$$

for all $t > 0$, $j = 1, \ldots, N$ and $i = 1, \ldots, K$ , where $\alpha_j$ is the same shift parameter used in (2.4) and only depends on the tenor,

**Definition 4.10.** The **default-risk factor** for credit rating $i$ at time $t$ for maturity $T_j$ is defined as:

$$D_j^i(t) = \frac{B_j^i(t)}{B_j^{i-1}(t)},\ j = 1, \ldots, N, i = 1, \ldots, K-1\ . \tag{4.5}$$

**Definition 4.11.** The **discrete-tenor forward default intensity** for credit rating $i$ over the period $[T_j, T_{j+1}]$ as seen from time $t$ is defined as

$$H_j^i(t) = \frac{1}{\delta_j}\left(\frac{D_j^i(t)}{D_{j+1}^i(t)} - 1\right),\ j = 1, \ldots, N, i = 1, \ldots, K-1. \tag{4.6}$$

The forward default intensities can be also written in terms of the bond prices in the form

$$H_j^i(t) = \frac{1}{\delta_j}\left(\frac{B_{j+1}^{i-1}(t)B_j^i(t)}{B_j^{i-1}(t)B_{j+1}^i(t)} - 1\right),\ j = 1, \ldots, N, i = 1, \ldots, K-1. \tag{4.7}$$

It is easy to establish the following relationships between the forward rates and default intensities:

$$H_j^i(t) = \frac{F_j^i(t) - F_j^{i-1}(t)}{1 + \delta_j F_j^{i-1}(t)},\ j = 1, \ldots, N, i = 1, \ldots, K-1. \tag{4.8}$$

therefore we get

$$\left(1 + \delta_j F_j^i(t)\right) = \left(1 + \delta_j H_j^i(t)\right)\left(1 + \delta_j F_j^{i-1}(t)\right),\ j = 1, \ldots, N, i = 1, \ldots, K-1. \tag{4.9}$$

By applying equation (4.9) recursively, we can express every rating-dependent forward rate as a product of the default-free forward rate and the forward default intensities as follows:

$$\left(1 + \delta_j F_j^i(t)\right) = \left(1 + \delta_j F_j^0(t)\right)\prod_{k=1}^{i}\left(1 + \delta_j H_j^k(t)\right),\ j = 1, \ldots, N, i = 1, \ldots, K-1. \tag{4.10}$$

**Definition 4.12.** The **forward inter-rating LIBOR spreads** $S_j^i$ are the spreads between LIBOR rates for two successive credit rating classes. Therefore we have,

$$S_j^i(t) = F_j^i(t) - F_j^{i-1}(t),\ j = 1, \dots, N, i = 1, \dots, K-1. \tag{4.11}$$

Since $\alpha_j$ does not depend on the credit rating we have,

$$S_j^i(t) = F_j^{i,D}(t) - F_j^{i-1,D}(t), i = 1, \dots, K-1. \tag{4.12}$$

Moreover, we have the following relation between the default-free and defaultable forward shifted rates:

$$F_j^{i,D}(t) = F_j^D(t) + \sum_{h=1}^{i} S_j^h(t),\ j = 1, \dots, N, i = 1, \dots, K-1. \tag{4.13}$$

There are two alternatives in the specification of the dynamics of the defaultable interest-rates. Either we model the spreads $S$ or the discrete intensities $H$. Both in Schönbucher [63] and Eberlein and Grbac [21], they model the discrete intensities as it is more convenient for pricing purposes. In this work, given that we intend to incorporate credit risk under the possibility of negative interest rates, we choose to model the inter-rating spreads directly according to a stochastic dynamics that guarantees the monotonicity of bond prices with respect to the credit ratings, i.e., $F_j(t) < F_j^1(t) < \dots < F_j^{K-1}(t)$, which is consistent with the fact that lower credit ratings are reflected by higher interest rates. Also, this specification allows us to capture the independent movements of each additional credit step that would not be so clear if we had chosen to model the spread over risk-free forward rate for each credit rating.

Thus, under the spot measure, $\mathbf{Q}$, we assume the following dynamics of the inter-rating spreads

$$\frac{dS_j^i(t)}{S_j^i(t)} = \mu_j^{S,i}(t)dt \ + \epsilon_j^{S,i}(t) \cdot dW^{S,i}(t),\ j = 1,\dots,N, \quad i = 1,\dots,K-1\ , \tag{4.14}$$

where: $\epsilon_j^{S,i}$ represents the $m_i$-dimensional volatility functions and $W^{S,i}$ is a multidimensional Brownian motion for each credit rating class $i$.

Next, we recall the adopted model for the default-free shifted LIBOR rates in from Chapter 2 as

$$\frac{dF_j^D(t)}{F_j^D(t)} = \epsilon_j(t) \cdot \sum_{i=m(t)}^{j} \beta_i^D(t)dt + \epsilon_j(t) \cdot dW^d(t)\ , \tag{4.15}$$

where:

$$\beta_i^D(t) = \frac{\epsilon_i(t)\delta_i F_i^D(t)}{1 + \delta_i(F_i^D(t) - \alpha_i)}\ .$$

Moreover, combining equations (4.13), (4.14) and (4.15), we obtain the shifted LIBOR rates for credit rating class $i$ dynamics under the spot measure as

$$dF_j^{i,D}(t) = \left( F_j^D(t)\epsilon_j(t) \cdot \sum_{l=m(t)}^{j} \beta_l^D(t) + \sum_{h=1}^{i} S_j^h(t)\mu_j^{S,h}(t) \right) dt + F_j^D(t)\epsilon_j(t) \cdot dW^d(t)$$

$$+ \sum_{h=1}^{i} S_j^h(t)\epsilon_j^{S,h}(t) \cdot dW^{S,h}(t),\ \ j = 1,\dots,N, \quad i = 1,\dots,K-1. \tag{4.16}$$

Consequently, the dynamics of the LIBOR rates for credit rating class $i$ is given by

$$dF_j^i(t) = \left( (F_j(t) - \alpha_j)\epsilon_j(t) \cdot \sum_{l=m(t)}^{j} \beta_l^D(t) + \sum_{h=1}^{i} S_j^h(t)\mu_j^{S,h}(t) \right) dt + (F_j(t) - \alpha_j)\epsilon_j(t) \cdot dW^d(t)$$

$$+ \sum_{h=1}^{i} S_j^h(t)\epsilon_j^{S,h}(t) \cdot dW^{S,h}(t),\ \ j = 1,\dots,N, i = 1,\dots,K-1. \tag{4.17}$$

# Chapter 5

# LIBOR market model with credit risk under the real world measure

In this section, we generalize the results presented in Chapter 3 by including credit risk in the default-free model.

In Section 5.1, we present the proposed dynamics inter-rating spreads under the real world measure and present the market price of risk process adopted for the real world setting. In Section 5.2, we show how to estimate the parameters based on historical information and Section 5.3 is dedicated to the simulation methodologies used. In the last section, we present the results of the projected rates one year ahead from the last data point by using historical data of sovereign European bonds.

The contents of this chapter have been published in [53].

## 5.1 The model

In order to have a more parsimonious model, we assume that the spreads for all credit ratings are dependent on the same stochastic factor. This means that Equation (4.14) can be rewritten and we consider the case where dynamics of the inter-rating spreads under the spot measure, **Q**, are given by :

$$\frac{dS_j^i(t)}{S_j^i(t)} = \mu_j^{S,i}(t)dt \ + \epsilon_j^{S,i}(t) \cdot dW^S(t), \ j = 1, \dots, N, i = 1, \dots, K-1. \tag{5.1}$$

where $\mu_j^{S,i}(t)$ are the drifts under the spot measure, $\epsilon_j^{S,i}(t)$ are the $m$-dimensional vectors of volatility functions and $dW^S(t)$ is a multidimensional Brownian motion.

From the adopted risk neutral model, defined by Equation (4.15) and (5.1), and using Girsanov's theorem [31], we can write the dynamics of the default-free LIBOR rates and inter-rating spreads under the real world measure using the following relations between the $r$-dimensional and the $m$-dimensional Brownian motions under the real world measure, $W^P$ and $W^{P,S}$, and the $r$-dimensional and $m$-dimensional Brownian motions under the spot measure $W^d$ and $W^S$:

$$\begin{aligned} dW^P(t) &= dW^d(t) - \theta(t)dt \\ dW^{P,S}(t) &= dW^S(t) - \overline{\theta}^S(t)dt. \end{aligned} \tag{5.2}$$

where $\theta(t)$ and $\overline{\theta}^S(t)$ are respectively the $r$-dimensional and $m$-dimensional market price of risk processes.

In this setting, the dynamics of shifted LIBOR risk-free forward rates under the real world measure are given by:

$$\frac{dF_j^D(t)}{F_j^D(t)} = \left\{ \epsilon_j(t) \cdot \sum_{l=m(t)}^{j} \beta_l^D(t) + \epsilon_j(t) \cdot \theta(t) \right\} dt + \epsilon_j(t) \cdot dW^P(t) \ , \tag{5.3}$$

where

$$\beta_l^D(t) = \frac{\epsilon_l(t)\delta_l F_l^D(t)}{1+\delta_l(F_l^D(t)-\alpha_l)} \ ,$$

and the dynamics of the inter-rating spreads are:

$$\frac{dS_j^i(t)}{S_j^i(t)} = \left(\overline{\theta}^S(t)\cdot\epsilon_j^{S,i}(t) + \mu_j^{S,i}(t)\right)dt \ + \ \epsilon_j^{S,i}(t)\cdot dW^{P,S}(t), \ j=1,...,N, \ i=1,...,K-1. \tag{5.4}$$

Therefore, we obtain the following SDE governing the shifted LIBOR rates for the credit rating class $i$ under the real world measure:

$$dF_j^{i,D}(t) = \left(F_j^D(t)\left(\epsilon_j(t)\cdot\sum_{l=m(t)}^{j}\beta_l^D(t) + \epsilon_j(t)\cdot\theta(t)\right) + \sum_{h=1}^{i} S_j^h(t)\left(\epsilon_j^{S,h}(t)\cdot\overline{\theta}^S(t) + \mu_j^{S,h}(t)\right)\right)dt$$
$$+ F_j^D(t)\epsilon_j(t)\cdot dW^P(t) + \sum_{h=1}^{i} S_j^h(t)\epsilon_j^{S,h}(t)\cdot dW^{P,S}(t), \ \ j=1,...,N, i=1,...,K-1. \tag{5.5}$$

Next we maintain the assumptions made in Chapter 3 regarding the volatility structure of the default-free forward rate, particularly, that the default-free interest rates volatilities depend only on the time-to-maturity:

$$\epsilon_j^k(t) = \lambda_{j-m(t)}^k \ ,$$

In addition, we consider the case when $S_j^i$ have piecewise constant instantaneous volatilities and depend also only on the time-to-maturity, i.e:

$$\epsilon_j^{i,S}(t) = \nu_{j-m(t)}^i \ ,$$

and that we can write the volatility functions in terms of the corresponding orthonormal basis[1] $\Lambda, \Pi^{i,S}$ :

$$\lambda_j^k = \sigma^k\Lambda_j^k, \quad \nu_j^{i,k} = \omega^k\Pi_{N\times(i-1)+j}^k \ .$$

[1] $\sum_{j=0}^{N-1}\Lambda_j^k\Lambda_j^l = \delta^{kl}$ and $\sum_{j=0}^{N-1}\Pi_j^{i,k}\Pi_j^{i,l} = \delta^{kl}$ for $i=1,..K-1$ where $N$ can be interpreted to be the number of maturities observable on the forward rate curve.

We assume that the $k$ component of the market price of risk for interest rates has the following structure presented already in chapter 3:

$$\theta^k(t) = \frac{a^k}{\sigma^k}\Big(b^k - \sum_{l=m(t)}^{N+m(t)-1} \Lambda^k_{l-m(t)} \ln(F^D_l(t))\Big) \ , \tag{5.6}$$

where $a^k$, $\sigma^k$ and $b^k$ are parameters to be estimated from historical data, while the vectors $\Lambda_i$ are the coefficients obtained from the principal component analysis of the historical covariance matrix of the logarithm of shifted rates.

Moreover, the same structure is proposed for the market price of risk of inter-rating spreads, with the appropriate modifications as follows:

$$\overline{\theta}^S(t) = \theta^S(t) + \rho(t) \ , \tag{5.7}$$

where $\theta^S(t)$ is a vector with components given by

$$\theta^{S,k}(t) = \frac{a^{S,k}}{\omega^k}\Big(b^{S,k} - \sum_{l=m(t)}^{N+m(t)-1} \sum_{h=1}^{K-1} \Pi^k_{N\times(h-1)+(l-m(t))} \ln(S^h_l(t))\Big), \tag{5.8}$$

where $a^{S,k}$, $\omega^k$ and $b^{S,k}$ are parameters to be estimated from historical data, the vectors $\Pi^i_l$ are the coefficients obtained from the principal component analysis of the historical covariance matrix of the logarithm of the inter-rating spreads and $\rho^k(t)$ are the compone-nents of the vector $\rho(t)$ defined as the solution of the following system of $(K-1)\times(N-1)$ linear equations:

$$\mu^{S,i}_j(t) + \nu^{i,k}_j \cdot \rho(t) = 0 \ , \tag{5.9}$$

such that $\theta^S(t)$ is the drift of the inter-rating spreads under the real world measure, and equation (5.4) can be simplified to:

$$\frac{dS^i_j(t)}{S^i_j(t)} = \theta^S(t) \cdot \epsilon^{S,i}_j(t)dt \ + \ \epsilon^{S,i}_j(t) \cdot dW^{P,S}(t), \ j = 1, ..., N, \ i = 1, ..., K-1. \tag{5.10}$$

## 5.2 Parameter estimation

In this section, the method for calibrating the real world model to historical data is described. We mainly adopt the methodology used in Chapter 3 with the appropriate modifications, since we are now modeling changes in the logarithm of the inter-rating spreads.

We consider a common constant accrual, i.e. $\delta = \delta_j = T_{j+1} - T_j$, and adopt the following notation for the constant maturity defaultable forward shifted rates:

$$L_j^{D,i}(t) = F^{D,i}(t, t + j\delta, t + (j+1)\delta)\ , \quad j = 1, \ldots, N. \tag{5.11}$$

More precisely $L_j^{D,i}(t)$ denotes the observed value at time $t$ of the forward shifted rates for the period $[t + \delta j, t + \delta(j+1)]$.

Under this notation, the inter rating spread observed at time $t$ is defined as:

$$S_j^i(t) = L_j^{D,i}(t) - L_j^{D,i-1}(t),$$

and the dynamics of $S_j^i$ satisfies the SDE:

$$dS_j^i(t) = S_j^i(t)\nu_j^i \cdot \theta^S(t)dt + S_j^i(t)\nu_j^i \cdot dW^{P,S}(t),\ j = 1, \ldots, N, i = 1, \ldots, K-1. \tag{5.12}$$

For convenience, we consider the stochastic differential equation for the dynamics of the logarithm of the inter-ratings spreads:

$$d \ln S_j^i(t) = \left(\nu_j^i \cdot \theta^S(t) - \frac{1}{2}\|\nu_j^i\|^2\right) dt + \nu_j^i \cdot dW^{P,S}(t)\ . \tag{5.13}$$

Next, we consider the covariance matrix $\Pi$ of the historical changes in the logarithms of forward inter-ratings spreads and, using Principal Components Analysis, we recover the $\Pi_i^k$'s and $\omega^{S,k}$'s as the eigenvectors and eigenvalues of $\Pi$, respectively. Also in this step, we define the dimension of the volatility functions and the number of components of the market price of risk process. A common criteria to select the number of components to retain is to use the proportion of total variance criteria where a predetermined threshold is defined so that the number of factors retained accounts for that selected value of total variation. Research on the term structure of interest rates, such as [51] and [19], suggests that the term structure can be explained by a small number of underlying factors: level, slope, bow, and higher order perturbations. Regarding the spreads, and to the best of our knowledge, this is the first PCA study on the observations of the inter-rating spreads. Previous studies of PCA applied to spreads (measured as the difference between yields of the defaultable bond and the default-free rates) include [17] who found evidence of one common factor for the corporate spread changes. In [62] PCA is applied to study sovereign spreads in Latin America countries and two components were able to explain 90% of total variance in this case

By considering a small time step, $\Delta t$, between observations and using the Euler-Maruyama scheme, we approximate the evolution of the logarithm of inter-rating spreads as follows:

$$\ln S_j^i(t+\Delta t) = \ln S_j^i(t) + \left(\nu_j^i \cdot \theta^S(t) - \frac{1}{2}\|\nu_j^i\|^2\right)\Delta t + \nu_j^i \cdot \Delta W^{P,S}(t)\ , \tag{5.14}$$

Next, we define

$$\ln \widetilde{S}_j^i(t) = \ln S_i^D(t) - \frac{1}{2}\|\nu_j^i\|^2 \Delta t, \tag{5.15}$$

so that

$$\ln S_j^i(t+\Delta t) = \ln \widetilde{S}_j^i(t) + \nu_j^i \cdot \theta^S(t)\Delta t + \nu_j^i \cdot \Delta W^{P,S}\ . \tag{5.16}$$

Thus, by multiplying (5.16) by $\Pi^l_{N\times(i-1)+j}$ and summing over $j$ and $i$, we obtain

$$g^l(t+\Delta t) = \widetilde{g}^l(t) + a^{S,l}\left(b^{S,l} - g^l(t)\right)\Delta t + \omega^{S,l}\Delta W^{S,l} \text{ for } l = 1, 2, \dots m \tag{5.17}$$

where we have introduced

$$g^l(t) = \sum_{j=1}^{N} \sum_{i=1}^{K-1} \Pi^l_{N\times(i-1)+j} \ln S^i_j(t) \ , \tag{5.18}$$

and

$$\widetilde{g}^l(t) = \sum_{j=1}^{N} \sum_{i=1}^{K-1} \Pi^l_{N\times(i-1)+j} \ln \widetilde{S}^i_j(t) \ . \tag{5.19}$$

In order to estimate the parameters $a^{S,l}$ and $b^{S,l}$, first from the historical data we define the following series of observations:

$$Z^l(t) = g^l(t+\Delta t) - \widetilde{g}^l(t) \ ,$$

and

$$U^l(t) = g^l(t) \ .$$

Next, we estimate the $l$ regression models obtained from equation (5.17):

$$Z^l(t) = c^{S,l} + m^{S,l} U^l(t) + e^{S,l}(t) \ , \tag{5.20}$$

where

$$e^{S,l}(t) = \omega^{S,k} \Delta W^{S,l} \ , \quad m^{S,l} = -a^{S,l}\Delta t \ , \quad c^{S,l} = a^{S,l} b^{S,l} \Delta t \ , \tag{5.21}$$

so that we can recover $a^{S,l}$ and $b^{S,l}$ from $m^{S,l}$ and $c^{S,l}$.

## 5.3 Real world scenarios simulation

After obtaining the real world measure parameters, we take the discretized model and the predictor-corrector method for our simulations. We consider the first approximation of the inter-rating spreads as:

$$\overline{S^i}_j(t+\Delta t) = S^i_j(t) \times \left\{ \left( v^i_j \cdot \theta^S(t) - \frac{1}{2}\|v^i_j\|^2 \right) \Delta t + \sqrt{\Delta t} v^i_j \cdot \Delta W^S \right\}, \tag{5.22}$$

where

$$\theta^{S,q}(t) = \frac{a^{S,q}}{\omega^{S,q}} \Big( b^{S,q} - \sum_{j=1}^{N} \sum_{h=1}^{K-1} \Pi^q_{N\times(h-1)+(l-1)} S^h_j(t) \Big) \text{ for } q = 1, \ldots, m \tag{5.23}$$

and $\Delta W^S \sim N(0,1)$

By repeating the previous step using the $\overline{S^i}_j$'s instead of the $S^i_j$'s to compute $\theta^{*S,q}(t)$, we obtain the final approximation of the spreads as:

$$S^i_j(t+\Delta t) = S^i_j(t) \times \exp \left\{ \left( v^i_j \cdot \frac{\theta^S(t) + \theta^{S*}(t)}{2} - \frac{1}{2}\|v^i_j\|^2 \right) \Delta t + \sqrt{\Delta t} v^i_j \cdot \Delta W^S \right\},$$

using the same values for $\Delta W^S$.

We can obtain the price at time $t + \Delta t$ of a zero-coupon bond with credit rating $i$ at time $t$ as follows:

$$B^i_j(t+\Delta t) = \sum_{l=1}^{K-1} B^l_j(t) P(C_{t+\Delta t} = l | C_t = i)(1 + \delta F^l_j(t+\Delta t)) + P(C_{t+\Delta t} = K | C_t = i) q_i,$$

where

$$F^l_j(t+\Delta t) = F_j(t+\Delta t) + S^l_j(t+\Delta t)$$

and $q_i$ is the recovery rate for credit rating $i$.

## 5.4 Results

In this section, we present the estimation and simulation results of the real world LIBOR market model with credit risk. We use a three-factor version of the model for the default-free[2] forward rates and a seven-factor model for the inter-rating spreads.

The parameter estimates are obtained from the historical observations of European government bond prices, ranging from 31 January 2016 up to 30 June 2018. **Table 5.1** presents the forward rate curve observed on 30 June 2018. At that time, negative forward rates of one year tenor for the maturities of one to three years were prevailing.

| Maturity | Forward Rate | Maturity | Forward Rate |
|---|---|---|---|
| 1 | -0.556% | 11 | 1.331% |
| 2 | -0.381% | 12 | 1.345% |
| 3 | -0.018% | 13 | 1.352% |
| 4 | 0.353% | 14 | 1.353% |
| 5 | 0.665% | 15 | 1.351% |
| 6 | 0.903% | 16 | 1.349% |
| 7 | 1.072% | 17 | 1.347% |
| 8 | 1.186% | 18 | 1,345% |
| 9 | 1.260% | 19 | 1,342% |
| 10 | 1.305% | 20 | 1,342% |

Table 5.1: One year forward rates observed on 30 June 2018.

For this study, we consider 23 European countries in order to construct the benchmark yield curves for each of the rating groups. The chosen countries are shown in **Table 5.2**. We group the last two rating groups, B and CCC-C, in order to have more than one country in each credit rating group.

| AAA | AA | A | BBB | BB | B | CCC-C |
|---|---|---|---|---|---|---|
| Denmark | Austria | Ireland | Bulgaria | Russia | Armenia | Greece |
| Germany | Finland | Malta | Hungary | Portugal | | Ukraine |
| Luxembourg | France | Latvia | Italy | Turkey | | |
| Netherlands | | Slovakia | Spain | | | |
| Switzerland | | | Romania | | | |

Table 5.2: Countries by rating group as of 31 January 2016.

After grouping the countries we compute the inter-rating spreads for each sucessive rating class. In **Figure** 5.1 the last observation of the inter-rating spreads is presented.

By using the historical forward rates, we compute the covariance matrix $\Sigma$ between

[2]We consider the AAA rated bonds as the default-free bonds.

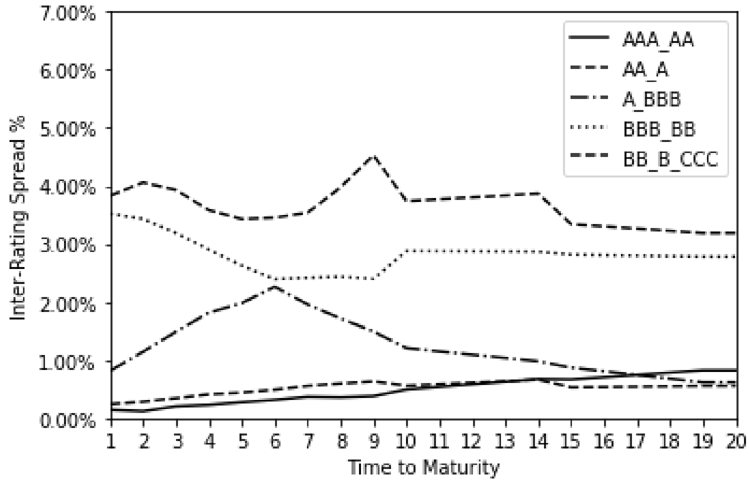


Figure 5.1: Inter-rating spreads observed as of 30 June 2018.

monthly changes in the logarithm of one year shifted forward rates of term $i$ and term $j$ $(i, j = 1, \ldots, 20)$. Since the lowest observed value of a forward rate for the period in study was -0.72% (one-year forward rate with one year maturity observed on 28 February of 2017) we defined $\alpha_k = \alpha = 0.8\%$ as the shift parameter for the following results to be presented. Note that this parameter can also be selected according to market expectations or risk manager views on the lower boundary for the forward rates in order to obtain scenarios with rates bellow the threshold we have selected.

The first three principal components corresponding to the decomposition of the covariance matrix are shown in **Figure 5.2**. In this figure we can identify the level, slope and bow factors, as usually in yield curve studies. The first three principal components explain a 98.84% of the covariance of the data.

Regarding the principal components obtained from the decomposition of the covariance matrix of the inter-rating spreads, we present in **Figure 5.3** the first seven principal components which explain a 90.58% of the covariance of the data.

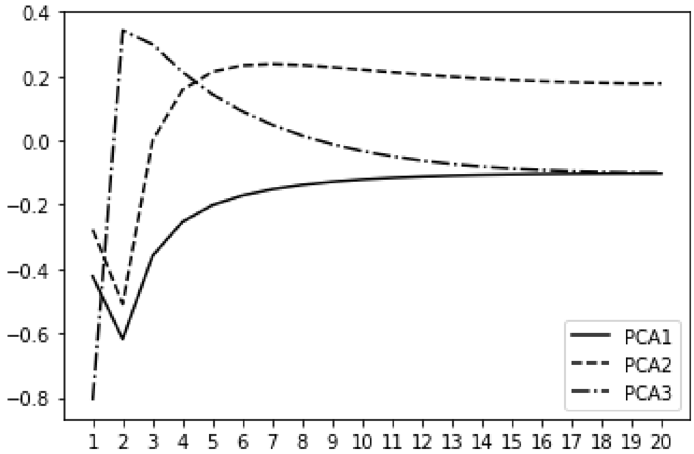


Figure 5.2: The first three principal components obtained from a principal components analysis of monthly observed AAA-rated European shifted forward rates.

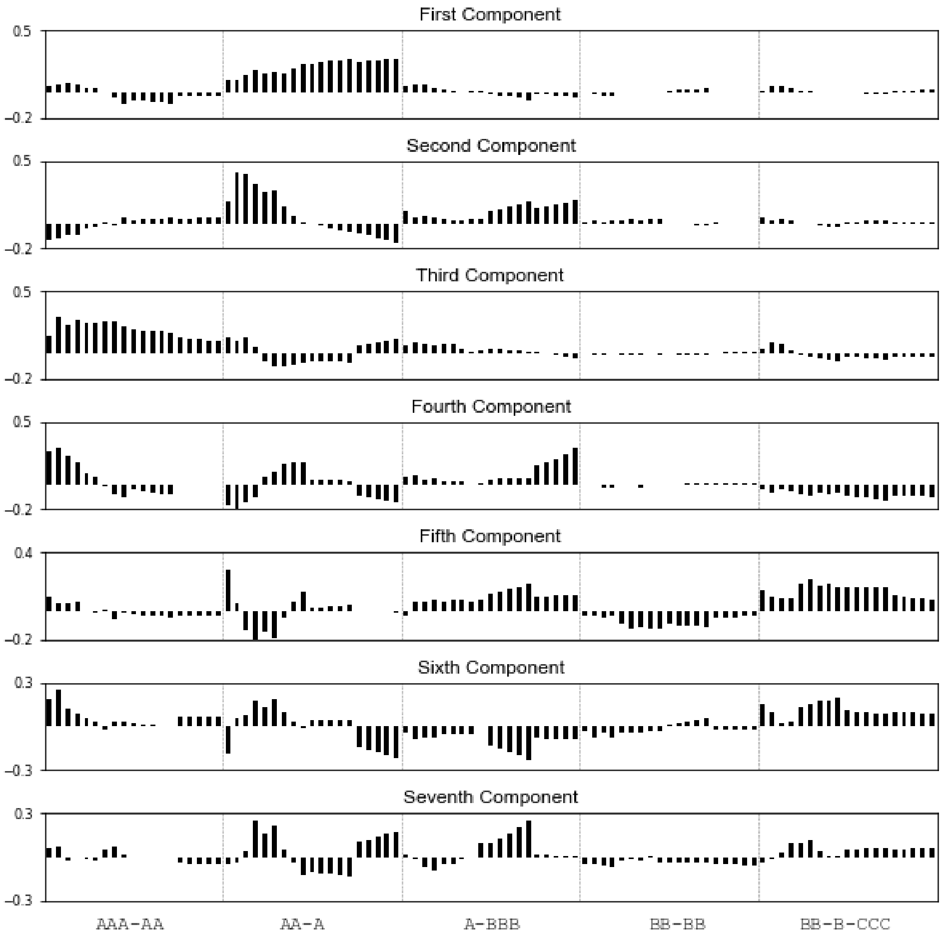


Figure 5.3: The first seven principal components obtained from a principal components analysis of monthly observed inter-rating spreads of European bonds.

The parameters of the market price of risk processes are shown in **Table 5.3** and **Table 5.4**. The results show low mean reversion rate parameters, meaning that any disturbance on the factors has a long term effect on the future rates. This can be explained by the strong influence that European Central Bank has had on current levels of the European bond interest rates, specifically with the Expanded Asset Purchase Programme and Covered Bond Purchase Programmes.

| Factor | $a^p$ | $b^p$ | $\sigma^p$ |
|---|---|---|---|
| 1 | 0.0045 | -5.9869 | 0.4795 |
| 2 | -0.0014 | -6.2500 | 0.0612 |
| 3 | 0.0013 | -4.0905 | 0.0141 |

Table 5.3: Estimated market price of risk parameters of default-free forward rates.

| Factor | $a^p$ | $b^p$ | $\sigma^p$ |
|---|---|---|---|
| 1 | 0.0072 | -5.2232 | 0.6130 |
| 2 | 0.0112 | -8.2517 | 0.4677 |
| 3 | 0.0080 | -12.1616 | 0.3043 |
| 4 | 0.0340 | -0.8974 | 0.2129 |
| 5 | 0.0108 | -10.4176 | 0.1440 |
| 6 | 0.0273 | -6.5070 | 0.0940 |
| 7 | 0.0241 | -0.4666 | 0.0861 |

Table 5.4: Estimated market price of risk parameters of inter-rating spreads.

In **Figure 5.4** we exhibit ten thousand paths of the forward rate curve in the 1st year of simulations and we can compare it with the observed forward curve on 30 June 2019. And in **Figures 5.5-5.9** we present the simulation results for the inter-rating spreads and defaultable forward rates for each credit rating.

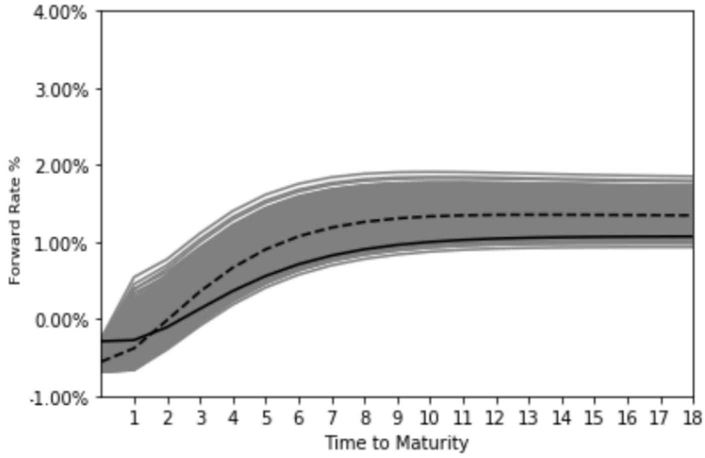


Figure 5.4: Ten thousand simulations of 1 year-ahead forward rate curve for AAA bonds (grey), observed forward curve on 30 June 2018 (dashed-black) and observed forward curve on 30 June 2019 (black).

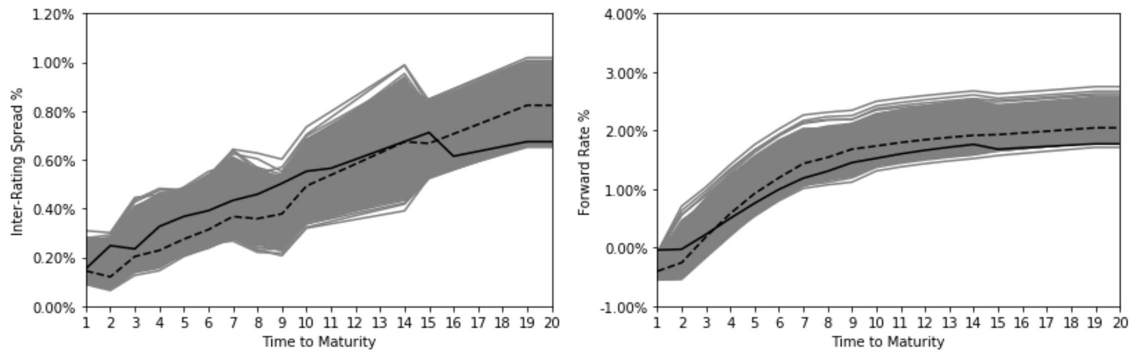


Figure 5.5: Ten thousand simulations of 1 year-ahead inter-rating spreads for credit ratings AAA and AA (grey), observed spread curve on 30 June 2018 (dashed-black) and observed spread curve on 30 June of 2019 (black) (left) and corresponding forward rates for rating AA (right).

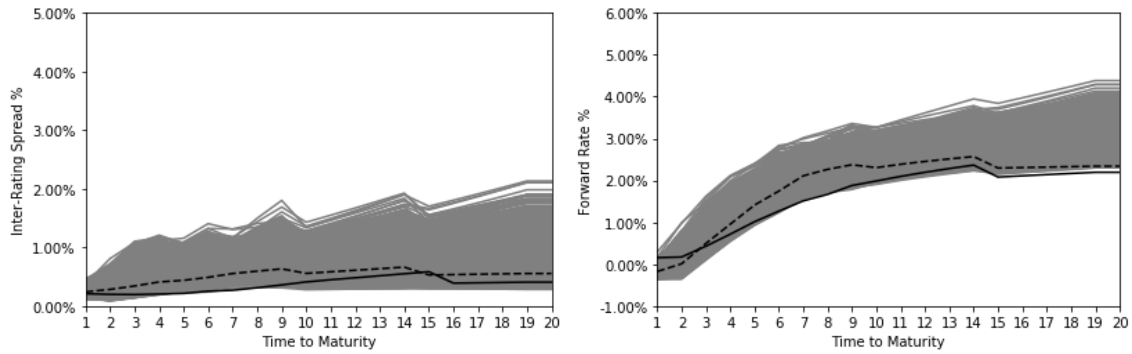


Figure 5.6: Ten thousand simulations of 1 year-ahead inter-rating spreads for credit ratings AAA and AA (grey), observed spread curve on 30 June 2018 (dashed-black) and observed spread curve on 30 June of 2019 (black) (left) and corresponding forward rates for rating AA (right).

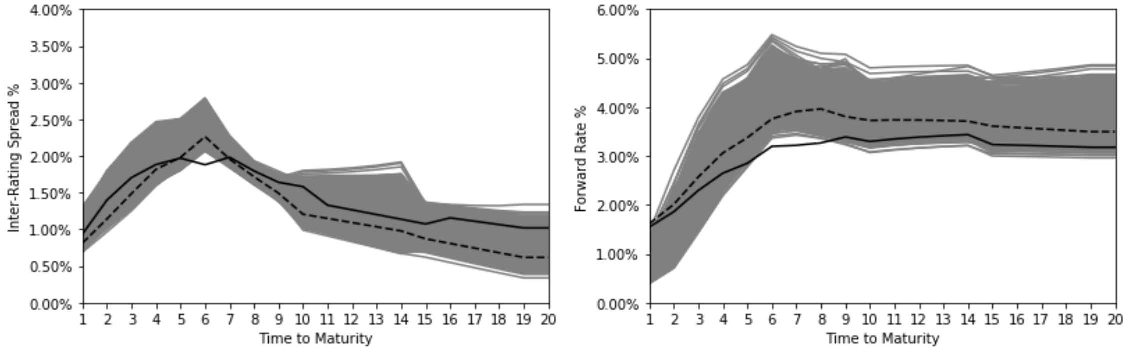


Figure 5.7: Ten thousand simulations of 1 year-ahead inter-rating spreads for credit ratings A and BBB (grey), observed spread curve on 30 June 2018 (dashed-black) and observed spread curve on 30 June of 2019 (black) (left) and corresponding forward rates for rating BBB (right).

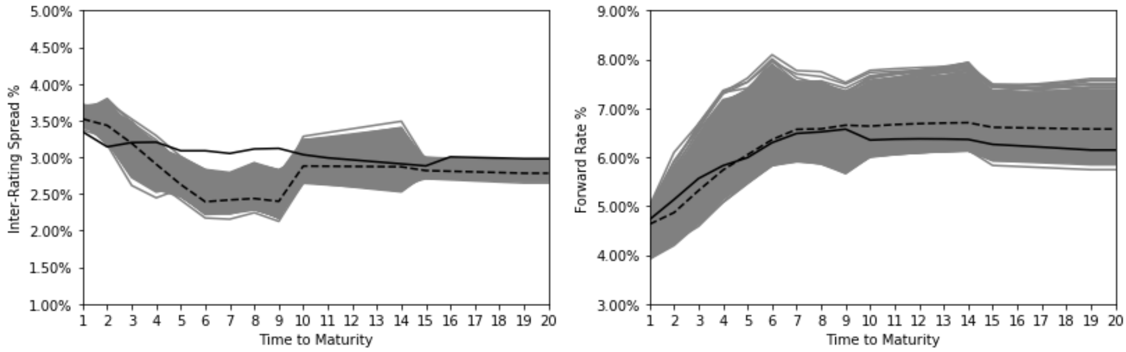


Figure 5.8: Ten thousand simulations of 1 year-ahead inter-rating spreads for credit ratings BBB and BB (grey), observed spread curve on 30 June 2018 (dashed-black) and observed spread curve on 30 June of 2019 (black) (left) and corresponding forward rates for rating BB (right).

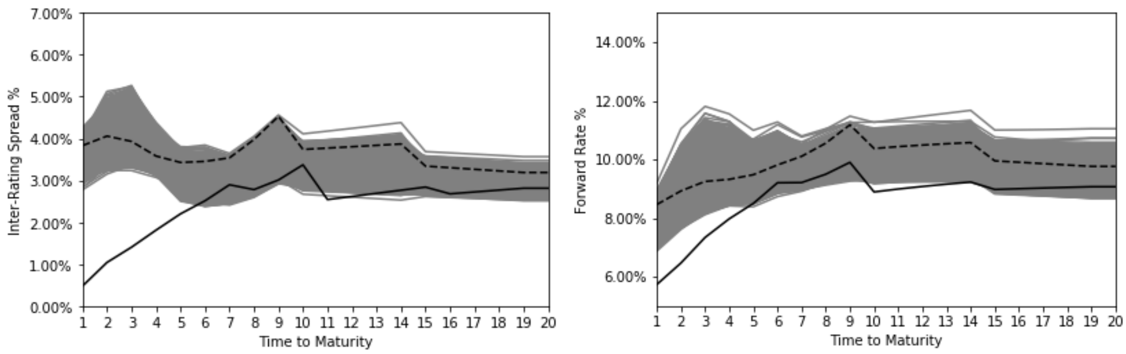


Figure 5.9: Ten thousand simulations of 1 year-ahead inter-rating spreads for credit ratings BB and B-CCC (grey), observed spread curve on 30 June 2018 (dashed-black) and observed spread curve on 30 June 2019 (black) (left) and corresponding forward rates for rating B-CCC (right).

Note that in order to simulate the paths of bond prices in a one-year horizon we need to take into account the rating-transitions occurring in one year and also the recovery rates in case of default. For this study we consider the sovereign average one-year rating migration rates from 1983 to 2018 provided in [56] and presented in **Table 5.5**.

| From/To | AAA | AA | A | BBB | BB | B | CC-C | Withdrawn | D |
|---|---|---|---|---|---|---|---|---|---|
| AAA | 96.774 | 3.077 | 0.033 | 0.083 | 0.000 | 0.000 | 0.000 | 0.033 | 0.000 |
| AA | 3.048 | 93.141 | 2.477 | 0.699 | 0.106 | 0.000 | 0.000 | 0.529 | 0.000 |
| AA | 0.000 | 3.878 | 91.879 | 3.059 | 1.120 | 0.065 | 0.000 | 0.000 | 0.000 |
| BBB | 0.000 | 0.000 | 5.920 | 88.818 | 4.787 | 0.439 | 0.037 | 0.000 | 0.000 |
| BB | 0.000 | 0.000 | 0.000 | 7.493 | 84.936 | 6.617 | 0.331 | 0.117 | 0.506 |
| B | 0.000 | 0.000 | 0.000 | 0.000 | 4.950 | 88.406 | 3.699 | 0.331 | 2.613 |
| CCC-C | 0.000 | 0.000 | 0.000 | 0.000 | 0.079 | 15.669 | 71.654 | 0.945 | 11.654 |
| D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 100.000 |

Table 5.5: One year transition matrix with withdrawl

Since in practice it can happen that a debt issuer disappears from the data sample because it is no longer rated and given that this fact is not considered in our modelling approach we assume that the diagonal entries absorbe the exits from the system in order to obtain a matrix without the class "Withdrawn" that corresponds to the fraction of the countries that migrated to the non-rated class.

The grouping of the last two classes is done by weighting the probabilities of both classes assuming a constant distribution provided by the report [56]. By using this technique, from the original data in **Table 5.5** we obtain in **Table 5.6** the one-year transition probability matrix that can be used for the simulations.

| From/To | AAA | AA | A | BBB | BB | B-CC-C | D |
|---|---|---|---|---|---|---|---|
| AAA | 96.807 | 3.077 | 0.033 | 0.083 | 0.000 | 0.000 | 0.000 |
| AA | 3.048 | 93.671 | 2.477 | 0.699 | 0.106 | 0.000 | 0.000 |
| AA | 0.000 | 3.878 | 91.879 | 3.059 | 1.120 | 0.065 | 0.000 |
| BBB | 0.000 | 0.000 | 5.920 | 88.818 | 4.787 | 0.475 | 0.000 |
| BB | 0.000 | 0.000 | 0.000 | 7.493 | 85.052 | 6.948 | 0.506 |
| B-CCC-C | 0.000 | 0.000 | 0.000 | 0.000 | 4.091 | 91.701 | 4.209 |
| D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 100.000 |

Table 5.6: One-year transition matrix without withdrawl.

For the rating classes with non-zero one year probability of default we use the average recovery rates for each credit rating using the sovereign recovery rates from 1983 to 2018, reported also in [56] as the average trading price in percentage of the par value of the bond at the time the initial default event occurred, and we obtained the results in **Table 5.7**.

| Rating Class | Recovery Rate |
|---|---|
| BB | 0.9515 |
| B or lower | 0.5977 |

Table 5.7: Average recovery rate by credit rating.

For the rest of this study we consider a portfolio with nominal value of 1,000,000 euros in zero coupon bonds distributed by all credit ratings for maturities of 2, 5, 7, 10, 12, 15, 18 and 20 years and an additional investment of equal weight in the two higher rated classes and maturity of 3 years. The present value of this portfolio as of 30 June 2018 was 747,929.95 euros. For this calculation, we have considered the observed forward rates for AAA-rated bonds presented in **Table 5.1** and the inter-rating spreads for each of the rating classes presented in **Figure 5.1** and observed on 30 June 2018. **Figure 5.10** presents the simulated profit and losses histogram for the portfolio.

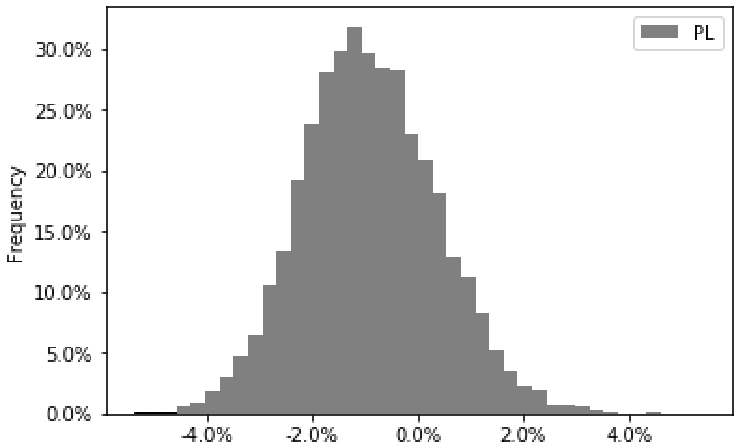


Figure 5.10: Histogram of % profit and loss.

Given the observed generalized drop in interest rates of all maturities and throughout all credit ratings, the one-year return on this portfolio was 3.569% on 30 June 2019.

Solvency II regulation defines that the capital requirement for interest rate risk is determined as the maximum change in the net value of assets and liabilities due to the revaluing of all interest rate sensitive items under two pre-defined scenarios: an instantaneous upward movement of interest rates and a downward movement. Note that for European government bonds, the standard formula does not model transitions or defaults. The scenarios for the term structures are obtained by multiplying the current interest rate curve by factors that are specified for each maturity in the Solvency II Technical Specifications [26].

A comparison of the SCR produced by the model and the SCR of the standard formula is shown in **Table 5.8**.

| Standard Formula SCR | Model VaR |
|---|---|
| 9.178% | 4.565 % |

Table 5.8: Comparison between the SCR of the standard formula and the simulated model.

Note that the value obtained by applying the standard formula is considerably higher than the one we obtain using the simulations of our model, even taking into account the rating transitions and possibility of default for lower-rated bonds. We have also considered the extreme case where the recovery rates for all the defaultable bonds is zero. For this case we obtained a simulated VaR of $4.887\%$, which is still considerable lower than the one given by the standard formula. By comparing the scenarios for the spot interest rates for the different rating classes,in **Figure 5.11**, we obtain two conclusions: first, the simulation results are lower when compared with the upward scenario obtained using Solvency II stress methodology. Second, for higher credit ratings, the downward shock is not penalizing enough, thus meaning that the negative setting of interest rates is still not completely accounted for. This last fact turns out to be consistent with the new EIOPA recommendations for 2020 review of Solvency II[3].

It should be noted that this exercise is purely theoretical since, for meaningful Solvency

[3] Formal request to EIOPA for technical advice on the review of the Solvency II Directive https://ec.europa.eu/info/files/190211-request-eiopa-technical-advice-review-solvency-2

capital requirement calculations, both the asset and liabilities side of an insurance company should be considered. Furthermore, SCR formulas and stress methodologies were designed to penalize mismatches between assets and liabilities.

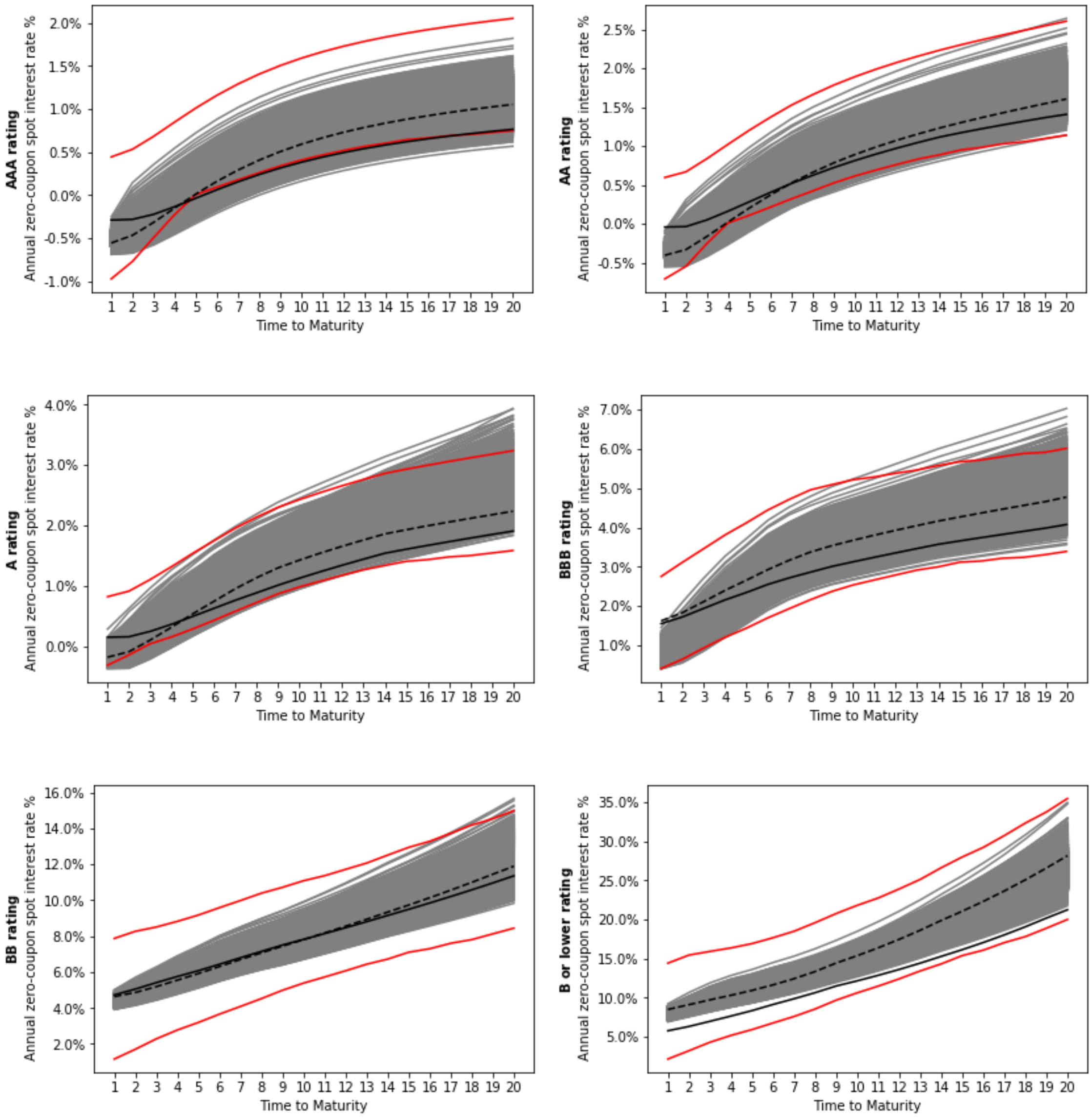


Figure 5.11: Ten thousand simulations of 1 year-ahead spot rate for each credit rating (grey), last observed curve (dashed-black), observed forward curve in 30 of June of 2019 (black) and upward and downward scenarios according to Solvency II methodology (red).

# Chapter 6

# Conclusion and future research

In this work, we have presented the shifted LIBOR market model for the simulation of interest rates under the real world measure. The model includes a market price of risk process, the parameters of which have been estimated by a principal component analysis (PCA) technique. The proposed model allows for negative interest rates, as the ones observed in current markets, and maintains realistic levels and shapes for the interest rate curves. Using the historical yield curves of AAA-rated European bonds, ranging from 31 January 2015 up to 30 June 2016, we have simulated the one-year trajectories for the forward rates and bond prices that are consistent with the observed forward rate curve on 30 June 2017.

After addressing the default-free framework, we have extended the setting to incorporate the credit risk to our model, so that we have developed a shifted LIBOR model for defaultable bonds with credit ratings under the real world measure. For this purpose, we have modeled the inter-rating spreads directly according to stochastic dynamics that guarantee the monotonicity of bond prices with respect to the credit ratings as expected in the market. In this setting, by using historical yield curves of bonds from 23 different European countries, ranging from 31 January 2016 up to 30 June 2018, we have compared the solvency capital requirements for an example portfolio of bonds with different credit ratings and maturities. From this, we have concluded that we obtain lower capital requirements than the when using the Solvency II methodology.

In the current economic environment of very low interest rates, for most of the credit ratings our results present forward rates that are lower than the stressed downward movement currently proposed by regulators. This current regulatory framework will be under revision during 2020. It is expected that both the upper and downward stress methodologies will become more extreme than the ones currently proposed.

Regarding the interest rate and spread models chosen for this work, this models could be extended to include stochastic volatilities by using the displaced diffusion stochastic volatility LIBOR market model from [46] for the interest rate dynamics, or using the Lévy LIBOR model proposed by [23] and [32], by adapting the risk neutral version to the real world setting.

In this work, we do not model the correlations between the default-free rates and inter-rating spreads directly. However, this could be achieved by performing a standard principal component analysis on both risk factors or by application of the principal component analysis on rolling windows in order to be able to estimate the historical correlation between the factors.

Another possible extension of this work could include the calibration of the model parameters to corporate spreads and compare the results with the SCR spread formula.

Also, the consideration of other additional sources of risk could be aimed, such as liquidity and currency risk that are present in global bond markets.

Finally, the ESG presented in this work can be used for many risk management purposes. For example, it could be used as a tool in Asset- Liability Management or for internal model calculations of interest rate and spread modules of SCR, as an alternative to the standard formula proposed by regulators. Nevertheless, the ESG should be improved in order to cover all the market risks an insurance company or a bank is subject to. Interest rates and spread rates components presented in this work can be combined with models to simulate future trajectories for other risk factors, such as inflation, equity returns, cross-currency rates, and real estate prices.

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# Appendix A

# Principal component analysis

Principal Component Analysis (PCA) is a technique to transform a set of $p$ correlated variables into a smaller set of uncorrelated called principal components (PCs). PCA is designed to retain as much as possible the variation present in the data set. By forming linear combinations of the original variables, PCA allows us to obtain a set of new uncorrelated variables.

The PCs can be used to discover and interpret the dependencies that may exist among the original variables and to reduce dimensionality. These characteristics contribute to the fact that PCA has already been applied in a wide variety of areas such as biology, psychology and genetics. In quantitative finance, PCA has been applied to the study of portfolio allocation strategies in order to form uncorrelated portfolios [59] and is frequently used to capture the variability in the movement of interest rates along the term structure. Furthermore, when applied to the term structure of interest rates, empirical analysis generally determines that a small number of principal components are enough to almost fully explain the dynamics observed in the markets. Litterman and Scheikman [51], analyzed the US treasury yield curve and found that with just three factor they could explain most of the movements in the yield curve. The interpretation of these principal components is that they represent the level, slope, and curvature movements of the yield curve. This factor decomposition of the term structure is rather stable through time [16] and common to many economies. Moreover, Driessen et al. in [19] show that a five-factor model explains 98.5% of the cross-sectional variation in the expected bond returns of different maturities in the US, Germany, and Japan.

We now present the method to obtain the principal components. Assuming that we have $p$ observable variables and define the random vector $\mathbf{X} = [X_1, X_2, \ldots, X_p]$ with mean $\mu$ and covariance matrix $\Sigma$ of full rank $p$.

Consider the linear combinations

$$\begin{aligned} Y_1 &= a_1 X = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1p}X_p \\ &\vdots \qquad\qquad\quad \vdots \\ Y_p &= a_p X = a_{p1}X_1 + a_{p2}X_2 + \ldots + a_{pp}X_p \end{aligned}$$

Then we have:

$$Var(Y_i) = \mathbf{a_i^T \Sigma a_i} \quad i = 1, \ldots, p \tag{A.1}$$

$$Cov(Y_i, Y_k) = \mathbf{a_i^T \Sigma a_k} \quad i, k = 1, \ldots, p \tag{A.2}$$

The principal components are the uncorrelated linear combinations that maximize (A.1) restricted to cases where the coefficients have unit length, since, without this restriction, we would be able to increase the variance of the linear combination arbitrarily. Then we can define the principal components as

- The first principal component is the linear combination $\mathbf{a_1 i^T X}$ that maximizes $Var(\mathbf{a_1^T X})$ subject to $\mathbf{a_1^T a_1} = 1$.

- The second principal component is the linear combination $\mathbf{a_2^T X}$ that maximizes

$Var(\mathbf{a_2^T X})$ subject to $\mathbf{a_2^T a_2} = 1$ and $Cov(\mathbf{a_1^T X}, \mathbf{a_2^T X}) = 0$.

- The i-th principal component is the linear combination $\mathbf{a_i^T X}$ that maximizes $Var(\mathbf{a_i^T X})$ subject to $\mathbf{a_j^T a_j} = 1$ and $Cov(\mathbf{a_j^T X}, \mathbf{a_k^T X}) = 0$ for $k < i$.

Given the eigenvalue-eigenvector pairs of the covariance matrix $\Sigma$, $(\lambda_1, e_1)$,$(\lambda_2, e_2), \ldots, (\lambda_p, e_p)$ with $\lambda_1, \lambda_2, \ldots, \lambda_p \geq 0$, then the ith principal component is given by:

$$Y_i = \mathbf{e_i^T X} = e_{i1}X_1 + e_{i2}X_2 + \ldots + e_{ip}X_p$$

and

$$Var(\mathbf{Y_i}) = \lambda_i \quad i = 1, 2, \ldots, p$$
$$Cov(Y_i, Y_k) = 0 \quad i, k = 1, 2, \ldots, p$$

The proof for this result can be found in [71]. A useful concept when performing PCA is the proportion of total variance explained by the ith principal component which is defined as

$$\frac{\lambda_i}{\lambda_1 + \lambda_2 + \ldots + \lambda_p} \quad i = 1, \ldots, p$$

This concept allow us to have a quantitative criteria to select the number of components to select, in order to retain a certain level of variance of the original data. The most common criteria to select the number of components include the total variance explained, relative sizes of eigenvalues and marginal increase in total variation. For a complete discussion on this topic we refer to [45], [68] and [71].

The magnitude of the coefficient vector $\mathbf{e_i} = [e_{i1}, e_{i2}, \ldots, e_{ip}]$ measures the importance of the variables $\mathbf{X}$ in the i-th principal component $\mathbf{Y_i}$ for this reason, the inspection of this coefficients are used to interpret the principal components.