Induced Binary Submatroids

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The notion of induced subgraphs is extensively studied in graph theory. An example is the famous Gyárfás-Sumner conjecture, which asserts that given a tree T and a clique K, there exists a constant c such that the graphs that omit both T and K as induced subgraphs have chromatic number at most c. This thesis aims to prove natural matroidal analogues of such graph-theoretic problems.

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Chapter 1

Introduction

1.1 Results

This thesis deals with the theory of induced submatroids in the context of simple binary matroids. Given two matroids M and N, we say that M *contains an induced N-restriction* if there exists a flat F of M for which the restriction of M to F is isomorphic to N.

Restrictions and induced restrictions are analogous to subgraphs and induced subgraphs respectively in graph theory. Graph theorists have studied subgraphs as well as induced subgraphs extensively. The notion of restrictions has been studied in matroid theory as well, but that of induced restrictions for matroids did not enjoy the same level of interest. The goal of this thesis is to consider and prove natural matroidal analogues of graph-theoretic problems involving induced subgraphs in the binary setting.

The first major result in this thesis, Theorem 3.1.1, is a full structure theorem for simple binary matroids with no three-element independent flat or, equivalently, no induced $U_{3,3}$-restriction. This theorem is a matroidal analogue of the structure theorem for claw-free graphs by Chudnovsky and Seymour [13]. The full statement is too technical to state here, but as one of its corollaries we obtain a description of such matroids in terms of their critical numbers.

Given a simple rank-r binary matroid M, viewed as a restriction of $G \cong \mathrm{PG}(r-1,2)$, the *critical number* $\chi(M)$ is the smallest integer c such that there exists a rank-$(r-c)$ flat F of G for which $F \cap E(M) = \varnothing$, and $\omega(M)$ is the rank of a largest flat of G contained in $E(M)$. The parameters χ and ω are analogous to the chromatic number and clique number for graphs.

Theorem 1.1.1. *There exists a function f such that if M is a simple binary matroid with no induced $U_{3,3}$-restriction or induced $M(K_5)$-restriction, then $\chi(M) \leq f(\omega(M))$.*

Borrowing from graph theory terminology, this shows that the class of matroids with no induced $U_{3,3}$-restriction or induced $M(K_5)$-restriction is 'χ-bounded'. We will later show that one can replace $M(K_5)$ with any 'even-plane matroid', and the theorem will continue to hold. The class of even-plane matroids, which we will define later, is relatively rich; in particular, it contains matroids of arbitrarily large χ. However, it is not possible to eliminate even-plane matroids from this theorem altogether. Bonamy, Kardoš, Kelly, Nelson and Postle [46] showed that for any given number $c \geq 0$, there exists a simple binary matroid with no induced $U_{3,3}$-restriction for which $\omega(M) \leq 2$ and $\chi(M) > c$.

Another result is concerned with a Gyárfás-Sumner-type problem. For any positive integer t, Bonamy, Kardoš, Kelly, Nelson and Postle [46] asked whether there exists a constant c_t such that every simple binary matroid M with no induced $U_{t,t}$-restriction or triangle satisfies $\chi(M) \leq c_t$. Excluding induced $U_{t,t}$-restrictions is analogous to excluding trees in the setting of graphs.

Conjecture 1.1.2 ([46])**.** *For all $t \geq 1$, there exists a constant $c_t > 0$ such that if a simple binary matroid M has no induced $U_{t,t}$-restriction or triangle, then $\chi(M) \leq c_t$.*

The conjecture is straightforward when $t = 1, 2, 3$. In this thesis, we settle the conjecture for the case $t = 4$. We can in fact obtain a full structure theorem for simple binary matroids with no induced $U_{4,4}$-restriction or triangle. As a corollary, we obtain the following.

Theorem 1.1.3. *If M is a simple binary matroid with no induced $U_{4,4}$-restriction or triangle, then $\chi(M) \leq 2$.*

The conjecture of Bonamy, Kardoš, Kelly, Nelson and Postle remains open when $t \geq 5$. In this thesis, we prove a weakening of this conjecture. A simple binary matroid M, viewed as a restriction of a binary projective geometry $G \cong \mathrm{PG}(n-1, 2)$, is *$I_{1,t}$-free* if $|F \cap E(M)| \neq 1$ for every rank-t flat of G. In particular, simple $I_{1,t}$-free binary matroids contain no induced $U_{t+1,t+1}$-restriction.

Theorem 1.1.4. *For any $t \geq 1$, there exists a constant c_t such that if M is a simple $I_{1,t}$-free binary matroid with no triangle, then $\chi(M) \leq c_t$.*

The final topic is an extremal problem for simple binary matroids. Nelson and Norin [45] determined the smallest simple matroids with no $(t+1)$-element independent flat. For integers $r \geq 1$ and $t \geq 1$, the matroid $N_{r,t}$ denotes the direct sum of t, possibly empty, binary projective geometries whose ranks sum to r and pairwise differ by at most 1. Their theorem is valid for all simple matroids, not just simple binary matroids.

Theorem 1.1.5 ([45]). *Let $r, t \geq 1$ be integers. If M is a simple rank-r matroid with no induced $U_{t+1,t+1}$-restriction, then $|E(M)| \geq |N_{r,t}|$. If equality holds and $r \geq 2t$, then $M \cong N_{r,t}$.*

In this thesis, we prove the following. For integers $r \geq 1$ and $t \geq 1$, the matroid $A_{r,t}$ denotes the direct sum of t, possibly empty, binary affine geometries whose ranks sum to r and pairwise differ by at most 1.

Theorem 1.1.6. *Let $r \geq 1$ be an integer. If M is a simple rank-r binary matroid with no induced $U_{5,5}$-restriction or triangle, then $|E(M)| \geq |A_{r,2}|$.*

This is a special case of a conjecture by Nelson and Norin [45], which states that if M is a simple rank-r matroid with no induced $U_{2t+1,2t+1}$-restriction or triangle, then $|E(M)| \geq |A_{r,t}|$. Note that the above theorem is for simple binary matroids, whereas Nelson and Norin's conjecture is for all simple matroids.

Although our problems are inspired by graph-theoretic problems, they do not imply much about graphs. Sometimes matroidal theorems generalise theorems in graph theory, but that is not generally the case with our problems in the 'binary restriction' setting. Furthermore, while the graph-theoretic flavours will remain in the theorem statements themselves, the proof ideas typically differ greatly, too. The goal of this thesis is not to generalise results in graph theory to those involving matroids, but is to develop a theory for induced submatroids, after using graph theory to motivate natural problems in our setting.

1.2 Matroids

Our formalism of matroids will somewhat deviate from standard matroid theory. In this section, we first give a limited introduction to standard matroid theory terminology, and then explain its connection with our adjusted formalism of matroids called *embedded matroids*.

1.2.1 Standard Definitions

In 1935, Whitney and Nakasawa independently introduced matroids [41, 62] as a way to provide a formal treatment of independence seen in linear algebra and graph theory. There are various equivalent ways to define matroids. Here, we define matroids using rank functions. A *matroid* M is a pair (E, r) where E is a finite *ground set* and $r : 2^E \mapsto \mathbb{Z}$ is a *rank function* that satisfies the following three axioms.

(R1) $0 \leq r(X) \leq |X|$ for all $X \subseteq E$,

(R2) $r(X) \leq r(Y)$ for all $X \subseteq Y \subseteq E$, and

(R3) $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$ for all $X \subseteq Y \subseteq E$.

We write $E(M)$ for the ground set of M. The *rank* of M is $r(E)$. A set $X \subseteq E$ is *independent* if $r(X) = |X|$; otherwise it is *dependent*. A *circuit* is a minimal dependent subset of E. We say that an element of E is a *loop* if it forms a single-element circuit of M. If a two-element set $\{f, g\}$ is a circuit of M, then f and g are a *parallel pair*. If a matroid has no loop or parallel pair, then it is *simple*.

The *closure* of $X \subseteq E$ is the set $\{e \in E : r(X \cup \{e\}) = r(X)\}$ and is denoted $\mathrm{cl}(X)$. A set $F \subseteq E$ is a *flat* of M if $F = \mathrm{cl}(F)$. For a set $Y \subseteq E$, the *restriction of M to Y* is the matroid with ground set Y and rank function $r' : Y \mapsto \mathbb{Z}$ defined by $r'(X) = r(X)$ where $X \subseteq Y$. If Y is a flat, then this matroid is called the *induced restriction of M to Y*.

Two matroids (E_1, r_1) and (E_2, r_2) are *isomorphic* if there exists a bijection $\psi : E_1 \mapsto E_2$ such that $r_1(X) = r_2(\psi(X))$ for all $X \subseteq E_1$.

Representable Matroids

Let $\mathbb{F}$ be a field and let A be a matrix with entries in $\mathbb{F}$, where the columns are indexed by a set E. For each subset $X \subseteq E$, let $A[X]$ be the submatrix of A which consists of all columns of A that are indexed by X. Define a function $r : E \mapsto \mathbb{Z}$ by $r(X) = \mathrm{rank}(A[X])$ for all $X \subseteq E$. Then (E, r) is the *matroid represented by* A, and is denoted by $M[A]$. A matroid M is $\mathbb{F}$-*representable* if $M = M[A]$ for some matrix A over $\mathbb{F}$, and A is a *representation* of M. Hence, a simple $\mathbb{F}$-representable matroid has a representation over $\mathbb{F}$ with no zero column and no pair of parallel columns. Matroids that are $\mathbb{F}_2$-representable are called *binary*.

If A is a representation of a matroid, one can perform elementary row operations and column scalings and delete zero rows to obtain another matrix A'. Since $\mathrm{rank}(A[X]) = \mathrm{rank}(A'[X])$ for any $X \subseteq E$, it follows that $M[A] \cong M[A']$. In particular, this means that if M is a simple rank-n $\mathbb{F}$-representable matroid, then it has various representations with varying numbers of rows (with at least n rows). Embedded matroids will deviate from standard matroid theory here; for us, matrices with different numbers of rows will give nonisomorphic matroids.

For each integer $n > 0$ and prime power q, the maximum number of nonzero, pairwise nonparallel columns of a matrix with n rows with entries in $\mathbb{F}_q$ is $\frac{q^n-1}{q-1}$. Hence, any simple rank-n $\mathbb{F}_q$-representable matroid has at most $\frac{q^n-1}{q-1}$ elements. A simple rank-n $\mathbb{F}_q$-representable matroid with precisely $\frac{q^n-1}{q-1}$ elements is called a *projective geometry over* $\mathbb{F}_q$, and if $q = 2$ then it is called a *binary projective geometry.*

Graphic Matroids

Let $G = (V, E)$ be a finite graph. For any subset $X \subseteq E$, let $c(X)$ denote the number of connected components of the graph (V, X). Then the *graphic matroid of* G is the matroid with ground set E whose rank function is $r(X) = |V| - c(X)$ and is denoted $M(G)$. A matroid that is isomorphic to $M(G)$ for some graph G is called *graphic*. By considering the signed incidence matrix of a graph, one can show that every graphic matroid is representable over all fields. A graphic matroid $M(G)$ is simple if and only if G has no loops and parallel edges in the graph-theoretic sense.

1.3 Embedded Matroids

In this thesis, we view a simple binary matroid as embedded in an explicit ambient geometry. This allows us to define induced substructures more naturally, in the same way an induced subgraph is defined by specifying a subset of the underlying vertex set of a graph. While a matroid is not typically equipped with an ambient set, this is not an entirely new concept either, as it is implicit in some matroidal concepts. An example is the definition of the critical number given earlier, in which we view a simple rank-r binary matroid as a restriction of a rank-r binary projective geometry. Having an ambient set will also allow us to take the complement of a matroid in the same way we may take the complement of a graph, which is an important operation in the theory of induced subgraphs.

Throughout this thesis, if V is an n-dimensional vector space over $\mathbb{F}_2$, we say that the set $G = V\backslash\{0\}$ is an *n-dimensional (binary) projective geometry* and write $G \cong \mathrm{PG}(n-1, 2)$. We write $[G]$ for the vector space $V = G \cup \{0\}$. An *embedded matroid* M is a pair (E, G) where G is an n-dimensional projective geometry, and E is a subset of G. The set E is the *ground set* of M. The *dimension of M* is the dimension n of the vector space $[G]$ and is denoted $\dim(M)$. We say that two embedded matroids $M_1 = (E_1, G_1)$ and $M_2 = (E_2, G_2)$ are *isomorphic* if there exists a linear bijection $\psi : [G_1] \to [G_2]$ for which $\psi(E_1) = E_2$; we call such a map ψ a *matroid isomorphism from M_1 to M_2*. Note that any given n-dimensional embedded matroid (E, G) is isomorphic to another embedded matroid of the form $(\psi(E), \mathbb{F}_2^n\backslash\{0\})$ by picking a linear bijection $\psi : [G] \mapsto \mathbb{F}_2^n$.

This definition deviates from standard terminology as it involves the notion of an explicit ambient set G. Embedded matroids distinguish simple binary matroids by respecting the numbers of rows in representations. Let $\mathcal{M}$ be the set of matrices with entries from $\mathbb{F}_2$ with distinct non-zero columns, and let $\mathcal{M}'$ be the set of isomorphism classes of embedded matroids. Define a map $\phi : \mathcal{M} \mapsto \mathcal{M}'$ where for each matrix $A \in \mathcal{M}$, $\phi(A)$ equals the isomorphism class containing the embedded matroid (E, G) where $G = \mathbb{F}_2^n\backslash\{0\}$, while n is the number of rows of A, and E equals the set of columns of A. This map is well-defined, non-injective and surjective. If $A_1, A_2 \in \mathcal{M}$ have different numbers of rows, then $\phi(A_1) \neq \phi(A_2)$, as the dimensions of their corresponding embedded matroids differ. If the numbers of rows agree, and n is the number of rows in A_1 (and A_2), then from linear algebra it follows that $M[A_1] \cong M[A_2]$ if and only if there exists an invertible matrix P for which $A_1 = PA_2$. The existence of the matrix P is equivalent to having a map $I_P : \mathbb{F}_2^n \mapsto \mathbb{F}_2^n$ for which $I_P(E_1) = E_2$ where E_i is the set of column vectors of A_i for $i = 1, 2$. Therefore, $\phi(A_1) = \phi(A_2)$ if and only if the numbers of rows of A_1 and A_2 agree and $M[A_1] \cong M[A_2]$.

The incorporation of an ambient set in the definition of embedded matroids allows us to consider matroidal problems in a slightly different sense. In particular, this definition leads to matroids $M = (E, G)$ that are *rank-deficient*, which means the span of E in the vector space $[G]$ does not equal $[G]$; otherwise the matroid is *full-rank*. We therefore distinguish the notion of the *dimension* of an embedded matroid M, which refers to the dimension of the vector space $[G]$, and that of the *rank* of M, which equals the dimension of the span of E in the vector space $[G]$. The dimension is denoted $\dim(M)$ and the rank is denoted $r(M)$. We make use of the additive notation and, given two elements $x, y \in [G]$, write $x+y$ for the element corresponding to the vector sum of x and y in the vector space $[G]$. Some of the results in this thesis will have much more natural statements in the context of our notion of embedded matroids, while those that are described only by full-rank matroids can just as easily be reformulated in standard terminology.

For the rest of this thesis, embedded matroids will be referred to as *matroids*, unless otherwise stated. We now define some important concepts related to embedded matroids; many of them correspond naturally to the definitions for standard matroids with minor adjustments. Given an n-dimensional matroid $M = (E, G)$, the *size* of M is $|E|$, which we also denote by $|M|$. If $|E| = 0$, then M is *empty* or an *empty matroid*. If $E = G$, then we abuse notation by writing $M \cong \mathrm{PG}(n-1, 2)$ and say that M is an *n-dimensional projective geometry*. The *complement* of matroid $M = (E, G)$ is the matroid $(G \backslash E, G)$ and is denoted M^c. These notions have natural analogues in graph theory. The dimension n corresponds to the number of vertices and the size $|E|$ corresponds to the number of edges. Complements of matroids are analogous to complements of graphs. Projective geometries are analogous to complete graphs; projective geometries of dimension n are the densest matroids of dimension n, in the same way the complete graph on n vertices is the densest graph on n vertices.

Certain matroids will play a key role in our binary setting. We will be especially interested in the matroids of the form (B, G), where $G \cong \mathrm{PG}(n-1, 2)$ and B is a basis of $[G]$. These matroids will play the role that trees do in graph theory. They are tree-like in the sense that they are maximally acyclic, meaning that the addition of any other element into the ground set will introduce a dependency. Note that, for any fixed integer n, any two n-dimensional matroids (B_1, G_1) and (B_2, G_2) where B_i is a basis of $[G_i]$ for $i = 1, 2$ are isomorphic. Hence, we write I_n to denote these matroids. We call I_3 a *claw*.

A *circuit of length n* is a full-rank $(n-1)$-dimensional matroid whose ground set consists of n elements that add to zero. If n is odd, it is called an *odd circuit*. A circuit of length n is a full-rank $(n-1)$-dimensional matroid (E, G) with the property that E is a dependent set in $[G]$ but $E \backslash \{e\}$ is an independent set for any $e \in E$. For fixed $n \geq 1$, all circuits of length n are isomorphic, and we write C_n to denote these matroids.

1.3.1 Flats

Given $G \cong \mathrm{PG}(n-1, 2)$, a *flat* is a subspace of $[G]$ with the zero vector removed. From here after in our formalism, the term 'flat' is always used in reference to some projective geometry G; we do not refer to a 'flat' of a matroid M itself unless we state otherwise explicitly. The *dimension* of a flat F is the dimension of $[F]$ as a vector space and is denoted $\dim(F)$, and its *codimension* is $\dim(G) - \dim(F)$. We call flats of dimension 2, 3 and $\dim(G) - 1$ *triangles*, *planes* and *hyperplanes* respectively. The *closure* or *span* of a given set $X \subseteq E$ is the minimal flat F that contains X and is denoted $\mathrm{cl}(X)$. The *rank* of X is the dimension of this F and is denoted $r(X)$. The rank $r(M)$ of M is $r(E)$.

Given $G \cong \mathrm{PG}(n-1, 2)$, a *basis* of G is a subset of G which is a basis of the vector space $[G]$. Given a subset $E \subseteq G$, a *basis of* E is a subset of E that is a basis of $\mathrm{cl}(E)$. Given a matroid $M = (E, G)$, a *basis of* M is a basis of $E \subseteq G$.

1.3.2 Restrictions

Given a matroid $M = (E, G)$, if F is a flat of G, then a matroid of the form (E', F), where $E' \subseteq E \cap F$ is a *restriction* of M. If, moreover, $E' = E \cap F$, then it is called the *induced restriction of M to F* and is denoted $M|F$. Given another matroid N, we say that M *contains an N-restriction* if there exists a restriction of M that is isomorphic to N. We say that M *contains an induced N-restriction* if there exists a flat F of G for which $M|F \cong N$. If M contains no induced N-restriction, then M is *N-free* or M *omits* N. If $\mathcal{N}$ is a set of matroids and M is N-free for all $N \in \mathcal{N}$, then M is $\mathcal{N}$-free. A matroid M is *triangle-free* if M contains no induced $\mathrm{PG}(1, 2)$-restriction. We also use the term *induced submatroid* in place of induced restriction.

The notion of induced restrictions and their containment can also be defined using linear injections. Given two matroids $M = (E, G)$ and $N = (E', G')$, it can be shown that M contains an N-restriction if there exists a linear injection $\psi : [G'] \to [G]$ for which $\psi(E') \subseteq E$. If, additionally, $\psi(E') = E \cap \psi(G')$, then M contains an induced N-restriction.

Note that we only say that M is N-free (or $\mathcal{N}$-free) when M omits N as an *induced* restriction, as opposed to as just a restriction. We opt for this terminology as much of our thesis will be devoted to excluding matroids as induced restrictions. The same convention will be taken with graphs. Note that if M has no N-restriction, then M has no induced N-restriction, but the converse is not necessarily true. One example in which the converse holds is when $N \cong \mathrm{PG}(n-1, 2)$. If $M = (E, G)$ contains no induced $\mathrm{PG}(n-1, 2)$-restriction, then $M|F \ncong \mathrm{PG}(n-1, 2)$ for every n-dimensional flat F. This means $E \cap F \neq F$ for every n-dimensional flat F of G, which means that M contains no $\mathrm{PG}(n-1, 2)$-restriction. This is analogous to the statement that a graph that contains no clique as an induced subgraph contains no clique as a subgraph.

1.3.3 Critical Number

Given a matroid $M = (E, G)$, the parameter $\omega(M)$ is the dimension of a largest projective geometry contained in E. It is analogous to the clique number for graphs. The *critical number* of M is the minimum nonnegative integer c for which G has a $(\dim(M) - c)$-dimensional flat disjoint from E. The critical number is denoted by $\chi(M)$. Note that the critical number of (E, G) equals the critical number of $(E, \mathrm{cl}(E))$.

In the introduction, given a simple rank-r binary matroid M, viewed as a restriction of $G \cong \mathrm{PG}(r-1, 2)$, we defined the critical number $\chi(M)$ to be the smallest integer c such that there exists a rank-$(r-c)$ flat F of G for which $F \cap E(M) = \varnothing$. This definition is independent of the embedding of M in G (in fact, there is only one unique embedding up to isomorphism in the binary case). Therefore the two notions of critical number agree.

The critical number was originally defined by Crapo and Rota [36] under the name *critical exponent*. From the above definition, it is not too hard to see that the critical number of $M = (E, G)$ also equals $n - \omega(M^c)$. The critical number of an n-dimensional matroid is always between 0 and n. In fact, $\chi(M) = 0$ if and only if $\omega(M^c) = n$, meaning M is an empty matroid. Also, $\chi(M) = n$ if and only if $\omega(M^c) = 0$, meaning that M is an n-dimensional projective geometry. Any matroid that is neither empty nor a projective geometry will give a critical number that is strictly greater than 0 and smaller than n. Lemma 2.2.1 will prove that the parameter χ does not increase under restrictions.

This notion of χ may appear to be unmotivated at first, but it can be seen as a matroidal analogue of the chromatic number for graphs. For example, if $\chi(G)$ is the chromatic number of a graph G, and $M = M(G)$ is the graphic matroid arising from G, then it can be shown [37] that $\chi(M) = \lceil \log_2(\chi(G)) \rceil$. The critical number may also be defined using the characteristic polynomial, which is a natural matroidal analogue of the chromatic polynomial for graphs ([36, 60]). We will later give the matroidal analogue of the Erdős–Stone Theorem, which also reveals a striking similarity between the graphic and matroidal notions of χ. Note that if the chromatic number of the graph G is 2, then the graphic matroid arising from G has critical number 1. Perhaps not surprisingly, the matroids for which $\chi = 1$ will sometimes behave very similarly to the graphs with chromatic number 2.

1.4 Excluding Restrictions

Many graph-theoretic problems involving subgraphs have been translated into matroidal problems. In this thesis, we mention two such examples. They are, respectively, matroidal analogues of Turán's Theorem and the Erdős–Stone Theorem. We note that these problems have been rephrased for embedded matroids; they are typically stated differently in the literature. We assume that all matroids are full-rank in this section.

1.4.1 The Bose–Burton Theorem

The Bose–Burton Theorem is a matroidal analogue of the classical theorem of Turán [57] in graph theory. The *Turán graph* $T(n,t)$ is the complete $(t-1)$-partite graph on n vertices whose parts are as equally-sized as possible. It is clear that $T(n,t)$ is K_t-free. Turán's Theorem, stated below, characterises the largest graph that is K_t-free.

Theorem 1.4.1 ([57])**.** *For all integers $n \geq t \geq 1$, if G is a simple K_t-free graph on n vertices, then $|E(G)| \leq |E(T(n,t))|$. Equality holds if and only if $G \cong T(n,t)$.*

For any $r \geq t \geq 0$, a *Bose–Burton geometry of order t* is a matroid (E,G) where $G \cong \mathrm{PG}(r-1,2)$ and E is obtained by removing a flat of dimension $r-t$ from G. Note that such matroids are determined up to isomorphism by r and t and we write $\mathrm{BB}(r-1,t)$ to denote these matroids. When $t=0$, then $E=\varnothing$, and when $t=r$, then $E=G$. Equivalently, Bose–Burton geometries are matroids (E,G) for which $G\backslash E$ forms a flat. An n-dimensional Bose–Burton geometry of order 1 is an *n-dimensional affine geometry* and is denoted $\mathrm{AG}(n-1,2)$.

Note that $\mathrm{BB}(r-1,t)$ is $\mathrm{PG}(t,2)$-free. If an order-t Bose–Burton geometry $M=(E,G)$ contained a $\mathrm{PG}(t,2)$-restriction, then there is a $(t+1)$-dimensional flat $F \subseteq G$ for which $F \subseteq E$. By the definition of Bose–Burton geometries, there is an $(r-t)$-dimensional flat $F' \subseteq G$ for which $F' \subseteq G\backslash E$. Hence $F \cap F' = \varnothing$. But note that $F \cap F' \neq \varnothing$, as $\dim(F)+\dim(F') > r$. This is a contradiction.

The following theorem of Bose and Burton [5] characterises the densest $\mathrm{PG}(t,2)$-free matroids. While we will not provide proofs for the other theorems in this section, we provide the proof for the Bose–Burton Theorem because of its importance to this thesis.

Theorem 1.4.2 ([5])**.** *For all integers $r-1 \geq t \geq 0$, if $M=(E,G)$ is a $\mathrm{PG}(t,2)$-free matroid of dimension r, then $|E| \leq |\mathrm{BB}(r-1,t)|$. Equality holds if and only if $M \cong \mathrm{BB}(r-1,t)$.*

Proof. The result is immediate when $t=0$. Let t be the smallest positive integer for which the claim fails, and let $M=(E,G)$ be a counterexample. Let $r=\dim(M)$.

We may assume that there exists a triangle $T \subseteq G$ for which $|T \cap E| = 1$. If not, then $G\backslash E$ forms a flat of G, and hence M is a Bose–Burton geometry of order at most t; it follows that $|E| \leq |\mathrm{BB}(r-1,t)|$ with equality if and only if M is a Bose–Burton geometry of order precisely t. Fix some triangle T such that $|T \cap E| = 1$, and let $a \in T \cap E$.

Now pick a hyperplane $H \subseteq G$ for which $a \notin H$. For each $i=0,1,2$, let F_i be the set of elements $x \in H$ for which $|\mathrm{cl}(\{x, a+x\}) \cap E| = i$. Note that the F_i partition H. Then the matroid (F_2, H) is $\mathrm{PG}(t-1,2)$-free; if $F' \subseteq F_2$ is a flat of dimension t, then $\mathrm{cl}(F' \cup \{a\}) \subseteq E$ is a flat of dimension $t+1$, a contradiction. By minimality, $|F_2| \leq |\,\mathrm{BB}(r-2, t-1)|$. Recall that $a \in T$ and $|(T \backslash \{a\}) \cap E| = 0$, so $F_0 \neq \varnothing$. Hence

$$|E| = 1 + |F_1| + 2|F_2| = 1 + |F_2| + |H| - |F_0| \leq |F_2| + |H| \leq |\,\mathrm{BB}(r-2, t-1)| + |H|.$$

But $|\,\mathrm{BB}(r-2, t-1)| = 2^{r-1} - 2^{r-t}$ and $|H| = 2^{r-1} - 1$. Hence

$$|\,\mathrm{BB}(r-2, r-1)| + |H| = 2^r - 2^{r-t} - 1 < |\,\mathrm{BB}(r-1, t)|$$

providing the required bound. □

The Bose–Burton Theorem is important to this thesis in many ways. From a practical point of view, we use it numerous times in later chapters as a proof ingredient. Also, the proof of the Bose–Burton Theorem is one example of a larger phenomenon that we tend to see repeatedly in the theory of excluded restrictions for matroids. That is, if the matroidal analogue of a graph-theoretic theorem holds (which, unfortunately, is not always true), the proof is often more straightforward than the graph-theoretic counterpart. While it is difficult to quantify the difficulty of a proof, this proof of the Bose–Burton Theorem seems conceptually simpler than any known proof of Turán's Theorem.

1.4.2 The Matroidal Erdős–Stone Theorem

Turán's Theorem characterises the densest K_t-free graphs on n vertices. The Erdős–Stone Theorem [22] considers an analogous question, but for general graphs other than K_t. Given a graph H, let $\mathrm{ex}(H, n)$ be the maximum number of edges in a graph on n vertices that contains no H as a subgraph. Turán's Theorem implies that $\mathrm{ex}(K_t, n) = |E(T(n,t))|$. The Erdős–Stone Theorem [22] states the following.

Theorem 1.4.3 ([22])**.** *Let H be a graph with chromatic number $\chi \geq 2$. Then*

$$\lim_{n\to\infty} \frac{\mathrm{ex}(H,n)}{|E(K_n)|} = \frac{\chi - 2}{\chi - 1}. \tag{1.1}$$

This is a remarkable theorem, as the quantity $\mathrm{ex}(H, n)$ seemingly has little to do with the chromatic number. The Erdős–Stone Theorem shows that $\mathrm{ex}(H,n) = \frac{\chi-2}{\chi-1}\binom{n}{2} + o(n^2)$, which determines the asymptotic behaviour of $\mathrm{ex}(H, n)$ unless $\chi = 2$. When $\chi = 2$, then determining the asymptotic behaviour of the Turán numbers is a difficult problem and is one of the major open problems in extremal graph theory.

Now, let $\text{ex}(N, r)$ be the maximum number of elements in a matroid of dimension r that contains no N-restriction. The following theorem by Geelen and Nelson [25] relates the quantity $\text{ex}(N, r)$ with the critical number.

Theorem 1.4.4 ([25]). *Let N be a matroid with critical number $\chi \geq 1$. Then*

$$\lim_{r \to \infty} \frac{\text{ex}(N, r)}{|\,\text{PG}(r-1, 2)|} = 1 - 2^{1-\chi}. \tag{1.2}$$

Theorem 1.4.4 is striking in its similarity to Theorem 1.4.3, and provides further evidence that the matroidal parameter χ is an appropriate analogue of the chromatic number for graphs. Theorem 1.4.4 shows that $\text{ex}(N, r) = 2^r - 2^{r-\chi+1} + o(2^r)$, which determines the asymptotic behaviour of the quantity $\text{ex}(N, r)$ when $\chi > 1$. When $\chi = 1$, it is again an open problem.

We will not give the proof of Theorem 1.4.4. However, as the authors of [25] themselves note [24], the proof of Theorem 1.4.4 is more straightforward than that of Theorem 1.4.3. It is another example in which a matroidal analogue goes more smoothly than its graph-theoretic counterpart.

The critical exponent extends in its similarity to the chromatic number far beyond the context of Theorem 1.4.4. As an example, we mention the well-studied notion of chromatic threshold in graph theory. The *chromatic threshold of a graph H*, defined in [39], is the infimum of all $\alpha > 0$ for which there exists a constant c such that graphs G with minimum degree at least $\alpha|V(G)|$ that contain no H as a subgraph have chromatic number at most c. Culminating in [1], the chromatic threshold has been determined for all graphs H (for results and related history about this problem, see [1, 6, 21, 29, 39, 55]).

Geelen and Nelson [26] define an analogous notion for matroids. Given a matroid N, the *critical threshold of* N is the infimum of all $\alpha > 0$ for which there exists a constant c such that matroids with $|M| > \alpha 2^{\dim(M)}$ that contain no N-restriction have critical number at most c. There exist classes of matroids for which the critical threshold has been determined (see [10, 26, 27, 56]), but unlike the graph counterpart, the problem of determining the critical threshold remains open in general.

Problems with other flavours, such as counting problems, have been translated into our matroidal setting as well. For example, Erdős made a conjecture about the maximum number of 5-cycles in triangle-free graphs, which was later settled in [33, 35]. There exists a matroidal analogue of this problem, which was partially settled in [7] recently.

1.5 Excluding Induced Restrictions

The theory of induced subgraphs is an active area of research with many known theorems and conjectures. One example, and one for which we will give a matroidal analogue, is the structure theorem of claw-free graphs by Chudnovsky and Seymour [13]; a *claw* is the bipartite graph $K_{1,3}$. Their theorem is too technical to state, but as a structure theorem it states that there are some well-understood basic classes of claw-free graphs and some operations that construct all claw-free graphs from the basic classes.

There are also known theorems when $\mathcal{F}$ consists of cycles. A famous example is the family of *Berge* graphs. Berge graphs are graphs with no induced subgraph isomorphic to a cycle of odd length at least five or its complement. In fact, a general decomposition theorem describing Berge graphs is the main ingredient in the proof of the Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour and Thomas [11]. A graph is *perfect* if, for every induced subgraph, the size of the maximum clique equals its chromatic number. A *hole* is an induced cycle of odd length at least five, and an *antihole* is the complement of an odd hole.

Theorem 1.5.1 (The Strong Perfect Graph Theorem, [11]). *A graph is perfect if and only if it has no odd holes or odd antiholes.*

The forward direction of this theorem is immediate, since odd holes and odd antiholes are not perfect. Chudnovsky, Robertson, Seymour and Thomas [11] obtain a decomposition theorem for all Berge graphs, in which all Berge graphs either belong to one of five well-understood classes or admitted certain special decompositions, which allows them to verify that all Berge graphs are perfect.

There also exist such structural theorems when $\mathcal{F}$ is the set of even-length cycles (a theorem by Conforti, Cornuéjols, Kapoor and Vušković [15]), and when $\mathcal{F}$ is the set of odd-length cycles of length at least five (a theorem by Conforti, Cornuéjols, and Vušković [16]).

There exists an abundance of open problems regarding induced subgraphs. Many of these problems are widely regarded as some of the most important problems in graph theory. One that is particularly relevant to our study is the Gyárfás-Sumner Conjecture [34, 53].

Conjecture 1.5.2 (Gyárfás-Sumner, [34, 53]). *For every tree T and complete graph K, there is an integer c such that if G is a graph containing neither T nor K as an induced subgraph, then $\chi(G) \leq c$.*

The Gyárfás-Sumner Conjecture is known to be true for certain trees. Amongst the simplest such examples are stars and paths [34]. For other known cases, [51] is a recent survey containing detailed information about this subject. The conjecture remains open in general. We will discuss the Gyárfás-Sumner Conjecture as it relates to matroids later in this chapter.

As we remarked before, many problems in graph theory have led to natural problems in matroid theory. It may therefore come as a surprise that, given the level of interest that graph theorists have in the theory of excluded *induced* subgraphs, there was little interest in an analogous notion in matroid theory. One of the few exceptions is the work of Mills in 1999 [40], who considered the notion of *perfect matroids* in the context of simple binary matroids. Another exception is the recent work by Bonamy, Kardoš, Kelly, Nelson and Postle [46], who considered a Gyárfás-Sumner-type problem for matroids. This thesis builds on their work.

For matroids $N = (E', G')$ with $\dim(G')$ very small, the class of N-free matroids is easy to describe in many cases. By appropriately taking complements, suppose that $|E'| > \frac{|G'|}{2}$. If G' is 1-dimensional, then $|E'| = 1$, so the only N-free graphs are empty. If G' is 2-dimensional, i.e., $|G'| = 3$, then $|E'| \in \{2, 3\}$. If $|E'| = 2$, then the N-free matroids are complements of Bose-Burton geometries, since E' forms a flat of G'. If $|E'| = 3$, we are considering triangle-free matroids. Triangle-free matroids do not seem to enjoy a pleasant structural description. Examples of triangle-free matroids are the graphic matroids of triangle-free graphs, but triangle-free graphs are already a wild class of graphs to describe. As such, we will later treat the class of triangle-free matroids (in fact the class of complements of triangle-free matroids) as a 'basic' class of matroids. When G' is 3-dimensional, the problem of describing N-free matroids becomes difficult.

1.5.1 Perfect Matroids

Let us now discuss prior work in the theory of induced submatroids.

Given a matroid $M = (E, G)$, recall that $\omega(M)$ and $\chi(M)$ are matroidal analogues of the clique number and chromatic number for graphs, respectively.

Mills [40] defines a notion for perfect matroids. Rephrased in our language, a matroid M is *perfect* if and only if $\chi(N) = \omega(N)$ for every induced restriction N of M. The following is true.

Theorem 1.5.3 ([40]). *A matroid M is perfect if and only if M^c is perfect.*

This theorem is a matroidal analogue of the Weak Perfect Graph Theorem of Lovász [38], which states that a graph is perfect if and only if its complement graph is perfect. While the proof of the Weak Perfect Graph Theorem is non-trivial, the proof of Theorem 1.5.3 is a direct manipulation of the definition of perfect matroids.

Mills also proves that the matroids M for which $\chi(M) = 1$ are perfect. This follows from the definition of the critical number. More generally, Mills proves that if $\chi(M) \leq 2$, then M is perfect if and only if M has no induced odd circuit of length five or more.

Theorem 1.5.4 ([40])**.** *Let M be a matroid.*

- *If $\chi(M) = 1$, then M is perfect.*
- *If $\chi(M) \leq 2$, then M is perfect if and only if M contains no induced odd circuit of length five or more.*

Theorem 1.5.1 characterises perfect graphs by the exclusion of holes and antiholes. Note that when $\chi(M) \leq 2$, the matroid M automatically omits the complements of odd circuits of length five or more and Theorem 1.5.4 might suggest that a similar statement to Theorem 1.5.1 might hold for matroids, too. But this is not true when $\chi(M) > 2$. Let H be the complement of the 9-cycle and let $M = M(H)$ (the graphic matroid of H). Then $\omega(M) = 2$, as H contains a triangle. On the other hand, $\chi(M) = \lceil \log_2(\chi(H)) \rceil = 3$ since $\chi(H) = 5$. But one can check that M omits odd circuits of length 5 or more and complements of odd circuits of length 5 or more. Hence, Theorem 1.5.1 would not hold for matroids.

1.5.2 Gyárfás-Sumner Conjecture

We now discuss a matroidal analogue of the Gyárfás-Sumner Conjecture (Conjecture 1.5.2) considered in [46].

Recall that, given $G \cong \mathrm{PG}(n-1, 2)$, if B is a basis of $[G]$, we write I_n for the matroid (B, G); the matroids I_n will play the role of trees in our setting. As an analogue for cliques, we consider projective geometries, as they are the densest objects of any given dimension. Bonamy, Kardoš, Kelly, Nelson and Postle [46] asked the following question.

Question 1.5.5 ([46])**.** *For which $s, t \geq 1$ does there exist a constant $k = k(s, t)$ such that every $M = (E, G)$ with no induced I_s-restriction or $\mathrm{PG}(t-1, 2)$-restriction satisfies $\chi(M) \leq k$?*

The answer to the above question is trivially affirmative when $s = 1, 2$ or $t = 1$. When $s = 1$ or $t = 1$, the only matroids permitted are empty. When $s = 2$, the set E forms a flat, so M is the complement of a Bose-Burton geometry. It is easy to check that Bose-Burton geometries satisfy $\chi(M) = \omega(M)$.

The case where $s = 3$ and $t = 2$ is less obvious, but one can in fact show that $(E, \mathrm{cl}(E))$ is an affine geometry, which implies that Question 1.5.5 is true in this case as well.

Lemma 1.5.6 ([46]). *If $M = (E, G)$ is I_3-free and triangle-free, then $(E, \mathrm{cl}(E))$ is an affine geometry.*

Proof. We may assume that M is full-rank. The condition is equivalent to saying that for any distinct $u, v, w \in E$, the vector $u + v + w \in E$. Fix $u_0 \in E$, and consider the set $W = E + u_0 = \{z + u_0 \mid z \in E\}$. Then $0 \in W$, and for any two distinct $x, y \in W$, we have $x + y = ((x - u_0) + (y - u_0) + u_0) + u_0 \in W$, so we see that W is a subspace. It then follows that (E, G) is either a projective geometry, or affine geometry. Because of triangle-freeness, (E, G) is an affine geometry. $\square$

However, the following theorem [46] shows that Conjecture 1.5.5 fails when $s = 3$ and $t = 3$. The authors found a family of (I_3, F_7)-free matroids, called the *even-plane matroids*, whose members have arbitrarily large critical number. The matroid F_7 is the *Fano matroid* which is simply $\mathrm{PG}(2, 2)$. We say that a matroid M is *even-plane* if all of its 3-dimensional induced restrictions have even-sized ground sets, and write $\mathcal{E}_3$ to denote the set of all even-plane matroids.

Theorem 1.5.7 ([46]). *For every integer k, there is a matroid $M \in \mathcal{E}_3$ containing no induced I_3-restriction or F_7-restriction such that $\chi(M) \geq k$.*

Theorem 1.5.7 implies that the only cases for which Conjecture 1.5.5 remains open are when $t = 2$, meaning that M is triangle-free. In light of this discussion, the authors of [46] make the following refined conjecture.

Conjecture 1.5.8 ([46]). *For all $s \geq 1$, there exists k such that if $M = (E, G)$ has no induced I_s-restriction or triangle, then $\chi(M) \leq k$.*

1.5.3 I_3-free and Fano-free Matroids

Bonamy, Kardoš, Kelly, Nelson and Postle [46] gave a full structure theorem for all I_3-free and F_7-free matroids.

To state their theorem, we need to define one operation. We give a short definition here and Chapter 2 will explain this operation more carefully. For now, if $M = (E, G)$ is an n-dimensional matroid, then $D(M)$ is the $(n+1)$-dimensional matroid (E', G') where $G' = (\{0,1\} \times [G])\backslash\{0\}$ and $E' = \{0,1\} \times E$. This operation 'doubles' M. One important feature of this operation is that it does not increase the value of χ.

Let $\mathrm{AG}^\circ(t-1,2)$ denote the unique (up to isomorphism) t-dimensional matroid obtained by adding a single element to the affine geometry $\mathrm{AG}(t-1,2)$. The theorem states that all I_3-free, F_7-free matroids are either even-plane, or isomorphic to $D^k(\mathrm{AG}^\circ(t-1,2))$ for some t and k. Hence, it is precisely the family of even-plane matroids that prevents one from bounding χ.

Theorem 1.5.9 ([46]). *If M is a full-rank matroid, then M does not contain an induced I_3-restriction or F_7-restriction if and only if*

- *M is an even-plane matroid, or*
- *there exist some $t \geq 3$ and $k \geq 1$ such that $M \cong D^k(\mathrm{AG}^\circ(t-1,2))$.*

In Chapter 3, we drop the F_7-free condition, and give a structure theorem of I_3-free matroids.

1.5.4 Extensions

While Question 1.5.5 has a negative answer in general for $t \geq 3$, it is possible to reconsider this problem with a different notion of χ. Although we will not discuss this notion or the following conjecture in the rest of this thesis, we mention them as a possible direction of future research.

We first mention the following characterisation of χ in terms of linear polynomials in [36] which will be useful for this discussion.

Lemma 1.5.10 ([36]). *For a matroid $M = (E, G)$ where G is identified with $\mathbb{F}_2^n\backslash\{0\}$, $\chi(M)$ is the smallest integer k such that there exist k linear polynomials $p_1, \dots, p_k \in \mathbb{F}_2[x_1, \cdots, x_n]$, with $p_i(0) = 0$ for all i, such that if $x \in E$ there exists some j for which $p_j(x) = 1$.*

Let $M = (E, G)$ be an n-dimensional matroid. Associate G with the vectors in $\mathbb{F}_2^n\backslash\{0\}$. Then Nelson and Norin define $\chi^*(M)$ to be the minimum integer k for which there exists a degree-k polynomial $p \in \mathbb{F}_2[x_1, \cdots, x_n]$ with $p(0) = 1$ such that $p(x) = 0$ for all $x \in E$.

First, we claim that $\chi^*(M) \leq \chi(M)$ for any matroid $M = (E, G)$. Let $k = \chi(M)$. By Lemma 1.5.10 there exist k linear polynomials $p_1, \ldots, p_k \in \mathbb{F}_2[x_1, \cdots, x_n]$, with $p_i(0) = 0$ for all i, such that if $x \in E$, then there exists some j for which $p_j(x) = 1$. Therefore, the degree-k polynomial $p = \prod_{i=1}^{k}(p_i + 1)$ certifies that $\chi^*(M) \leq k = \chi(M)$. In light of this relationship, Nelson and Norin conjecture the following.

Conjecture 1.5.11 (Nelson, Norin)**.** *For any $s, t \geq 1$, there exists a constant c such that if M is an I_t-free matroid with $\omega(M) < s$, then $\chi^*(M) \leq c$.*

For (I_3, F_7)-free matroids, by Theorem 1.5.9, the *only* obstruction to bounding χ is the even-plane matroids. But, as a consequence of a structural description of even-plane matroids given in [46], it can then be shown that all even-plane matroids have χ^* at most 2. In fact, due to Nelson (via personal communication), more can be said. It can be shown that the even-plane matroids are precisely the solutions to the equations $p(x) = 0$ where $p \in \mathbb{F}_2[x_1, \cdots, x_n]$ is quadratic with $p(0) = 1$.

Theorem 1.5.12 (Nelson)**.** *Let $M = (E, G)$ be an n-dimensional matroid. Then $M \in \mathcal{E}_3$ if and only if $E = \{x \in G \mid p(x) = 0\}$ for some quadratic polynomial $p \in \mathbb{F}_2[x_1, \cdots, x_n]$ for which $p(0) = 1$.*

1.5.5 Thesis Structure

This thesis contains four main contributions to the theory of induced submatroids.

The first result is a full structure theorem for claw-free matroids. This result is analogous to the structure theorem for claw-free graphs by Chudnovsky and Seymour [13]. We will mention some corollaries as a consequence of this structure theorem, which provide interesting conjectures for future study as well.

The second result is a full structure theorem for I_4-free, triangle-free matroids. As a corollary, we prove Conjecture 1.5.8 when $s = 4$, which is the first interesting case of this conjecture.

Our third result deals with a certain weakening of Conjecture 1.5.8. The proof will rely on results from additive combinatorics. Part of the motivation for studying such a weakening is to explore the use of additive combinatorics in the theory of induced submatroids.

While Conjecture 1.5.8 remains out of reach when $s > 4$, we can also consider a related extremal question. Our final result is an extremal result for I_5-free triangle-free matroids, in which we determine the smallest matroids that are I_5-free and triangle-free.

Chapter 2

Preliminaries

In this chapter, we collect some facts that will be used repeatedly in this thesis. In what follows, given a vector space V, we extend the sum operation $+\colon V^2 \to V$ to sets, writing $X + Y$ for $\{x + y \colon x \in X, y \in Y\}$ and $x + Y$ for $\{x\} + Y$. Also recall that, given a flat F of a projective geometry $G \cong \mathrm{PG}(n-1, 2)$, $[F]$ is the set $F \cup \{0\}$.

2.1 Cosets

Given a projective geometry G, a *coset* of a flat $F \subseteq G$ is any proper translate of the subspace $[F]$ of $[G]$, i.e. a set of the form $x + [F]$ for some $x \in G \backslash F$. The set $G \backslash F$ has a partition into cosets of F. We do not consider the set F itself a coset, which is nonstandard in group theory. We will often use the term *translate* when we wish to include F itself in the definition.

Cosets are useful to us because we will encounter many scenarios in which we understand only a small part of a given matroid $M = (E, G)$, say on a flat F, and wish to explore the how the rest of the matroid is influenced by the existence of such an induced restriction $M|F$. In fact, many of our proofs come down to picking the right restriction $M|F$ so that the structure of M on the cosets of F, as well as the interactions between the cosets, are highly restricted.

Now, we state and prove a lemma that gives a global structure in a matroid for which certain types of triangles are forbidden. This is a particularly important result in the proof of the structure theorem for claw-free matroids.

Lemma 2.1.1 (Coset Lemma)**.** *Let (P, Q, R) be a partition of a projective geometry G for which no triangle T of G satisfies $|T \cap P| \geq 1$ and $|T \cap R| = 1$. Then*

- $\mathrm{cl}(P) \subseteq P \cup Q$*, and*
- *All cosets of* $\mathrm{cl}(P)$ *in* G *are contained in* Q *or* R*.*

Furthermore, if (R_1, R_2) is a partition of R and G has no triangle that intersects P, R_1 and R_2, then all cosets of $\mathrm{cl}(P)$ *in G are contained in Q, R_1 or R_2.*

Proof. For all $X \subseteq [G]$ and $k \geq 0$, let $kX = \{x_1 + \cdots + x_k \mid x_1, \ldots, x_k \in X\}$. Note that $[\mathrm{cl}(X)] = \cup_{k \geq 1}(kX)$. Let $P' = P \cup \{0\}$. The triangle condition given implies that $P' + (P' \cup Q) \subseteq P' \cup Q$. An easy inductive argument gives that $kP' \subseteq P' \cup Q$ for all $k \geq 1$; thus $[\mathrm{cl}(P)] = \cup_{k \geq 1}(kP') \subseteq P' \cup Q$ and so $\mathrm{cl}(P) \subseteq P \cup Q$ as required.

Let A be a coset of $\mathrm{cl}(P)$; note that $A \subseteq G \backslash \mathrm{cl}(P) \subseteq Q \cup R$. If A contains a vector $w \in Q$, then a similar inductive argument gives that $w + kP' \subseteq P' \cup Q$ for all $k \geq 0$ and so $A = [\mathrm{cl}(P)] + w = \cup_{k \geq 0}(w + kP') \subseteq Q$. Otherwise $A \subseteq R$, as required.

Finally, if (R_1, R_2) is a partition of R as in the hypothesis, then for each coset $A \subseteq R$ of $\mathrm{cl}(P)$, we have $(A \cap R_1) + P' \subseteq A \cap (G \backslash R_2) = A \cap R_1$. If A contains some $u \in R_1$, then induction gives $u + kP' \subseteq R_1$ for all $k \geq 0$; it follows that $A = \cup_{k \geq 0}(u + kP') \subseteq R_1$. So each coset of $\mathrm{cl}(P)$ that is contained in R is contained in either R_1 or R_2, as required. □

2.1.1 Mixed Sets

A set $Y \subseteq G$ is *mixed* with respect to a matroid $M = (E, G)$ if Y intersects both E and $G \backslash E$. Otherwise it is *unmixed*. If Y intersects E in at least one element, then it is *intersecting*. While these notions are defined for general sets Y, Y will typically be a coset of a flat in this thesis.

2.2 Critical Number and Circuits

Recall that given a matroid $M = (E, G)$, $\omega(M)$ is the dimension of a largest projective geometry contained in M. The critical number of M is the minimum nonnegative integer c for which G has a $(\dim(M) - c)$-dimensional flat disjoint from E and is denoted $\chi(M)$. By using the notion of matroid complements, we see that $\chi(M) = \dim(M) - \omega(M^c)$. It is also possible to show that the critical number equals the least number of affine geometries that are required to cover E.

We note that the critical number does not increase under taking restrictions.

Lemma 2.2.1. *If $M_1 = (E_1, G_1)$ and $M_2 = (E_2, G_2)$ are matroids where $G_2 \subseteq G_1$ and $E_2 \subseteq G_2 \cap E_1$, then $\chi(M_2) \leq \chi(M_1)$.*

Proof. Let $U \subseteq G_1$ be a flat for which $U \cap E_1 = \varnothing$ and $\chi(M_1) = \dim(G_1) - \dim(U)$. Note that $\dim(U \cap G_2) \geq \dim(U) + \dim(G_2) - \dim(G_1)$. Note that $(G_2 \cap U) \cap E_2 \subseteq U \cap E_1 = \varnothing$. Therefore, $\chi(M_2) \leq \dim(G_2) - \dim(G_2 \cap U) \leq \dim(G_1) - \dim(U) = \chi(M_1)$. □

Given a matroid $M = (E, G)$, let F be a flat for which $F \cap E = \varnothing$ and $\chi(M) = \dim(G) - \dim(F)$, and let F' be a flat for which $F' \subseteq E$ and $\dim(F') = \omega(M)$. Since $F \cap F' = \varnothing$, it follows that $\dim(F) + \dim(F') \leq \dim(G)$, which implies that $\omega(M) \leq \chi(M)$. The other direction is not necessarily true. As with graphs, a class $\mathcal{M}$ of matroids is *χ-bounded* by a function f if $\chi(M) \leq f(\omega(M))$ for all $M \in \mathcal{M}$, or simply *χ-bounded* if such an f exists.

Any matroid $M = (E, G)$ for which $\chi(M) = 1$ is called *affine*. These are the non-empty matroids for which we can find a hyperplane H for which $E \cap H = \varnothing$. There is a classical characterisation of affine matroids that is analogous to the characterisation of bipartite graphs in terms of odd cycles. For $k \geq 3$, recall that C_k is the full-rank, $(k-1)$-dimensional matroid with k elements that add to zero. When k is odd, C_k is an odd circuit. Note that odd circuits have critical number exactly 2.

Theorem 2.2.2 (Folklore)**.** *A non-empty matroid $M = (E, G)$ is affine if and only if it has no induced odd circuits.*

Proof. We may assume that M is full-rank. The forward direction follows by the observation that the critical number does not increase under taking induced restrictions by Lemma 2.2.1 and odd circuits have critical number 2.

Conversely, note that if M contains no odd circuit (as a restriction), then M is affine; we may pick a basis B of E, then if we let H be the set of elements of G that are expressed as the sum of an even number of elements of B, then H is a hyperplane and $H \cap E = \varnothing$, certifying that M is affine. Hence it suffices to show that M has no odd circuit restriction. For a contradiction, suppose not, and let $X = \{x_1, \ldots, x_k\}$ be a shortest odd circuit contained in M. Since M has no induced odd circuit restriction, and by the minimality of k, it follows that there exist l elements, l odd and $3 \leq l < k$, $y_1, \ldots, y_l \in \{x_1, \ldots, x_k\}$

for which $y_1 + \cdots + y_l \in E$; otherwise $M|\,\mathrm{cl}(X)$ is an induced odd circuit. But then $(X \backslash \{y_1, \ldots, y_l\}) \cup \{y_1 + \cdots + y_l\}$ is a shorter odd circuit restriction contained in M, contradicting the minimality of k. □

Lastly, we make the following easy observation.

Lemma 2.2.3. *Given $t, k \geq 1$, I_t is an induced restriction of C_k if and only if $k \geq t + 2$.*

In later chapters, we will consider triangle-free matroids that are either I_4-free or I_5-free. Combined with Lemma 2.2.2, Lemma 2.2.3 implies that, for $t = 4, 5$, I_t-free, triangle-free matroids are affine if and only if they contain no induced C_5-restriction. Because of this reason, the matroid C_5 will play an important role in this thesis. Note that, if $M = (E, G) \cong C_5$, then any element $x \in G \backslash E$ can be written as $x = y + z$ for unique $y, z \in E$. This in particular implies that C_5 is a maximally triangle-free 4-dimensional matroid.

2.3 Direct Sums

2.3.1 Direct Sums of Projective Geometries

For disjoint projective geometries G_1 and G_2, we write $G_1 \oplus G_2$ for the projective geometry $([G_1] \oplus [G_2]) \backslash \{0\}$ where the '$\oplus$' in the second expression denotes the vector space direct sum. For disjoint projective geometries G_1, G_2, and G_3, because $G_1 \oplus (G_2 \oplus G_3)$ and $(G_1 \oplus G_2) \oplus G_3$ correspond in a natural way, we omit the brackets and informally write $G_1 \oplus G_2 \oplus G_3$ to refer to these projective geometries. We use additive notation to denote the elements of $G_1 \oplus G_2$; every element in $G_1 \oplus G_2$ is uniquely expressed as $x_1 + x_2$ where $x_i \in [G_i]$ for $i = 1, 2$. W can also think of G_1 and G_2 as flats of $G_1 \oplus G_2$. The $\oplus$ operation forms the basis of the operations that appear in this thesis; we will often form a new matroid from two existing matroids $M_1 = (E_1, G_1)$ and $M_2 = (E_2, G_2)$ and the ambient set of the resulting matroid will be $G_1 \oplus G_2$.

Note that if G_1 and G_2 are disjoint flats of a projective geometry G whose union spans G, then there is a unique linear bijection ψ from $[G_1 \oplus G_2]$ to $[G]$ that fixes G_1 and G_2 point-wise; we call this the *canonical isomorphism from* $[G_1 \oplus G_2]$ *to* $[G]$. We state this fact as a lemma for future reference.

Lemma 2.3.1. *If G_1 and G_2 are disjoint flats of a projective geometry G for which $\mathrm{cl}(G_1 \cup G_2) = G$, then there exists a unique linear bijection $\psi : [G_1 \oplus G_2] \mapsto [G]$ that fixes G_1 and G_2 point-wise.*

The canonical isomorphism $\psi : [G_1 \oplus G_2] \mapsto [G]$ is more or less the identity map because of the additive notation we use to denote the elements in both of the projective geometries $G_1 \oplus G_2$ and G. More specifically, we have that $\psi(x_1 + x_2) = x_1 + x_2$ for any element $x_1 + x_2 \in G_1 \oplus G_2$, although the element $x_1 + x_2$ on the right hand side of the previous equation belongs to a different projective geometry, which is G.

Often, we wish to describe the structure of a given matroid M in terms of various operations. We wish to say that a given matroid M is 'constructed' using some operation from submatroids, say $M|G_1$ and $M|G_2$, where G_1 and G_2 are disjoint flats of G whose union spans G. However, the operation, with input $M|G_1$ and $M|G_2$, will produce a matroid with ambient set $G_1 \oplus G_2$, which does not formally equal G. Hence, the resulting matroid can only be isomorphic to the target matroid M. If they are isomorphic, then we want such a matroid isomorphism to be captured in a canonical fashion, meaning that the flats G_1 and G_2 in the ambient set $G_1 \oplus G_2$ should map naturally to G_1 and G_2 in G.

Below we describe *direct sums of matroids* as a trivial example to illustrate this notion of construction; other less trivial operations in this thesis will follow a similar template in how they are used to describe the structure of a given matroid.

2.3.2 Direct Sums of Matroids

Given two matroids $M_1 = (E_1, G_1)$ and $M_2 = (E_2, G_2)$, we define $M_1 \oplus M_2$ to be the matroid $(E_1 \cup E_2, G_1 \oplus G_2)$. Now, given a matroid $M = (E, G)$ and two disjoint flats G_1 and G_2 for which $\mathrm{cl}(G_1 \cup G_2) = G$, we say that M *is the direct sum of* $M|G_1$ *and* $M|G_2$ if the canonical isomorphism ψ from $[G_1 \oplus G_2]$ to $[G]$ is a matroid isomorphism from $M|G_1 \oplus M|G_2$ to M. Note that if M is the direct sum of $M|G_1$ and $M|G_2$, then in particular $M \cong (M|G_1) \oplus (M|G_2)$.

Note that our abuse of notation regarding projective geometries, in which we refer to matroids of the form (G, G), where G is a projective geometry, also as projective geometries, is slightly inconvenient; given a projective geometry G_i, and the matroid $M_i = (G_i, G_i)$ for $i = 1, 2$, the matroid $M_1 \oplus M_2$ is different from the matroid we obtain by treating the projective geometry $G_1 \oplus G_2$ as a matroid. In such cases, we will explicitly state which matroid we are referring to.

Given these definitions, we have the following; it allows us to recognise when a matroid M is the direct sum of $M|G_1$ and $M|G_2$.

Lemma 2.3.2. *Let* $M = (E, G)$ *be a matroid, and let* $G_1, G_2 \subseteq G$ *be disjoint flats for which* $\mathrm{cl}(G_1 \cup G_2) = G$. *Then the following are equivalent.*

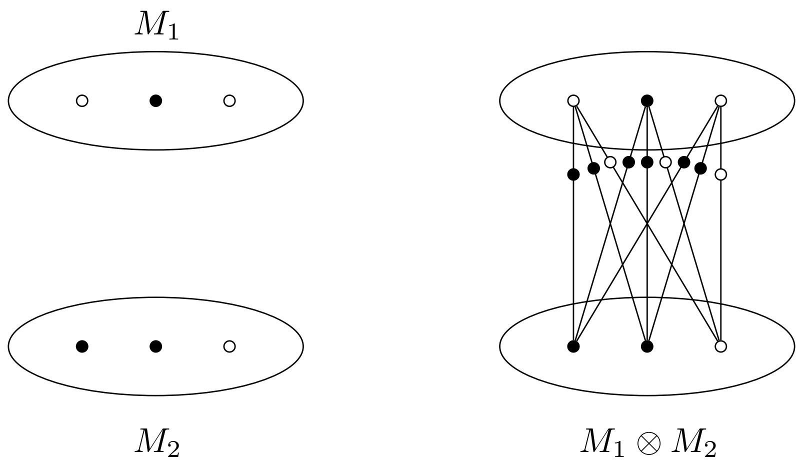

Figure 2.1: The matroid $M_1 \otimes M_2$. Solid dots denote matroid elements and empty dots denote nonelements.

1. *M is the direct sum of $M|G_1$ and $M|G_2$.*
2. *$E \subseteq G_1 \cup G_2$.*

Proof. Let ψ be the canonical isomorphism from $[G_1 \oplus G_2]$ to $[G]$. It follows that $\psi((E \cap G_1) \cup (E \cap G_2)) = (E \cap G_1) \cup (E \cap G_2) = E \cap (G_1 \cup G_2)$. Hence $\psi((E \cap G_1) \cup (E \cap G_2)) = E$ if and only if $E \subseteq G_1 \cup G_2$. $\square$

2.4 Lift-joins

Given two matroids $M_1 = (E_1, G_1)$ and $M_2 = (E_2, G_2)$, we define the matroid $M_1 \otimes M_2$ as

$$M_1 \otimes M_2 = (E_1 \cup (E_2 + [G_1]), G_1 \oplus G_2),$$

where we see G_1 and G_2 as subsets of $G_1 \oplus G_2$. See Figure 2.1. Note that $M_i \cong (M_1 \otimes M_2)|G_i$ for each $i \in \{1, 2\}$. The $\otimes$ operation is easily seen to be noncommutative, but as we will see later, it is associative. The $\otimes$ operation is, in some sense, analogous to the substitution operation in graph theory.

The $\otimes$ operation plays a crucial role in the next chapter, which is devoted to the structure of claw-free matroids. Moreover, certain special cases of this operation, most notably when M_1 is a 1-dimensional empty matroid, will appear frequently in this thesis. We will return to this case later.

Given disjoint flats $F, J \subseteq G$ for which $\text{cl}(F \cup J) = G$, we say that $M = (E, G)$ *is the lift-join of $M|F$ and $M|J$* if the canonical isomorphism ψ from $[F \oplus J]$ to $[G]$ is a matroid isomorphism from $M|F \otimes M|J$ to M. Note that if M is the lift-join of $M|F$ and $M|J$, then in particular $M \cong (M|F) \otimes (M|J)$.

The following lemma provides a way to recognise when a matroid is the lift-join of its submatroids in terms of the cosets of a proper flat. Recall that a set $Y \subseteq G$ is mixed with respect to a matroid $M = (E, G)$ if Y intersects both E and $G \backslash E$.

Lemma 2.4.1. *Let $M = (E, G)$ be a matroid and F be a flat of G. The following are equivalent.*

1. *F has no mixed cosets with respect to M.*
2. *$E = (E \cap F) \cup ((E \cap J) + [F])$ for every maximal flat J of G that is disjoint from F.*
3. *M is the lift-join of $M|F$ and $M|J$ for some maximal flat J of G that is disjoint from F.*
4. *M is the lift-join of $M|F$ and $M|J$ for every maximal flat J of G that is disjoint from F.*

Proof. Suppose that (1) holds. Let J be a maximal flat of G that is disjoint from F; note that each coset of F intersects J in exactly one element. For each $x \in J$, the fact that the coset $[F]+x$ is unmixed implies that $([F]+x) \subseteq E$ if and only if $x \in E$, and $([F]+x) \cap E = \varnothing$ if and only if $x \notin E$. Therefore $E = (E \cap F) \cup \bigcup_{x \in E \cap J}([F]+x) = (E \cap F) \cup ((E \cap J) + [F])$. So (2) holds.

Let ψ be the canonical isomorphism from $[F \oplus J]$ to $[G]$. Then $\psi(E(M|F \otimes M|J)) = \psi((E \cap F) \cup ((E \cap J) + [F])) = (E \cap F) \cup ((E \cap J) + [F])$. Therefore ψ is a matroid isomorphism if and only if $E = (E \cap F) \cup ((E \cap J) + [F])$ (in the projective geometry G). So M is the lift-join of $M|F$ and $M|J$ if and only if $E = (E \cap F) \cup ((E \cap J) + [F])$. Thus (2) and (4) are equivalent.

Clearly (4) implies (3); Suppose that (3) holds, so M is the lift-join of $M|F$ and $M|J$ for some flat J of G that is disjoint from F. Thus from the same argument from the above it follows that $E = (E \cap F) \cup ((E \cap J) + [F])$. Let $x, y \in G \backslash F$, distinct, belong to the same coset of F in G, so $x + y \in F$ (note that $[F]$ is not considered a coset). If $x \in E$ then $x \in (E \cap J) + [F]$ and so $y \in (E \cap J) + [F] \subseteq E$. Therefore $x \in E$ implies that $y \in E$; it follows that F has no mixed cosets. So (3) implies (1). $\square$

This lemma motivates a definition. If $M = (E, G)$ is a matroid and F is a nonempty proper flat of G that has no mixed cosets with respect to M, then we call F a *decomposer* of M, and say that F *decomposes* M. The above lemma shows that M is the lift-join of two submatroids if and only if M has a decomposer. Note that if F decomposes M and F' decomposes $M|F$, then F' also decomposes M, since each coset of F partitions into cosets of F'.

Given a matroid $M = (E, G)$, the existence of a decomposer $F \subseteq G$ implies the following about the induced restrictions of the form $M|K$ where $K \cap F = \varnothing$.

Lemma 2.4.2. *If F is a decomposer of a matroid $M = (E, G)$, and K, K' are flats of G disjoint from F for which $\mathrm{cl}(F \cup K) = \mathrm{cl}(F \cup K')$, then there is an isomorphism ψ from $M|K$ to $M|K'$ for which each $x \in K$ satisfies $x + \psi(x) \in [F]$.*

Proof. Since K and K' are each disjoint from F while $\mathrm{cl}(F \cup K) = \mathrm{cl}(F \cup K')$, we have $\dim(K) = \dim(K')$, and each coset of F in $\mathrm{cl}(F \cup K)$ meets K and K' in a single element each. For each $x \in K$, let $\psi(x)$ be the unique element of $([F] + x) \cap K'$. Since F is a decomposer we have $\psi(x) \in E$ if and only if $x \in E$. For each $x' \in K'$ the coset $[F] + x'$ intersects K in some element x for which $x' = \psi(x)$, so ψ is surjective and thus bijective. Finally, for distinct $x, y \in K$ the elements $\psi(x) + \psi(y)$ and $\psi(x + y)$ are both in $([F] + (x + y)) \cap K'$, so are equal. Thus ψ is an isomorphism. $\square$

We now prove basic properties of the $\otimes$ operation.

Lemma 2.4.3. *If $M = M_1 \otimes M_2$, where $M_i = (E_i, G_i)$ then*

- $M^c = M_1^c \otimes M_2^c$,
- $M|\,\mathrm{cl}(F_1 \cup F_2) = (M_1|F_1) \otimes (M_2|F_2)$ *for all flats F_1, F_2 of G_1, G_2,*
- $\omega(M) = \omega(M_1) + \omega(M_2)$, *and*
- $\chi(M) = \chi(M_1) + \chi(M_2)$.

Proof. Let $M = (E, G)$, where $G = G_1 \oplus G_2$ and $E = E_1 \cup ([G_1] + E_2)$. To see the first part, note that

$$\begin{aligned} G \backslash E &= G \backslash (E_1 \cup ([G_1] + E_2)) \\ &= (G_1 \backslash E_1) \cup ([G_1] + G_2) \backslash ([G_1] + E_2) \\ &= (G_1 \backslash E_1) \cup ([G_1] + (G_2 \backslash E_2)), \end{aligned}$$

where we use the fact that every $z \in [G_1]+G_2$ is uniquely expressible in the form $z = x+y$ for $x \in [G_1]$ and $y \in G_2$. This gives $M^c = M_1^c \otimes M_2^c$.

For the second part, let $F = \mathrm{cl}\,(F_1 \cup F_2)$. We have

$$\begin{aligned} E \cap F &= ((E_1 \cup E_2) \cap F) \cup ((F_1 + (E_2 \cap F_2)) \cap F) \\ &= (E_1 \cap F_1) \cup (E_2 \cap F_2) \cup (F_1 + (E_2 \cap F_2)) \\ &= (E_1 \cap F_1) \cup ([F_1] + (E_2 \cap F_2)), \end{aligned}$$

from which it follows that $M|F = (M_1|F_1) \otimes (M_2|F_2)$.

For each $i \in \{1, 2\}$ let $\omega_i = \omega(M_i)$ and let $K_i \subseteq E_i$ be a flat of dimension ω_i. Then $K_1 \cup ([K_1] + K_2) = \mathrm{cl}(K_1 \cup K_2)$ is a flat of G contained in E by the previous part, giving $\omega(M) \geq \omega_1 + \omega_2$.

Now let F be a flat of G for which $F \subseteq E$. Let K be a maximal flat of F that is disjoint from G_1; note that $\dim(F) = \dim(K) + \dim(F \cap G_1) \leq \dim(K) + \omega_1$. Since $G = G_1 \oplus G_2$, the flat $\mathrm{cl}(G_1 \cup K)$ intersects G_2 in a flat K' for which $\mathrm{cl}(G_1 \cup K) = \mathrm{cl}(G_1 \cup K')$. By Lemma 2.4.2 we have $M|K \cong M|K' = M_2|K'$, and since $K \subseteq E$ this implies that $K' \subseteq E_2$ and so $\dim(K') \leq \omega_2$. Thus $\dim(F) \leq \omega_1 + \omega_2$, and the third part follows.

Finally, let $n = \dim(G)$ and $n_i = \dim(G_i)$ for each i. We have

$$\chi(M) = n - \omega(M^c) = (n_1 + n_2) - \omega(M_1^c) - \omega(M_2^c) = \chi(M_1) + \chi(M_2)$$

giving the last part. $\square$

As previously remarked, the projective geometries $(G_1 \oplus G_2) \oplus G_3$ and $G_1 \oplus (G_2 \oplus G_3)$ correspond in a natural way, and therefore we remove the parentheses and informally think of them as being the same. The following lemma shows that we may take the same convention with the $\otimes$ operation.

Lemma 2.4.4. *$\otimes$ is associative.*

Proof. Let $M_i = (E_i, G_i)$ for $i \in \{1, 2, 3\}$ with the G_i disjoint. Both the matroids $(M_1 \otimes M_2) \otimes M_3$ and $M_1 \otimes (M_2 \otimes M_3)$ have ambient space $G = G_1 \oplus G_2 \oplus G_3$. The first has ground set

$$(E_1 \cup (E_2 + [G_1])) \cup (E_3 + [G_1 \oplus G_2]) = E_1 \cup (E_2 + [G_1]) \cup (E_3 + [G_1] + [G_2]).$$

The second has ground set

$$E_1 \cup ((E_2 \cup (E_3 + [G_2])) + [G_1]) = E_1 \cup (E_2 + [G_1]) \cup (E_3 + [G_1] + [G_2]),$$

giving the lemma. $\square$

Lemma 2.4.5. *Let N be a full-rank matroid of dimension at least 3, containing no four distinct elements that sum to zero. If M_1 and M_2 are N-free matroids, then so is $M_1 \otimes M_2$.*

Proof. Let $M_i = (E_i, G_i)$ for $i \in \{1, 2\}$ and let $M = M_1 \otimes M_2 = (E, G)$. By definition, no coset of G_1 is mixed with respect to M. Let F be a flat of G for which $M|F \cong N$. Since M_1 is N-free, we have $F \not\subseteq G_1$; since $\mathrm{cl}(E \cap F)$ has dimension $\dim(F) > \dim(F \cap G_1)$, we have $(E \cap F)\backslash G_1 \neq \varnothing$; let $x \in (E \cap F)\backslash G_1$.

If $\dim(F \cap G_1) \geq 2$ then let $v, w \in F \cap G_1$ be distinct. Since $[G_1] + x$ is unmixed and intersects E, we have

$$E \supseteq [G_1] + x \supseteq \{x, x + w, x + v, x + v + w\}.$$

These four distinct elements all belong to $E \cap F$; this is a contradiction, since they sum to zero. Thus $\dim(F \cap G_1) \leq 1$.

If $|F \cap G_1| = 1$ then let w be its element; since $\dim(E \cap F) \geq 3$ there is some $y \in (F \cap E)\backslash \mathrm{cl}(\{x, w\})$. Now $y \in E\backslash G_1$ and so $[G_1] + y \subseteq E$; it follows that $\{x, x + w\} \subseteq E$ and $\{y, y + w\} \subseteq E$. But $x, x + w, y, y + w$ are distinct elements of $F \cap E$ with sum zero, again a contradiction.

If $F \cap G_1 = \varnothing$ then, for each element z of F, the coset $[G_1] + z$ intersects G_2 in exactly one element z' and moreover, the map $\psi\colon z \mapsto z'$ is a linear injection from F to G_2. Since G_1 has no mixed cosets, we have $\psi(z) \in E$ if and only if $z \in E$, and so $M|F$ and $M|\psi(F)$ are isomorphic. The latter is an induced restriction of the N-free matroid $M|G_2$, giving a contradiction to $M|F \cong N$. □

For $t \geq 3$, the matroid I_t satisfies the hypotheses of the above lemma. This gives the following.

Corollary 2.4.6. *If $t \geq 3$ is an integer then the class of I_t-free matroids is closed under the $\otimes$ operation.*

2.5 Doublings

The matroids of the form $O_1 \otimes M$ where O_1 is an 1-dimensional empty matroid play a special role in this thesis. Given a matroid M, we write $D(M)$ for the matroid $O_1 \otimes M$ where O_1 is some fixed 1-dimensional empty matroid.

We think of the D notation as an operation acting on the matroid M. This operation is, in some sense, analogous to cloning a vertex in a graph, as it extends the matroid by creating a copy of itself; it can also be seen as a special case of the $\otimes$ operation in the same way that vertex cloning can be seen as a special case of substitution for graphs. For $k \geq 0$, let $D^k(M)$ denote the matroid obtained from M by applying the operation D k times; note that if $M_1 \cong M_2$, then $D^k(M_1) \cong D^k(M_2)$ for every $k \geq 0$. Equivalently, $D^k(M)$ is the matroid $O_k \otimes M$ where O_k is some fixed k-dimensional empty matroid.

The operation D satisfies various nice properties, as discussed in [46].

Lemma 2.5.1 ([46])**.** *Let M be a matroid. Then the following hold.*

- *$\chi(M) = \chi(D(M))$, and*
- *if N is a matroid such that $N \cong D(N')$ for no matroid N', and M contains no induced N-restriction, then neither does $D(M)$.*

The first statement also follows from Lemma 2.4.3 by setting M_1 to be the 1-dimensional empty matroid O_1. Also, note that if a matroid N satisfies $N \cong D(N')$ for some matroid N', then $|N|$ is even. Hence, the above lemma always applies when N has an odd-sized ground set.

The next lemma allows us to recognise when a matroid M is isomorphic to $D^k(M')$ for some matroid M'.

Lemma 2.5.2. *Let $M = (E, G)$ and $M' = (E', G')$ be matroids. Let n be the dimension of M. For all $k \geq 0$, the following are equivalent.*

1. *$M \cong D^k(M')$.*
2. *G has a k-dimensional flat $F \subseteq G \backslash E$ and an $(n-k)$-dimensional flat $F' \subseteq (G \backslash F)$ such that $M|F' \cong M'$ and $E = [F] + (E \cap F')$.*

Proof. For the forward direction, write $O_k = (\varnothing, G_k)$ where G_k is a k-dimensional projective geometry. Let $\phi : [G] \mapsto [G_k \oplus G']$ be a linear bijection for which $\phi(E) = E' + [G_k]$. Let $F = \phi^{-1}(G_k)$ and $F' = \phi^{-1}(G')$. Then $F \cap E = \varnothing$ and $F \cap F' = \varnothing$ in the projective geometry G. Since $(O_k \otimes M')|G' \cong M'$, it follows that $M|F' \cong M'$. Finally, $E = \phi^{-1}(E' + [G_k]) = E \cap \phi^{-1}(G') + [F] = E \cap F' + [F]$.

Conversely, let $\phi : [F'] \mapsto [G']$ be a linear bijection for which $\phi(E \cap F') = E'$. We extend the map by defining $\phi : [F] \mapsto [G_k]$ to be any linear bijection, and extending ϕ linearly. Then $\phi(E) = \phi([F] + (E \cap F')) = \phi([F]) + \phi(E \cap F') = [G_k] + E' = E(D^k(M'))$. $\square$

Given a matroid $M = (E, G)$ and disjoint flats $F, J \subseteq G$ for which $\mathrm{cl}(F \cup J) = G$, we say that M *is the doubling of* $M|J$ *by* F if $E \cap F = \varnothing$ and the canonical isomorphism from $[F \oplus J]$ to $[G]$ is a matroid isomorphism from $M|F \otimes M|J$ to G. Equivalently, M is the doubling of $M|J$ by F if and only if it is the lift-join of $M|F$ and $M|J$ and $E \cap F = \varnothing$.

If M is the doubling of $M|J$ by F for some flats F, J, then we sometimes omit the flat F and say that M is the *doubling of* $M|J$ or omit J and say that M *is the doubling by* F. We also say that M is a *doubling* if such F and J exist.

Since doublings are a special case of lift-joins, we can recognise a doubling by checking whether the cosets of a proper flat F are mixed or not. In the case of doublings, since there is an extra requirement that $F \cap E = \varnothing$, there are also other equivalent formulations.

Lemma 2.5.3. *Let $M = (E, G)$ be a matroid and let $F \subseteq G$ be a flat. Then the following are equivalent.*

1. *F has no mixed cosets with respect to M and $E \cap F = \varnothing$.*
2. *$F + E = E$.*
3. *$E = [F] + (E \cap J)$ for every maximal flat J disjoint from F.*
4. *M is the doubling of $M|J$ by F for every maximal flat J that is disjoint from F.*

Proof. Note that (1) (3) and (4) are equivalent by definition and Lemma 2.4.1.

If (2) holds, then $E \cap F = \varnothing$. Moreover, if $x \in E$, then by assumption $x + F \subseteq F + E = E$, so every coset of F is unmixed with respect to M. So (1) holds.

Suppose that (1) holds. If $x \in E$, then since $E \cap F = \varnothing$, $x \notin F$. This means that $x + F \subseteq E$ since no coset of F is mixed with respect to M, so $x \subseteq F + E$ and therefore $E \subseteq F + E$. But $F + E = \cup_{x \in E}(x + F) \subseteq E$. So $F + E = E$ so (2) follows. $\square$

When we apply the above lemma in this thesis, the flat F will typically be a 1-dimensional flat $\{w\}$; in this case, we simply write w in place of $\{w\}$. By the above lemma, M is a doubling if and only if there exists an element $w \in G$ for which $w + E = E$.

2.6 Shorthand for Induced Restrictions

In many of our proofs, we will often obtain a contradiction by finding certain induced restrictions, such as claws and triangles. Given matroids $M = (E, G)$ and N, it is sometimes cumbersome to specify the correct flat F on which $M|F \cong N$. Instead, for the sake of brevity, when N is full-rank, we say that a set $Z \subseteq E$ is an *induced N-restriction* if $M|\operatorname{cl}(Z)$ is an induced N-restriction. This shorthand will not be used when N is not full-rank; in such a case, we will fully specify the flat F on which $M|F \cong N$.

Furthermore, we will also often simply assert that such a set Z is an induced N-restriction without performing all necessary checks explicitly. For example, when $N = I_n$, then in order to check that $Z = \{z_1, z_2, \ldots, z_n\} \subseteq E$ is an induced I_n-restriction, one needs to check that $\sum_{i \in I} z_i \notin E \cup \{0\}$ for all $I \subseteq \{1, 2, \ldots, n\}$ and $|I| > 1$. Writing down these checks explicitly is often cumbersome. Instead, we will provide enough information prior to such a claim so that these checks can be performed easily.

Chapter 3

Claw-free Matroids

This chapter is based on joint work with Peter Nelson [48].

3.1 Introduction

A seminal result in the theory of excluded subgraphs is Chudnovsky and Seymour's classification of claw-free graphs [13]. Their classification of claw-free graphs is highly complex, as it requires a series of papers with technically defined basic classes and sporadic examples of claw-free graphs. Due to its complexity, we will not state their theorem in this thesis.

In this chapter, we consider and prove a classification of I_3-free matroids. Recall that the matroids I_t, being maximally acyclic, play the role of trees in our study of induced restrictions. Our theorem can also be seen as a natural extension of Theorem 1.5.9, in which we drop the F_7-freeness condition. What we obtain is a structure theorem which shows that claw-free matroids can all be constructed from matroids in one of three 'basic classes' of claw-free matroids via a single 'join' operation that preserves the property of being claw-free. The join operation is the $\otimes$ operation defined in the preceding chapter. We now define these basic classes.

We say a matroid $M = (E, G)$ is a *PG-sum* if E is the disjoint union of two (possibly empty) flats of G. Recall from the previous chapter that we call a matroid M *even-plane* if all of its 3-dimensional induced restrictions have even-sized ground sets. It is easy to see that PG-sums and even-plane matroids are claw-free, and, since the complement of a claw contains a triangle, that the complements of triangle-free matroids are claw-free. Recall from the previous chapter that given two matroids $M_1 = (E_1, G_1)$ and $M_2 = (E_2, G_2)$, $M_1 \otimes M_2$ is the matroid (E, G), where $G = G_1 \oplus G_2$ and $E = E_1 \cup (E_2 + [G_1])$.

We showed in Lemma 2.4.4 that the $\otimes$ operation is associative and in Corollary 2.4.6 that it preserves the property of being claw-free. The main result of this chapter, stated up to isomorphism, is the following.

Theorem 3.1.1. *A matroid M is claw-free if and only if M can be obtained via the $\otimes$ operation from matroids that are either PG-sums, even-plane matroids, or the complements of triangle-free matroids.*

Notice that two of our basic classes are similar to natural classes of claw-free graphs; PG-sums are analogous to the graphs G having a vertex v adjacent to every other vertex, for which $G - v$ is the union of two cliques, and the complements of triangle-free matroids generalise the complements of triangle-free graphs, which are clearly claw-free. On the other hand, the class of even-plane matroids does not enjoy such a direct graph-theoretic analogue; this class becomes much more natural if one takes a more algebraic view and considers them as the solutions of some quadratic equations, as in Theorem 1.5.12. This algebraic perspective is not needed in the proof.

In the introduction, we discussed some matroidal analogues of graph-theoretic theorems which, in many cases, had shorter proofs that their graph-theoretic counterparts. Our structure theorem for claw-free matroids falls in this category. As we shall see, the proof is by no means straightforward, yet when compared with Chudnovsky and Seymour's classification of claw-free graphs, Theorem 3.1.1 is much easier to state, and the proof is certainly much shorter.

The proof of Theorem 3.1.1 will require some preliminaries before we can give the actual proof. We begin by studying each component of Theorem 3.1.1 in detail.

3.2 Preliminaries

Lift-joins

Recall that, given two matroids $M_1 = (E_1, G_1)$ and $M_2 = (E_2, G_2)$, we define the matroid $M_1 \otimes M_2$ as

$$M_1 \otimes M_2 = (E_1 \cup (E_2 + [G_1]), G_1 \oplus G_2).$$

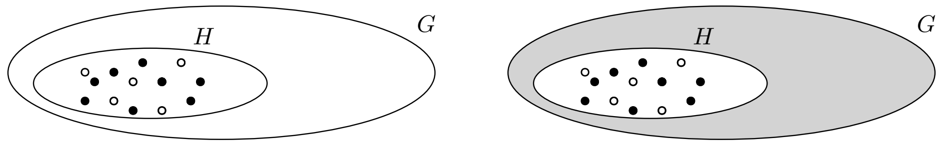

Figure 3.1: H is a hyperplane of G. In the first figure, $E \subseteq H$. In the second figure, $G \backslash H \subseteq E$. In both cases, H decomposes $M = (E, G)$.

Theorem 3.1.1 claims that a matroid M is a claw-free matroid if and only if $M \cong M_1 \otimes \cdots \otimes M_k$, where each M_i belongs to one of the basic classes of claw-free matroids. The backward direction is easy; it amounts to checking that the matroids in the basis classes are claw-free, and Corollary 2.4.6 implies the result. To prove the forward direction, we will show that any given claw-free matroid $M = (E, G)$ is either a PG-sum, an even-plane matroid, the complement of a triangle-free matroid, or has a decomposer; if it has a decomposer, then from Lemma 2.4.1, it will follow that M is the lift-join of $M|F$ and $M|J$. This means that if ψ is the canonical isomorphism from $[F \oplus J]$ to $[G]$, then ψ is a matroid isomorphism from $(M|F) \otimes (M|J)$ to M. This in particular implies that $M \cong (M|F) \otimes (M|J)$, and Theorem 3.1.1 will follow by induction.

As an aside, note that, in concluding that $M \cong (M|F) \otimes (M|J)$, we did not use the fact that ψ is a canonical isomorphism; it is in fact possible to retain this information, and write down a statement more precise than Theorem 3.1.1. We say that a matroid $M = (E, G)$ is the *lift-join of* $M|F_1, \ldots, M|F_k$ where $F_1, \ldots, F_k$ are disjoint flats for which $\sum_{i=1}^{k} \dim(F_i) = \dim(G)$ and $\mathrm{cl}(\cup_{i=1}^{k} F_i) = G$ if the canonical isomorphism from $[F_1 \oplus \cdots \oplus F_k]$ to $[G]$ is a matroid isomorphism from $(M|F_1) \otimes \ldots \otimes (M|F_k)$ to M. Then, our proof of Theorem 3.1.1 can easily be adjusted appropriately to show that a given matroid $M = (E, G)$ is claw-free if and only if there exist disjoint flats $F_1, \ldots, F_k \subseteq G$ whose union spans G and $\sum_{i=1}^{k} \dim(F_i) = \dim(G)$ for which the canonical isomorphism from $[F_1 \oplus \cdots \oplus F_k]$ to $[G]$ is a matroid isomorphism from $(M|F_1) \otimes \cdots \otimes (M|F_k)$ to M where each $M|F_i$ is a PG-sum, an even-plane matroid or the complement of a triangle-free matroid.

The bulk of our proof will be devoted to considering special decomposers. Since they are important in our proof, it is worthwhile to spend some time understanding them explicitly. Let $M = (E, G)$. The first type of decomposer we will be particularly interested in is a hyperplane decomposer. If H is a hyperplane of G, it is easy to see that H decomposes M if and only if $G \backslash H$ (the only coset of H) is either contained in E or disjoint from E. In the latter case we have $\mathrm{cl}(E) \subseteq H$, so M is not full-rank; conversely, if M is not full-rank, then any hyperplane containing $\mathrm{cl}(E)$ is a decomposer. See Figure 3.1.

Another type of decomposer we will be interested in is a one-element decomposer. Note

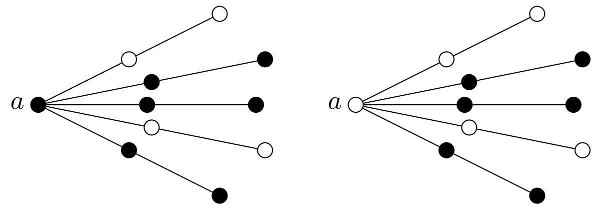

Figure 3.2: a is an element of G. In the first figure, $a+(E\cup\{0\}) = E\cup\{0\}$. In the second figure, $a+E=E$. In both cases, $\{a\}$ composes $M=(E,G)$.

that if an element a of G satisfies $a+E=E$, then $a\notin E$ and $\{a\}$ is a decomposer of M. If a satisfies $a+(E\cup\{0\}) = E\cup\{0\}$, then $a\in E$ and $\{a\}$ is a decomposer of M. It is easy to see that $\{a\}$ is a decomposer of M if and only if one of these two statements holds. Note that if $a+E=E$, then we are simply saying that M is a doubling, so every matroid that is a doubling has a decomposer. See Figure 3.2.

Three-dimensional matroids

In this chapter, we will often consider three-dimensional matroids. There are 10 three-dimensional matroids, up to isomorphism, 6 of which are full-rank. Define $M_{5,3}$ and $M_{6,3}$ to be the unique three-dimensional matroids on 5 and 6 elements respectively. Then I_3, C_4, I_3^c, $M_{5,3}$, $M_{6,3}$ and F_7 provide a complete list of full-rank three-dimensional matroids. The 4 rank-deficient three-dimensional matroids are $F_7^c, M_{6,3}^c, M_{5,3}^c$ and C_4^c.

PG-sums

A matroid $M=(E,G)$ is a PG-sum if E is the disjoint union of at most two flats of G. We say that M is a *strict PG-sum* if M is full-rank and its ground set is the union of exactly two disjoint nonempty flats F_1, F_2. If this is the case, then $G\backslash E = F_1+F_2$, and in fact $|F_1+F_2| = |F_1||F_2|$; i.e. every $x\in G\backslash E$ is uniquely expressible as $x=x_1+x_2$ where $x_1\in F_1$ and $x_2\in F_2$.

The next lemma, along with the easy observation that PG-sums are closed under taking induced restrictions, shows that PG-sums are perfect.

Lemma 3.2.1. *If M is a PG-sum then $\chi(M)=\omega(M)$.*

Proof. It suffices to consider the case where $M=(E,G)$ is a strict PG-sum, since otherwise either M is not full-rank and we can pass to the restriction $(E,\mathrm{cl}(E))$, or $E=G$ and the conclusion is obvious. Let F_1, F_2 be disjoint flats of G whose union is E, and let $n_i = \dim(F_i)$ and $n = \dim(M)$, so $n = n_1 + n_2$. Assume that $n_1 \le n_2$, so $\omega(M) = n_2$.

Let $F_1' \subseteq F_2$ be an n_1-dimensional flat and let ψ be an isomorphism from $M|F_1$ to $M|F_1'$. Let $K = \{x + \psi(x) \colon x \in F_1\}$. Note that $K \subseteq F_1 + F_2 = G \backslash E$. Since F_1 and F_1' are flats, for distinct $x, y \in K$ we have $x + y \in K$. Finally, since $|F_1||F_2| = |F_1 + F_2|$, we have $|K| = |F_1|$ so $\dim(K) = n_1 = n - n_2$. It follows that $\chi(M) \le n_2 = \omega(M)$, and the result follows. □

It is easy to see that the PG-sums are claw-free. Given a PG-sum $M = (E, G)$ where $E = F_1 \cup F_2$ and F_1 and F_2 are flats of G for which $F_1 \cap F_2 = \varnothing$, if we take any three-element subset $I \subseteq E$ for which $\dim(\mathrm{cl}(I)) = 3$, there exists $i \in \{1, 2\}$ for which $|I \cap F_i| \ge 2$. If we let $x, y \in I \cap F_i$, then $x + y \in F_i \subseteq E$ as F_i is a flat. Hence $M|\,\mathrm{cl}(I)$ has at least 4 elements, so M is claw-free. We can in fact characterize this class by forbidding four particular three-dimensional induced restrictions.

Lemma 3.2.2. *M is a PG-sum if and only if it is $(I_3, C_4, M_{5,3}, M_{6,3})$-free.*

Proof. The forwards direction follows from the fact that the class of PG-sums is closed under taking induced restrictions, and that the matroids I_3, C_4, $M_{5,3}$ or $M_{6,3}$ are not PG-sums.

Conversely, suppose that $M = (E, G)$ is $(I_3, C_4, M_{5,3}, M_{6,3})$-free. Let F be a largest flat of G that is contained in E. We may assume that $3 \le |E| < |G|$ and that $F \ne E$, as otherwise M is a PG-sum.

Suppose first that $|F| = 1$. Let $v_1 \in F$ and $v_2, v_3 \in E \backslash F$. By maximality, E contains none of the elements $v_1 + v_2, v_1 + v_3, v_2 + v_3$. But then $M|\,\mathrm{cl}(\{v_1, v_2, v_3\})$ is isomorphic to either I_3 or C_4, a contradiction. So $|F| \ge 3$. Let $v_1 \in E \backslash F$. Then for any $v_2 \in F$, we must have $v_1 + v_2 \notin E$; otherwise, by maximality there exists $v_3 \in F$ such that $v_1 + v_3 \notin E$, but then $M|\,\mathrm{cl}(\{v_1, v_2, v_3\})$ has either 5 or 6 elements, so is either an induced $M_{5,3}$ or $M_{6,3}$-restriction. Now, let $v_1, v_2 \in E \backslash F$ be distinct. We claim that $v_1 + v_2 \in E \backslash F$, which will imply that $E \backslash F$ is a flat of G and hence that M is a PG-sum. Suppose not, and let $v_3 \in F$. Then $M|\,\mathrm{cl}(\{v_1, v_2, v_3\})$ is isomorphic to either I_3 or C_4, a contradiction. □

Even-plane matroids

In addition to the discussion on even-plane matroids in the previous chapter, we now give extra properties of even-plane matroids that will be required for the proof of the structure theorem and related corollaries. The class of even-plane matroids was introduced in [46] in which the authors studied the class of I_3-free, F_7-free matroids. For further discussions about the even-plane matroids, see [46].

Recall that a matroid $M = (E, G)$ is *even-plane* if $|E \cap P|$ is even for every plane P of G, and we write $\mathcal{E}_3$ for the class of even-plane matroids. We saw before that the even-plane matroids have arbitrarily large χ. The next result shows that the even-plane matroids are just one even-plane matroid away from bounding χ for *any* even-plane matroid we choose to exclude.

Theorem 3.2.3 ([46], Theorem 1.2)**.** *If $M, N \in \mathcal{E}_3$ and M has no induced N-restriction, then $\chi(M) \leq \dim(N) + 4$.*

There is a way to describe even-plane matroids through doublings and what we refer to as *semidoublings*. The treatment of semidoublings in this thesis is minimal; for a thorough treatment, see [46].

Let $M = (E, G)$ be a matroid, and let $H' \subseteq H \subseteq G$ be nested hyperplanes and $a \in G \backslash H$. Then we say that M is the *semidoubling of $M|H$ by a with respect to H'* if $a \in G \backslash E$ and $E = (E \cap H) \cup (a + ((H \backslash E) \Delta H'))$. This condition says that for each $x \in H$, we have $a + x \in E$ if and only if either $x \in H' \cap E$ or $x \in (H \backslash H') \backslash E$. Note that $M|\operatorname{cl}(H' \cup \{a\})$ is simply the doubling of $M|H'$.

The significance of semidoublings is that they preserve the property of being even-plane.

Theorem 3.2.4 ([46], Corollary 3.4)**.** *Let $M = (E, G)$ be a matroid such that $M|H \in \mathcal{E}_3$ where H is a hyperplane of G. Then the following hold.*

- *If M is the doubling of $M|H$, then $M \in \mathcal{E}_3$*
- *If M is the semidoubling of $M|H$ (with respect to some hyperplane $H' \subseteq H$ and $a \in G \backslash H$), then $M \in \mathcal{E}_3$.*

In fact, a lot more can be said. It is shown in [46] that *all* even-plane matroids arise via a sequence of doublings and semidoublings of 2-dimensional matroids. This fact provides a decomposition theorem that further justifies treating $\mathcal{E}_3$ as a basic class; however, we will not need this fact in our proof. Finally, the following lemma is immediate.

Lemma 3.2.5. *If $M = (E, G) \in \mathcal{E}_3$ and H is a hyperplane of G, then $(E \Delta (G \backslash H), G) \in \mathcal{E}_3$.*

Proof. Let P be a plane of G. Note that $|P \cap (G \backslash H)|$ is either 0 or 4, and that $|E \cap P|$ is even. Working modulo 2, we have $|(E \Delta (G \backslash H)) \cap P| \equiv |E \cap (P \cap H)| + |(P \cap (G \backslash H)) \backslash E| \equiv |E \cap (P \cap H)| + |P \cap (G \backslash H) \cap E| \equiv |E \cap P| \equiv 0$, as required. $\square$

Complements of triangle-free matroids

A matroid $M = (E, G)$ is the *complement of a triangle-free matroid* if M^c is triangle-free. Equivalently, $M = (E, G)$ is the complement of a triangle-free matroid if, for any two distinct elements $x, y \in G \backslash E$, $x + y \in E$. It is easy to see that the complements of triangle-free matroids are claw-free, since the matroid I_3^c contains a triangle.

The class of triangle-free matroids includes the graphic matroids of triangle-free graphs. It is known from [61] that as long as two graphs are nonisomorphic and 3-connected, we obtain nonisomorphic graphic matroids arising from these graphs.

Theorem 3.2.6 ([61])**.** *If G_1 and G_2 are 3-connected graphs, then $M(G_1) \cong M(G_2)$ if and only if $G_1 \cong G_2$.*

This means that the class of triangle-free matroids is at least as complicated as the class of 3-connected triangle-free graphs. As triangle-free graphs themselves are known to be a difficult class to describe, we treat the class of triangle-free matroids (or, in our case, the class of the complements of triangle-free matroids) as a basic class.

3.3 Large Decomposers

We have the necessary preliminaries to now prove Theorem 3.1.1. Before proceeding, however, we restate Theorem 3.1.1 in terms that will be more convenient. The equivalence follows from Corollary 2.4.6 and Lemma 2.4.1, as well as the obvious fact that the three basic classes are claw-free.

Theorem 3.3.1. *If $M = (E, G)$ is a claw-free matroid, then either*

- *M is even-plane,*
- *M^c is triangle-free,*
- *M is a strict PG-sum, or*
- *M has a decomposer.*

We will prove Theorem 3.3.1 by induction on the dimension of M, and we are thus interested in when a decomposer F of some induced restriction $M|H$ of M extends to a decomposer of $M = (E, G)$ itself. This section and the next address a few special cases of this that turn out to be all we need: namely, where H is a hyperplane of G, and the decomposer F is minimal and has dimension either 1 or $\dim(H) - 1$.

In many lemmas to come, we will consider a partition (X_0, X_1) of a hyperplane H defined by $X_1 = (a + E) \cap H$ and $X_0 = H \backslash X_1$ for some element a of $G \backslash H$. Hence, if we pick $x \in H$, then $x \in X_1$ if and only if $x + a \in E$.

We will first consider the case in which some hyperplane H has a minimal decomposer F which is itself a hyperplane of H. This implies that the coset $H \backslash F$ is either contained in E or disjoint from E; the next two lemmas deal with these subcases.

Lemma 3.3.2. *Let $M = (E, G)$ be a claw-free matroid, let H be a hyperplane of G and let F be a hyperplane of H. If $H \backslash F \subseteq E$ and $M|F$ has no decomposer, then either*

- *M^c is triangle-free, or*
- *M has a decomposer F' containing F.*

Proof. Let $A = H \backslash F$. Suppose for a contradiction that M^c has a triangle, and that no flat F' containing F is a decomposer of M. Thus F does not decompose M, so F has a mixed coset B. Fix an element $a \in B \backslash E$, and let $X_1 = (a + E) \cap H$ and $X_0 = H \backslash X_1$. Note that $X_1 \cap F \neq \varnothing$ since B is mixed. Note that F has three cosets, namely B, A and $A + a$

We may assume that $A \cap X_0 \neq \varnothing$, as otherwise $A + a \subseteq E$, which implies that the coset $G \backslash (F \cup B) = A \cup (A + a)$ of $F \cup B$ is contained in E, and so $F \cup B$ decomposes M. Let $v \in A \cap X_0$, so $a + v \notin E$.

3.3.2.1. *Let $T \subseteq H$ be a triangle. Then*

- *if $T \subseteq F$ and $|T \cap X_0| = 2$ then $T \subseteq E$, and*
- *T does not intersect all three of $X_0 \backslash E, X_1 \backslash E$ and $X_1 \cap E$.*

Subproof: For the first part, let $T = \{v_1, v_2, v_3\}$ with $T \cap X_0 = \{v_1, v_2\}$. Note that $\{v\} \cup (T + v) \subseteq A \subseteq E$. We now perform a case analysis.

Suppose first that $v_1 \notin E$. If $v + v_1 \in X_1$, then $\{v + v_1, a + v + v_1, v\}$ is a claw, and if $v + v_1 \in X_0$, then $\{v + v_2, v + v_3, a + v_3\}$ is a claw; either case gives a contradiction, so $v_1 \in E$. Symmetrically we must also have $v_2 \in E$.

Suppose now that $v_3 \notin E$. If $v + v_3 \in X_0$ then $\{v, v + v_3, a + v_3\}$ is a claw, and if $v + v_3 \in X_1$ $\{v + v_1, v + v_2, a + v + v_3\}$ is a claw. Thus $v_3 \in E$. So $T \subseteq E$ which proves the first part.

For the second part, let $T = \{v_1, v_2, v_3\} \subseteq H$ be a triangle for which $v_1 \in X_0 \backslash E$, $v_2 \in X_1 \backslash E$ and $v_3 \in X_1 \cap E$. Then $\{a + v_2, a + v_3, v_3\}$ is claw; thus, there are no such triangles. ■

We may now apply Lemma 2.1.1 to conclude the following. The proof is simply stating a suitable partition of the flat F, and checking that the conditions for Lemma 2.1.1 are met.

3.3.2.2. *$F \cap X_0 \subseteq E$.*

Subproof: Suppose not, so that $(F \backslash E) \cap X_0 \neq \varnothing$. Let $(P, Q, R) = ((F \backslash E) \cap X_0, (F \cap E) \cap X_0, F \cap X_1)$. The choice of a implies that $R \neq \varnothing$. This is a partition of F, and by the first part of the previous claim, no triangle T of F satisfies $|T \cap P| \geq 1$ and $|T \cap R| = 1$, as such a triangle contains exactly two elements of X_0 but is not contained in E. Moreover, if $(R_1, R_2) = ((F \backslash E) \cap X_1, (F \cap E) \cap X_1)$, then no triangle of F intersects P, R_1 and R_2 by the second part of the claim.

Thus, we can apply Lemma 2.1.1 to the flat F with the given partition, so the flat $F' = \mathrm{cl}(P)$ satisfies $F' \subseteq P \cup Q \subseteq X_0$, and every coset of F' in F is contained in either Q, R_1 or R_2; none of these sets is mixed with respect to M, so it follows that either $F' = \varnothing$, $F' = F$, or F' is a decomposer of $M|F$. The last case contradicts the hypotheses (recall that we are assuming that $M|F$ has no decomposer). The fact that R is nonempty and $F' \subseteq P \cup Q = F \backslash R$ implies that $F' \neq F$, and so $F' = \varnothing$. This yields $P = \varnothing$, giving the claim. ■

This claim, together with the fact that $H \backslash F \subseteq E$, implies that each triangle of G containing a must intersect E (note that triangles containing a must intersect A or F). Recall that by assumption, M^c has a triangle T_0. As just observed, we have $a \notin T_0$. For each $x \in T_0$ we therefore have $x + a \in E$, as otherwise $\{a, x + a, x\}$ does not intersect E. It follows that $T_0 + a \subseteq E$, so $E \cap \mathrm{cl}(T_0 \cup \{a\}) = T_0 + a$ and $T_0 + a$ is thus a claw, giving a contradiction. □

Next, we consider the case in which the coset of $H \backslash F$ is disjoint from E. Although the statement is not self-complementary, the first part of its proof is similar to the proof of the previous case.

Lemma 3.3.3. *Let $M = (E, G)$ be a claw-free matroid, let H be a hyperplane of G and let F be a hyperplane of H with $|F| > 1$. If $(H \backslash F) \cap E = \varnothing$ and $M|F$ has no decomposer, then either*

- *M and $M|F$ are both strict PG-sums, or*
- *M has a decomposer F' containing F.*

Proof. Let $A = H\backslash F$, so $A \cap E$ is empty. Suppose that neither outcome holds. Thus, F is not a decomposer, so has a mixed coset B. Fix an element $a \in B \cap E$, and let $X_1 = (a + E) \cap H$ and $X_0 = H\backslash X_1$. Note that the cosets of F are A, B and $A + a$. That B is mixed implies that $X_0 \cap F \neq \varnothing$. If $A \subseteq X_0$, then the set $A \cup (A + \{a\}) = G\backslash(F \cup B)$ is disjoint from E and so $F \cup B$ is a decomposer of M; thus $A \cap X_1 \neq \varnothing$. Let $v \in A \cap X_1$, so $a + v \in E$. We claim the following.

3.3.3.1. *Let $T \subseteq H$ be a triangle. Then*

(i) if $T \subseteq F$ and $|T \cap X_1| = 2$, then $T \subseteq E$, and

(ii) T does not intersect all three of $X_1\backslash E, X_0\backslash E$ and $X_0 \cap E$.

Subproof: For (i), suppose that $T \subseteq F$ and $|T \cap X_1| = 2$; let $\{v_1, v_2\} = T \cap X_1$ and $\{v_3\} = T \cap X_0$. Note that $v + [T]$ is disjoint from E as $v + [T] \subseteq A$. Now

$$\mathrm{cl}(\{a, v, v_3\}) \cap E = \{a, a + v\} \cup (E \cap \{v_3, a + v + v_3\}),$$

and

$$\mathrm{cl}(\{v_3, a + v, a + v_1\}) \cap E = \{a + v_1, a + v_2, a + v\} \cup (E \cap \{v_3, a + v + v_3\}),$$

which implies that $\{v_3, a + v + v_3\} \subseteq E$, as otherwise one of these planes gives a claw in M. We have

$$\mathrm{cl}(\{a + v, v_1, a + v_2\}) \cap E = \{a + v, a + v_2\} \cup (\{v_1, a + v + v_1\} \cap E)$$

and

$$\mathrm{cl}(\{a, v, v_1\}) \cap E = \{a, a + v_1, a + v\} \cup (\{v_1, a + v + v_1\} \cap E),$$

so, similarly, $\{v_1, a + v + v_1\} \subseteq E$. Thus $v_1 \in E$ and, symmetrically, $v_2 \in E$; therefore $T \subseteq E$.

For the second part, note that if $T = \{v_1, v_2, v_3\}$ is a triangle with $v_1 \in X_1\backslash E$, $v_2 \in X_0\backslash E$ and $v_3 \in X_0 \cap E$, then $\mathrm{cl}(T \cup \{a\}) \cap E = \{a, a + v_1, v_3\}$ and so M has a claw. Thus, there are no such triangles. ■

We may now apply Lemma 2.1.1 to conclude the following.

3.3.3.2. *$F \cap X_1 \subseteq E$.*

Subproof: Suppose not, so that $(F \backslash E) \cap X_1 \neq \varnothing$. Define a partition of F by $(P, Q, R) = ((F \backslash E) \cap X_1, F \cap E \cap X_1, F \cap X_0)$ and let $(R_1, R_2) = ((F \backslash E) \cap X_0, F \cap E \cap X_0)$. Recall that $R \neq \varnothing$ by the choice of a.

The previous claim gives that no triangle T of F satisfies $|T \cap R| = 1$ and $|T \cap P| \geq 1$, and that no triangle of F intersects all three of P, R_1, R_2; thus, Lemma 2.1.1 implies that the flat $F' = \mathrm{cl}(P)$ satisfies $F' \subseteq P \cup Q$, while each coset of F is contained in Q, R_1 or R_2. As before, this implies that F' has no mixed cosets in F, so either $F' = \varnothing, F' = F$, or F' is a decomposer of $M|F$; the last case contradicts the hypothesis, and we have $F' \neq F$ because $R \neq \varnothing$; it follows that $F' = \varnothing$ and so $P = \varnothing$, which gives the claim. ■

We now diverge from the techniques in Lemma 3.3.2. The previous claim gives that the sets $X_0 \cap E, X_0 \backslash E$ and $X_1 \cap E$ induce a partition of F. We now show that this partition naturally gives rise to a partition of $A = v + [F]$.

3.3.3.3. *For all $u \in F$,*

- *If $u \in X_0 \cap E$, then $v + u \in X_1$.*
- *If $u \in X_0 \backslash E$, then $v + u \in X_0$.*
- *If $u \in X_1 \cap E$, then $v + u \in X_0$.*

Subproof: The first two are immediate; if $u \in X_0 \cap E$ and $v + u \in X_0$, then $\{a, u, a + v\}$ is a claw. Moreover, if $u \in X_0 \backslash E$ and $v + u \in X_1$, then $\{a, a + u + v, a + v\}$ is a claw.

Finally, suppose for a contradiction that $u \in X_1 \cap E$ but $v + u \in X_1$. Then we claim that $\{u\}$ is a decomposer of $M|F$; since $\{u\} \neq F$ and all cosets of $\{u\}$ are pairs of the form $\{w, w + u\}$, it suffices to show that there does not exist $w \in F \backslash \{u\}$ for which $w \notin E$ and $u + w \in E$. Consider such a w. By 3.3.3.2, we have $w \in X_0$. If $u + w \in X_1$, then this is a contradiction to 3.3.3.1(i), so $u + w \in X_0$. The first two statements of the current claim imply that $v + u + w \in X_1$, and $v + w \in X_0$, but then $\{a, a + v + u + w, a + v + u\}$ is a claw. Thus, $\{u\}$ is a decomposer of $M|F$, contradicting the hypothesis. ■

3.3.3.4. *$F \cap X_1$ and $(F \cap E) \cap X_0$ are flats.*

Subproof: Suppose that there is a triangle T of F such that $|T \cap X_1| = 2$. Let $\{v_1, v_2\} = T \cap X_1$ and $\{v_3\} = T \cap X_0$. Then 3.3.3.1 gives $T \subseteq E$, and 3.3.3.3 gives $\{v + v_1, v + v_2\} \subseteq X_0$

and $v + v_3 \in X_1$. But this implies that $\{a + v, a + v_1, v_2\}$ is a claw. So there are no such triangles. This implies that $F \cap X_1$ is a flat in F.

Now, suppose that F has a triangle T for which $|T \cap (E \cap X_0)| = 2$. Let $\{v_1, v_2\} = T \cap E \cap X_0$ and $\{v_3\} = T \backslash \{v_1, v_2\}$. If $v_3 \in X_0 \backslash E$, then $\{a, v_1, v_2\}$ is a claw. If $v_3 \in X_1$, then $v_3 \in E$, and 3.3.3.3 gives $\{v + v_1, v + v_2\} \subseteq X_1$ and $v + v_3 \in X_0$. Now $\{a, v_1, a + v + v_2\}$ is a claw. It follows that there are no such triangles, so $F \cap E \cap X_0$ is a flat. ■

Let $F_1 = F \cap X_1$ and $F_2 = F \cap E \cap X_2$; we know from the above and 3.3.3.2 that F_1, F_2 are flats with $F_1 \subseteq E$. Since $E \cap A = \varnothing$, we have $E \cap H = E \cap F = F_1 \cup F_2$. Moreover,

$$\begin{aligned} E \backslash H &= \{a\} \cup (a + X_1) \\ &= \{a\} \cup (a + (X_1 \cap F)) \cup (a + (\{v\} \cup (v + X_0 \cap F \cap E))) \\ &= (a + [F_1]) \cup ((a + v) + [F_2]), \end{aligned}$$

where the second line uses 3.3.3.3. It follows that

$$E = (E \cap H) \cup (E \backslash H) = F_1 \cup F_2 \cup (a + [F_1]) \cup ((a + v) + [F_2]),$$

which is the union of the disjoint flats $\mathrm{cl}(F_1 \cup \{a\})$ and $\mathrm{cl}(F_2 \cup \{a + v\})$, neither of which is contained in F. It follows that M and $M|F$ are PG-sums.

It remains to check that both M and $M|F$ are strict PG-sums. By hypothesis, $\dim(F) > 1$. If $M|F$ is not a strict PG-sum then by definition of strict PG-sums, either $F \subseteq E$, or $E \cap F$ is contained in a hyperplane of F. In either case, some hyperplane decomposes $M|F$, a contradiction (in the latter case, the hyperplane described decomposes $M|F$). So $M|F$ is a strict PG-sum. From this, it also follows that $\mathrm{cl}(E \cap F) = F$. We now check that M is also a strict PG-sum. Notice that the flat $\{a, v, a + v\}$ is disjoint from F, so $\mathrm{cl}(E)$ contains $\mathrm{cl}(F \cup \{a, a + v\}) = G$, and thus $\mathrm{cl}(E) = G$. Since $F \not\subseteq E$ we have $E \neq G$ and so M is therefore also a strict PG-sum. □

3.4 Small Decomposers

We now handle the cases where M has a hyperplane H for which $M|H$ has a one-element decomposer. These turn out to be harder. We first consider the case where this decomposer is contained in E.

Lemma 3.4.1. *Let $M = (E, G)$ be a claw-free matroid. If H is a hyperplane of G, and $\{b\} \subseteq E$ is a decomposer of $M|H$, then either*

- *M^c is triangle-free, or*

- *M has a decomposer containing b.*

Proof. Let $E^c = G \backslash E$. Suppose that M has no decomposer; thus, $\{b\}$ has a mixed coset (which, in this case, forms a triangle with b) B in M; let $a \in B \cap E^c$. Note that $a \notin H$ since $B \not\subseteq H$; let $X_1 = (a+E) \cap H$ and $X_0 = H \backslash X_1$. By construction we have $b \in X_1$. Fix a hyperplane F of H for which $b \notin F$. Let $F_j = \{v \in F \colon |\{a+v, a+b+v\} \cap E| \equiv j \pmod 2\}$ for each $j \in \{0,1\}$.

We may assume that $F_1 \subseteq E$; indeed, if there exists $v \in F_1 \cap E^c$, then $\{b, a+b, a+v\}$ gives a claw provided $v \in X_1$, and $\{b, a+b, a+b+v\}$ gives a claw if $v \in X_0$ (note that $b+v \notin E$ since $\{b\}$ is a decomposer for $M|F$). We now argue that we can apply Lemma 2.1.1 to a certain partition of F; the proof is highly routine involving numerous cases.

3.4.1.1. *The sets $Q = X_1 \cap F_0 \cap E$, $R = X_0 \cap F_0 \cap E^c$ and $P = F \backslash (Q \cup R)$ satisfy the hypotheses of Lemma 2.1.1 in the flat F.*

Subproof: Since $F_1 \subseteq E$, the four sets

$$(P_1, P_2, P_3, P_4) = (X_0 \cap F_1 \cap E, X_0 \cap F_0 \cap E, X_1 \cap F_1 \cap E, X_1 \cap F_0 \cap E^c)$$

form a partition of P. Suppose that the claim fails; then F has a triangle $\{v_1, v_2, v_3\}$ for which $v_1 \in P$, $v_2 \in P \cup Q$ and $v_3 \in R$. The fact that $v_3 \in R$ implies that $v_3, v_3+a, v_3+b, v_3+a+b \notin E$.

Suppose first that $v_2 \in Q$, which gives $v_2, b+v_2, a+v_2, a+b+v_2 \in E$. If $v_1 \in P_1 \cup P_2 = X_0 \cap E$, then $\{v_1, v_2, a+v_2\}$ is a claw and if $v_1 \in X_1 \cap F_1 \cap E$, then $\{b+v_1, b+v_2, a+b+v_2\}$ is a claw. If $v_1 \in X_1 \cap F_0 \cap E^c$, then $\{a+b+v_1, v_2, a+v_2\}$ is a claw. Thus $v_2 \in P$.

If $v_1, v_2 \in P$ then there exist i, j so that $v_1 \in P_i$ and $v_2 \in P_j$; we may assume by symmetry that $i \leq j$. A careful check shows that

- $\{b, a+b+v_1, a+b+v_2\}$ is a claw if $(i,j) = (1,1)$,

- $\{v_2, b+v_1, a+b\}$ is a claw if $i \in \{1,2\}$ and $j = 2$,

- $\{v_1, v_2, a+v_2\}$ is a claw if $i \in \{1,2\}$ and $j = 3$,

- $\{b, a+v_1, a+v_2\}$ is a claw if $(i,j) = (3,3)$,

- $\{a+b, a+v_2, b+v_1\}$ is a claw if $(i,j) = (1,4)$,
- $\{a+b, v_1, a+v_2\}$ is a claw if $j = 4$ and $i \in \{2,3\}$,
- $\{a+b, a+b+v_1, a+b+v_2\}$ is a claw if $(i,j) = (4,4)$,

giving a contradiction in all cases. ■

It follows from Lemma 2.1.1 that every coset of $\mathrm{cl}(P)$ is contained in Q or in R, and that $\mathrm{cl}(P) \subseteq P \cup Q$.

Suppose that $R \neq \varnothing$, so $\mathrm{cl}(P) \neq F$. We argue that the flat $D = \mathrm{cl}(\{a,b\} \cup P)$ is a decomposer in M; let A be a coset of D in G. Note that A has to have the form $A_0 + \{0, a, b, a+b\}$ for some coset A_0 of $\mathrm{cl}(P)$ in F. Also, we either have $A_0 \subseteq Q$ or $A_0 \subseteq R$. If $A_0 \subseteq Q$ then $A = A_0 + \{0,a,b,a+b\} \subseteq Q + \{0,a,b,a+b\} \subseteq E$ by definition of Q (recall $Q = X_1 \cap F_0 \cap E$). If $A_0 \subseteq R$ then $A = A_0 + \{0,a,b,a+b\} \subseteq R + \{0,a,b,a+b\} \subseteq E^c$ by the definition of R (recall $R = X_0 \cap F_0 \cap E^c$). In either case, A is not mixed in M. Furthermore, since $\mathrm{cl}(P) \neq F$ we have $\dim(\mathrm{cl}(P)) < \dim(G) - 2$ and so $\dim(D) \leq \dim(\mathrm{cl}(P)) + 2 < \dim(G)$. Thus D is a decomposer of G containing b, giving a contradiction.

So $R = \varnothing$; we now argue that M^c is triangle-free. Indeed, the fact that $F = P \cup Q$ implies that for each $v \in F$, each triangle in the plane $\mathrm{cl}(\{v,a,b\})$ contains an element of E (to see this, note that it is enough to find a triangle in the plane that is contained in E; hence, if $v \in E$, then the triangle $\{v, b+v, b\} \subseteq E$ certifies this statement, and if $v \notin E$, then $v \in F_0$ and since $R = \varnothing$, we have $v \in X_1$, so the triangle $\{b, a+v, a+b+v\} \subseteq E$ certifies this statement). Every triangle of G containing a is contained in such a plane, so each triangle of G containing a contains an element of E. Thus if $T \subseteq E^c$ is a triangle, then $a \notin T$, and for each $x \in T$ we must have $x + a \in E$. This implies that $T + \{a\}$ is a claw in M. So M^c is triangle-free. □

We now handle the complementary case where $M|H$ has a one-element decomposer that is not contained in E. This case turns out to be extremely intricate, and very different from the previous case, even though the matroids involved ostensibly only differ in a single element. We first need a lemma that recognises even-plane matroids.

Lemma 3.4.2. *Let $M = (E, G)$ be a claw-free matroid, let H be a hyperplane of G, and let $b \in H \backslash E$ be such that, for all $u \in G \backslash \{b\}$, we have $|E \cap \{u, u+b\}| = 1$ if and only if $u \notin H$. Then, for every hyperplane F of H not containing b, either*

- *$M|F$ is even-plane, or*

- *$M|F$ is a Bose-Burton geometry.*

Proof. Let F be a hyperplane of H not containing b, and suppose that $M|F$ is not even-plane. Let $a \in G \backslash H$; we have $|E \cap \{a, a+b\}| = 1$ by hypothesis; we may assume that $a \in E$ and $a + b \notin E$.

Let $X_1 = (a+E) \cap H$ and $X_0 = H \backslash X_1$. The hypotheses imply that if $v \in H \backslash \{b\}$, then $\{v, b+v\} \cap E$ has even size, and $\{v, b+v\} \cap X_0$ has odd size. The next claim essentially states that given a flat K of H, there is another flat K' where $M|K = M|K'$, and the elements of K' have some desired intersection with X_0 and X_1.

3.4.2.1. *If K is a flat of H with $b \notin K$, and I_0, I_1 are disjoint subsets of K for which $I_0 \cup I_1$ is linearly independent, then there is a flat K' of H, not containing b, and an isomorphism ψ from $M|K$ to $M|K'$ for which $\psi(I_0) \subseteq X_0$ and $\psi(I_1) \subseteq X_1$.*

Subproof: By replacing I_0, I_1 with supersets if necessary, we may assume that $I_0 \cup I_1$ is a basis for K. For each $i \in \{0, 1\}$ and $x \in I_i$, let $\psi(x)$ be the unique element of $\{x, x+b\} \cap X_i$, and extend ψ linearly to all $x \in K$. Let $K' = \psi(K)$. The linear independence of $I_0 \cup I_1 \cup \{b\}$ implies that ψ is injective. Moreover, it is clear that $\psi(v) \in \{v, v+b\}$ for all $v \in K$ and so, since $|E \cap \{v, v+b\}|$ is even, ψ is an isomorphism from $M|K$ to $M|K'$ that has the required property by construction. ■

3.4.2.2. *If P is a plane of H not containing b, then either $P \subseteq E$, or $|P \cap E|$ is even.*

Subproof: Let P be a counterexample. Suppose first that $|P \cap E| = 5$, so $P \cap E = \{v_1, v_2, v_3, v_1+v_2, v_1+v_3\}$ for some linearly independent v_1, v_2, v_3; by 3.4.2.1 with $(I_0, I_1) = (\{v_1, v_2, v_3\}, \varnothing)$, we may assume that $v_1, v_2, v_3 \subseteq X_0$. Now

$$E \cap \mathrm{cl}(\{a, v_2, v_3\}) = \{a, v_2, v_3\} \cup (E \cap \{a + v_2 + v_3\}),$$

which implies that $a + v_2 + v_3 \in E$, and

$$E \cap \mathrm{cl}(\{a, v_1, a + v_2 + v_3\}) = \{a, v_1, a + v_2 + v_3\} \cup (E \cap \{a + v_1 + v_2 + v_3\}),$$

which gives $a + v_1 + v_2 + v_3 \in E$. But now $v_1 + v_2, v_1 + v_3$, and $a + v_1 + v_2 + v_3$ give a claw in M.

Suppose now that $|P \cap E| = 3$. Since $M|P$ is not a claw, we must have $E \cap P = \{v_1, v_2, v_1 + v_2\}$ for some v_1, v_2; let $v_3 \in P \backslash E$. By 3.4.2.1 we may assume that $v_1, v_3 \in X_0$ and $v_2 \in X_1$. This implies that $b + v_2 \in X_0$ and $b + v_1, b + v_3 \in X_1$. Now

$$E \cap \mathrm{cl}(\{a, v_1, v_3\}) = \{a, v_1\} \cup (E \cap \{a + v_1 + v_3\}),$$

which implies that $a + v_1 + v_3 \notin E$, and

$$E \cap \mathrm{cl}(\{a, b + v_2, v_3\}) = \{a, b + v_2\} \cap (E \cap \{a + b + v_2 + v_3\})$$

giving $a + b + v_2 + v_3 \notin E$; thus $a + v_2 + v_3 \in E$. Now the set $\{v_1 + v_2, a + v_2, a + v_2 + v_3\}$ is a claw.

Finally, suppose that $|P \cap E| = 1$; by 3.4.2.1 we may assume that $P = \mathrm{cl}(\{v_1, v_2, v_3\})$ where $\{v_1\} = P \cap E$ and $v_1 \in X_0$ while $v_2, v_3 \in X_1$. Then $\{v_1 + v_2, v_1 + v_3, v_2 + v_3\} \subseteq X_1$, as otherwise one of $\{a, v_1, a + v_2\}$, $\{a, v_1, a + v_3\}$ or $\{a, a + v_2, a + v_3\}$ is a claw. Hence $v_1 + v_2 + v_3 \in X_1$, as otherwise $\{a, v_1, a + v_2 + v_3\}$ is a claw. But now the triple $\{a + v_1 + v_2, a + v_1 + v_3, a + v_1 + v_2 + v_3\}$ is a claw, completing the contradiction. ■

By the above claim and the assumption that $M|F$ is not even-plane, we can conclude that E contains a plane of F. Let K be a largest flat of F for which $K \subseteq E$; by the above, $\dim(K) \geq 3$. If $K = F$ then $M|F$ is a Bose-Burton geometry, as required. Otherwise, let $v \in F \backslash E$. Let $K_1 = \{(v + E) \cap K\}$ and $K_0 = K \backslash K_1$. If some triangle T of K has even intersection with K_1, then the plane $\mathrm{cl}(T \cup \{v\})$ contains an odd number of elements of E and contains the nonelement v of E, contradicting 3.4.2.2. Thus, every triangle of K has odd intersection with K_1; it follows that either K_1 is a hyperplane of K, or $K_1 = K$.

If K_1 is a hyperplane of K, then since $\dim(K) \geq 3$, there is some triangle $T \subseteq K_1$ and some $w \in K_0$. Now $T + w \subseteq K_0$ and so $E \cap \mathrm{cl}(T \cup \{v + w\}) = T$, so $|E \cap \mathrm{cl}(T \cup \{v + w\})| = 3$, contradicting 3.4.2.2. So $K_1 = K$, and thus $K + v \subseteq E$. This argument applies for every $v \in F \backslash E$, and by the maximality of K, every coset of K contains such a v. Therefore every coset A of K satisfies $|A \cap E| = |A| - 1 = |K|$. It follows that $|E \cap F| = 2^{\dim(F)}(1 - 2^{-\dim(K)})$. Since $E \cap F$ contains no flat of dimension larger than $\dim(K)$, Theorem 1.4.2 implies that $M|F$ is a Bose-Burton geometry, as required. □

We now deal with the case where $M|H$ has a one-element decomposer disjoint from M.

Lemma 3.4.3. *Let $M = (E, G)$ be a claw-free matroid and let H be a hyperplane of G. If $\{b\} \subseteq H \backslash E$ is a decomposer of $M|H$, then either*

- *M^c is triangle-free,*
- *M is even-plane, or*
- *M has a decomposer.*

Proof. Suppose that M^c has a triangle and M has no decomposer, so $\{b\}$ has a coset $\{a, a+b\}$ in M where $\{a, a+b\} \cap E = \{a\}$. We argue through a series of claims that M is even-plane. Since $\{b\}$ is a decomposer of H, we have $a \notin H$; as usual, let $X_1 = (a+H) \cap E$ and $X_0 = H \backslash X_1$. Note that $b \in X_0$. Let $E^c = G \backslash E$ and define a partition (H_0, H_1) of $H \backslash \{b\}$ by

$$H_i = \{v \in H \backslash \{b\} \colon |\{a+v, a+b+v\} \cap E| \equiv i \pmod 2\}.$$

If $H \backslash \{b\} \subseteq E$ then recall that E^c contains a triangle T of G. This T intersects H as H is a hyperplane, and so $b \in T$, which implies that $a \notin T$, giving $\mathrm{cl}(T \cup \{a\}) \cap E = \{a+b\} + T$, which yields a claw in M. This is a contradiction, so $\{b\}$ has a coset contained in H that is disjoint from E.

If $v \in H_0 \cap X_0 \cap E$ then $\{a, v, b+v\}$ is a claw. If $v \in H_0 \cap X_1 \cap E^c$ then $\{a, a+b+v, a+v\}$ is a claw; it follows that $H_0 \subseteq E \Delta X_0$, and so for every $v \in H_0$ the set $v + \{0, a, b, a+b\}$ is unmixed in M.

3.4.3.1. *$H_1 \not\subseteq E$.*

Subproof: Suppose that $H_1 \subseteq E$. Then the sets

$$(P, Q, R) = (H_1 \cap E, H_0 \cap X_1 \cap E, H_0 \cap X_0 \cap E^c)$$

partition H by the observation just made. Let F be a hyperplane of H not containing b, and let $(P', Q', R') = (P \cap F, Q \cap F, R \cap F)$; we just saw that $H \backslash E$ contains a coset of $\{b\}$ which implies that R contains such a coset, so $R' \neq \varnothing$. We show that the partition (P', Q', R') of F satisfies the hypotheses of Lemma 2.1.1. If it does not, then F has a triangle $\{v_1, v_2, v_3\}$ with $v_1 \in P, v_2 \in P \cup Q$ and $v_3 \in R$. If $v_2 \in Q$ and $v_1 \in X_0$, then $\{a+v_2, a+b+v_2, a+b+v_1\}$ is a claw. If $v_2 \in Q$ and $v_1 \in X_1$, then $\{a+v_2, a+b+v_2, a+v_1\}$ is a claw. If $v_2 \in P$ then $\{a, i_1 b + v_1, i_2 b + v_2\}$ is a claw, where i_1 and i_2 are the binary scalars for which $v_1 \in X_{i_1}$ and $v_2 \in X_{i_2}$. So Lemma 2.1.1 applies to (P', Q', R') in F, giving $\mathrm{cl}(P') \subseteq P' \cup Q'$, while every coset of $\mathrm{cl}(P')$ in F is contained in either Q' or R'.

We now argue that $D = \mathrm{cl}(\{a, b\} \cup P')$ is a decomposer of M. Clearly $D \neq \varnothing$, and since $\mathrm{cl}(P') \subseteq (P' \cup Q')$ and $R' \neq \varnothing$, we have $\mathrm{cl}(P') \neq F$ which implies that $\dim(\mathrm{cl}(P')) < \dim(F)$ and $\dim(D) < \dim(M)$. Consider a coset A of D; now $A = A_0 + \{0, a, b, a+b\}$ for some coset A_0 of $\mathrm{cl}(P')$ in F. If $A_0 \subseteq Q'$ then $A = A_0 + \{0, a, b, a+b\} \subseteq Q + \{0, a, b, a+b\} \subseteq E$ by the definition of Q, and if $A_0 \subseteq R'$ we have $A \subseteq E^c$ by the definition of R. Therefore D has no mixed cosets in G, and is thus a decomposer of M; this is a contradiction. ■

3.4.3.2. *For each flat K of H with $K \subseteq H_1$, either $M|K \in \mathcal{E}_3$ or $M|K$ is a Bose-Burton geometry.*

Subproof: Let $G' = \mathrm{cl}(K \cup \{a, b\})$ and $H' = G' \cap H$. Let $M' = M|G'$; we apply Lemma 3.4.2 to b and H' in G'. Let $v \in G' \backslash \{b\}$. If $v \in H'$ then $|\{v, b+v\} \cap E| \in \{0, 2\}$ since no coset of $\{b\}$ in H is mixed. If $v \in \{a, a+b\}$ then $|\{v, b+v\} \cap E| = 1$ and, if $v \in (G' \backslash H') \backslash \{a, a+b\}$, then the plane $\mathrm{cl}(\{a, b, v\})$ intersects K in an element w for which $v \in \{w+a, w+a+b\}$; since $w \in H_1$ this implies that $|\{v, b+v\} \cap E| = 1$. Thus, for all $v \in G' \backslash \{b\}$, we have $|\{v, b+v\} \cap E| = 1$ if and only if $v \notin H$, and the claim follows from Lemma 3.4.2. ■

Define a partition (P, Q, R_1, R_2) of $H \backslash \{b\}$ by

$$(P, Q, R_1, R_2) = (H_1 \cap E^c, H_1 \cap E, H_0 \cap X_0 \cap E^c, H_0 \cap X_1 \cap E).$$

Note that we have argued $H_1 \not\subseteq E$ above. So in particular we have $P \neq \varnothing$.

3.4.3.3. *G has no triangle T with $|T \cap P| \geq 1$ and $|T \cap (R_1 \cup R_2)| = 1$, and there is no triangle of G that intersects P, R_1 and R_2.*

Subproof: Let $R = R_1 \cup R_2$. The definition of the H_i and the fact that $\{b\}$ is a decomposer imply that each set $W \in \{P, Q, R_1, R_2, R_1 \cup R_2\}$ satisfies $W = \{b\} + W$ (from the definition of H_i, for an element $v \in H \backslash \{b\}$, $v \in H_i$ if and only if $v + b \in H_i$, and we have observed before that $H_0 \subseteq E \Delta X_0$).

To see the first part, let $\{v_1, v_2, v_3\}$ be a triangle of G with $v_1 \in P$, $v_2 \in P \cup Q$ and $v_3 \in R$. For each $i \in \{1, 2\}$, the set $\{v_i, v_i + b\}$ intersects X_0 in exactly one element w_i, and we have $w_1 + w_2 \in \{v_1 + v_2, v_1 + v_2 + b\} \subseteq R$, so $\{w_1, w_2, w_3 = w_1 + w_2\}$ is a triangle with $w_1 \in P \cap X_0, w_2 \in (P \cup Q) \cap X_0$ and $w_3 \in R$.

Suppose that $w_2 \in Q$. If $w_3 \in R_1$ then $\{a, a+b+w_1, w_2\}$ is a claw. If $w_3 \in R_2$ then $\{a+w_3, a+b+w_3, a+b+w_2\}$ is a claw. Suppose now that $w_2 \in P$. If $w_3 \in R_1$ then $\{a, a+b+w_1, a+b+w_2\}$ is a claw. If $w_3 \in R_2$ then $\{a+w_3, a+b+w_3, a+b+w_2\}$ is a claw, completing the contradiction.

For the second part, consider a triangle $\{v_1, v_2, v_3\}$ of G where $v_1 \in P$, $v_2 \in R_1$ and $v_3 \in R_2$. Let $\{w_1\} = \{v_1, v_1 + b\} \cap X_0$; let $w_2 = v_2$ and $w_3 = w_1 + v_2$; since $w_i \in \{v_i, v_i + b\}$, the set $\{w_1, w_2, w_3\}$ is a triangle with $w_1 \in P \cap X_0$ while $w_2 \in R_1$ and $w_3 \in R_2$. Now $\{a+b+w_1, a+w_3, b+w_3\}$ is a claw. ■

Since $P \neq \varnothing$, let $w \in P$ and let z be the element of $\{w, b+w\} \cap X_1$, noting that it is also the case that $z \in P$. Let F be a hyperplane of H containing z but not b. Let (P', Q', R_1', R_2') be the partition of F induced by (P, Q, R_1, R_2). Let $F_i = H_i \cap F$ for $i = 0, 1$. By Lemma 2.1.1 and 3.4.3.3, we have $\mathrm{cl}(P') \subseteq P' \cup Q' = F_1$, and every coset of $\mathrm{cl}(P')$ in F is contained in Q', R_1' or R_2'. Let $F' = \mathrm{cl}(P')$.

3.4.3.4. *$M|F'$ is not a Bose-Burton geometry.*

Subproof: Suppose that $M|F'$ is a Bose-Burton geometry. Let $F'' = P' \cap X_1$. This set is nonempty because $z \in F''$. We will show that F'' is a decomposer of M; we first argue that it is a flat. Indeed, if $x, y \in F''$ are distinct then, since $x, y \in E^c$, the fact that $M|F'$ is a Bose-Burton geometry gives $x + y \in E^c$, and since F' is a flat we have $x + y \in F'$, so $x + y \in F' \cap E^c$, but this gives $x + y \in X_1$, as otherwise $\{a, a + x, a + y\}$ is a claw. Therefore $x + y \in F' \cap E^c \cap X_1 = F''$, and thus F'' is a nonempty flat.

Let A be a coset of F'' in F; we now show that A is contained in E or E^c, is contained in X_i for some i, and is contained in F_j for some j. To see this, we consider two cases:

- If $A \not\subseteq F'$ then A is contained in some coset of F' in F and is thus contained in R_1, R_2 or Q; in the first two cases the conclusion is clear, in the last case we have $A \subseteq E$ and $A \subseteq F_1$, and if $x_0 \in X_0 \cap A$ and $x_1 \in X_1 \cap A$ then $x_0 + x_1 \in F'' \subseteq X_1 \cap E^c$ and it follows that $\{a, x_0, b + x_1\}$ is a claw; thus $A \subseteq X_0$ or $A \subseteq X_1$.

- If $A \subseteq F'$ then $A \subseteq F_1$ (since $F' = \mathrm{cl}(P') \subseteq F_1$), and since $M|F'$ is a Bose-Burton geometry while $F'' \subseteq E^c \cap F'$, we clearly have $A \subseteq E$ or $A \subseteq E^c$. If $A \subseteq E^c$ then $A \subseteq P'$; since A is disjoint from $F'' = P' \cap X_1$ we have $A \subseteq X_0$. If $A \subseteq E$ and A intersects X_0 in x_0 and X_1 in x_1, then $\{a, x_0, b + x_1\}$ is a claw; thus $A \subseteq X_0$ or $A \subseteq X_1$.

Finally, we show that F'' is a decomposer in M. For this, we need to show that every coset of F'' in M is contained in E or in E^c; let $B = [F''] + u$ be such a coset, where $u \in G \backslash F''$. Since $\mathrm{cl}(F \cup \{a, b\}) = G$ we have $u = v + h$ for some $v \in F$ and $h \in \{0, a, b, a + b\}$. Therefore $B = ([F''] + v) + h$.

If $v \notin F''$ then $[F''] + v$ is a coset A of F'' in F, so A is contained in either E or E^c, is contained in X_i for some $i \in \{0, 1\}$, and is contained in F_i for some $i \in \{0, 1\}$. The fact that $H \cap E = (H \cap E) + b$, together with the definition of the X_i and F_i, thus implies that each of the sets $A, A + a, A + b, A + (a + b)$ is contained in either E or E^c. But B is one of these sets, so $B \subseteq E$ or $B \subseteq E^c$.

If $v \in F''$ then $B = [F''] + h$ for some $h \in \{a, b, a + b\}$. Recall that $F'' \subseteq F_1 \cap X_1 \cap E^c$. If $h = a$ then the fact that $F'' \subseteq X_1$ and $a \in E$ gives $B \subseteq E$. If $h = b$ then the fact that $\{b\}$ is a decomposer of $M|H$ and $b \notin H$ gives $B \subseteq E^c$. If $h = a + b$ then, since $F'' \subseteq F_1 \cap X_1$ and $a + b \notin E$, we have $B \subseteq E^c$. Therefore F'' is a decomposer of M, giving the needed contradiction. ∎

Let A be a coset of $\text{cl}(P')$ in F, so A is contained in Q, R_1 or R_2. We now claim that A is contained in R_1 or R_2. If, for a contradiction, $A \subseteq Q$, then $A \subseteq F_1 \cap E$ and so the flat $\text{cl}(P') \cup A$ is contained in F_1. By 3.4.3.2 it follows that $M|(\text{cl}(P') \cup A)$ is either a Bose-Burton geometry or is in $\mathcal{E}_3$. In the first case, $M|\,\text{cl}(P')$ is a Bose-Burton geometry, contradicting 3.4.3.4. In the second case, 3.4.3.4 gives that there is a triangle $T \subseteq \text{cl}(P')$ for which $|T \cap E| = 1$ (if no such triangle exists, then it would imply that $M|\,\text{cl}(P')$ is a Bose-Burton geometry, again contradicting 3.4.3.4), but then the fact that $A \subseteq E$ yields $|\,\text{cl}(T \cup \{u\}) \cap E| = 5$ for every $u \in A$, contradicting $M|(\text{cl}(P') \cup A) \in \mathcal{E}_3$. Therefore every coset of $\text{cl}(P')$ in F is contained in R_1 or R_2.

3.4.3.5. $F' = F = F_1$.

Subproof: Since $F' \subseteq F_1 \subseteq F$, it suffices to show that $F' = F$; suppose not, so $\dim(F') \leq \dim(G) - 3$, and therefore $\text{cl}(F' \cup \{a, b\}) \neq G$. We show that $\text{cl}(F' \cup \{a, b\})$ is a decomposer of M. Let C be a coset of $\text{cl}(F' \cup \{a, b\})$, so C has the form $C = [F'] + \{0, a, b, a+b\} + u$ for some $u \notin \text{cl}(F' \cup \{a, b\})$; by replacing u by $u + a$, $u + b$ or $u + a + b$ if necessary, we may assume that $u \in F$, so $C = ([F'] + u) + \{0, a, b, a+b\}$. The coset $[F'] + u$ of F' is contained in R_1 or in R_2 by the above observations, so $[F'] + u$ is contained in E or E^c, and it follows from that fact that $\{b\}$ is a decomposer and the definition of the X_i and F_i that for all $x \in [F'] + u$, we have $x + \{0, a, b, a+b\} \subseteq E$ for all $x \in E$, and $x + \{0, a, b, a+b\} \subseteq E^c$ for all $x \in E^c$. Therefore the coset C is contained in either E or E^c. Since F' is a proper nonempty flat of G, $\text{cl}(F' \cup \{a, b\})$ is thus a decomposer of M, giving a contradiction. ■

By 3.4.3.2 and 3.4.3.4, we have $M|F = M|F' \in \mathcal{E}_3$. Since $M|H$ is the doubling of $M|F$, it follows that $M|H \in \mathcal{E}_3$ also (recall that doublings preserve the property of being even-plane by Theorem 3.2.4). The fact that $F = F_1$ implies that $H \backslash \{b\} = H_1$ and so $X_0 + b = X_1$. Let $J_i = (H \backslash \{b\}) \cap (E \Delta X_i)$ for each $i \in \{0, 1\}$, so (J_0, J_1) is a partition of $H \backslash \{b\}$ for which $J_0 + b = J_1$.

3.4.3.6. *If T is a triangle of H with $b \notin T$, then $|T \cap J_0|$ is even.*

Subproof: Let $T = \{v_1, v_2, v_3\}$ be a triangle of H with $b \notin T$ for which $T \cap J_0$ is odd and as large as possible. If $v_2, v_3 \notin J_0$ then $\{v_1, v_2 + b, v_3 + b\}$ is a triangle contained in J_0, contradicting maximality. Thus $T \subseteq J_0$.

If $v_1 \in X_0 \backslash E$ then $\{a, b + v_2 + (1 - i_2)a, b + v_3 + (1 - i_3)a\}$ is a claw, where i_2, i_3 are the binary scalars for which $v_2 \in X_{i_2}$ and $v_3 \in X_{i_3}$. Therefore $T \subseteq X_1 \cap E$.

Since $M|H \in \mathcal{E}_3$, it contains no F_7-restriction, and since by 3.4.3.4 it is not a Bose-Burton geometry, Theorem 1.4.2 gives that $|E \cap H| < \frac{3}{4} \cdot 2^{\dim(H)}$. The triangle T is contained in exactly $2^{\dim(H)-2} - 1$ planes of H; since $3 + 3(2^{\dim(H)-2} - 1) > |E \cap H|$, a majority argument gives that there is a plane W of H containing T for which $|(W \backslash T) \cap E| < 3$; thus $|W \cap E| = |T| + |(W \backslash T) \cap E| < 6$; since $M|H \in \mathcal{E}_3$ this gives $|W \cap E| \leq 4$, and so $W \cap E$ contains exactly one element w outside T, and $W = \mathrm{cl}(T \cup \{w\})$. Every element of $W \cap E^c$ lies in a triangle of W containing exactly one element of E, so $b \notin W$ (since $\{b\}$ is a decomposer for $M|H$). The three-element set $T + w$ intersects either X_0 or X_1 in two elements; say they are $v_1 + w$ and $v_2 + w$. If $v_1 + w, v_2 + w \in X_0$ then $\{a, a + b + v_2 + w, b + v_3\}$ is a claw. If $v_1 + w, v_2 + w \in X_1$ then $\{a, a + w + v_1, b + v_3\}$ is a claw. ■

The above claim implies that each triangle of F has even intersection with J_0, so $F \cap J_1$ is either equal to F, or is a hyperplane of F. The definitions of J_0 and J_1 imply that if $v \in F$, then $\{v, v + a\} \cap E$ has even size if and only if $v \in F \cap J_0$. Let $M' = (E', \mathrm{cl}(F \cup \{a\}))$ be the doubling of $M|F$ by a if $F \cap J_1 = F$, or the semidoubling of $M|F$ by a with respect to the hyperplane $F \cap J_1$ if $F \cap J_1$ is a hyperplane of F; since $M|F \in \mathcal{E}_3$, the matroid M' is even-plane by Theorem 3.2.4. Since $E \cap \mathrm{cl}(F \cup \{a\}) = E' \Delta (\mathrm{cl}(F \cup \{a\}) \backslash F)$, by Lemma 3.2.5 it follows that $M|\,\mathrm{cl}(F \cup \{a\}) \in \mathcal{E}_3$.

Moreover, if $x \in F$ then $\{x, b + x\} \cap E$ has even size, and if $x \in [F] + a$ then $\{x, b + x\} \cap E = \{(x + a) + a, (x + a) + a + b\} \cap E$, which has odd size because $x + a \in F = F_1$. Since $b \notin \mathrm{cl}(F \cup \{a\})$ and F is a hyperplane of $\mathrm{cl}(F \cup \{a\})$, it follows that M is the semidoubling of $M|\,\mathrm{cl}(F \cup \{a\})$ by b with respect to the hyperplane F. Thus $M \in \mathcal{E}_3$, as required. □

3.5 General Decomposers

We now combine the results in the previous two sections to completely describe the claw-free matroids having a hyperplane that admits a decomposer.

Theorem 3.5.1. *Let $M = (E, G)$ be a claw-free matroid and let H be a hyperplane of G for which $M|H$ has a decomposer. Then either*

- *M is even-plane,*
- *M is a PG-sum,*
- *M^c is triangle-free, or*
- *M has a decomposer.*

Proof. Suppose that none of the outcomes hold. Let $F \subseteq H$ be a minimal decomposer of $M|H$. The minimality of F implies

3.5.1.1. *$M|F$ has no decomposer.*

If F is a hyperplane of H or $|F| = 1$, then one of Lemmas 3.3.2, 3.3.3, 3.4.1 or 3.4.3 yields a contradiction. Therefore $\dim(F) \geq 2$, and F has more than one coset in H. Since F does not decompose M, it has a mixed coset B in G.

We say that a set $A \subseteq G$ is *vacant* if $A \cap E = \varnothing$, and *full* if $A \subseteq E$. The fact that F decomposes $M|H$ implies that every coset of F in H is either vacant or full. Note that if A and A' are distinct cosets of F in some flat F^+ containing F, then $A + A'$ is also a coset of F in F^+.

Call a coset A of F in $M|H$ *good* if the cosets A and $A + B$ are either both vacant or both full, and say A is *bad* otherwise.

3.5.1.2. *F has a bad coset in H.*

Subproof: Suppose not; we argue that the flat $F \cup B$ decomposes M. Indeed, if A is a coset of $F \cup B$ then $A = A' \cup (A' + B)$ for some coset A' of F in H; since A' is good this implies that A is vacant or full. Thus $F \cup B$ decomposes M, a contradiction. ■

3.5.1.3. *Let A be a bad coset of F in H and let $F_A = F \cup A \cup B \cup (A + B)$.*

- *If A is vacant, then $M|F_A$ and $M|F$ are strict PG-sums, and*
- *if A is full, then $(M|F_A)^c$ is triangle-free.*

Subproof: Note that F_A is a flat of G such that $F \cup A$ is a hyperplane of F_A, and F is a hyperplane of $F \cup A$. Let F^+ be any proper flat of F_A that contains F; we will now try to show that F^+ is not a decomposer of $M|F_A$. It is easy to see that F^+ has a coset containing B or $A \cup (B + A)$. But B is mixed, and the fact that A is bad implies that $A \cup (B + A)$ is mixed. So F^+ is not a decomposer of $M|F_A$, and thus no decomposer of $M|F_A$ contains F. By 3.5.1.1, we can apply Lemma 3.3.2 (if A is full) or Lemma 3.3.3 (if A is vacant) to obtain the desired conclusion. ■

3.5.1.4. *If A_1, A_2 are distinct full cosets of F in H and $A_1 + A_2$ is vacant, then A_1 and A_2 are good.*

Subproof: Suppose otherwise; we may assume that A_1 is bad, so there exists $u_1 \in (A_1 + B)\backslash E$. Let $A_3 = A_1 + A_2$ and $F_i = F \cup A_i \cup B \cup (A_i + B)$ for each $i \in \{1, 2, 3\}$. By 3.5.1.3, the matroid $(M|F_1)^c$ is triangle-free. Let $a \in B\backslash E$ and let $v_1 = a + u_1 \in A_1$. Since $\dim(F) > 1$ and $M|F$ has no decomposer, there exists $x \in F\backslash E$. Since $(M|F_1)^c$ is triangle-free we have $x + a \in E$ and $x + a + v_1 \in E$ (to see this, observe that $x \notin E$, $a \notin E$ and $a + v_1 = u_1 \notin E$, and apply the triangle-freeness of $(M|F_1)^c$ to obtain this statement).

Let $v_3 \in A_3$. If both $a + v_3$ and $a + x + v_3$ are nonelements of E, then $\{v_1 + v_3, x + v_1 + v_3, a + x + v_1\}$ is a claw, and if both are elements of E, then $\{a + v_3, a + x + v_3, a + x\}$ is a claw (to check that this is a claw, recall that $A_3 = A_1 + A_2$ is vacant by assumption, so $v_3, x + v_3 \notin E$). Thus exactly one is an element of E; by possibly replacing v_3 by $v_3 + x$, we may assume that $a + v_3 \in E$ and $a + x + v_3 \notin E$. If $a + x + v_1 + v_3 \in E$, then $\{v_1, x + v_1 + v_3, a + x + v_1 + v_3\}$ is a claw, so $a + x + v_1 + v_3 \notin E$ (again, to check that this is a claw, observe that A_3 is vacant so that $v_3, x + v_3 \notin E$).

Since A_3 is vacant and $a + v_3 \in E$, the set A_3 is bad. By 3.5.1.3 the matroid $M|F_3$ is a PG-sum. Thus $E \cap F_3$ is the disjoint union of two flats K_1, K_2. Note that the elements $a + x, a + v_3 \in F_3$ belong to E, yet their sum, $(a + x) + (a + v_3) = x + v_3 \in A_3$, is a nonelement of E (recall A_3 is vacant). Hence, one of these two flats (say K_1) contains $a + x$, and the other (say K_2) contains $a + v_3$. Since K_1, K_2 are disjoint flats with union $E \cap F_3$, we have $(K_1 + K_2) \cap E = \varnothing$.

If $K_1 \cap F = \varnothing$, then $F \cap E = F \cap K_2$ which is a flat of F. But since $\dim(F) \geq 2$, it follows that either $F \cap E = \varnothing$, in which case every 1-dimensional flat of F decomposes $M|F$, or $F \cap E$ is a nonempty flat of F, in which case every singleton in this flat decomposes F. Either case contradicts the hypothesis, so it follows that $K_1 \cap F \neq \varnothing$. Hence we may pick an element $w \in K_1 \cap F$.

So $w \in E$ and, since K_1 is a flat, we have $w + a + x \in E$. Moreover, we have $a + w + v_3 = w + (a + v_3) \in K_1 + K_2$, so $a + w + v_3 \notin E$. Now using the fact that $a + x + v_1 + v_3 \notin E$ as observed earlier, the set $\{a + x + w, x + w + v_1, w + v_1 + v_3\}$ is a claw. ■

3.5.1.5. *Let A_1, A_2 be distinct vacant cosets of F in H. Then A_1 or A_2 is good. Moreover, if $A_1 + A_2$ is full, then A_1 and A_2 are both good.*

Subproof: Let $a \in B \cap E$. Let I be the set of $i \in \{1, 2\}$ for which A_i is bad. It suffices to show that if $|I| \geq 1$, then $|I| = 1$ and $A_1 + A_2$ is vacant; let $A_3 = A_1 + A_2$, and suppose that $|I| \geq 1$; we may assume that $1 \in I$. For each $i \in I$, let $F_i = F \cup B \cup A_i \cup (A_i + B)$. By 3.5.1.3 the matroids $M|F_i$ and $M|F$ are strict PG-sums. Let K_i, L_i be the summands of $M|F_i$, where $a \in K_i$; since $M|F$ is strict, the sets $K_i \cap F$ and $L_i \cap F$ are nonempty.

Specialising to $i = 1$, let $x \in (K_1 + L_1) \cap F$; thus, $x \notin E$, and, since $x + a \in K_1 + L_1$, we have $x + a \notin E$.

Again, consider a general $i \in I$. Since A_i is bad, the set $A_i + B = A_i + \{a\}$ contains an element $v_i + a$ of E, where $v_i \in A_i \subseteq G \backslash E$. Since $a \in K_i$ and $v_i \notin K_i$, we have $v_i + a \notin K_i$, so $v_i + a \in L_i$. The element $u = a + v_i + x$ satisfies $u + a \notin E$ and $u + v_i + a \notin E$, so $u \notin K_i \cup L_i$, giving $a + v_i + x \notin E$.

Assume that A_3 is full, and let $v_3 \in A_3$. If $a + v_3$ and $a + v_3 + x$ are both not in E, then $\{v_3, v_3 + x, a\}$ is a claw; by possibly replacing v_3 with $v_3 + x$ we may assume that $v_3 + a \in E$. Note that $v_1 + v_3$ and $x + v_1 + v_3$ are in the vacant set A_2; if $a + v_1 + v_3 \in E$ then $\{x + v_3, a + v_3, a + v_1 + v_3\}$ is a claw, so $a + v_1 + v_3 \notin E$.

Let $w \in F \cap L_1$; we have $a + w \in K_1 + L_1$ so $a + w \notin E$. Since $a + v_1 \in L_1$ we also have $a + w + v_1 \in E$. Now $w + v_3 \in A_3 \subseteq E$ while $w + v_1 + v_3 \in A_2 \subseteq G \backslash E$; it follows that $\{w + v_3, a + v_3, a + w + v_1\}$ is a claw. This contradiction shows that A_3 is vacant.

Now assume that $|I| = 2$. Recall that $w \in F \cap L_1$ and $a + w \notin E$; it follows that $w \in L_2$ also, as $w \in L_2 \cup K_2$, and $w \in K_2$ would imply that $w + a \in K_2 \subseteq E$. For each $i \in \{1, 2\}$ we have shown that $a + v_i \in L_i$, so $a + w + v_i \in L_i$, giving $a + w + v_i \in E$.

We have $a + w + v_1 + v_2 \in E$, as otherwise $\{a, a + v_1, a + w + v_2\}$ is a claw. But now $\{a + w + v_1, a + w + v_2, a + w + v_1 + v_2\}$ is a claw, giving a contradiction. Thus $|I| = 1$ as required. ■

3.5.1.6. *F has no bad vacant coset in H.*

Subproof: Let A be such a coset. We first argue that A is the only bad coset; indeed, if A' is another bad coset then (A, A') contradicts 3.5.1.5 if A' is vacant, the pair $(A', A + A')$ contradicts 3.5.1.4 if A' and $A + A'$ are both full, and the pair $(A, A + A')$ contradicts 3.5.1.5 if A' is full and $A + A'$ is vacant. Thus A is the only bad coset.

We now argue that the flat $F' = \mathrm{cl}(F \cup A \cup B) = F \cup A \cup B \cup (A + B)$ decomposes M. Since $F' \neq \varnothing$ while $\dim(F') = \dim(F) + 2 < \dim(G)$, it suffices to show that F' has no mixed cosets in G. Let C be a coset of F'; we have

$$C = C_0 \cup (A + C_0) \cup (B + C_0) \cup (A + B + C_0)$$

for some coset C_0 of F in H. If exactly one of C_0 and $A + C_0$ is full, then either (A, C_0) or $(A, A + C_0)$ contradicts 3.5.1.5. Thus C_0 and $A + C_0$ are either both empty or both full. Since they are both good, this implies that C is either empty or full. So F' is a decomposer of M, contrary to assumption. ■

Thus, all bad cosets are full.

3.5.1.7. *F has a vacant coset in H.*

Subproof: Suppose not; we show that M^c is triangle-free. Let T be a triangle in M^c. Since $T \cap H$ is nonempty while $H \backslash F \subseteq E$, we have $T \cap F \neq \varnothing$. If $T \subseteq F \cup B$, let $F' = F \cup B \cup A \cup (A+B)$ for some bad coset A. Otherwise, let $F' = \mathrm{cl}(F \cup B \cup T)$. In either case, we have $T \subseteq F'$, and $F' = F \cup B \cup A \cup (A+B)$ for some coset A.

If A is good, then by construction we have $T \nsubseteq F \cup B$, so T intersects $A \cup (A+B)$. But A is a full good coset, so $A \cup (A+B) \subseteq E$, contradicting the fact that $T \subseteq G \backslash E$. If A is bad, then 3.5.1.3 implies that $(M|F')^c$ is triangle-free. This contradicts $T \subseteq F'$. ■

Let K be a maximal flat of H that is disjoint from F, so each coset A of F in H intersects K in a unique element v_A, with $v_A \in E$ if and only if A is full, while $A = F + v_A$. Let $P \subseteq K$ be the set of all v_A for which A is bad, let Q be the set of all v_A for which A is good and full, and R be the set of all v_A for which A is good and vacant. Now 3.5.1.4 implies that there is no triangle T of K with $|T \cap R| = 1$ and $|T \cap P| \geq 1$; by Lemma 2.1.1, each coset of $\mathrm{cl}(P)$ in K is contained in Q or R. We have $R \neq \varnothing$ by 3.5.1.7, so $\mathrm{cl}(P) \neq K$.

We now argue that the flat $F' = \mathrm{cl}(F \cup \{a\} \cup P)$ decomposes M. Clearly $F' \neq \varnothing$, and the fact that $\mathrm{cl}(P) \neq K$ implies that $F' \neq G$. Consider a coset C of F'. We have $C = [F'] + u = [(F \cup (F+B))] + [\mathrm{cl}(P)] + u$ for some $u \in G \backslash F'$; since K contains a flat that is maximally disjoint from F', we can take $u \in K \backslash \mathrm{cl}(P)$.

The set $C_0 = [\mathrm{cl}(P)] + u$ is a coset of $\mathrm{cl}(P)$ in K, so is contained in either Q or R. Therefore $C \subseteq [F \cup (F+B)] + Z$ for some $Z \in \{Q, R\}$. Now

$$[F \cup (F+B)] + Z = \bigcup_{z \in Z} \left(([F] + z) \cup ((F+B) + z) \right).$$

If $Z = Q$ then each coset $[F] + z$ and $(F+B) + z$ is full by definition of Q, so C is full. Similarly, if $Z = R$ then $[F] + z$ and $(F+B) + z$ are empty for all z, so C is empty. Thus F' is a decomposer of M, yielding a final contradiction. □

At this point, we can easily reduce Theorem 3.3.1 to a finite computation, showing that it suffices to verify the result in dimension at most 8. Although actually performing this check computationally would likely be impossible, we include the argument here for interest.

Theorem 3.5.2. *If Theorem 3.3.1 holds for all matroids of dimension at most* 8*, then it holds in general.*

Proof. Let $M = (E, G)$ be a minimal counterexample to Theorem 3.5.2. Note that $\dim(M) \geq 9$. Since M^c is not triangle-free, there is a triangle $T \subseteq G \backslash E$. Since M is not a PG-sum, there is a plane P for G for which $M|P$ is not a PG-sum by Lemma 3.2.2. Since $M \notin \mathcal{E}_3$, there is a plane Q of G for which $|E \cap Q|$ is odd.

Note that $T \cup P \cup Q$ is contained in a flat F of dimension at most $2 + 3 + 3 = 8$. Let H be a hyperplane of G containing F. The existence of T, P and Q certifies that $(M|H)^c$ is not triangle-free, and that $M|H$ is not even-plane or a PG-sum; since $M|H$ is not a counterexample, it follows that $M|H$ has a decomposer F. Theorem 3.5.1 thus implies that M has a decomposer, contrary to assumption. $\square$

3.6 The Main Theorem

In this section, we finally prove Theorem 3.3.1 in its full generality. The strategy is an adaptation of the proof of Theorem 3.5.2, where we reduce the size of the base case from 8 to something more manageable. The 8 appears in the argument above because, naively, if a matroid is not in one of our three basic classes, then it contains a certificate of this fact in dimension at most 8. Our argument below is essentially reducing the size of such a certificate in the important cases.

Recall that $M_{5,3}$ is the three-dimensional matroid with five elements. This matroid is of particular interest at this point because it is neither even-plane nor a PG-sum, so any matroid having an induced $M_{5,3}$-restriction belongs to neither of these two basic classes.

Lemma 3.6.1. *Let $M = (E, G)$ be a claw-free matroid and let H be a hyperplane of G for which $M|H$ is a strict PG-sum such that $M|H \notin \mathcal{E}_3$ and $(M|H)^c$ is not triangle-free. Then either*

- *M has an induced $M_{5,3}$-restriction,*
- *M is a strict PG-sum, or*
- *M has a decomposer.*

Proof. Suppose that none of the conclusions hold. Let K_0, K_1 be the disjoint nonempty flats of H whose disjoint union is $E \cap H$. If one of the K_i has dimension 1, then since $M|H$

is strict, the other is a hyperplane of H, which meets every triangle of H; this contradicts the hypothesis that $H \backslash E$ contains a triangle. Therefore each K_i has dimension at least 2. If they both have dimension 2 then $M|H \in \mathcal{E}_3$; thus K_0 or K_1 has dimension at least 3, and therefore $\dim(H) \geq 5$ and $\dim(G) \geq 6$.

Since M has no decomposer, there exists $a \in E \backslash H$. Let $X_1 = (a + E) \cap H$ and $X_0 = H \backslash X_1$. Think of the indices of X_0 and X_1 as belonging to $\mathbb{Z}_2$.

Since M has no induced $M_{5,3}$-restriction, no plane P of G contains precisely five elements of E. For every triangle T of H, the plane $\mathrm{cl}(T \cup \{a\})$ contains exactly $1 + |T \cap E| + |T \cap X_1|$ elements of E, so we have

3.6.1.1. $|T \cap E| + |T \cap X_1| \neq 4$ *for every triangle* T *of* H*.*

For any triangle T of K_i, it follows that $|T \cap X_1| \neq 1$; therefore

3.6.1.2. *For each* $i \in \{0, 1\}$ *the set* $X_0 \cap K_i$ *is a flat of* K_i*.*

This also imposes structure on the elements of $K_0 + K_1$.

3.6.1.3. *For each* $i, j \in \mathbb{Z}_2$ *we have* $(X_i \cap K_0) + (X_j \cap K_1) \subseteq X_{1+i+j}$*.*

Subproof: Let $x \in K_0 \cap X_i$ and $y \in K_1 \cap X_j$ and $x + y \in X_\ell$, where $\ell \in \mathbb{Z}_2$. If $i = j = 0$ then, since $\{a, x, y\}$ is not a claw, we have $a + x + y \in E$ and so $\ell = 1$. Otherwise the triangle $T = \{x, y, x + y\}$ satisfies $|T \cap E| = 2$ and $1 \leq |T \cap X_1| \leq 3$, so since $|T \cap E| + |T \cap X_1| \neq 4$ we have $|T \cap X_1|$ odd. It follows that $i + j + \ell = 1$ in $\mathbb{Z}_2$ and so $\ell = 1 + i + j$, giving the claim. ■

3.6.1.4. *For each* $i \in \{0, 1\}$*, either* $K_i \subseteq X_0$ *or* $|K_i \cap X_0| = 1$*.*

Subproof: Suppose first that $K_i \subseteq X_1$. If $K_{1-i} \subseteq X_0$ then by 3.6.1.3 we have $K_0 + K_1 \subseteq X_0$, and it follows that E is the union of the disjoint flats $\mathrm{cl}(K_i \cup \{a\})$ and K_{1-i}, so is a PG-sum, contrary to assumption. If there is some $v \in K_{1-i} \cap X_1$, then let $T = \{u_1, u_2, u_3\}$ be a triangle of K_i. We have $T + v \subseteq X_1$ by 3.6.1.3, so now the triangle $T' = \{u_2, u_1 + v, u_3 + v\}$ satisfies $|T' \cap E| = 1$ and $T' \subseteq X_1$, contradicting 3.6.1.1. Therefore $X_0 \cap K_i$ is nonempty for both i.

If the conclusion fails, then there exist $z_1, z_2 \in K_i \cap X_0$ and $v \in K_i \cap X_1$; since $K_{i-1} \nsubseteq X_1$ there also exists $y \in K_{1-i} \cap X_0$. Since $X_0 \cap K_i$ is a flat we have $z_1 + z_2 \in X_0$ and $v + z_1 + z_2 \in X_1$. By 3.6.1.3 we have $y + z_2 \in X_1$ and $y + v + z_1 + z_2 \in X_0$. Now $\{z_1 + v, a + y + z_2, a + v\}$ is a claw, a contradiction. ■

3.6.1.5. $|K_i \cap X_0| = 1$ *for both* i.

Subproof: If not, then there is some $i \in \{0, 1\}$ for which $K_i \subseteq X_0$ by 3.6.1.4.

If $K_{1-i} \subseteq X_0$, then 3.6.1.3 implies that $K_0 + K_1 \subseteq X_1$. Since the dimensions of K_0 and K_1 are at least 2 and sum to at least 5, we may assume by symmetry that $\dim(K_1) \geq 3$. Let $\{x_1, x_2\} \subseteq K_0$ and $\{y_1, y_2, y_3\} \subseteq K_1$ be linearly independent sets. Using $K_0 + K_1 \subseteq X_1$ together with $K_0 + K_1 \subseteq G \backslash E$, the set

$$\{a + x_1 + y_1 + y_3, a + x_2 + y_2 + y_3, a + x_1 + x_2 + y_1 + y_2 + y_3\}$$

is a claw, giving a contradiction.

Otherwise, the set $K_{1-i} \cap X_0$ has size 1 by 3.6.1.4; let w be its element. Now 3.6.1.3 gives $X_1 = (K_{1-i} - \{w\}) \cup (K_i + w)$ and so

$$E = K_i \cup (K_i + a + w) \cup K_{1-i} \cup (K_{1-i} + a + w)$$

which implies that $E + a + w = E$, and so $\{a + w\}$ decomposes M, a contradiction. ■

Assume now by symmetry that $\dim(K_1) \geq 3$; thus $K_1 \cap X_1$ contains a triangle $T = \{x, y, x+y\}$. Let $u \in K_0 \cap X_1$; now $u + T \subseteq X_1$ by 3.6.1.3. Therefore the plane $\mathrm{cl}(\{a, x, y + u\})$ contains exactly five elements of E, giving a contradiction. □

We now restate and prove Theorem 3.3.1.

Theorem 3.6.2. *If M is a claw-free matroid, then either*

- *M is even-plane,*
- *M^c is triangle-free,*
- *M is a strict PG-sum, or*
- *M has a decomposer.*

Proof. Let $M = (E, G)$ be a counterexample of smallest possible dimension. Clearly $\dim(M) \geq 3$, since otherwise $M \in \mathcal{E}_3$. It is easy to check the following:

3.6.2.1. *Every 3-dimensional, odd-sized claw-free matroid has a one-element decomposer.*

This gives $\dim(M) \geq 4$. Also observe that for every hyperplane H of G, the matroid $M|H$ has no decomposer, as otherwise we obtain a contradiction from Theorem 3.5.1.

Since $M \notin \mathcal{E}_3$, there is a plane P of G for which $|P \cap E|$ is odd, and since M^c is not triangle-free, there is a triangle $T \subseteq G \backslash E$. Choose P and T so that their intersection is as large as possible.

3.6.2.2. $\dim(M) = 5$.

Subproof: Suppose not, so either $\dim(M) = 4$ or $\dim(M) \geq 6$. If $\dim(M) = 4$, then P is a hyperplane of G, and 3.6.2.1 implies that $M|P$ has a decomposer, giving a contradiction.

Therefore $\dim(M) \geq 6$. Let H be a hyperplane of G containing $P \cup T$. By construction the matroid $M|H$ is not even-plane, while $(M|H)^c$ contains a triangle; since $M|H$ has no decomposer but is not a counterexample, it is a strict PG-sum. Since M is not a strict PG-sum and has no decomposer, Lemma 3.6.1 implies that M has a $M_{5,3}$-restriction $M|P'$. Let H' be a hyperplane of G containing T and P'. Now the existence of P' certifies that $M|H' \notin \mathcal{E}_3$ and $M|H'$ is not a PG-sum, and T certifies that $(M|H')^c$ is not triangle-free. Since $M|H'$ has no decomposer, this contradicts the minimality in the choice of M. ■

3.6.2.3. $P \subseteq E$.

Subproof: We first argue that $P \cap T$ is empty. If $P \cap T \neq \varnothing$ then $\dim(\mathrm{cl}(P \cup T)) \leq 4$; let H be a hyperplane containing $P \cup T$. Since $M|H$ is not a counterexample, but $M|H \notin \mathcal{E}_3$ while $T \subseteq H \backslash E$ and $M|H$ has no decomposer, we conclude that $M|H$ is a strict PG-sum.

However, for each 4-dimensional strict PG-sum, the ground set is either the disjoint union of an element and a hyperplane, or of two triangles. If $M|H$ has the former structure then every triangle of H intersects the hyperplane, contradicting the existence of T. If $M|H$ has the latter structure, then every plane of H has odd intersection with each of the two triangles so has even intersection with E; this contradicts the existence of P.

Therefore $P \cap T$ is empty. We now argue that $P \subseteq E$. If not, let $v \in P \backslash E$. If $u + v \notin E$ for some $u \in T$, then $T' = \{u, v, u + v\}$ is a triangle contained in $G \backslash E$ that intersects P in more elements than T does; this contradicts the choice of T and P. Thus $v + T \subseteq E$. But this implies that $\mathrm{cl}(T \cup \{v\}) \cap E = v + T$ and so M has a claw, a contradiction. Thus $P \subseteq E$. ■

Since $T \subseteq G \backslash E$ and $P \subseteq E$, we have $T \cap P = \varnothing$ and so the fact that $\dim(G) = 5$ implies that $G = \mathrm{cl}(P \cup T)$. Let $T = \{u_1, u_2, u_3\}$ and for each $u \in T$, let $A_u = P \cap (u + E)$.

3.6.2.4. *$A_{u_1} \Delta A_{u_2} \Delta A_{u_3} = P$, and each of $A_{u_1}, A_{u_2}, A_{u_3}$ is either a triangle of P or equal to P.*

Subproof: If the first conclusion fails, then there is some $v \in P$ for which $|\{u \in T : v \in A_u\}|$ is even. Then the plane $Q = \mathrm{cl}(\{v\} \cup T)$ has odd intersection with E and intersects T, contradicting the choice of P and T.

To see the second conclusion, it suffices to show that each triangle T' of P has odd intersection with A_u; indeed, if $|T' \cap A_u|$ is even, then the plane $P' = \mathrm{cl}(T' \cup \{u\})$ has odd intersection with E and also intersects T; this contradicts the choice of P and T. ■

If at least two of the sets A_u are equal to P (say the first two), then $A_{u_3} = P \Delta P \Delta P = P$ and so $E = G \backslash T$ which implies that T decomposes M, a contradiction.

If exactly one of the A_u (say A_{u_1}) is equal to P, then A_{u_2} and A_{u_3} are triangles with symmetric difference $P \Delta P = \varnothing$, so $A_{u_2} = A_{u_3} = T'$ for some triangle T' of P. It follows that

$$E = ([u_1] + P) \cup (\{u_2\} + T') \cup (\{u_3\} + T') = ([u_1] + P) \cup (\{u_2, u_3\} + T').$$

Since $u_1 + [u_1] = [u_1]$ and $u_1 + \{u_2, u_3\} = \{u_2, u_3\}$, this implies that $u_1 + E = E$, and therefore $\{u_1\}$ decomposes M, a contradiction.

Finally, if all three A_u are triangles, then since they have symmetric difference P, it is easy to see that they are exactly the three triangles T_1, T_2, T_3 through some element z of P. Let $\{y_1, y_2, y_3\}$ be a triangle of P not containing z for which $y_i \in A_{u_i}$ for each $i \in \{1, 2, 3\}$, so by construction, if $x \in P$ and $i \in \{1, 2, 3\}$, then $u_i + x \in E$ if and only if $x \in \{z, y_i, z + y_i\}$. Using this, we see that $\{z + y_2, u_1 + z + y_1, u_3 + z\}$ is a claw. □

As discussed earlier, the above theorem, together with an inductive argument, implies Theorem 3.1.1.

3.7 Corollaries

In the rest of this chapter, we prove some corollaries of Theorem 3.1.1 regarding claw-free matroids.

χ-boundedness

A class of graphs or matroids is χ-bounded if χ is bounded above by a function of ω. The class of claw-free graphs is known to be χ-bounded, as shown in [53]. However, the same cannot be said of claw-free matroids. Theorem 1.5.7 shows that the even-plane matroids have arbitrarily large χ, and as even-plane matroids are claw-free while having $\omega < 3$, this implies that the claw-free matroids are not χ-bounded. However, as a consequence of our structure theorem for claw-free matroids, we can prove that even-plane matroids present the only obstruction to χ-boundedness; it turns out that the classes of PG-sums and the matroids whose complements are triangle-free are in fact χ-bounded, and the $\otimes$ operation behaves nicely. In fact, as soon as we exclude just *one* even-plane matroid, we obtain χ-boundedness.

Theorem 3.7.1. *If N is an even-plane matroid, then the class of N-free, claw-free matroids is χ-bounded.*

We will in fact show that for every $N \in \mathcal{E}_3$, the class of claw-free, N-free matroids is χ-bounded by some function f that grows exponentially. A function $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is *superadditive* if $f(x+y) \geq f(x) + f(y)$ for all $x, y > 0$.

Lemma 3.7.2. *Let $\mathcal{M}$ be a class of matroids and let $\mathcal{M}'$ be its closure under $\otimes$. If $\mathcal{M}$ is χ-bounded, then so is $\mathcal{M}'$. Moreover, if $\mathcal{M}$ is χ-bounded by a superadditive function f, then $\mathcal{M}'$ is χ-bounded by f.*

Proof. Let g be a function for which $\chi(M) \leq g(\omega(M))$ for all $M \in \mathcal{M}$. Define g' by $g'(k) = g(k)$ for $k \leq 1$, and $g'(k) = \max(g(k), \max_{1 \leq i < k}(g'(i) + g'(k-i)))$ for all $k > 1$. By construction we have $g'(k) \geq g(k)$ for all $k \geq 1$, and g' is superadditive. Since $g \leq g'$, the class $\mathcal{M}$ is χ-bounded by g'. Note also that if g is superadditive, then by induction it follows that $g = g'$.

It now suffices to argue that $\mathcal{M}'$ is χ-bounded by g'. Suppose not, and let $M \in \mathcal{M}'$ have minimal dimension for which $\chi(M) > g'(\omega(M))$. If $M \in \mathcal{M}$ we have a contradiction. Otherwise, $M = M_1 \otimes M_2$ where $M_1, M_2 \in \mathcal{M}'$ of smaller dimension, and now Lemma 2.4.3 gives

$$\begin{aligned}
\chi(M) &= \chi(M_1) + \chi(M_2) \\
&\leq g'(\omega(M_1)) + g'(\omega(M_2)) \\
&\leq g'(\omega(M_1) + \omega(M_2)) = g'(\omega(M)),
\end{aligned}$$

a contradiction. $\square$

The above theorem allows us to reduce the problem to checking χ-boundedness within each of the three basic classes of claw-free matroids. Let us argue that the class of complements of triangle-free matroids is χ-bounded. We can do this in two ways. The second method is more ideal as it gives a better bound. First, observe that if M^c is triangle-free then $\chi(M)$ is equal to $\dim(M)$ or $\dim(M)-1$. Hence, what we are trying to show is that $\dim(M)$ is bounded by a function of $\omega(M)$. This is in fact a special case of a Ramsey theorem for projective geometries; the following is a consequence of Corollary 2 of [31], rephrased in our language.

Theorem 3.7.3 ([31])**.** *For all $s,t \geq 0$ there is an integer $n = n(s,t)$ such that every n-dimensional matroid M satisfies $\omega(M) \geq s$ or $\omega(M^c) \geq t$.*

This theorem would suffice for the purpose of showing that the complements of triangle-free matroids are χ-bounded. The methods used in [31], however, give an extremely large value for n, even when $t = 2$. In order to obtain a more reasonable χ-bounding function, we turn to a theorem of Sanders [50].

Theorem 3.7.4 ([50], Theorem 4.1)**.** *Let $G \cong \mathrm{PG}(n-1,2)$, let $X \subseteq [G]$, and let $\alpha = |X|/2^n$. If $\alpha \leq \frac{1}{2}$, then $X+X$ contains a subspace of $[G]$ of dimension $n - \lceil n/\log_2\left(\frac{2-2\alpha}{1-2\alpha}\right)\rceil$.*

Corollary 3.7.5. *Let $s \geq 2$ be an integer. If $M = (E,G)$ is a matroid for which M^c is triangle-free and $\dim(M) \geq 2^s(s+1)$, then $\omega(M) \geq s$.*

Proof. Let $\dim(M) = n$. Suppose for a contradiction that $\omega(M) < s$; Theorem 1.4.2 implies that $|E| \leq 2^n - 2^{n-s}$. If equality holds then M is an order-s Bose-Burton geometry so $G \backslash E$ is a flat of dimension $n-s \geq 2$ and thus contains a triangle, contrary to hypothesis. Therefore $|E| \leq 2^n - 2^{n-s} - 1$.

Let $E^c = G\backslash E$. We have $|E^c| = 2^n - 1 - |E| \geq 2^{n-s}$. Let X be a 2^{n-s}-element subset of E^c; we have $\alpha = |X|/2^n = 2^{-s} \leq \frac{1}{2}$. Now

$$\log_2\left(\tfrac{2-2\alpha}{1-2\alpha}\right) = 1 + \tfrac{1}{\ln(2)}\ln(1+\tfrac{1}{1/\alpha-2}) = 1 + \tfrac{1}{\ln(2)}\ln(1+\tfrac{1}{2^s-2})$$

Using $\ln(1+x) \geq x - \frac{1}{2}x^2$ and the fact that $\frac{1}{2^s-2} - \frac{1}{2(2^s-2)^2} \geq 2^{-s}$ for $s \geq 2$, this gives $\log_2\left(\frac{2-2\alpha}{1-2\alpha}\right) \geq 1 + \frac{1}{\ln(2)}2^{-s}$. By Theorem 3.7.4, the set $X+X$ thus contains a subspace F of dimension $n - \lceil n/(1+\frac{1}{2^s \ln 2})\rceil$. Using $\lceil x \rceil \leq x+1$ we get

$$\dim(F) \geq n - \lceil n/(1+\tfrac{1}{2^s\ln 2})\rceil \geq \tfrac{n}{2^s \ln 2 + 1} - 1 \geq \tfrac{n}{2^s} - 1 \geq s,$$

where we use $2^s \ln 2 + 1 \leq 2^s$. But now $F \subseteq E^c + E^c$, so since E^c contains no triangle, we have $F \subseteq E$. This contradicts the fact that $\omega(M) < s$. $\square$

In the language of Theorem 3.7.3, this result states that $n(s, 2) \leq (s+1)2^s$. Let us now complete the proof of Theorem 3.7.1. We restate it here with an explicit function f, which grows exponentially.

Theorem 3.7.6. *Let $N \in \mathcal{E}_3$. The class of N-free, claw-free matroids is χ-bounded by the function $f(k) = (k+2)2^{k+1} + k(\dim(N) + 4)$.*

Proof. Let $\mathcal{M}_1$ denote the class of N-free matroids in $\mathcal{E}_3$, let $\mathcal{M}_2$ denote the class of matroids whose complement is triangle-free, and let $\mathcal{M}_3$ denote the class of PG-sums. Let $\mathcal{M}' = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$. By Theorem 3.1.1, every claw-free matroid M is obtained via the $\otimes$ operation from 'basic' matroids in $\mathcal{M}_1$, $\mathcal{M}_2$ or $\mathcal{M}_3$, and since all the basic matroids are induced restrictions of M, if M is N-free, then so are all the basic matroids. Thus M lies in the closure under $\otimes$ of $\mathcal{M}'$.

The function $h(k) = (k+2)2^{k+1}$ is superadditive since

$$(x+y+2)2^{x+y+1} \geq (x+y+4)2^{x+y} \geq (x+2)2^{x+1} + (y+2)2^{y+1}$$

for all $x, y \geq 1$. Therefore f is superadditive as it is the sum of two superadditive functions. By Lemma 3.7.2 it is thus enough to show that $\mathcal{M}'$ is χ-bounded by f. Indeed, if $M \in \mathcal{M}_1$ then $\chi(M) \leq \dim(N) + 4 \leq f(\omega(M))$ by Theorem 3.2.3. If $M \in \mathcal{M}_2$ with $\omega(M) = s$, then $\chi(M) \leq \dim(M) < 2^{s+1}(s+2) \leq f(s)$ by Corollary 3.7.5. Finally, if $M \in \mathcal{M}_3$ then $\chi(M) = \omega(M) \leq f(\omega(M))$ by Lemma 3.2.1; the theorem follows. □

We can now characterise exactly which down-closed classes of claw-free matroids are χ-bounded.

Corollary 3.7.7. *If $\mathcal{M}$ is a class of claw-free matroids that is closed under taking induced restrictions, then $\mathcal{M}$ is χ-bounded if and only if $\mathcal{E}_3 \nsubseteq \mathcal{M}$.*

Proof. If $\mathcal{E}_3 \subseteq \mathcal{M}$, then Theorem 1.5.7 implies that $\mathcal{M}$ is not χ-bounded. Otherwise, there is some $N \in \mathcal{E}_3$ with $N \notin \mathcal{M}$, so all matroids in $\mathcal{M}$ are N-free. Theorem 3.7.6 now gives the result. □

In order to extend the χ-boundedness result to I_t-free matroids for higher values of $t > 3$, it will be necessary to consider generalisations of $\mathcal{E}_3$. In [46], $\mathcal{E}_t$ is defined to be the class of matroids whose t-dimensional restrictions have even-sized ground sets. A natural starting point is to consider whether one can obtain a theorem similar to Theorem 3.2.3 for the class $\mathcal{E}_t$, $t \geq 4$.

Rough structure

Theorem 3.7.1 gives a description of claw-free matroids with bounded ω subject to the exclusion of some even-plane matroid $N \in \mathcal{E}_3$. We can in fact more generally describe claw-free matroids with bounded ω as in the following theorem (which does not assume the exclusion of any even-plane matroid). Combined with Theorem 3.2.3, it implies Theorem 3.7.1.

Theorem 3.7.8. *For all $s \geq 1$ there exists $k \geq 2$ so that, if M is a claw-free matroid and $\omega(M) \leq s$, then $M \cong M_1 \otimes \cdots \otimes M_{2s+1}$, each of M_i either is even-plane, or has dimension at most k.*

Observe that the matroids described in the conclusion of the theorem must have bounded ω as well. By Lemma 2.4.3, we have that $\omega(M_1 \otimes M_2) = \omega(M_1) + \omega(M_2)$ for all matroids M_1, M_2. Since even-plane matroids have $\omega \leq 2$ while $\omega(N) \leq \dim(N)$ for each N, the matroids in the conclusion of the theorem above satisfy $\omega(M) \leq (2s+1)k$. Hence we can understand Theorem 3.7.8 to be a qualitative structure theorem for claw-free matroids with bounded ω.

Proof. Let $k = k(s) = 2^{s+1}(s+2)$ for each s. Recall that Lemma 2.4.4 shows that $\otimes$ is associative. By Theorem 3.1.1, there is some t for which there are matroids $M_1, \ldots, M_t$ with $M \cong \otimes_{i=1}^{t} M_i$, for which each M_i is either an even-plane matroid, the complement of a triangle-free matroid, or a strict PG-sum. By including dimension-zero matroids, we may assume that $t \geq 2s+1$; choose t so that if $t > 2s+1$, then t is as small as possible.

Each M_i that is not even-plane is either a strict PG-sum, or the complement of a triangle-free matroid. It is clear that a strict PG-sum with $\omega \leq s$ has dimension at most $2s$. If M_i^c is triangle-free and $\omega(M_i) \leq s$, then by Corollary 3.7.5 we have $\dim(M_i) < 2^{s+1}(s+2)$; in either case, $\dim(M_i) \leq k$.

It remains to argue that $t = 2s+1$; suppose not. Using $\omega(N \otimes N') = \omega(N) + \omega(N')$, it is routine to show by induction that $\omega(M) \geq c$, where c is the number of M_i that have a nonempty ground set. Since $\omega(M) \leq s$, this implies that $c \leq s < \frac{1}{2}(t-1)$. Therefore there are two consecutive matroids M_i, M_{i+1} that are both empty. But if this is the case, then one could replace M_i, M_{i+1} with the empty (and thus even-plane) matroid $M_i \otimes M_{i+1}$ in the sequence to shorten its length, contradicting the minimality of t. $\square$

We also give an alternative version of the statement in Theorem 3.7.8.

Theorem 3.7.9. *For all $s \geq 1$ there exists $k \geq 2$ such that, for every claw-free matroid $M = (E, G)$ with $\omega(M) \leq s$, there is a flat F of G whose codimension is at most k, such that $M|F \cong M_1 \otimes \cdots \otimes M_{2s+1}$, where each of M_i is an even-plane matroid.*

Observe that $\omega(M) \leq \omega(M|F) + k$ for every codimension-k flat, so the outcome of this theorem still certifies bounded ω as in Theorem 3.7.8.

Proof. Let $s \geq 1$. Let k' be the constant depending on s given by the previous theorem, and let $k = (2s+1)k'$. By the previous theorem we have $M \cong \otimes_{i=1}^{2s+1} M_i$ where each $M_i = (E_i, G_i)$ either has dimension at most k' or is even-plane. For each $1 \leq i \leq 2s+1$, let $F_i = G_i$ if M_i is even-plane, and let $F_i = \varnothing$ otherwise. Let $F = \mathrm{cl}(\cup_i F_i)$. By the second part of Lemma 2.4.3 the matroid $M|F \cong M_1|F_1 \otimes \cdots \otimes M_{2s+1}|F_{2s+1}$ where each $M_i|F_i$ is even-plane, and since $\dim(F) = \sum_{i=1}^{2s+1} \dim(F_i) \geq \sum_{i=1}^{2s+1} (\dim(G_i) - k') \geq \dim(M) - k$, the result follows. □

Excluding anticlaws

As a final application of our structure theorem for claw-free matroids, we will give a structure theorem for excluding claws as well as anticlaws. An *anticlaw* is simply the complement of a claw. We say that a matroid $M = (E, G)$ is a *target* if there are distinct flats $F_0 \subset \ldots \subset F_k$ of G for which E is the union of $F_{i+1} \backslash F_i$ over all even $i < k$. Note that F_0 is allowed to be empty.

Theorem 3.7.10. *A matroid M is claw-free and anticlaw-free if and only if M is a target.*

Let us first show that targets are closed under some basic properties.

Lemma 3.7.11. *The class of targets is closed under taking induced restrictions, complementations, and the $\otimes$ operation.*

Proof. We first observe that if $F_0 \subseteq \ldots \subseteq F_k$ are flats of G, not necessarily distinct, for which E is the union of $F_{i+1} \backslash F_i$ over all even $i < k$, then (E, G) is a target. This holds because, given such a sequence, we can remove consecutive pairs of equal flats to obtain such a sequence where the flats are distinct, certifying that $M = (E, G)$ is a target. Thus we can allow the F_i in the definition of target to be equal. It follows from this fact that targets are closed under taking induced restrictions and under complementation (consider the sequence $(\varnothing, F_0, \ldots, F_k, G)$ if k is odd and $(\varnothing, F_0, \ldots, F_{k-1}, G)$ if k is even).

Finally, suppose that $M_1 = (E_1, G_1)$ and $M_2 = (E_2, G_2)$ are targets; let $F_0 \subseteq \ldots \subseteq F_s$ be flats of G_1 and $K_0 \subseteq \ldots \subseteq K_t$ be flats of G_2 certifying this. By possibly deleting the last element of the first sequence, we may assume that s is odd. Let $K_i' = G_1 \oplus K_i$ for each $i \in \{1, \ldots, t\}$, so $G_1 \subseteq K_0'$ and $K_0', \ldots, K_t'$ is a nested sequence of flats of $G_1 \oplus G_2$. If $M = (E, G_1 \oplus G_2) = M_1 \otimes M_2$, then

$$\begin{aligned} E &= E_1 \cup ([G_1] + E_2) \\ &= \cup_i (F_{i+1} \backslash F_i) \cup ([G_1] + (\cup_j (K_{j+1} \backslash K_j))) \\ &= \cup_i (F_{i+1} \backslash F_i) \cup \left(\cup_j (K_{j+1}' \backslash K_j')\right), \end{aligned}$$

where the unions are taken over all even i and j with $i < s$ and $j < t$. Now, since s is odd while $F_s \subseteq G_1 \subseteq K_1'$, the sequence $(F_0, F_1, \ldots, F_s, K_0', \ldots, K_t')$ certifies that M is a target, as required. □

We can now prove Theorem 3.7.10.

Proof. It is easy to see that a claw is not a target. By Lemma 3.7.11, targets are closed under complementation and induced restrictions; it follows that targets are claw-free and anticlaw-free. It remains to show that claw-free, anticlaw-free matroids are targets; suppose otherwise and let $M = (E, G)$ be a counterexample of smallest possible dimension.

By minimality, M is full-rank, and since M is claw-free and anticlaw-free, both E and $G \backslash E$ contain triangles, as otherwise we can apply Lemma 1.5.6 to M or M^c to conclude that M is a target, giving a contradiction. By Theorem 3.3.1 and the fact that $G \backslash E$ contains a triangle, M is either even-plane, a strict PG-sum, or obtained from two matroids of smaller dimension using the $\otimes$ operation. Let T be a triangle contained in E.

Suppose first that M is even-plane. Let $\mathcal{P}$ be the collection of planes of G containing T. For each $P \in \mathcal{P}$, we have $|E \cap P| \in \{4, 6\}$, and since $E \cap P$ contains a triangle but $M|P$ is not an anticlaw, we have $|E \cap P| = 6$ and $|E \cap (P \backslash T)| = 3$. Therefore

$$|E| = |T| + \sum_{P \in \mathcal{P}} |E \cap (P \backslash T)| = 3(1 + |\mathcal{P}|) = 3 \cdot 2^{\dim(M) - 2}.$$

But E contains no plane of G, so Theorem 1.4.2 implies that M is a Bose-Burton geometry of order 2 and is thus a target, giving a contradiction.

Suppose that M is a strict PG-sum, so E is the disjoint union of two nonempty flats F_1 and F_2. One of these flats, say F_1, must contain T, but then $M|\operatorname{cl}(T \cup \{v\})$ is a anticlaw for each $v \in F_2$, a contradiction.

Finally, suppose that $M \cong M_1 \otimes M_2$ for matroids M_1, M_2 of smaller dimension than M. By minimality, both M_1 and M_2 are targets; by Lemma 3.7.11, it follows that M is a target, giving a contradiction. $\square$

Chapter 4

I_4-free and Triangle-free Matroids

This chapter is based on joint work with Peter Nelson [44].

4.1 Introduction

We now return to Conjecture 1.5.8, restated below.

Conjecture 4.1.1 ([46]). *For all $s \geq 1$, there exists k such that if $M = (E, G)$ has no induced I_s-restriction or triangle, then $\chi(M) \leq k$.*

The cases $s \leq 3$ are straightforward. Our main result in this chapter is a full structure theorem for I_4-free, triangle-free matroids. As a corollary of our exact structure theorem, it will follow that I_4-free, triangle-free matroids have χ at most 2, answering Conjecture 1.5.8 in the case $s = 4$.

We need a few definitions to state our result. Given a matroid $M = (E, G)$ and a hyperplane H, we say that *M is the 0-expansion of $M|H$* if $E \subseteq H$; this in particular implies that M is rank-deficient. We say that *M is the 1-expansion of $M|H$* if there exists $x \in G \backslash H$ and a hyperplane $H_0 \subseteq H$ for which $E \cap H \subseteq H \backslash H_0$ and $E = (E \cap H) \cup (x + [H_0])$. In the definition of 1-expansions, it is required $M|H$ have critical number at most 1; we will later verify that this means that M has critical number 1.

We write $\mathrm{AG}^{\text{\ding{73}}}(n-1, 2)$ for the $(n+1)$-dimensional matroid $M = (E, G)$ with nested hyperplanes $H_0 \subseteq G_0 \subseteq G$ and $x \in G \backslash G_0$, $y \in G_0 \backslash H_0$ such that $E = (G_0 \backslash H_0) \triangle \{x, y, x +$

$y\}$. We note that $\mathrm{AG}^{\star}(n-1,2)$ is simply the *series extension* of $\mathrm{AG}(n-1,2)$ in standard matroid terminology.

Our main result of this chapter is as follows.

Theorem 4.1.2. *A full-rank matroid $M = (E, G)$ is I_4-free and triangle-free if and only if either*

- *there exist nested flats $H_0 \subseteq \cdots \subseteq H_l$ with $H_l = G$ and $\dim(H_0) = 1$ so that $M|H_i$ is either the 0-expansion or 1-expansion of $M|H_{i-1}$ for $i = 1, \ldots, l$, or*
- *$M \cong D^k(\mathrm{AG}^{\star}(n-1,2))$ for some $n \geq 3$ and $k \geq 0$.*

The first outcome in Theorem 4.1.2 results in a class of matroids with critical number ≤ 1, whereas the last outcome consists of matroids with critical number 2. It is also worth noting that the last outcome is a rather restrictive class of matroids; up to isomorphism, the number of r-dimensional matroids in the latter case is $r-3$ for any given dimension $r \geq 4$. We remark that Theorem 4.1.2 can also be phrased as a constructive theorem using appropriately defined operations that correspond to the notions of 0-expansions and 1-expansions.

The proof of Theorem 4.1.2 falls in two parts, based on the existence of a C_5-restriction. By Lemma 2.2.3, an I_4-free, triangle-free matroid M is affine if and only if it does not contain a C_5-restriction. As we will see, having a C_5-restriction greatly restricts the structure of I_4-free, triangle-free matroids.

The case when M is affine is more difficult. In fact, we will first consider the class of *AI_4-free* matroids; we say that a matroid $M = (E, G)$ is AI_4-free if for any four elements $x_1, x_2, x_3, x_4 \in E$ for which $\mathrm{rank}(x_1, x_2, x_3, x_4) = 4$, there exists some $i \in \{1, 2, 3, 4\}$ such that $\sum_{j \neq i} x_j \in E$. It turns out that if $M = (E, G)$ is AI_4-free, then we can always find a special hyperplane H of G such that either E or $G \backslash E$ is contained in either H or $G \backslash H$. Once we understand this feature of AI_4-free matroids, it will be straightforward to derive the outcome corresponding to the affine matroids in Theorem 4.1.2.

One may hope for a similar result for I_5-free, triangle-free matroids, yet it seems much more difficult. It is still the case that the only obstruction to an I_5-free, triangle-free matroid being affine is the existence of a C_5-restriction. In a later chapter, we will consider an extremal problem for I_5-free, triangle-free matroids, in which we will uncover some potentially useful information about the structure of I_5-free, triangle-free matroids with a C_5-restriction.

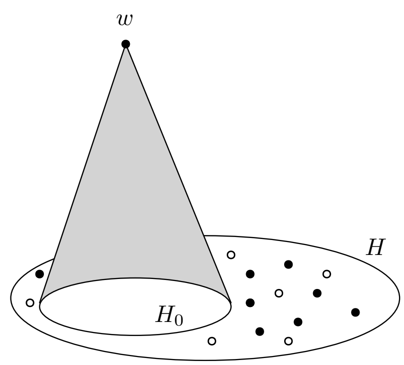

Figure 4.1: In this figure, $M = (E, G)$ is the 1-expansion of $M|H$ for hyperplane $H \subsetneq G$. This means that there exists a hyperplane H_0 of H and $w \in G\backslash H$ for which $E \cap H \subseteq H\backslash H_0$ and $E = (E \cap H) \cup (w + [H_0])$.

In order to prove Conjecture 1.5.8, it is likely that we will need a different approach. In a later chapter, we prove a weakening of Conjecture 1.5.8 using an additive combinatorial method which, we hope, will shed light on the types of techniques required for proving Conjecture 1.5.8.

4.2 Preliminaries

Expansions

Recall that, given a matroid $M = (E, G)$ and a hyperplane H, M is the 0-expansion of $M|H$ if $E \subseteq H$. The following is immediate.

Lemma 4.2.1. *If $M = (E, G)$ is the 0-expansion of $M|H$ for some hyperplane $H \subsetneq G$, then the following hold.*

- *If $M|H$ is I_t-free for $t \geq 1$, then M is I_t-free, and*
- *$\chi(M) = \chi(M|H)$.*

Recall that $M = (E, G)$ is the 1-expansion of $M|H$ if there exists $w \in G\backslash H$ and a hyperplane $H_0 \subseteq H$ for which $E \cap H \subseteq H\backslash H_0$ and $E = (E \cap H) \cup (w + [H_0])$. See Figure 4.1.

Lemma 4.2.2. *If $M = (E, G)$ is the 1-expansion of $M|H$ for some hyperplane $H \subseteq G$, then the following hold.*

- *If $M|H$ is I_t-free for $t \geq 4$, then M is I_t-free, and*
- $\chi(M) = 1$.

Proof. By definition there exists $w \in G \backslash H$ and a hyperplane $H_0 \subseteq H$ for which $E \cap H \subseteq H \backslash H_0$ and $E = (E \cap H) \cup (w + [H_0])$.

Suppose for a contradiction that $M|H$ is I_t-free but M is not, so that there exists $F = \{v_1, \cdots, v_t\} \subseteq E$ for which $M|\operatorname{cl}(F) \cong I_t$. Note that for any distinct three elements $x, y, z \in (w + [H_0])$, we have that $x + y + z \in (w + [H_0]) \subseteq E$, so $|(w + [H_0]) \cap F| \leq 2$. Hence $|H \cap F| \geq 2$. Let $x, y \in H \cap F$. Also, note that we may pick some $z \in (w + [H_0]) \cap F$; otherwise $\operatorname{cl}(F) \subseteq H$ but $M|\operatorname{cl}(F) \cong I_t$, contrary to $M|H$ being I_t-free. But then $x + y + z \in w + [H_0] \subseteq E$, a contradiction.

Note that $|M| > 0$, so $\chi(M) > 0$. On the other hand, pick any $x \in H \backslash H_0$ and let $H' = \operatorname{cl}(H_0 \cup \{w + x\})$. Then $H' \cap E = \varnothing$ and H' is a hyperplane. So $\chi(M) = 1$. $\square$

Series Extension of Affine Geometries

Let $n \geq 3$. Then recall that $\mathrm{AG}^{\star}(n-1, 2)$ is the $(n+1)$-dimensional matroid $M = (E, G)$ with nested hyperplanes $H_0 \subseteq G_0 \subseteq G$ and $x \in G \backslash G_0$, $y \in G_0 \backslash H_0$ for which $E = (G_0 \backslash H_0) \triangle \{x, y, x + y\}$. Note that when $n = 3$, then $\mathrm{AG}^{\star}(n-1, 2)$ is simply the matroid C_5.

In standard matroid terminology, $\{x, x+y\}$ is in fact a *series pair*, meaning that neither x nor $x + y$ is a coloop and any basis of E must include either x or $x + y$, and the matroid $\mathrm{AG}^{\star}(n-1, 2)$ is a *series extension* of $\mathrm{AG}(n-1, 2)$, meaning when we *contract* either the element x or $x + y$, the resulting matroid is in fact $\mathrm{AG}(n-1, 2)$. In the special case $n = 3$, i.e., when the matroid is C_5, then any two elements in E form a series pair. However, when $n \geq 4$, then $\{x, x + y\}$ is the unique series pair.

Note that $\mathrm{AG}^{\star}(n-1, 2)$ always contains an induced C_5-restriction. It is routine to check that $\mathrm{AG}^{\star}(n-1, 2)$ is I_t-free for any $t \geq 4$, triangle-free, and has critical number exactly 2.

Lemma 4.2.3. *Let $M = \mathrm{AG}^{\star}(n-1, 2)$ where $n \geq 3$. The following holds.*

- *M is I_t-free for $t \geq 4$,*
- *M is triangle-free and*
- *M has critical number 2.*

Proof. Write $M = (E, G)$ with nested hyperplanes $H_0 \subseteq G_0 \subseteq G$ and $x \in G \backslash G_0$, $y \in G_0 \backslash H_0$ for which $E = (G_0 \backslash H_0) \triangle \{x, y, x + y\}$. Note that, given three distinct elements $t_1, t_2, t_3 \in G_0 \backslash H_0$, $t_1 + t_2 + t_3 \in G_0 \backslash H_0$.

To show that M is I_t-free for every $t \geq 4$, for a contradiction let $F \subseteq G$ be a t-dimensional flat such that $M|F \cong I_t$. If $x, x + y \in F \cap E$, then we may take any two $t_1, t_2 \in G_0 \cap E \cap F$ so that $x + (x + y) + t_1 + t_2 = y + t_1 + t_2 \in E$, a contradiction. If only one of $x \in F \cap E$ or $x + y \in F \cap E$ holds, then without loss of generality suppose that $x \in F \cap E$, and pick any three $t_1, t_2, t_3 \in G_0 \cap E \cap F$; if $t_1 + t_2 + t_3 \in E$, then it is a contradiction, and if $t_1 + t_2 + t_3 \notin E$, then $t_1 + t_2 + t_3 = y$, and hence $x + t_1 + t_2 + t_3 = x + y$, contradicting the linear independence of $E \cap F$. If $x, x + y \notin E \cap F$, then pick any two $t_1, t_2 \in G_0 \cap E \cap F$; we may then pick $t_3 \in (G_0 \cap E \cap F) \backslash \{t_1 + t_2 + y\}$ so that $t_1 + t_2 + t_3 \neq y$ and hence $t_1 + t_2 + t_3 \in E$, again a contradiction. Thus we obtain contradictions in all cases and we conclude that M is I_t-free for every $t \geq 4$.

Let T be a two-dimensional flat of G so that $|T \cap E| = 3$. If $T \subseteq G_0$, then since H_0 is a hyperplane of G_0, $T \cap H_0 \neq \varnothing$; since $H_0 \cap E = \varnothing$, $|T \cap E| < 3$, a contradiction. So $T \not\subseteq G_0$, so that $|T \backslash G_0| = 2$. Since $E \backslash G_0 = \{x, x + y\}$, it follows that $x, x + y \in T$; but then $y \notin E$, so $|T \cap E| < 3$, again a contradiction. So M is triangle-free.

Finally, note that M contains an induced C_5-restriction (take $\{x, x+y, t_1, t_2, y+t_1+t_2\}$ for any two distinct $t_1, t_2 \in G_0 \cap E$), so that $\chi(M) > 1$. On the other hand, H_0 has dimension $\dim(M) - 2$ and $H_0 \cap E = \varnothing$, so $\chi(M) \leq 2$. Hence $\chi(M) = 2$. $\square$

AI_4-freeness

Given a matroid $M = (E, G)$, recall that M is AI_4*-free* if for any four elements $x_1, x_2, x_3, x_4 \in E$ for which $\text{rank}(x_1, x_2, x_3, x_4) = 4$, there exists some $i \in \{1, 2, 3, 4\}$ such that $\sum_{j \neq i} x_j \in E$.

We consider the class of AI_4-free matroids as a way to understand the affine I_4-free matroids. If we already know that a matroid M is affine, then the only information about I_4-freeness that is useful is how the sums of three elements in E behave, since the sums of even numbers of elements in E are guaranteed to be nonelements of E by the affineness. Namely, if M is affine, then M is I_4-free if and only if it is AI_4-free.

It is useful to note that AI_4-freeness is preserved under complementation.

Lemma 4.2.4. *A matroid $M = (E, G)$ is AI_4-free if and only if M^c is AI_4-free.*

Proof. Due to symmetry, it suffices to prove the backward direction. We prove its contrapositive. Suppose that M is not AI_4-free, so that there exists $B = \{x_1, x_2, x_3, x_4\} \subseteq E$, which is independent and such that $\sum_{j \neq i} x_j \notin E$ for all $1 \leq i \leq 4$. Then consider the independent set $B' = \{w_1, w_2, w_3, w_4\}$ of $G \backslash E$ where $w_i = \sum_{j \neq i} x_j$. Then $\sum_{j \neq i} w_j \in E$ for $1 \leq i \leq 4$. Hence M^c is not AI_4-free. $\square$

We will also need the following easy lemma.

Lemma 4.2.5. *If a matroid $M = (E, G)$ is AI_4-free and triangle-free, then M is affine.*

Proof. As M is AI_4-free, it contains no induced odd circuit of length 5 or more. Additionally M is triangle-free by assumption. Therefore M contains no induced odd circuit and is affine by Theorem 2.2.2. $\square$

Checking I_4-freeness

In this chapter, we will often claim that a given set $I = \{x_1, x_2, x_3, x_4\} \subseteq E$ is an induced I_4-restriction for a matroid $M = (E, G)$. As most of the matroids we study in this chapter are triangle-free, this amounts to checking that I is an independent set in $[G]$ and that the sums of three and four elements of I are nonelements of E. The following is immediate.

Lemma 4.2.6. *If $M = (E, G)$ is triangle-free and $x_1, x_2, x_3, x_4 \in E$ with $(\sum_{i=1}^{4} x_i + \{x_1, x_2, x_3, x_4, 0\}) \cap (E \cup \{0\}) = \varnothing$, then $M|\operatorname{cl}(x_1, x_2, x_3, x_4) \cong I_4$.*

While we will not quote this lemma explicitly, it will be implicit when we assert that a given set $I = \{x_1, x_2, x_3, x_4\}$ is an induced I_4-restriction.

4.3 The Non-affine Case

In this section, we will consider the non-affine I_4-free, triangle-free matroids. The goal will be to prove the following. For the purpose of showing that there exists a constant c such that the I_4-free, triangle-free matroids have χ at most c, this is in fact the only theorem we need.

Theorem 4.3.1. *If $M = (E, G)$ is an I_4-free, triangle-free matroid, then either M is affine, or $(E, \mathrm{cl}(E)) \cong D^k(\mathrm{AG}^{\star}(n-1, 2))$ for some $n \geq 3$ and $k \geq 0$.*

The proof is by induction on $\dim(M)$. We will first prove the following lemma, which describes the case when we can find a hyperplane H such that $M|H$ contains an induced $D(C_5)$-restriction. This condition turns out to be very strong, as it implies that the matroid M is a doubling.

Lemma 4.3.2. *Let $M = (E, G)$ be an I_4-free, triangle-free matroid with a flat F for which $M|F \cong C_5$. Let $w \in G\backslash(F \cup E)$. If $M|\,\mathrm{cl}(F \cup \{w\})$ is the doubling of $M|F$ by w, then M is the doubling by w.*

Proof. If $\dim(M) = 5$, then the result follows trivially, so suppose that $\dim(M) > 5$.

Suppose for a contradiction that M is not the doubling by w. This implies that there exists $z \in E\backslash\,\mathrm{cl}(F \cup \{w\})$ such that $w + z \notin E$. We will now show that the 6-dimensional matroid $M|\,\mathrm{cl}(F \cup \{w, z\})$ contains an induced I_4-restriction.

4.3.2.1. *Let $\{x_1, x_2, x_3\}$ be any three distinct elements of $F \cap E$. Then $x_1 + x_2 + x_3 + z \notin E$.*

Subproof: Suppose not, so that $x_1 + x_2 + x_3 + z \in E$; then $\{x_1 + x_2 + x_3 + z, w + x_1, w + x_2, w + x_3\}$ is an induced I_4-restriction. ■

We now fix a linearly independent set $\{x_1, x_2, x_3, x_4\}$ of $F \cap E$. Let $x_5 = x_1 + x_2 + x_3 + x_4$, so that $F \cap E = \{x_1, x_2, x_3, x_4, x_5\}$. We may apply the above claim with $\{x_1, x_2, x_3\}$, $\{x_1, x_4, x_5\}$, $\{x_2, x_4, x_5\}$ and $\{x_3, x_4, x_5\}$ to obtain that $x_1 + x_2 + x_3 + z, x_2 + x_3 + z, x_1 + x_3 + z, x_1 + x_2 + z \notin E$. Since $x_1 + x_2 + x_3 \notin E$, this implies that $\{x_1, x_2, x_3, z\}$ is an induced I_4-restriction, a contradiction. This completes the proof. □

The next lemma describes the situation in which there is a hyperplane H of G such that $M|H$ is $\mathrm{AG}^{\star}(n-1, 2)$. This case turns out to be harder and requires a detailed case analysis.

Lemma 4.3.3. *Let $M = (E, G)$ be an n-dimensional, I_4-free, triangle-free matroid, $n \geq 5$. If G has a hyperplane H so that $M|H \cong \mathrm{AG}^{\star}(n-3, 2)$, then either*

- *$E \subseteq H$,*
- *M is the doubling of $M|H$, or*

- $M \cong \mathrm{AG}^{\star}(n-2, 2)$.

Proof. Since $M|H \cong \mathrm{AG}^{\star}(n-3, 2)$, there exist a hyperplane F of H, a hyperplane F' of F, and elements $W = \{w_0, w_1\} \subseteq (H \cup E)\backslash F$ so that $M|H = (W \cup (y + F'), H)$ where $w_0 + w_1 = y$. We consider the following cases.

Case 1: There exists $z \in E\backslash H$ for which $y + z \in E$.

We will make a series of straightforward observations to help understand the structure of M.

4.3.3.1. $(z + F) \cap E = \{y + z\}$

Subproof: If $x \in F\backslash(F' \cup \{y\})$, then clearly $x + z \notin E$ as $\{x, z, x + z\}$ would be a triangle. If there exists $x \in F'$ such that $x + z \in E$, then $\{y + z, x + z, x + y\}$ is a triangle. ■

4.3.3.2. *For any triangle $T = \{x_0, x_1, x_0 + x_1\} \subseteq F'$ and $w \in W$, $|(z + w + \{x_0 + y, x_1 + y, x_0 + x_1\}) \cap E| > 0$.*

Subproof: If not, then $\{w, z, x_0 + y, x_1 + y\}$ is an induced I_4-restriction. ■

4.3.3.3. *For any distinct $x, x' \in F'$, $|\{x + z + w_0, x' + z + w_1\} \cap E| < 2$.*

Subproof: If not, then $\{x + z + w_0, x' + z + w_1, x + x' + y\}$ is a triangle. ■

When $\dim(F') > 2$ we obtain the following observation.

4.3.3.4. *Provided that $\dim(F') > 2$, if there exists $w \in W$ for which $x_0 + w + z \in E$ for some $x_0 \in F'$, then $F' + w + z \subseteq E$.*

Subproof: Let $x \in F'\backslash\{x_0\}$. Since $\dim(F') > 2$, we may select $x_1, x_2 \in F'$ such that $x_0 \notin \mathrm{cl}(\{x_1, x_2\})$ and $x = x_0 + x_1 + x_2$. Then the set $\{z, x_0 + w + z, x_1 + y, x_2 + y\}$ is an induced I_4-restriction if $x + w + z \notin E$. Therefore $F' + w + z \subseteq E$. ■

At this point, it is helpful to consider the cases when $\dim(F') > 2$ and $\dim(F') = 2$ separately.

Case 1.1: $\dim(F') > 2$.

We claim that M is the doubling of $M|H$. By 4.3.3.2, there exists $x \in F'$ and $w \in W$ for which $x + z + w \in E$, and by 4.3.3.4, $w + z + F' \subseteq E$. By 4.3.3.3, $(F' + (y + w) + z) \cap E = \varnothing$.

Hence, along with 4.3.3.1, we have that $E\backslash H = \{z, y+z\} \cup (w+z+F')$. Recall that $E \cap H = \{w, w+y\} \cup (y+F')$. Therefore we have

$$\begin{aligned} E &= (E \cap H) \cup (E\backslash H) \\ &= \{w, w+y\} \cup (y+F') \cup \{z, y+z\} \cup (w+z+F') \\ &= [y+w+z] + (\{w, w+y\} \cup (y+F')) = [y+w+z] + (E \cap H). \end{aligned}$$

Since $y+w+z \notin E$, we conclude that M is the doubling of $M|H$.

Case 1.2: $\dim(F') = 2$.

By 4.3.3.2, there exists $x \in F'$ such that $x+z+w \in E$ for some $w \in W$. Write $F' = \mathrm{cl}(x, x')$ for $x' \in F'$. Note that $x+z+w \in E$ implies by 4.3.3.3 that $(z+y+w+(F'\backslash\{x\})) \cap E = \varnothing$.

If $x+z+y+w \in E$, then 4.3.3.3 gives that $(z+w+(F'\backslash\{x\})) \cap E = \varnothing$. Hence $E\backslash H = \{z\} \cup (z+\{y, x+w, x+y+w\})$. Therefore

$$\begin{aligned} E &= (E \cap H) \cup (E\backslash H) \\ &= \{w, y+w\} \cup (y+\{x, x', x+x'\}) \cup \{z\} \cup (z+\{y, x+w, x+y+w\}) \\ &= \{y+x', y+x+x'\} \cup (x+\mathrm{cl}(y, w+z, x+z)). \end{aligned}$$

Since $F'' = \mathrm{cl}(y, w+z, x+z)$ satisfies $F'' \cap E = \varnothing$, it follows that $M \cong \mathrm{AG}^{\text{\ding{73}}}(n-2, 2)$

We may therefore assume that $x+z+y+w \notin E$. But then repeated application of 4.3.3.3 implies that $x+w+F' \subseteq E$. The analysis is then identical to Case 1.1. This concludes Case 1.

Case 2: There exist no $z \in G\backslash H$ for which $\{z, y+z\} \subseteq E$.

Suppose that one of the conclusions, $E \subseteq H$, does not hold. We may select $z \in E\backslash H$.

Again, we collect a series of straightforward facts to help understand the structure of M.

4.3.3.5. *For any triangle $T \subseteq F'$, $(T+z) \cap E \neq \varnothing$.*

Subproof: If not, then $(y+T) \cup \{z\}$ is an induced I_4-restriction. ■

4.3.3.6. *For any $x \in F'$, $x+z \notin E$ if and only if $(x+z+W) \cap E \neq \varnothing$. Moreover, in this case we have $|(x+z+W) \cap E| = 1$.*

Subproof: To show the forward statement, if $(x+z+W)\cap E=\varnothing$, $\{x+y,w_0,w_1,z\}$ is an induced I_4-restriction.

For the reverse direction, note that if $x+z\in E$, then $\{x+z,x+w+z,w\}$ would be a triangle for some $w\in W$.

Finally, since there exists no $t\in G\backslash H$ for which $\{t,y+t\}\subseteq E$, it follows that $(x+z+W)\cap E\neq\varnothing$ if and only if $|(x+z+W)\cap E|=1$. ■

4.3.3.7. *For any distinct $x,x'\in F'$, $|\{x+w_0+z,x'+w_1+z\}\cap E|<2$.*

Subproof: If not, then $\{x+w_0+z,x'+w_1+z,x+x'+y\}$ is a triangle. ■

Let us write $X_0=\{x\in F'\mid x+w_0+z\in E\}$, $X_1=\{x\in F'\mid x+w_1+z\in E\}$ and $X_2=\{x\in F'\mid x+z\in E\}$. By 4.3.3.5, $X_2\neq\varnothing$, and 4.3.3.6 implies that (X_0,X_1,X_2) partitions F'. Moreover, by 4.3.3.7, if $|X_i|>0$ for some $i\in\{0,1\}$, then $|X_{1-i}|=0$. Moreover, when we restrict to a triangle, we have the following.

4.3.3.8. *If $T\subseteq F'$ is a triangle, then*

1. *$|T\cap X_2|=3$, or*
2. *$|T\cap X_2|=1$ and $|T\cap X_i|=2$ for some $i\in\{0,1\}$.*

Subproof: By 4.3.3.5, $|T\cap X_2|>0$. Let $x\in T\cap X_2$. By 4.3.3.6, $(x+z+W)\cap E=\varnothing$.

Suppose that (1) does not hold, so that there exists $x'\in T$ such that $x'+z\notin E$. By 4.3.3.6, we know that there exists exactly one $w\in W$ for which $x'+z+w\in E$.

If $x+x'+z\in E$ holds, then by 4.3.3.6, $(x+x'+z+W)\cap E=\varnothing$. But then, it follows that $\{x+y,x'+y,x+x'+y,x'+w+z\}$ is an induced I_4-restriction. Hence $x+x'+z\notin E$.

By 4.3.3.6 and 4.3.3.7, it follows that $x+x'+z+w\in E$ and $x+x'+z+y+w\notin E$, which is (2). ■

The above claim will be enough to settle the case where $\dim(F')=2$. To handle the case $\dim(F')>2$, the following observation will be useful.

4.3.3.9. *For either $i\in\{0,1\}$, there exist no 3-dimensional flat $F_0\subseteq F'$ and a triangle $T\subseteq F_0$ for which $T\subseteq X_2$ and $F_0\backslash T\subseteq X_i$.*

Subproof: Assume for a contradiction that such a triangle $T = \mathrm{cl}(x_1, x_2)$ and a flat $F_0 = \mathrm{cl}(x_0, x_1, x_2)$ exist, $x_0 \notin T$. But then $\{x_0 + y, x_1 + y, z, x_0 + x_2 + w_i + z\}$ is an induced I_4-restriction. ■

Case 2.1: $\dim(F') = 2$.

We will apply 4.3.3.8 to give a case analysis depending on the value of $|F' \cap X_2|$.

If $|F' \cap X_2| = 3$, then we have that

$$E = W \cup (y + \mathrm{cl}(F' \cup \{y + z\})),$$

so that $M \cong \mathrm{AG}^{\star}(n-2, 2)$.

If $|F' \cap X_2| = 1$ and $|F' \cap X_i| = 2$ for $i \in \{0, 1\}$, let $x' \in F' \backslash \{x\}$, and $F'' = \mathrm{cl}(\{x, x' + w_i, y + w_i + z\})$. We then have

$$E = \{w_i, x + y\} \cup ((x + y + w_i) + F''),$$

where $F'' \cap E = \varnothing$. Hence $M \cong \mathrm{AG}^{\star}(n-2, 2)$.

Case 2.2: $\dim(F') > 2$.

We claim that $F' \subseteq X_2$.

We know from 4.3.3.8 that $X_2 \neq \varnothing$. Fix $v \in X_2$.

Suppose towards a contradiction that there exists $v' \in F' \backslash X_2$, so that $v' \in X_i$ for some $i \in \{0, 1\}$

By 4.3.3.8 it follows that $v + v' \in X_i$. Since $\dim(F') > 2$, there exists $v'' \notin \mathrm{cl}(\{v, v'\})$. If $v'' \in X_i$, then 4.3.3.8 applied to triangles implies that $v + v' + v'' \in X_2$, $v' + v'' \in X_2$ and $v + v'' \in X_i$, but this contradicts 4.3.3.9. Similarly, if $v'' \in X_2$, then applying 4.3.3.8 again implies that $v + v' + v'' \in X_i$, $v' + v'' \in X_i$, and $v + v'' \in X_2$. But then this contradicts 4.3.3.9. This shows that $F' \subseteq X_2$.

Since $F' \subseteq X_2$, we have $E \backslash H = (z + [F'])$, so that

$$\begin{aligned} E &= (E \cap H) \cup (E \backslash H) \\ &= W \cup (y + F') \cup (z + [F']) \\ &= W \cup (y + F''), \end{aligned}$$

where $F'' = \mathrm{cl}(\{y + z\} \cup F')$ and $F'' \cap E = \varnothing$. Therefore $M \cong \mathrm{AG}^{\star}(n-2, 2)$. □

We can now prove the main theorem of this section, restated below.

Theorem 4.3.4. *If $M = (E, G)$ is an I_4-free, triangle-free matroid, then either M is affine, or $(E, \mathrm{cl}(E)) \cong D^k(\mathrm{AG}^{\star}(n-1, 2))$ for some $n \geq 3$ and $k \geq 0$.*

Proof. We proceed by induction on $\dim(M)$. The cases where $\dim(M) = 1, 2, 3$ are routine to check, so we may assume that $\dim(M) \geq 4$. We may assume without loss of generality that M is full-rank.

Suppose that M is not affine, so that it contains an induced C_5-restriction. If $\dim(M) = 4$, then $M \cong C_5 = \mathrm{AG}^{\star}(2, 2)$, so suppose that $\dim(M) > 4$. By extending a basis of this C_5-restriction, we can select a hyperplane H of G such that $M|H$ contains a C_5-restriction, and $M|H$ is full-rank. By the inductive hypothesis, we have that $M|H \cong D^k(\mathrm{AG}^{\star}(n-1, 2))$ for some $n \geq 3$ and $k \geq 0$.

If $k \geq 1$, then $M|H$ contains an induced $D(C_5)$-restriction. By Lemma 4.3.2, we have that M is the doubling, say by w, of $M|H'$ for some hyperplane H'. Note that $M|H'$ is not affine; otherwise M is affine by Lemma 2.5.1. Hence we may apply the inductive hypothesis to $M|H'$ to obtain the required result in this case.

If $k = 0$ then Lemma 4.3.3 gives the required result. □

As an immediate corollary, this shows that the I_4-free, triangle-free matroids have critical number at most 2.

Corollary 4.3.5. *If M is I_4-free and triangle-free, then $\chi(M) \leq 2$.*

Proof. By Theorem 4.3.1, it follows that $\chi(M) = 1$, or $(E, \mathrm{cl}(E)) \cong D^k(\mathrm{AG}^{\star}(n-1, 2))$ where $n \geq 3$ and $k \geq 0$. Note that $\mathrm{AG}^{\star}(n-1, 2)$ has critical number 2 for $n \geq 3$. Since the D operation preserves critical number, it follows in the latter case that $\chi(M) = 2$. □

4.4 AI_4-freeness

In order to understand the structure of I_4-free, triangle-free matroids with critical number exactly 1, it is helpful to consider the notion of AI_4-freeness and our goal of this section is to prove the following.

Lemma 4.4.1. *If $M = (E, G)$ is AI_4-free, then there exists a hyperplane H of G such that either E or $G \backslash E$ is contained in either H or $G \backslash H$; that is, either*

- $E \subseteq H$ *(i.e., M is rank-deficient),*
- $G - E \subseteq H$ *(i.e., M^c is rank-deficient),*
- $E \cap H = \varnothing$*, or*
- $H \subseteq E$*.*

We first state an important lemma used repeatedly in this section. We remind the reader that for a given flat U, U itself is not considered a 'coset' of U, whereas it is considered a 'translate' of itself.

Lemma 4.4.2. *For every matroid $M = (E, G)$, there is a flat U of G for which $U^c = E + E^c$ and E is a union of translates of U. Moreover, if $|E| \geq 2$ then $U \subseteq E + E$, and if $|E| \leq |G| - 2$ then $U \subseteq E^c + E^c$.*

Proof. For each set $A \subseteq [G]$, let $S_A = \{s \in [G] : s + A = A\}$; this is a subspace of $[G]$, so has cardinality a power of 2, and since A is a disjoint union of translates of S_A the ratio $|A|/|S_A|$ is an integer. Thus, if $|A|$ is odd, the subspace S_A is trivial.

Since $|G|$ is odd, by switching to E^c if necessary, we may assume that $|E|$ is even; thus $|E^c|$ is odd, so S_{E^c} is trivial. We show that the flat $U = S_E \backslash \{0\}$ (i.e., U is the stabiliser of E) satisfies the lemma.

First, let $e + f \in E + E^c$, where $e \in E$ and $f \in E^c$. If $e + f \in [U]$, then $f = (e + f) + e \in (e + f) + E = E$, a contradiction. Therefore $E + E^c \subseteq G \backslash [U] = U^c$.

Now, let $a \in U^c$. If $a \in E$, then since S_{E^c} is trivial, we have $a + E^c \neq E^c$, so there is some $f \in E^c$ such that $a + f \in G \backslash E^c$; since $a \neq f$ we have $a + f \neq 0$ and so $a + f \in E$. Therefore $a \in E + f \subseteq E + E^c$. If $a \in E^c$, then since $a \notin U$, we have $a + E \neq E$ and so there is some $e' \in E$ for which $a + e' \notin E$, Since $a \neq e'$ we have $a + e' \neq 0$ and so $a + e' \in E^c$; this gives $a \in E + E^c$.

The last two arguments give $U^c = E + E^c$ as required. Since $U = S_E \backslash \{0\}$, E is a union of some cosets of U. By switching back to E if necessary (if we replaced E by E^c), we conclude that E is a union of some translates of U.

If $|E| \geq 2$, then either there exists a coset A of U contained in E, in which case $U \subseteq A + A \subseteq E + E$, and if no coset of U is contained in E, then $U = E$, in which case $U \subseteq U + U \subseteq E + E$ as U has dimension at least 2. Similarly, if $|E| \leq |G| - 2$ then $U \subseteq E^c + E^c$. □

We also use the following easy-to-verify lemma.

Lemma 4.4.3. *Let U_1, U_2, U_3 be flats of a projective geometry G such that for any distinct $i, j, k \in \{1, 2, 3\}$, $U_i \subseteq U_j \cup U_k$. Then there exist two distinct $i, j \in \{1, 2, 3\}$ for which $U_i = U_j$, and $U_k \subseteq U_i$ where $k \in \{1, 2, 3\} \backslash \{i, j\}$.*

Proof. Without loss of generality, suppose that $|U_1| \leq |U_2| \leq |U_3|$. Note that it suffices to show that $U_2 = U_3$, as it would imply $U_1 \subseteq U_2 \cup U_3 = U_2$.

We first argue that $|U_2| = |U_3|$. Write $|U_3| = 2^m - 1$ for some $m \geq 1$. If $|U_2| < |U_3|$, then since $U_3 \subseteq U_1 \cup U_2$, we have $2^m - 1 = |U_3| \leq |U_1| + |U_2| \leq 2(2^{m-1} - 1) = 2^m - 2$, a contradiction. Hence $|U_2| = |U_3|$. If $U_2 \neq U_3$, then there exist $x \in U_2 \backslash U_3$ and $y \in U_3 \backslash U_2$. Because $U_2 \subseteq U_1 \cup U_3$, $x \in U_1$, and similarly $y \in U_1$. This implies $x + y \in U_1 \subseteq U_2 \cup U_3$. Without loss of generality, suppose that $x + y \in U_2$. But then $y = (x + y) + x \in U_2$, a contradiction. $\square$

The proof of Lemma 4.4.1 will follow from the following lemmas.

Lemma 4.4.4. *Let $M = (E, G)$ be an AI_4-free matroid of dimension at least 5, and H be a hyperplane of G such that $|E \backslash H| \leq 1$. Then G has a hyperplane H' such that either $E \subseteq H'$, $E \cap H' = \varnothing$, or $H' \subseteq E$.*

Proof. We may assume that M is full-rank and that $|E \backslash H| = 1$, as otherwise the result is trivial. Let $\{v\} = E \backslash H$. Note that $M|H$ must be full-rank, as otherwise, $r(E) \leq r(E \cap H) + 1 \leq \dim(G) - 1$, so M is not full-rank. Also, observe that we have $\sum_{x \in J} x \in E$ for each three-element linearly independent subset J of $E \cap H$, since otherwise $J \cup \{v\}$ would violate AI_4-freeness. In particular, the matroid $M|H$ is I_3-free.

Hence, if $M|H$ is triangle-free, it follows from Lemma 1.5.6 that $M|H$ is an affine geometry, so that $E \cap H = H \backslash K$ where K is a hyperplane of H. Then for any $x \in H \backslash K$, we have that $E \cap H' = \varnothing$ where $H' = \text{cl}(K \cup \{v + x\})$ is a hyperplane of G, giving us the required result. Therefore, we may assume that $M|H$ contains a triangle. For any triangle $T \subseteq E \cap H$ and any $x \in (E \cap H) \backslash T$ it follows from the above observation that $T + \{x\} \subseteq E$, as each element of $T + \{x\}$ is the sum of three linearly independent elements of $E \cap H$. Now, we claim that $H \subseteq E$. If not, then pick a largest flat F of H for which $F \subseteq E$; since $M|H$ contains a triangle, $\dim(F) \geq 2$. We may also pick $x \in (H \cap E) \backslash F$ since $M|H$ is full-rank. Since every element $y \in F$ is contained in a triangle $T \subseteq F$, it follows that $y + x \in T + x \subseteq E$. Hence $\text{cl}(F \cup \{x\}) \subseteq E \cap H$, contradicting maximality of F. Hence it follows that $H \subseteq E$. $\square$

Lemma 4.4.5. *Let $M = (E, G)$ be an AI_4-free matroid of dimension at least 5, let H be a hyperplane of G, and let F be a hyperplane of H such that $E \cap H \subseteq F$. Then there exists a hyperplane H' of G such that either E or $G\backslash E$ is contained in either H' or $G\backslash H'$.*

Proof. Suppose that no such H' exists. In particular, M is full-rank (since otherwise any hyperplane H' containing $\text{cl}(E)$ satisfies $E \subset H'$), and $|E \cap F| \geq 1$ (since otherwise $E \subseteq G\backslash H$). We will first prove the lemma in the case where $|E \cap F| > 1$.

Let $A_0 = H\backslash F$, and let A_1, A_2 denote the two remaining cosets of F in G. By assumption, we have $A_0 \cap E = \varnothing$.

Case 1: $|E \cap F| > 1$

4.4.5.1. *Let $v_1 \in E \cap A_1$ and $v_2 \in E \cap A_2$. If u, u' are distinct elements of F with $u' \in E$, then $|\{u + v_1, u + v_2, u + u'\} \cap E| \neq 1$.*

Subproof: Note that $E \cap (F + v_1 + v_2) = E \cap A_0 = \varnothing$. If $u + u' \in E$ while $u + v_1, u + v_2 \notin E$, then $\{v_1, v_2, u', u + u'\}$ would violate AI_4-freeness. If $u + v_i \in E$ for some $i \in \{1, 2\}$ while $u + u', u + v_{3-i} \notin E$, then $\{u', v_1, v_2, v_i + u\}$ would violate AI_4-freeness. ■

We may assume that $|A_1 \cap E| > 1$; if not, we can take a hyperplane $H' = \text{cl}(F \cup A_2)$, so that $|E\backslash H'| \leq 1$, and Lemma 4.4.4 gives a contradiction. Similarly, we may assume $|A_2 \cap E| > 1$. If $A_i \cap E \neq A_i$, select $v_i \in A_i\backslash E$, and otherwise, choose $v_i \in A_i \cap E = A_i$, for $i = 1, 2$. Note that at most one of $v_1 \in E$ and $v_2 \in E$ holds; otherwise, $A_1, A_2 \subseteq E$, so that choosing $H' = F \cup A_0$ gives $G\backslash E \subseteq H'$.

Let $X_i = (v_i + (A_i \cap E))\backslash\{0\}$ for $i = 1, 2$. Note that $X_i \subseteq F$ and $|X_i| > 1$ for each $i = 1, 2$ by our choice of v_i. Write $X_0 = F \cap E$. Note that $|X_0| > 1$.

Now, 4.4.5.1 implies

$$(X_0 + X_0) \cap (X_1 + X_1^c) \cap (X_2 + X_2^c) = \varnothing$$
$$(X_0 + X_0^c) \cap (X_1 + X_1) \cap (X_2 + X_2^c) = \varnothing$$
$$(X_0 + X_0^c) \cap (X_1 + X_1^c) \cap (X_2 + X_2) = \varnothing$$

We may now apply Lemma 4.4.2 (with F being the ambient space), to obtain flats U_i for which $X_i + X_i^c = U_i^c$, and X_i is a union of translates of U_i for each $i = 0, 1, 2$ provided U_i is not empty. Moreover, since $|X_i| > 1$, we have $U_i \subseteq X_i + X_i$ for $i = 0, 1, 2$.

Hence, for any distinct $i, j, k \in \{1, 2, 3\}$, we have

$$(X_i + X_i) \subseteq U_j \cup U_k.$$

Recall that $U_i \subseteq X_i + X_i$ for $i = 0, 1, 2$. Therefore, for any distinct $i, j, k \in \{1, 2, 3\}$ we obtain

$$U_i \subseteq U_j \cup U_k.$$

Note that each of U_1, U_2, U_3 is a flat. By Lemma 4.4.3, this implies that at least two of U_0, U_1, U_2 are identical and the third is contained in the other two. Let us write $U_i \subseteq U_j = U_k$ for some $i, j, k \in \{0, 1, 2\}$. Let $U = U_j = U_k$.

Since $U \subseteq X_j + X_j$ and $X_j + X_j \subseteq U_i \cup U = U$, it follows that $U = X_j + X_j$, and similarly $U = X_k + X_k$. We also have $X_i + X_i \subseteq U$. In particular, since $|X_j| > 1$, U is non-empty. Since $U = X_j + X_j$ and X_j is a union of translates of U, it follows that X_j, and similarly X_k, equal a translate of U in F. In a similar vein, since $X_i + X_i \subseteq U$, X_i is contained in a translate of U in F. To summarise, we have the following.

4.4.5.2. *There exists a non-empty flat $U \subseteq F$ and $j, k \in \{0, 1, 2\}$, $j \neq k$, such that*

- *X_j and X_k equal a translate of U.*
- *X_i is contained in a translate of U.*

where $i \in \{0, 1, 2\} \backslash \{j, k\}$.

We say that a set X is full if it equals a translate of U. Hence, at least two of X_0, X_1, X_2 are full.

We now consider two cases depending on whether $v_1 \in E$ and $v_2 \in E$ (recall that at most one of the two can hold at the same time). In the case $v_1 \in E$ or $v_2 \in E$, we may assume, by symmetry, that $v_2 \in E$ and $v_1 \notin E$.

Case 1.1: $v_1, v_2 \notin E$

4.4.5.3. *At most two of X_0, X_1, X_2 are contained in U. Moreover, if precisely two of X_0, X_1, X_2 are contained in U, then $X_0 \subseteq U$.*

Subproof: Suppose first for a contradiction that $X_i \subseteq U$ for each $i = 0, 1, 2$. If X_1, X_2 are both full, then since $|X_0| > 1$, we may select two elements $y_1, y_2 \in X_0$, then $\{y_1, y_2, v_1 + y_1 + y_2, v_2 + y_1 + y_2\}$ would violate AI_4-freeness. If X_1 is not full, then we may select $x_1 \in X_1 \subseteq U$, and $y_1 \in U \backslash X_1$, so that $\{x_1, y_1, x_1 + v_1, x_1 + y_1 + v_2\}$ would violate AI_4-freeness. Similarly, if X_2 is not full, a symmetrical argument shows that it would violate AI_4-freeness, giving a contradiction.

Suppose next that precisely two of X_0, X_1, X_2 are contained in U. For a contradiction, suppose that X_0 is not contained in U, so that $X_1, X_2 \subseteq U$, and X_0 is contained in a coset of U. If X_1, X_2 are full, then select two elements $y_1, y_2 \in X_0$, then $\{y_1, y_2, v_1 + y_1 + y_2, v_2 + y_1 + y_2\}$ would violate AI_4-freeness. If X_1 is not full, then select $x_1 \in X_1$, $y_1 \in U\backslash X_1$ and $z_1 \in X_0$. Then $\{x_1 + v_1, x_1 + y_1 + v_2, z_1, x_1 + y_1 + z_1\}$ would violate AI_4-freeness. By symmetry, the case where X_2 is not full follows, giving a contradiction in all cases. ■

We are now ready to complete the analysis of Case 1.1. In each of the possible outcomes resulting from 4.4.5.3, we will show that we can select a hyperplane H' of G that satisfies the theorem or find an induced I_4-restriction, giving a contradiction.

Suppose first that none of X_0, X_1, X_2 is contained in U. Let $X_i \subseteq B_i$ where B_i is a coset of U for $i = 0, 1, 2$. If $B_0 = B_1 = B_2$, then we may assume that there is no other coset of U, as otherwise M is rank-deficient. But then, we may select $H' = \mathrm{cl}(U \cup \{v_1, v_2\})$, and we have $E \subseteq G\backslash H'$. Therefore we may assume that B_0, B_1, B_2 are not identical, so we may assume without loss of generality that $B_0 \neq B_1$. But then $B_1 \cap \mathrm{cl}(E) = \varnothing$ (to see this, note that a general element z of $\mathrm{cl}(E)$ has the form $v_1 + v_2 + x_0 + x_1 + x_2 + y$ where $x_0 \in B_0 \cup \{0\}$, $x_1 \in B_1 \cup \{v_1\}$, $x_2 \in B_2 \cup \{v_2\}$, $y \in U \cup \{0\}$, and hence $z \notin B_1$ as otherwise it would force $x_1 = v_1$, $x_2 = v_2$, giving $z \in U \cup B_0$). Therefore M is rank-deficient.

Suppose next that precisely one of X_0, X_1, X_2 is contained in U. First, suppose that $X_0 \subseteq U$. Then, it follows in a similar way that $B_1 \cap \mathrm{cl}(E) = \varnothing$. Therefore, M is rank-deficient. Hence we may assume without loss of generality that $X_1 \subseteq U$, and $X_0 \subseteq B_0$ and $X_2 \subseteq B_2$ for (possibly identical) cosets B_0, B_2 of U. Note that we may assume that the only cosets are $B_0, B_2, B_0 + B_2$, as otherwise M is rank-deficient. Select $x \in B_0$, and let $H' = \mathrm{cl}(U \cup (B_0 + B_2) \cup \{v_1 + x, v_2\})$. Then we have that $E \subseteq G\backslash H'$.

Finally, we consider the case where precisely two of X_0, X_1, X_2 are contained in U. Suppose without loss of generality that $X_0, X_1 \subseteq U$ and $X_2 \subseteq B_2$ where B_2 is a coset of U. But then it follows that $B_2 \cap \mathrm{cl}(E) = \varnothing$. So M is rank-deficient.

Case 1.2: $v_1 \notin E$**,** $v_2 \in E$.

The fact that $v_2 \in E$ means that $A_2 \subseteq E$ by our choice of v_2. Hence $X_2 = F$, and therefore $U_2 = F$. Hence, either $U_0 = F$ or $U_1 = F$. If $U_0 = F$, then choosing the hyperplane $H' = \mathrm{cl}(F \cup A_2)$, we have that $G\backslash E \subseteq G\backslash H'$. Hence we may assume that $U_1 = F$. We may also assume that $F \cap E \neq E$, as otherwise $H' = \mathrm{cl}(F \cup A_2)$ satisfies $G\backslash E \subseteq G\backslash H'$ again. Let $w_1 \in F\backslash E$, $w_2 \in F \cap E$. Then $\{v_1 + w_1 + w_2, w_2, v_2, v_2 + w_1 + w_2\}$ would violate AI_4-freeness.

Case 2: $|E \cap F| = 1$

Choose F'' to be a hyperplane of F such that $F'' \cap E = \varnothing$. Let $F' = \mathrm{cl}(F'' \cup \{z\})$ for any $z \in A_0$ so that that $F' \cap E = \varnothing$, and consider the three cosets of F' in G, denoted A_0', A_1', A_2' where we take A_0' so that $|A_0' \cap E| = 1$. Let $v \in A_0' \cap E$.

If $|A_1' \cap E| > 1$ and $M|(A_1' \cup F')$ is not full-rank, then we may select a hyperplane H' of $F' \cup A_1'$ such that $E \cap (F' \cup A_1') \subseteq H'$, and $|E \cap H'| > 1$. Case 1 then applies, and the same holds with A_2'. So we may assume without loss of generality that $|A_i' \cap E| = 1$ or $M|(A_i' \cup F')$ is full-rank for each $i = 1, 2$.

If $|A_i' \cap E| = 1$ for $i = 1, 2$, then because $\dim(M) \geq 5$, M is rank-deficient, so we may assume without loss of generality that $M|(A_1' \cup F')$ is full-rank. Given three linearly independent vectors $v_1, v_2, v_3 \in A_1' \cap E$, we must have that $v_1 + v_2 + v_3 \in E$, as otherwise $\{v, v_1, v_2, v_3\}$ would violate AI_4-freeness. Therefore, $A_1' \subseteq E$, and let $H' = A_1' \cup F'$. It is then easy to check that the conditions are met to apply Case 1 with the matroid M^c and the hyperplane H' to give the required result. $\square$

We can now prove Lemma 4.4.1.

Proof of Lemma 4.4.1. Let $M = (E, G)$ be a counterexample of smallest dimension. If $\dim(M) = 1, 2, 3$, then we obtain a contradiction from a routine check, hence we may assume $\dim(M) \geq 4$.

4.4.5.4. $\dim(M) \neq 4$.

Proof. This is a tedious check. If $\dim(M) = 4$, then replacing M with M^c if necessary, we may assume that $|E| \leq 7$. We may also assume that M is full-rank. Note that M needs to contain a C_4-restriction, on a hyperplane H, since M is AI_4-free.

Suppose first that it is an induced C_4-restriction. Then $M|H \cong C_4$ and write $E \cap H = \{v_1, v_2, v_3, v_1 + v_2 + v_3\}$. Let $v_4 \in E \backslash H$. Then there exists $v \in E \cap H$ such that $v + v_4 \in E$, as otherwise $M^c|\operatorname{cl}(\{v_1 + v_2, v_1 + v_3, v_1 + v_2 + v_3 + v_4\})$ is an F_7-restriction. Without loss of generality, suppose that $v_1 + v_4 \in E$. Now, we must have that $v_2 + v_3 + v_4 \in E$ or $v_1 + v_2 + v_3 + v_4 \in E$, as otherwise, $\{v_2, v_3, v_4, v_1 + v_4\}$ would violate AI_4-freeness. If $v_2 + v_3 + v_4 \in E$, then $\{v_1, v_2, v_1 + v_4, v_2 + v_3 + v_4\}$ would violate AI_4-freeness. If $v_1 + v_2 + v_3 + v_4 \in E$, then $\{v_1, v_2, v_4, v_1 + v_2 + v_3 + v_4\}$ would violate AI_4-freeness.

Hence we may assume that it has no induced C_4-restriction. Suppose that $|H \cap E| = 6$. Recall that M is full-rank and $|E| \leq 7$. We may take $v_4 \in E \backslash H$, and $v_1, v_2, v_3 \in E \cap H$ for which $v_1 + v_2 + v_3 \notin E$, and $\{v_1, v_2, v_3, v_4\}$ would violate AI_4-freeness. Hence we may assume $|E \cap H| = 5$. Let $E \cap H = \{v_1, v_2, v_3, v_1 + v_2, v_1 + v_2 + v_3\}$, and pick $v_4 \in E \backslash H$. By symmetry and the fact that $|E| \leq 7$, we may assume that $v_1 + v_4 \notin E$. We may also assume that $v_2 + v_3 + v_4 \notin E$; otherwise $E \backslash H = \{v_4, v_2 + v_3 + v_4\}$ and so $\{v_1, v_1 + v_2, v_3, v_4\}$ would violate AI_4-freeness. Now it follows that $v_1 + v_2 + v_3 + v_4 \in E$, as otherwise $\{v_2, v_3, v_1 + v_2, v_4\}$ would violate AI_4-freeness. But then $\{v_1, v_2, v_4, v_1 + v_2 + v_3 + v_4\}$ violates AI_4-freeness. $\square$

Hence we have that $\dim(M) \geq 5$. Let $k = \dim(M)$. By minimality, for every hyperplane H of G, H contains a hyperplane that satisfies one of the four outcomes. If, for any hyperplane H of G, the conditions of Lemma 4.4.5 are satisfied, then Lemma 4.4.5 provides a contradiction. Hence, we may assume that, for every hyperplane H of G, either $M|H$ or $M^c|H$ contains a $\mathrm{PG}(k-3,2)$-restriction.

Moreover, since $\dim(M) = k \geq 5$, if M contains a $\mathrm{PG}(k-3,2)$-restriction, then M^c cannot contain a $\mathrm{PG}(k-3,2)$-restriction, as otherwise we would have $\dim(M) \geq 2(k-2)$, which implies $k \leq 4$. By switching to M^c if necessary, we may therefore suppose that M^c contains a $\mathrm{PG}(k-3,2)$-restriction in every hyperplane. Now, observe that M is triangle-free, since otherwise any restriction to a hyperplane containing such a triangle cannot contain a $\mathrm{PG}(k-3,2)$-restriction in M^c. Therefore, M is both AI_4-free and triangle-free, so by Lemma 4.2.5, it follows that M is affine. □

4.5 The Main Theorem

We are now ready to prove the main structure theorem for I_4-free, triangle-free matroids, restated below.

Theorem 4.5.1. *For a full-rank matroid $M = (E, G)$, M is I_4-free and triangle-free if and only if*

- *there exist nested flats $H_0 \subseteq \cdots \subseteq H_l$ with $H_l = G$ and $\dim(H_0) = 1$ so that $M|H_i$ is either the 0-expansion or 1-expansion of $M|H_{i-1}$ for $i = 1, \ldots, l$, or*
- *$M \cong D^k(\mathrm{AG}^{\star}(n-1,2))$, for some $n \geq 3$ and $k \geq 0$.*

Proof. The backward direction follows from Lemmas 4.2.3, 4.2.2 and 4.2.1.

We now prove the forward direction. By Theorem 4.3.1, $M \cong D^k(\mathrm{AG}^{\star}(n-1,2))$ for some $n \geq 3$ and $k \geq 0$, or M is affine. If the former case holds then we are done, so we may suppose that M is affine, so that there exists a hyperplane H of G for which $E \subseteq G \backslash H$.

If M is the empty matroid, then the result is trivially true, so suppose that $E \neq \varnothing$. Pick $z \in E$, and consider the matroid $M_0 = (F, H)$ where $F = \{v \mid v + z \in E\}$. Since M is I_4-free, it follows that M_0 is I_3-free. Moreover, since M is affine, it follows that M_0 is AI_4-free.

Let H' be the hyperplane of H from the conclusion of Lemma 4.4.1. We will now go through each conclusion of Lemma 4.4.1 to see that in each of the cases, we obtain a 0-expansion or a 1-expansion, proving the result.

Case 1: $F \subseteq H'$.

In this case, let $H'' = \mathrm{cl}(H' \cup \{z\})$, so that H'' is a hyperplane of G. Then $M|H''$ is an affine matroid with H' satisfying $E \cap H'' \subseteq H'' \backslash H'$. Then

$$\begin{aligned} E &= \{z\} \cup (z + F) \\ &\subseteq \{z\} \cup (z + H') \\ &\subseteq H''. \end{aligned}$$

Therefore, M is the 0-expansion of the affine matroid $M|H''$.

Case 2: $H \backslash F \subseteq H'$.

Let $H'' = \mathrm{cl}(H' \cup \{z\})$ as before. Let $w \in H \backslash H'$. Then

$$\begin{aligned} E &= \{z\} \cup (z + F) \\ &= \{z\} \cup (z + F \cap H') \cup (z + (F \backslash H')) \\ &= \{z\} \cup (z + F \cap H') \cup (z + (H \backslash H')) \\ &= (E \cap H'') \cup \{z + w\} \cup (z + w + H') \end{aligned}$$

Therefore, M is the 1-expansion of the affine matroid $M|H''$.

Case 3: $F \cap H' = \varnothing$.

Note that if M_0 is rank-deficient, then we are in Case 1, so assume that M_0 is full-rank. Observe that M_0 is I_3-free and triangle-free (since it is affine). Therefore, Lemma 1.5.6 implies that M_0 is a full-rank affine geometry. We are in Case 2.

Case 4: $H' \subseteq F$.

Let $w \in F \backslash H'$ (if no such w exists, then M_0 is rank-deficient and we are in Case 1), and let $H'' = \mathrm{cl}(H' \cup \{z + w\})$.

Then $M|H''$ is an affine matroid with H' as its hyperplane such that $H'' \cap E \subseteq H'' \backslash H'$. Then

$$\begin{aligned} E &= \{z\} \cup (z + F) \\ &= \{z\} \cup (z + F \cap H') \cup (z + (F \backslash H')) \\ &= \{z\} \cup \{z + H'\} \cup (E \cap H'') \end{aligned}$$

Therefore, M is the 1-expansion of the affine matroid $M|H''$.

Thus M is either the 0-expansion or 1-expansion of another affine matroid of smaller dimension. The result now follows by induction on $\dim(M)$. $\square$

Chapter 5

$I_{1,t}$-free and Triangle-free Matroids

This chapter is based on joint work with Peter Nelson [42].

5.1 Introduction

So far, we have primarily focused on obtaining a complete structure theorem for various families of matroids. But in order to prove Conjecture 1.5.8 (the matroidal version of Gyárfás-Sumner), or other problems with similar flavours, it may be difficult to prove a full structure theorem and then deduce properties such as χ-boundedness as a corollary. For the case $t = 5$, some preliminary calculations that we have performed indicate that the general structure of I_5-free and triangle-free matroids is rather complex, and hard to describe. Nor would we expect it to be simple; for all but very small graphs H, determining the structure of H-free graphs is very difficult.

What one wishes to obtain is a set of tools that can be employed to answer questions of this nature. In this chapter, we present a result that is a significant weakening of Conjecture 1.5.8, but its proof circumvents the need to obtain a full structural theorem.

To state our result, let us first consider the following more general problem. For $1 \leq s \leq t$, $I_{s,t}$ is the t-dimensional matroid (E, G) for which $(E, \mathrm{cl}(E)) \cong I_s$.

Conjecture 5.1.1. *For any $1 \leq s \leq t$, there exists a constant $c_{s,t} > 0$ such that if M is an $I_{s,t}$-free, triangle-free matroid, then $\chi(M) \leq c_{s,t}$.*

It is immediate that Conjecture 1.5.8 is a special case of Conjecture 5.1.1, by setting $s = t$. But these two conjectures are in fact equivalent; for any given s, t, $1 \le s \le t$, there exists a sufficiently large s' such that if M is an $I_{s,t}$-free matroid, then M is $I_{s'}$-free.

Our main result in this chapter is a proof of Conjecture 5.1.1 for the case $s = 1$, making partial progress towards Conjecture 1.5.8.

Theorem 5.1.2. *For any $t \ge 1$, there exists a constant $c_t > 0$ such that if M is an $I_{1,t}$-free, triangle-free matroid, then $\chi(M) \le c_t$.*

The main ingredient in the proof is a consequence of Green's regularity lemma for abelian groups [32], and will be explained in detail in a later section. We will also use a few standard results from matroid theory, such as Ramsey's Theorem for matroids.

5.2 Preliminaries

In this section, we state the results we need for our main theorem. The main ingredient is the following key lemma, which is an application of Green's regularity lemma for abelian groups [32]. We will derive this key lemma in the next section. The *codimension* of a subspace U in a vector space V is $\dim(V) - \dim(U)$. An *affine subspace* of a vector space V is a set of the form $a + U$ where U is a subspace and $a \notin U$, and its *codimension* is $\dim(V) - \dim(U)$.

Lemma 5.2.1. *For every $0 < \alpha \le 1$, there exists an integer $l \ge 0$ such that for every $X \subseteq \mathbb{F}_2^n$ with $|X| \ge \alpha 2^n$, $X + X + X$ contains an affine subspace of codimension l.*

We will use the following matroidal analogue of Ramsey's Theorem for graphs. It is in fact the multicolour version of Theorem 3.7.3, which follows immediately from Theorem 3.7.3 itself.

Theorem 5.2.2 ([31])**.** *For any $c \ge 1$, $r_1, r_2, \cdots r_c \ge 1$, there exists an integer $N = GR(r_1, \cdots, r_c)$ such that for all $n \ge N$, if the elements of $G \cong \mathrm{PG}(n-1, 2)$ are coloured with c different colours, there exists a flat $G_i \cong \mathrm{PG}(r_i - 1, 2)$ of G that is monochromatic in colour i for some $1 \le i \le c$.*

In our application, the r_i's will always be the same. We therefore write $GR(c, r)$ for the Ramsey number by only specifying the number of colours c and the dimension r of a monochromatic projective geometry we expect to find. The constant $GR(c, r)$ guaranteed is very large, however, leading to a large bound on c_t in Theorem 5.1.2.

5.3 Regularity

In this section, we will derive our key lemma from Green's regularity lemma for abelian groups [32].

Let $V = \mathbb{F}_2^n$ and $X \subseteq V$. We say that X is *ε-uniform* if for each hyperplane W of V

$$||W \cap X| - |X - W|| \leq \varepsilon |V|.$$

Note that a hyperplane W slices the space V into two equally-sized sets; hence being ε-uniform means that the elements of X are distributed about equally across these two sets no matter how one slices V.

Let H be a subspace of V and $v \in V$. As before, we call the sets of the form $H + v = \{x + v \mid x \in H\}$ the translates (note that H is considered a translate of H itself). We say that H is *ε-regular* with respect to V and X if for all but an ε-fraction of translates $H' = H + v$ of H, the set $(H' \cap X) + v$ is ε-uniform. Note that the set $(H' \cap X) + v$ is a subset of H; we are projecting the elements of $(H' \cap X)$ back onto the subspace H by adding v to each point. Regularity measures how evenly the elements of X are distributed across the different translates of H in V.

The following lemma by Green ensures that there is an ε-regular subspace of bounded codimension for any given ε. The lemma holds in a more general context but we state the version of his theorem for $V = \mathbb{F}_2^n$, which is all we need for our main theorem. We note that in this lemma, l is guaranteed to be at most $T(\varepsilon^{-3})$, where $T(\alpha)$ is an exponential tower of 2's of height $\lceil \alpha \rceil$; this also contributes to a large bound on c_t in Theorem 5.1.2.

Lemma 5.3.1 ([32])**.** *For every $0 < \varepsilon < \frac{1}{2}$, there exists $l \in \mathbb{N}$ such that for every $X \subseteq V$, there is a subspace $H \subseteq V$ of codimension at most l that is ε-regular with respect to X and V.*

We also need the following counting lemma for ε-uniform sets from [32, 54]. We simply state a special version that is adequate for our main theorem.

Lemma 5.3.2 ([32, 54])**.** *Let $X \subseteq V$ with $|X| = \alpha |V|$. If $0 < \varepsilon < \frac{1}{2}$ and X is ε-uniform, then for any $u \in V$,*

$$|\{(x_1, x_2, x_3) \in X^3 \mid x_1 + x_2 + x_3 = u\}| \geq (\alpha^3 - \varepsilon)|V|^2.$$

Note that the quantity $\alpha^3 2^{2n}$ is the number of solutions to $x_1 + x_2 + x_3 = u$ one would expect to find if the set X was constructed randomly by choosing elements of V independently at random with probability α.

We can now prove our key lemma, restated below.

Lemma 5.3.3. *For every $0 < \alpha \leq 1$, there exists an integer $l \geq 0$ such that for every $X \subseteq \mathbb{F}_2^n$ with $|X| \geq \alpha 2^n$, $X + X + X$ contains an affine subspace of codimension l.*

Proof. Apply Lemma 5.3.1 with $\varepsilon = \frac{\alpha^3}{9}$ to obtain an integer l', so that we obtain a subspace H of codimension $k \leq l'$ that is ε-regular with respect to X and V.

There are exactly 2^k translates of H. Pick $A \subseteq V$ so that $H + a$, $a \in A$ are the distinct translates of H. Let

$$A_{sparse} = \{a \in A : |X \cap (H + a)| < \frac{\alpha}{2}|H|\}$$

$$A_{bad} = \{a \in A : (X \cap (H + a)) - a \text{ is not } \varepsilon\text{-uniform}\}$$

Since H is ε-regular with respect to X and V, it follows that $|A_{bad}| \leq \varepsilon 2^k$. Note

$$\begin{aligned}\alpha 2^k |H| = \alpha 2^n \leq |X| &= \sum_{a \in A} |X \cap (H + a)| \\ &\leq \frac{\alpha}{2}|H||A_{sparse}| + |H|(|A| - |A_{sparse}|).\end{aligned}$$

Therefore, it follows that $\alpha 2^k \leq \frac{\alpha}{2}|A_{sparse}| + 2^k - |A_{sparse}|$. Hence, $|A_{sparse}| \leq 2^k \frac{1-\alpha}{1-\frac{\alpha}{2}} = 2^k(1 - \frac{\alpha}{2-\alpha})$. Therefore

$$|A| - |A_{sparse}| \geq \frac{\alpha}{2} 2^k > \varepsilon 2^k.$$

So there exists some $a_0 \in A \backslash (A_{sparse} \cup A_{bad})$, which means that $|X \cap (H + a_0)| \geq \frac{\alpha}{2}|H|$ and $(X \cap (H + a_0)) - a_0$ is ε-uniform.

Let $u \in H + a_0$. We argue that $u \in X + X + X$. We apply Lemma 5.3.2 with H as the vector space. Then the number of solutions to the equation $x_1 + x_2 + x_3 = u + a_0$ with $x_1, x_2, x_3 \in (X \cap (H + a_0)) - a_0$ is at least

$$\begin{aligned}&\left(\frac{|(X \cap (H + a_0)) - a_0|}{|H|}\right)^3 |H|^2 - \varepsilon |H|^2 \\ &\geq \left(\frac{\alpha}{2}\right)^3 |H|^2 - \varepsilon |H|^2 \geq \left(\frac{\alpha^3}{8} - \frac{\alpha^3}{9}\right) |H|^2 > 0.\end{aligned}$$

So there exists $x_1, x_2, x_3 \in (X \cap (H + a_0)) - a_0$ such that $x_1 + x_2 + x_3 = u + a_0$. Therefore, $(x_1 + a_0) + (x_2 + a_0) + (x_3 + a_0) = u$, as required. If $a_0 \in H$, $X + X + X$ contains a subspace, otherwise $X + X + X$ contains an affine subspace, of codimension l'. In both cases, we conclude that $X + X + X$ contains an affine subspace of codimension at least $l = l' + 1$. $\square$

5.4 Tripods

We define a class of matroids called *tripods* (not to be confused with tripods in graph theory). Let T_0, the *0-th order tripod*, be the 1-dimensional matroid with a one-element ground set. The tripods are constructed as follows. The *k-th order tripod* $T_k = (E_k, G_k)$, $k \geq 1$, is the matroid with a codimension-3 flat H of G_k and $x, y, z \notin H$ where $\mathrm{cl}(H \cup \{x, y, z\}) = G_k$ so that $E_k = (\{x, y, z, 0\} + (H \cap E_k)) \cup \{x + y + z\}$ and $(E_k \cap H, H) \cong T_{k-1}$.

The properties of the tripods we will need are as follows. For $t \geq 4$, we write $C_{5,t}$ for the t-dimensional matroid (E, G) such that $(E, \mathrm{cl}(E)) \cong C_5$. Note that $C_{5,4} \cong C_5$.

Lemma 5.4.1. *Let $T_k = (E_k, G_k)$ be the k-th order tripod. Then the following hold.*

- *$T_k = (E_k, G_k)$ has dimension $3k + 1$,*
- *For $k \geq 1$, there is a flat F_k of dimension $2k + 2$ for which $T_k|F_k \cong C_{5,2k+2}$ and $F_k \subseteq E_k \cup (E_k + E_k)$.*

Proof. It follows straight from the definition that T_k has dimension $3k + 1$.

Note that when $k = 1$, then $T_1 \cong C_5$, so all the statements are true for $k = 1$ as well. From here, we work by induction on k. Take $T_k = (E_k, G_k)$ where $k \geq 2$ and assume that the statements are true for smaller values of k.

By definition, we can write $E_k = (\{x, y, z, 0\} + H \cap E_k) \cup \{x + y + z\}$ where H is a codimension-3 flat and $x, y, z \notin H$ such that $(E_k \cap H, H) \cong T_{k-1}$ and $\mathrm{cl}(H \cup \{x, y, z\}) = G_k$. By induction, we can find a flat $F_{k-1} \subseteq H$ of dimension $2k$ for which $T_k|F_{k-1} \cong C_{5,2k}$ and $F_{k-1} \subseteq (E_k \cap H) \cup ((E_k \cap H) + (E_k \cap H))$. We claim that $F_k = \mathrm{cl}(F_{k-1} \cup \{x + y, x + z\})$ suffices.

Note that $T_k|F_k \cong C_{5,2k+2}$ is immediate; $T_k|F_{k-1} \cong C_{5,2k}$ and $E_k \cap (F_k \backslash F_{k-1}) = \varnothing$. It remains to show that $F_k \subseteq E_k \cup (E_k + E_k)$. Let $v \in F_k \backslash E_k$. We now perform a case analysis.

If $v \in F_{k-1}$, then $v \in (E_k \cap H) + (E_k \cap H)$ by induction, so $v \in E_k + E_k$ so suppose not. If $v \in x + y, x + z, y + z$, then by symmetry we may assume $v = x + y$. Then we may pick any element $t \in E_k \cap H$, and $v = (x + t) + (y + t) \in E_k + E_k$. The only elements that remain to be checked are of the form $v + w$ where $v \in \{x + y, x + z, y + z\}$ and $w \in F_{k-1}$. By symmetry we may assume that $v = x + y$. If $w \notin F_{k-1} \cap E_k$, then by induction, it follows that there exist two elements $a, b \in H \cap E_k$ for which $w = a + b$. But then $v + w = (a + x) + (b + y) \in E_k + E_k$. Hence, we may assume that $w \in F_{k-1} \cap E_k$. But then $v + w = (x + y + z) + (z + w) \in E_k + E_k$. This finishes the proof. $\square$

5.5 The Main Theorem

We are now ready to prove our main theorem.

Theorem 5.5.1. *For any $t \geq 1$, there exists a constant $c_t > 0$ such that if M is an $I_{1,t}$-free, triangle-free matroid, then $\chi(M) \leq c_t$.*

Proof. The result is trivial when $t = 1$, so suppose that $t \geq 2$. We will work by induction on t. Suppose that the $I_{1,t-1}$-free, triangle-free matroids have critical number at most c_{t-1}.

Let $M = (E, G)$ be an $I_{1,t}$-free, triangle-free matroid. We may assume that $E \neq \varnothing$, as otherwise $\chi(M) = 0$. Hence M contains a T_0-restriction. Let $k \geq 0$ be the largest integer for which M contains a T_k-restriction (not necessarily induced); let G_1 be a $(3k+1)$-dimensional flat for which $M|G_1$ contains a T_k-restriction. We claim that $k < \frac{t}{2}$.

For a contradiction, suppose not. In particular, $k \geq 1$. Let $F_k \subseteq G_1$ be the $(2k+2)$-dimensional flat obtained from Lemma 5.4.1, so that $M|F_k$ contains a $C_{5,2k+2}$-restriction. Let $E' \subseteq E \cap F_k$ be the set of five elements corresponding to the ground set of this $C_{5,2k+2}$-restriction, so that $(E', F_k) \cong C_{5,2k+2}$. We now claim $M|F_k \cong C_{5,2k+2}$. To show this, it suffices to check that $(F_k \backslash E') \cap E = \varnothing$. If we take $v \in F_k \backslash E'$, then there exists $v_1, v_2 \in E \cap G_1$ by Lemma 5.4.1 such that $v = v_1 + v_2$, but since M is triangle-free, this implies that $v \notin E$. Hence $(F_k \backslash E') \cap E = \varnothing$ and we conclude that $M|F_k \cong C_{5,2k+2}$. But note that $C_{5,2k+2}$ contains an induced $I_{1,2k+1}$-restriction. But we assumed $t \leq 2k$, so this contradicts $I_{1,t}$-freeness.

Now, fix a flat G_2 such that $G_1 \cap G_2 = \varnothing$ and $\mathrm{cl}(G_1 \cup G_2) = G$. For each $v \in G_2$, we assign a colour S where $S = v + ((v + [G_1]) \cap E)$. Note that $S \subseteq G_1 \cup \{0\}$. Hence we have at most $c = 2^{|G_1|+1} = 2^{2^{3k+1}}$ colours.

We now define a new matroid $N = (X, G_2)$, where X consists of elements of G_2 for which their colour S satisfies $G_1 \cap E \subseteq S$ and $0 \notin S$. We first claim that N is dense.

5.5.1.1. $|X| \geq 2^{\dim(N) - GR(c,t)}$.

Subproof: If $G_2 \backslash X$ contains a $GR(c,t)$-dimensional flat, then by Theorem 5.2.2, we have a monochromatic t-dimensional flat F'contained in $G_2 \backslash X$, in colour S. If $0 \in S$, then $F' \subseteq E$, which contradicts triangle-freeness since $t \geq 2$. So $0 \notin S$, meaning $F' \cap E = \varnothing$. But there also exists $v \in G_1 \cap E$ for which $(F' + v) \cap E = \varnothing$. This means that $M|\,\mathrm{cl}(\{v\} \cup F')$ is an induced $I_{1,t}$-restriction, a contradiction.

Now, we apply Theorem 1.4.2 to conclude that $|G_2 \backslash X| \leq 2^{\dim(N)}(1 - 2^{-GR(c,t)+1})$. Hence, $|X| \geq 2^{\dim(N)} 2^{-GR(c,t)}$ ■

We make one further observation which follows immediately from the definition of tripods.

5.5.1.2. *$(X + X + X) \cap E = \varnothing$.*

Subproof: For a contradiction, suppose that there exist $v_1, v_2, v_3 \in X$ for which $v_1 + v_2 + v_3 \in E$. Note that the v_i are distinct; if two of them are identical, say $v_1 = v_2$, then $v_1 + v_2 + v_3 = v_3 \in E$, contradicting the fact that $v_3 \in X$. Also, the v_i are independent; otherwise $v_1 + v_2 + v_3 = 0 \in E$, a contradiction.

But then since $M|G_1$ contains a T_k-restriction, $M|(\{v_1, v_2, v_3\} \cup G_1)$ contains a T_{k+1}-restriction, contradicting the maximality of k. ■

Let $l \geq 0$ be the integer obtained from applying Lemma 5.3.3 with $\alpha = 2^{-GR(c,t)}$. So $X+X+X$ contains an AG$(\dim(N)-l, 2)$-restriction. We can find a flat F' of codimension $l-1$ in N, and a hyperplane F'' of F' for which $F' \backslash F'' \subseteq X+X+X$. In particular, by 5.5.1.2, $(F' \backslash F'') \cap E = \varnothing$. Hence $M|F''$ is $I_{1,t-1}$-free (and triangle-free). Since F'' has codimension at most $l + (3k+1)$ in M, we conclude by induction that $\chi(M) \leq c_{t-1} + 3k + l + 1$. Note that k and l are both functions of t. This completes the proof by induction on t. □

Chapter 6

I_5-free and Triangle-free Matroids

This chapter is based on joint work with Peter Nelson [43].

6.1 Introduction

In this chapter, we consider an extremal problem concerning the class of I_5-free, triangle-free matroids.

We first recall the problem of determining the smallest simple matroids with no large independent flat, mentioned in the introduction. This problem was first considered by Nelson and Norin [45] for general simple matroids; we return to standard matroid terminology for this discussion. For integers $r \geq 1$ and $t \geq 1$, $N_{r,t}$ denotes the direct sum of t, possibly empty, binary projective geometries whose ranks sum to r and pairwise differ by at most 1 (note that Nelson and Norin used $M_{r,t}$ instead of $N_{r,t}$ to denote these matroids, but to avoid confusion with the $M_{5,3}$ and $M_{6,3}$ matroids defined in an earlier chapter, we us $N_{r,t}$ instead). Nelson and Norin determine the smallest simple matroids with no t-element independent flat.

Theorem 6.1.1 ([45]). *Let $r, t \geq 1$ be integers. If M is a simple rank-r matroid with no $(t+1)$-element independent flat, then $|E(M)| \geq |N_{r,t}|$. If equality holds and $r \geq 2t$, then $M \cong N_{r,t}$.*

The tight examples $N_{r,t}$ are binary and, being a direct sum of binary projective geometries, contain many triangles. In the same paper [45], Nelson and Norin made the following natural conjecture, in which it is required that the matroids have no three-element circuit.

Conjecture 6.1.2 ([45], Conjecture 1.3)**.** *Let t, r be integers with $t \geq 1$ and $t|r$. If M is a simple rank-r matroid with no $(2t+1)$-element independent flat or three-element circuit, then $|M| \geq t2^{r/t-1}$.*

The bound in the above conjecture is based on their prediction that the tight examples are direct sums of t binary affine geometries.

Our main result in this chapter is the case $t = 2$ in the context of embedded matroids; note that we now return to the adjusted formalism for embedded matroids.

Theorem 6.1.3. *Let $M = (E, G)$ be a full-rank, I_5-free, and triangle-free matroid of dimension r. Then $|E| \geq 2^{\lfloor r/2 \rfloor - 1} + 2^{\lceil r/2 \rceil - 1}$. When $r \geq 6$, equality holds if and only if $M \cong M_1 \oplus M_2$, where M_1 and M_2 are affine geometries of dimensions $\lfloor r/2 \rfloor$ and $\lceil r/2 \rceil$ respectively*

The proof largely falls into two parts. Note that the tight examples are affine. The first step is to show that the tight examples must be affine by showing that they cannot contain a C_5-restriction. In the second part, we consider the class of I_5-free and affine matroids to determine the tight examples. The main idea of our proof will be to reduce a problem involving I_5-free and triangle-free matroids to one that involves I_3-free matroids.

6.2 Preliminaries

In the proof of Theorem 6.1.3, it will be necessary to perform contraction. Contraction is a standard operation in matroid theory; here we give a corresponding notion for embedded matroids.

Contraction

Suppose that we are given an n-dimensional matroid $M = (E, G)$ and a k-dimensional flat F of G. Let F_1 be an $(n-k)$-dimensional flat of G for which $F \cap F_1 = \varnothing$. Then the matroid (E_1, F_1) where $E_1 = F_1 \cap (E + [F])$ is called the *contraction of M by F onto F_1*.

Note that if F_2 is another $(n-k)$-dimensional flat of G for which $F \cap F_2 = \varnothing$, then the contraction of M by F onto F_2, denoted by (E_2, F_2) where $E_2 = F_2 \cap (E + [F])$, is isomorphic to (E_1, F_1). To see this, let $U = F_1 \cap F_2$. Note that U is a flat of both F_1 and F_2. Let $x_1 \in F_1 \backslash U$, and let x_2 be the unique element in $F_2 \cap (x_1 + [F])$. Define a map

$\psi : [F_1] \mapsto [F_2]$ by setting $\psi(x) = x$ for $x \in U$, $\psi(x_1) = x_2$ and extending ψ linearly. It is then easy to check that ψ is a linear bijection with the property that $\psi(E_1) = E_2$. We therefore omit the flat F_1 and write M/F to refer to any such matroid, and call it the *contraction of* M *by* F.

We can view the contraction at the coset level as well. Recall that a coset of F is called *intersecting* with respect to $M = (E, G)$ if F contains at least one element of E. Note that the elements of the ground set of the matroid M/F correspond precisely to the cosets A of F for which $A \cap E \neq \varnothing$. Hence the number of elements of M/F equals the number of intersecting cosets of F with respect to M. We will often refer to the elements of the ambient set of M/F by identifying them with the cosets of F. To give an example in how this equivalence is used, note that, given a matroid $M = (E, G)$ and a flat $F \subseteq G$, M/F has an induced I_3-restriction if and only if there exist distinct intersecting cosets A_1, A_2, A_3 of F with respect to M for which $A_1 + A_2 \neq A_3$ and $A_1 + A_2$, $A_1 + A_3$, $A_2 + A_3$ and $A_1 + A_2 + A_3$ all have no intersection with E.

6.2.1 Checking I_5-freeness

In this chapter, we will often assert that a given five-element set $I \subseteq E$ in a matroid $M = (E, G)$ is an induced I_5-restriction. In many cases, the matroid M will be assumed to be triangle-free, which reduces the number of necessary checks as we only need to verify that I is an independent set in $[G]$, and that $\sum_{x \in J} x \notin E$ for all $J \subseteq I$ for which $|J| \in \{3, 4, 5\}$. If, additionally, M is also C_5-free, then we only need to check these conditions for $J \subseteq I$ for which $|J| \in \{3, 5\}$. As before, we will not write down these checks explicitly.

In some cases, some elements of I belong to the cosets of a given flat F in a special way which makes it easier to verify that I is an induced I_5-restriction. For example, given a triangle-free matroid $M = (E, G)$ and a flat $F \subseteq G$, suppose that there are cosets A_1, A_2, A_3 of F for which $A_1 + A_2 \neq A_3$, $(A_i + A_j) \cap E = \varnothing$ for every pair of $i, j \in \{1, 2, 3\}$ where $i \neq j$, and $(A_1 + A_2 + A_3) \cap E = \varnothing$; this is indeed the setting of our first lemma (see Lemma 6.3.1). Suppose further that $|I \cap A_i| = 1$ for each $i = 1, 2, 3$, and let $a_i \in I \cap A_i$ and $\{x, y\} = (I \cup F) \backslash \{a_1, a_2, a_3\}$. Then in order to verify that I is an induced I_5-restriction, we only need to verify that $a_i + x + y \notin E$ for every $i = 1, 2, 3$.

6.2.2 I_t-free Matroids

Here, we give the proof for Theorem 6.1.1 by Nelson and Norin, adjusted for embedded matroids. The tight examples are harder to determine, so we will not discuss them.

Theorem 6.2.1 ([45]). *For any $t \geq 1$, if M is an r-dimensional, full-rank, I_{t+1}-free matroid of dimension r, then $|M| \geq |N_{r,t}|$.*

Proof. We work by induction on t, and then on r. The case $t = 1$ is immediate, and note that the case $r = 1$ is immediate for any $t \geq 1$. Let M be a full-rank, I_{t+1}-free matroid of dimension r. Then we may assume that there exists an induced I_t-restriction, as otherwise the inductive hypothesis applies. Let F be a flat for which $M|F \cong I_t$. Then M/F is I_{t+1}-free. By induction on r, we have that $|M/F| \geq |N_{r-t,t}|$. But note that each intersecting coset A of F contains at least two elements of E; otherwise, $M|(A \cup F)$ is an induced I_{t+1}-restriction. Therefore, $|M| \geq 2 \cdot |M/F| + t \geq 2|N_{r-t,t}| + t = |N_{r,t}|$. The last equality follows from the fact that $N_{r,t}$ is the direct sum of t projective geometries (individually viewed as matroids) whose ranks are as equal as possible and add to r; from this it is easy to deduce that $2|N_{r-t,t}| + t = |N_{r,t}|$. □

This proof is difficult to extend to our case since M/F will not, in general, be triangle-free when M is also triangle-free. This makes it difficult to perform induction as in the proof of Theorem 6.2.1. The main idea in our proof will be to take advantage of triangle-freeness early on to deduce, in some instances, that the contraction by an appropriate choice of flat F is I_3-free, instead of I_5-free.

6.3 C_5-restriction

In this section, our goal is to show that the smallest I_5-free and triangle-free matroids are affine when $r \geq 6$. The strategy is to start with a C_5-restriction and show that matroids containing a C_5-restriction are too large. Note that I_5-free matroids are automatically C_k-free for odd $k \geq 7$ by Lemma 2.2.3.

The argument hinges on the following observation, which will allow us to reduce our problem to one involving I_3-free matroids, which we understand well. We note that this lemma could be useful for solving Conjecture 1.5.8 in the case $s = 5$.

Lemma 6.3.1. *Let $M = (E, G)$ be a full-rank, I_5-free and triangle-free matroid. Let F be a flat of G so that $M|F \cong C_5$. Then M/F is I_3-free.*

Proof. Suppose for a contradiction that M/F contains an induced I_3-restriction. This means there exist three intersecting cosets A_1, A_2, A_3 of F with respect to M for which $(A_i + A_j) \cap E = \varnothing$ for every pair of $i, j \in \{1, 2, 3\}$ where $i \neq j$, and $(A_1 + A_2 + A_3) \cap E = \varnothing$.

Fix $a_k \in A_k \cap E$. Denote the five elements of $E(M|F)$ by $\{x_1, x_2, x_3, x_4, x_5\}$, and consider the complete graph H on 5 vertices with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$. For $k = 1, 2, 3$, let $S_k \subseteq E(H)$ be such that the edge $v_i v_j \in S_k$ if and only if $a_k + x_i + x_j \in E$ (note that $E(H)$ denotes the edge set of the graph H).

We now make the following sequence of observations.

6.3.1.1. *Every edge of H belongs to S_k for some $k \in \{1, 2, 3\}$.*

Subproof: If there exists an edge $e = v_i v_j$ that belongs to no S_k for $k \in \{1, 2, 3\}$, then $\{a_1, a_2, a_3, x_i, x_j\}$ is an induced I_5-restriction, a contradiction (as remarked in Subsection 6.2.1, to check that this is indeed an induced I_5-restriction, we only need to verify that $a_k + x_i + y_j \notin E$ for $k = 1, 2, 3$). ■

6.3.1.2. *For every $k \in \{1, 2, 3\}$, S_k contains no two-edge matching of H.*

Subproof: Suppose otherwise, so that S_k for some $k \in \{1, 2, 3\}$ contains a two-edge matching. Without loss of generality, suppose that $v_1 v_2, v_3 v_4 \in S_k$. Then $\{a_k + x_1 + x_2, a_k + x_3 + x_4, x_5\}$ is a triangle, a contradiction. ■

6.3.1.3. *For every $k \in \{1, 2, 3\}$, S_k contains no triangle of H.*

Subproof: Suppose for a contradiction that, say, S_3 contains a triangle of H. Denote this triangle $T = \{v_3 v_4, v_4 v_5, v_3 v_5\}$. Note that every edge in $E(H) \backslash T$ is in a two-edge matching with an edge in T. Hence by 6.3.1.1 and 6.3.1.2, we have $E(H) \backslash T \subseteq S_1 \cup S_2$.

Let $H' = H - T - \{v_1 v_2\}$ (H' is the graph H with the edges in $T \cup \{v_1 v_2\}$ removed). Then $H' \cong K_{2,3}$. Since $E(H') \subseteq S_1 \cup S_2$ and neither S_1 nor S_2 contains a two-edge matching, we must have $\{S_1 \cap E(H'), S_2 \cap E(H')\} = \{\{v_1 v_3, v_1 v_4, v_1 v_5\}, \{v_2 v_3, v_2 v_4, v_2 v_5\}\}$. Without loss of generality, suppose that $S_k \cap E(H') = \{v_k v_3, v_k v_4, v_k v_5\}$ for $k = 1, 2$.

Now, either $v_1 v_2 \in S_1$ or $v_1 v_2 \in S_2$. Hence, $\{x_1, x_3, x_1 + x_2 + a_1, a_2, a_3\}$ or $\{x_2, x_3, x_1 + x_2 + a_2, a_1, a_3\}$ is an induced I_5-restriction (to quickly check that they would give an induced I_5-restriction, see Subsection 6.2.1). ■

Now we can finish the proof. By 6.3.1.2, we have that, for each $k \in \{1, 2, 3\}$, S_k is either a triangle or a star in H. By 6.3.1.3, we have that each S_k is a star. Let c_k be the centre of the star corresponding to S_k. Since $|V \backslash \{c_1, c_2, c_3\}| \geq 2$, pick two distinct vertices $v_i, v_j \in V \backslash \{c_1, c_2, c_3\}$. But then the edge $v_i v_j$ does not belong to S_k for any $k \in \{1, 2, 3\}$, contradicting 6.3.1.1. □

We will make one more observation about the cosets of F.

Lemma 6.3.2. *Let $M = (E, G)$ be a full-rank, I_5-free and triangle-free matroid. Let A, B, C be three cosets of flat F where $M|F \cong C_5$ for which $|A \cap E| = 1$ and $B + C = A$, $|B \cap E| \leq |C \cap E|$ and $|C \cap E| \neq 0$. If $|B \cap E| \leq 2$, then $|C \cap E| \geq 3$.*

Proof. Suppose for a contradiction that $|B \cap E| \leq 2$ but $|C \cap E| \leq 2$.

We first consider the case $|B \cap E| = 0$. Fix $a_1 \in A \cap E$ and $c_1 \in C \cap E$. Then $|C \cap E| = 2$; otherwise $\{a_1, c_1, x_1, x_2, x_3\}$ for any three elements $x_1, x_2, x_3 \in E \cap F$ gives an induced I_5-restriction. Let $c_2 \in C \cap E$, $c_2 \neq c_1$. Since M is triangle-free, we have that $c_1 + c_2 \in F \backslash E$. Because $M|F \cong C_5$, $c_1 + c_2 = x_1 + x_2$ for some two $x_1, x_2 \in E \cap F$. Pick any other two elements $x_3, x_4 \in E \cap F$. Then $\{a_1, c_1, x_1, x_3, x_4\}$ is an induced I_5-restriction, a contradiction.

Hence we may assume that $|B \cap E| > 0$. Now fix $b_1 \in B \cap E$ and $c_1 \in C \cap E$. Denote the five elements of $E(M|F)$ by $\{x_1, x_2, x_3, x_4, x_5\}$, and consider the complete graph H on five vertices with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$. Let $S_A \subseteq E(H) \cup V(H)$ consist of vertices v_i for which $b_1 + c_1 + x_i \in E$ and edges $v_i v_j$ for which $b_1 + c_1 + x_i + x_j \in E$. Let $S_B \subseteq E(H)$ consist of the edges $v_i v_j$ for which $b_1 + x_i + x_j \in E$, and let $S_C \subseteq E(H)$ consist of the edges $v_i v_j$ for which $c_1 + x_i + x_j \in E$. We now make a sequence of claims.

6.3.2.1. *$|S_A| = 1$, $|S_B| \leq 1$ and $|S_C| \leq 1$.*

Subproof: Since $0 < |B \cap E|, |C \cap E| \leq 2$, we have that $|S_B|, |S_C| \leq 1$. By assumption, $|E \cap A| = 1$ so that $|S_A| = 1$. ■

6.3.2.2. *If an edge $e \in E(H)$ satisfies $e \in S_B \cup S_C$, then $e \notin S_A$.*

Subproof: Suppose not, so that some edge $e = v_i v_j$ satisfies $e \in S_A$ and $e \in S_B \cup S_C$. If $e \in S_B$, then $\{b_1 + c_1 + x_i + x_j, b_1 + x_i + x_j, c_1\}$ is a triangle, a contradiction. The case $e \in S_C$ is symmetrical. ■

6.3.2.3. *If four distinct vertices $\{v_i, v_j, v_k, v_l\}$ of H satisfy $v_i v_j \in S_B$ and $v_k v_l \in S_C$, then $v_t \notin S_A$ where $v_t \in V \backslash \{v_i, v_j, v_l, v_k\}$.*

Subproof: If not, then $\{b_1 + c_1 + x_t, b_1 + x_i + x_j, c_1 + x_k + x_l\}$ is a triangle. ■

6.3.2.4. *There are no distinct vertices v_i, v_j, v_l such that*

- *$\{v_i, v_j, v_l\} \cap S_A = \varnothing$ and*

- $\{v_iv_j, v_iv_l, v_jv_l, v_kv_p\} \cap (S_A \cup S_B \cup S_C) = \varnothing$,

where $\{v_k, v_p\} = V \backslash \{v_i, v_j, v_l\}$.

Subproof: Suppose for a contradiction that such vertices exist. Then $\{b_1, c_1, x_i, x_j, x_l\}$ is an induced I_5-restriction. ■

6.3.2.5. *There is no triangle H where each of the three edges of T belongs to a distinct S_k, $k \in \{A, B, C\}$.*

Subproof: Suppose not, so that, without loss of generality, $v_1v_2 \in S_A$, $v_2v_3 \in S_B$, and $v_1v_3 \in S_C$. Then $\{b_1 + c_1 + x_1 + x_2, b_1 + x_2 + x_3, c_1 + x_1 + x_3\}$ is a triangle. ■

It is now possible to perform a case analysis to show that there is no valid assignment of the vertices and edges of the complete graph H to S_A, S_B, S_C under these constraints. Recall that, since S_A is a one-element set, S_A either consists of one vertex or one edge.

Case 1: S_A consists of one edge.

Consider the graph H' on the vertex set V with edge set $S_A \cup S_B \cup S_C$. By 6.3.2.1, this graph has at most 3 edges. Note that a graph on 5 vertices with at most 3 edges either contains a triangle, or $K^c_{2,3}$ in its complement. If H' contains a triangle, then by 6.3.2.1 again, each edge of this triangle belongs to a distinct S_k, $k \in \{A, B, C\}$, contradicting 6.3.2.5. If H' contains $K^c_{2,3}$ in its complement, then this contradicts 6.3.2.4.

Case 2: S_A consists of one vertex.

Consider the graph H' on the vertex set V with edge set $S_B \cup S_C$. Observe that a graph on 5 vertices with a fixed vertex v and at most 2 edges either contains two vertex-disjoint edges that do not intersect v, or the disjoint union of an edge that contains v and a triangle in its complement. Let v be the unique vertex contained in S_A. Then this observation implies that H' contains two vertex-disjoint edges that do not intersect v, contradicting 6.3.2.3, or the disjoint union of an edge that contains v and a triangle in its complement, contradicting 6.3.2.5. □

Combining Lemma 6.3.1, Lemma 6.3.2 and Theorem 6.1.1 yields the following result, which shows that the smallest I_5-free, triangle-free matroids are affine.

Lemma 6.3.3. *Let $M = (E, G)$ be an r-dimensional, full-rank, I_5-free and triangle-free matroid with a C_5-restriction on a flat F, so that $M|F \cong C_5$. Then, $|E| \geq 2^{\lfloor r/2 \rfloor - 1} + 2^{\lceil r/2 \rceil - 1}$. Moreover, when $r \geq 6$, this is a strict inequality.*

Proof. By Lemma 6.3.1, M/F is I_3-free. It is also full-rank; otherwise, M is rank-deficient. By Theorem 6.1.1, this means $|M/F| \geq 2^{\lfloor r-4/2 \rfloor} + 2^{\lceil r-4/2 \rceil} - 2$. We may assume that there exists some intersecting coset of F with respect to M that contains exactly 1 element; otherwise,

$$\begin{aligned} |E| \geq |M|F| + 2|M/F| &= 5 + 2(2^{\lfloor r-4/2 \rfloor} + 2^{\lceil r-4/2 \rceil} - 2) \\ &= 1 + 2^{\lfloor r/2 \rfloor - 1} + 2^{\lceil r/2 \rceil - 1} \\ &> 2^{\lfloor r/2 \rfloor - 1} + 2^{\lceil r/2 \rceil - 1}. \end{aligned}$$

which already satisfies the lemma.

Let A be a coset of F for which $|A \cap E| = 1$. Let $\mathcal{C}$ be the set of cosets of F. We will partition the intersecting cosets of F as follows by T_1 and T_0.

$$T_1 = \{B \mid B \neq A, |B \cap E| > 0, |(A + B) \cap E| > 0\}$$

$$T_0 = \{B \mid B \neq A, |B \cap E| > 0, |(A + B) \cap E| = 0\}.$$

Note that a coset B belongs to T_1 if and only if $A + B$ belongs to T_1. In particular, $|T_1|$ is even, and we have that $\sum_{B \in T_1} |B \cap E| = \sum_{B \in T_1} |(A + B) \cap E|$.

By Lemma 6.3.2 , we have that if $B \in T_1$, then $|B \cap E| + |(A + B) \cap E| \geq 4$. Therefore

$$\sum_{B \in T_1} |B \cap E| = \frac{1}{2} \sum_{B \in T_1} (|B \cap E| + |(A + B) \cap E|) \geq 2|T_1|.$$

By Lemma 6.3.2 again, for every $B \in T_0$, we have that $|B \cap E| \geq 3$. Hence, we have

$$\sum_{B \in T_0} |B \cap E| \geq 3|T_0|.$$

Therefore we have

$$
\begin{aligned}
|E| &= |M|F| + \sum_{B\in\mathcal{C}} |B \cap E| \\
&= |M|F| + |A \cap E| + \sum_{B\in T_1} |B \cap E| + \sum_{B\in T_0} |B \cap E| \\
&\geq 5 + 1 + 2|T_1| + 3|T_0| \\
&\geq 6 + 2(|T_1| + |T_0|) \\
&= 6 + 2(|M/F| - 1) \\
&\geq 6 + 2(2^{\lfloor (r-4)/2 \rfloor} + 2^{\lceil (r-4)/2 \rceil} - 3) \\
&= 2^{\lfloor r/2 \rfloor - 1} + 2^{\lceil r/2 \rceil - 1},
\end{aligned}
$$

This proves the bound. Moreover, note that equality can only be achieved when T_0=0 and $|M/F| = 2^{\lfloor (r-4)/2 \rfloor} + 2^{\lceil (r-4)/2 \rceil} - 2$, which implies $|T_1| = 2^{\lfloor (r-4)/2 \rfloor} + 2^{\lceil (r-4)/2 \rceil} - 3$. But note that T_1 is always even by definition, which is a contradiction, unless $r = 5$. This completes the proof. □

6.4 Largest Affine Geometries

In the previous section, we showed that the smallest matroids with no I_5-restriction or triangle must be affine when $r \geq 6$ by showing that they cannot contain a C_5-restriction (which is the only obstruction to such matroids being affine). The goal of this section is to study the matroids that are I_5-free, triangle-free and C_5-free.

We divide this section into three subsections. Given an I_5-free, triangle-free and C_5-free matroid $M = (E, G)$, suppose that $M|F$ is a largest affine geometry contained in M. We show that M/F is close to being I_3-free in some sense. In the second subsection, we show that M/F is always $2T$-free ($2T$ is the four-dimensional matroid consisting of two disjoint triangles). In the third subsection, we determine the smallest I_3-free and $2T$-free matroids. Combining these pieces in the next section will yield our result.

6.4.1 I_3-freeness and Kites

In this subsection, we will use the following simple lemma.

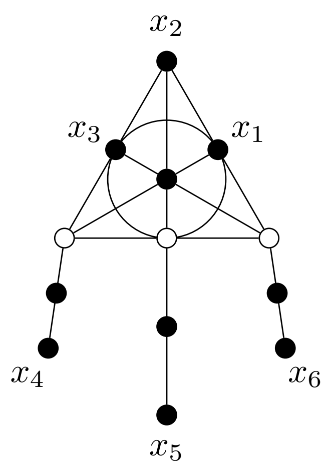

Figure 6.1: Kite

Lemma 6.4.1. *If G is a projective geometry of dimension at least 2, and W_1, W_2, W_3 are proper flats of G for which $W_1 \cup W_2 \cup W_3 = G$, then*

- *W_i is a hyperplane of G for each $i = 1, 2, 3$, and*
- *$W_1 \cap W_2 \cap W_3$ is a flat of G of codimension 2.*

Doubled Kites

The *kite* is the 6-dimensional matroid (E, G) with basis $B = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ of $[G]$ such that $E = B \cup \{x_1+x_2+x_3, x_2+x_3+x_4, x_1+x_3+x_5, x_1+x_2+x_6\}$ and is denoted M_K. Note that $|M_K| = 10$. See Figure 6.1. The kite can also be obtained from the matroid F_7 by picking a triangle and series-extending each element of the triangle. Note that the matroid $D^k(M_K)$ has dimension $(k+6)$. A matroid that is isomorphic to $D^k(M_K)$ for some $k \geq 0$ is called a *doubled kite*.

We remark that the matroid M_K contains an induced $D(I_3)$-restriction and an induced C_6-restriction; this fact will become important later. More specifically, in the above representation, $M_K|\operatorname{cl}(x_4, x_1, x_2, x_2+x_3) \cong D(I_3)$, and $M_K|\operatorname{cl}(x_4, x_5, x_6, x_1+x_2+x_6, x_1+x_3+x_5) \cong C_6$.

Suppose that M is an I_5-free, triangle-free and C_5-free matroid, and that $M|F$ is a largest affine geometry restriction contained in M. We show below that if M/F fails to be I_3-free, then M contains an induced doubled kite.

Lemma 6.4.2. *Let $M|F$ be an affine geometry restriction contained in an I_5-free, triangle-free and C_5-free matroid M of dimension at least 3. Let A_1, A_2, A_3 be three intersecting cosets of F with respect to M for which $A_1 + A_2 \neq A_3$ and $A_1 + A_2$, $A_1 + A_3$, $A_2 + A_3$ and $A_1 + A_2 + A_3$ all have no intersection with E. Then either*

- *$M|\operatorname{cl}(F \cup A_i)$ is an affine geometry for some $i \in \{1, 2, 3\}$, or*
- *$M|\operatorname{cl}(F \cup A_1 \cup A_2 \cup A_3) \cong D^{\dim(F)-3}(M_K)$.*

Furthermore, if the second outcome holds, then there exists a flat $W \subseteq F$ such that for any transversal (a_1, a_2, a_3) of $(E \cap A_1, E \cap A_2, E \cap A_3)$, there exist distinct hyperplanes W_1, W_2, W_3 of W for which

- *$E \cap \operatorname{cl}(F \cup A_1 \cup A_2 \cup A_3) = (F \backslash W) \cup \bigcup_{i=1}^{3}(a_i + [W_i])$, and*
- *$W_1 \cap W_2 \cap W_3$ is a codimension-2 flat of W.*

Proof. Let W be the hyperplane of F for which $E \cap W = \varnothing$. Suppose that $M|\operatorname{cl}(F \cup A_i)$ is not an affine geometry for any $i = 1, 2, 3$. For each i, let $\operatorname{Stab}(A_i \cap E) = \{w \in W | w + A_i \cap E = A_i \cap E\}$. Note that $\operatorname{Stab}(A_i \cap E)$ is a flat of W.

6.4.2.1. $W = \displaystyle\bigcup_{i=1}^{3} \operatorname{Stab}(A_i \cap E)$.

Subproof: Let $w \in W$. If $w \notin \cup_{i=1}^{3} \operatorname{Stab}(A_i \cap E)$, then there exists $a_i \in A_i \cap E$ for $i = 1, 2, 3$ for which $w + a_i \notin A_i \cap E$. Pick any $z \in E \cap F$. Then $\{z, z + w, a_1, a_2, a_3\}$ is an induced I_5-restriction (by the remark made in Subsection 6.2.1, checking that this is an induced I_5-restriction amounts to having $a_i + z + (z + w) = a_i + w \notin E$ for $i = 1, 2, 3$). ■

6.4.2.2. *For $i = 1, 2, 3$, $\operatorname{Stab}(A_i \cap E)$ is a hyperplane of W. Moreover, the three hyperplanes intersect at a codimension-2 flat of W.*

Subproof: By Lemma 6.4.1 and 6.4.2.1, it suffices to show that $W_i = \operatorname{Stab}(A_i \cap E)$ is a proper flat of W for each $i = 1, 2, 3$. But since $M|\operatorname{cl}(F \cup A_i)$ is not an affine geometry for any $i = 1, 2, 3$, it follows that W_i is a proper flat of W. ■

Let $W_i = \operatorname{Stab}(A_i \cap E)$. Now we claim that $M|\operatorname{cl}(F \cup A_1 \cup A_2 \cup A_3)$ is a doubled kite. Let $W' = W_1 \cap W_2 \cap W_3$. By the above, W' has dimension $\dim(W) - 2$. Pick a triangle $T = \{t_1, t_2, t_3\} \subseteq W \backslash W'$ where $t_i \in W_i$, and fix $a_i \in A_i \cap E$ for $i = 1, 2, 3$, and $x \in F \backslash W$.

Write $F' = \operatorname{cl}(T \cup \{a_1, a_2, a_3, x\})$. Then $E \cap F' = (x + [T]) \cup \bigcup_{i=1}^{3}\{a_i, a_i + t_i\}$.

6.4.2.3. *$M|F' \cong M_K$.*

Subproof: Write $B = (x+T) \cup \{a_1, a_2, a_3\} \subseteq E$. From the definition of kites, it is easy to check that $M|F' \cong M_K$. ■

6.4.2.4. *$M|\,\mathrm{cl}(F \cup A_1 \cup A_2 \cup A_3)$ is a doubled kite.*

Subproof: We use Lemma 2.5.2. Note that $W' \cap F' = \varnothing$, and $\mathrm{cl}(F' \cup W') = \mathrm{cl}(F \cup A_1 \cup A_2 \cup A_3)$. Note that $([W'] + t_i) \cup W' = W_i$ for every $i = 1, 2, 3$.

$$\begin{aligned}
[W'] + (E \cap F') &= \bigcup_{i=1}^{3}([W'] + a_i) \cup \bigcup_{i=1}^{3}([W'] + t_i + a_i) \cup \bigcup_{i=1}^{3}([W'] + x + t_i) \cup ([W'] + x) \\
&= \bigcup_{i=1}^{3}([W_i] + a_i) \cup \bigcup_{i=1}^{3}(W_i \backslash W' + x) \cup ([W'] + x) \\
&= \bigcup_{i=1}^{3}([W_i] + a_i) \cup ([W] + x) \\
&= \bigcup_{i=1}^{3}([W_i] + a_i) \cup (F \backslash W) \\
&= E \cap \mathrm{cl}(F \cup A_1 \cup A_2 \cup A_3).
\end{aligned}$$

■

If the second statement in the theorem statement holds, then $E \cap \mathrm{cl}(F \cup A_1 \cup A_2 \cup A_3) = (F \backslash W) \cup \bigcup_{i=1}^{3}(a_i + [W_i])$ follows from the above calculation. □

In the proof of the main theorem, given a matroid $M = (E, G)$ and a flat F for which $M|F$ is a largest affine geometry restriction contained in M, we will consider the matroid M/F. The case where M/F is I_3-free is pleasant, as we understand I_3-free matroids well. In the case M/F leads to a doubled kite, then we can use the existence of a doubled kite to conclude that M contains too many elements. First, we make the following observation; it follows from the fact that the matroid M_K contains an induced $D(I_3)$-restriction and an induced C_6-restriction.

Lemma 6.4.3. *The matroid $D^k(M_K)$ contains an induced $D^{k+1}(I_3)$-restriction and an induced $D^k(C_6)$-restriction.*

We first focus on the role of the $D^{k+1}(I_3)$-restriction contained in the matroid $D^k(M_K)$.

Lemma 6.4.4. *For an I_5-free, C_5-free and triangle-free matroid $M = (E, G)$, suppose that $k + 3$ is the dimension of a maximum affine geometry restriction contained in M, where $k \geq 0$. If F' is a flat for which $M|F' \cong D^{k+1}(I_3)$, then M/F' is claw-free.*

Proof. By Lemma 2.5.2, there exist disjoint flats D and F of F' where D is $(k+1)$-dimensional, $F' = \mathrm{cl}(D \cup F)$, $M|F \cong I_3$ and $E \cap \mathrm{cl}(D \cup F) = [D] + (E \cap F)$ (so that $M|F' \cong D^{k+1}(I_3)$). Write $F \cap E = \{x_1, x_2, x_3\}$.

Suppose for a contradiction that M/F' contains a claw, corresponding to three intersecting cosets A_1, A_2, A_3 of F' with respect to E for which $A_1 + A_2 \neq A_3$, and $A_1 + A_2$, $A_1 + A_3$, $A_2 + A_3$, $A_1 + A_2 + A_3$ have no intersection with E. Pick an element $y_i \in E \cap A_i$ for $i = 1, 2, 3$. For $i = 1, 2, 3$, let $U_i = \{v \in F' \backslash E \mid y_i + v \in E\} = (y_i + E) \cap F'$.

6.4.4.1. $U_i \cap (x_1 + x_2 + x_3 + [D]) = \varnothing$*, for* $i = 1, 2, 3$.

Subproof: If there exists $v \in U_i \cap (x_1 + x_2 + x_3 + [D])$, then $\{y_i, v + y_i, x_1, x_2, v + x_1 + x_2\}$ is a C_5-restriction (to check this, because of triangle-freeness, it suffices to note that the five elements sum to zero). ■

6.4.4.2. $x_1 + x_2 + [D] \subseteq \bigcap_{i=1}^{3} U_i$

Subproof: Note that $\mathrm{cl}(D \cup \{x_1, x_2\})$ is a largest affine geometry restriction of M since it has dimension $k+3$. Hence by applying Lemma 6.4.2 to the cosets $A_i' = \mathrm{cl}(D \cup \{x_1, x_2, y_i\}) \subseteq A_i$ for $i = 1, 2, 3$, the matroid $M|\,\mathrm{cl}(D \cup \{x_1, x_2, y_1, y_2, y_3\})$ is a doubled kite. Hence, by the last statement of Lemma 6.4.2, for every element $v \in x_1 + x_2 + [D]$, either there exists a unique $l \in \{1, 2, 3\}$ for which $v \in U_l$, or $v \in U_i$ for every $i = 1, 2, 3$. If, for some $v \in x_1 + x_2 + [D]$, there exists a unique $l \in \{1, 2, 3\}$ for which $v \in U_l$, then $\{y_1, y_2, y_3, v + y_l, x_3\}$ is an induced I_5-restriction, a contradiction. Hence, $x_1 + x_2 + [D] \subseteq \bigcap_1^3 U_i$. ■

This is enough to derive a contradiction. Note again, by Lemma 6.4.2 and the fact that $\mathrm{cl}(D \cup \{x_1, x_2\})$ is a largest affine geometry restriction of M, that $M|\,\mathrm{cl}(D \cup \{x_1, x_2, y_1, y_2, y_3\})$ is a doubled kite, and by the last statement of Lemma 6.4.2, this implies that $J = \mathrm{cl}(D \cup \{x_1, x_2\}) \cap U_1 \cap U_2 \cap U_3$ is a flat of codimension 2 of $\mathrm{cl}(D \cup \{x_1 + x_2\})$. Since $\dim(\mathrm{cl}(D \cup \{x_1 + x_2\})) = \dim(D) + 1 = k + 2$, this means that $\dim(J) = \dim(\mathrm{cl}(D \cup \{x_1 + x_2\})) - 2 = k$. In particular, this implies that $|J| = 2^k - 1$. But by 6.4.4.2, we have that $x_1 + x_2 + [D] \subseteq J$. But $|(x_1 + x_2 + [D])| = 2^{k+1}$, which implies $2^{k+1} \leq 2^k - 1$, a contradiction. □

The next goal is to estimate the number of elements in a matroid that contains an induced $D^k(C_6)$-restriction. To do so, we estimate their sizes at the coset level. For each $k \geq 0$, let d_k be the smallest positive integer such that there exists a I_5-free, triangle-free and C_5-free matroid $M = (E, G)$ and a hyperplane $H \subseteq G$ with $M|H \cong D^k(C_6)$ and $|E \backslash H| = d_k$. We will provide a lower bound on d_k in two steps. In the first step we will show that $d_0 \geq 4$.

Lemma 6.4.5. *$d_0 \geq 4$.*

Proof. Let $M = (E, G)$ be a 6-dimensional matroid with a hyperplane $H \subseteq G$ for which $M|H \cong C_6$. Suppose that $|E\backslash H| > 0$. Our aim is to show that $|E\backslash H| \geq 4$. Fix $v \in E\backslash H$. We note that $|E\backslash H| \geq 2$; otherwise $\{v, v_1, v_2, v_3, v_4\}$ is an induced I_5-restriction for any four elements $v_i \in E \cap H$, $i = 1, 2, 3, 4$. Because of C_5-freeness, $v + v_1 + v_2 + v_3 \notin E$ for any three distinct $v_1, v_2, v_3 \in E \cap H$. Hence we have some element $v' \in E\backslash H$ of the form $v' = v + v_1 + v_2$ for some two $v_1, v_2 \in E \cap H$.

Now, consider the flat $F = \mathrm{cl}(v_1, v_3, v_4, v_5)$ for any three $v_3, v_4, v_5 \in E \cap H$ other than v_2. Now note that $M|\,\mathrm{cl}(F \cup \{v\})$ is an induced I_5-restriction unless $|E \cap (v + F)| \neq 0$, and similarly $M|\,\mathrm{cl}(F \cup \{v'\})$ is an induced I_5-restriction unless $|E \cap (v' + F)| \neq 0$. But $(v + F') \cap (v' + F) = \varnothing$. Hence $|E\backslash H| \geq 4$. $\square$

Lemma 6.4.6. *$d_k \geq 3 \cdot 2^k + 1$ for $k \geq 0$.*

This will follow directly from the following lemma by setting $\mathcal{P}$ to be the family of matroids that are I_5-free, C_5-free and triangle-free, and $N = C_6$. A class of matroids $\mathcal{P}$ is called *hereditary* if it is closed under taking induced restrictions.

Lemma 6.4.7. *Let $\mathcal{P}$ be a hereditary class of matroids, and let $N \in \mathcal{P}$. Suppose that for every $M = (E, G) \in \mathcal{P}$ having a hyperplane H with $M|H \cong N$ and $|E\backslash H| > 0$, we have $|E\backslash H| > t$.*

Then, if $M = (E, G) \in \mathcal{P}$ and there exists a hyperplane H for which $M|H \cong D^k(N)$ and $|E\backslash H| > 0$, then $|E\backslash H| > t \cdot 2^k$.

Proof. Suppose that $M = (E, G) \in \mathcal{P}$ and there is a hyperplane $H \subseteq G$ for which $M|H \cong D^k(N)$. Using Lemma 2.5.2, let H' be a flat of H such that $M|H' \cong N$, and let D be a k-dimensional flat for which $\dim(D) + \dim(H') = \dim(H)$ and $[D] + (E \cap H') = H \cap E$. Let $\mathcal{F}$ be the set of $dim(N)$-dimensional flats F of H such that $F \cap D = \varnothing$ (and hence $M|F \cong N$ by Lemmas 2.4.2 and 2.5.3).

Fix $v \in E \backslash H$. Let $S = \{(F, e) \mid F \in \mathcal{F}, e \in (F + v) \cap E\}$. By assumption, we know that $|S| \geq t|\mathcal{F}|$. Observe that given an element $y \in H \backslash D$, the number of $\dim(N)$-dimensional flats $F \in \mathcal{F}$ that contain y is $|\mathcal{F}|/2^k$. This can be seen as follows. Let X be the number of elements of $\mathcal{F}$ that contains a given element $y \in H \backslash D$ (note that this does not depend on the choice of y by symmetry). Then counting pairs of an element F of $\mathcal{F}$ and an element $y \in F$ in two different ways, we obtain that $(2^{k+\dim(N)} - 2^k)X = |\mathcal{F}|(2^{\dim(N)} - 1)$, giving us $X = |\mathcal{F}|/2^k$.

Therefore, we have the following.

$$
\begin{aligned}
|S| &= \sum_{e \in ((H \backslash D)+v) \cap E} |\{F \in \mathcal{F} \mid e \in F + v\}| \\
&= \sum_{e \in ((H \backslash D)+v) \cap E} |\{F \in \mathcal{F} \mid e + v \in F\}| \\
&= \sum_{e \in ((H \backslash D)+v) \cap E} |\mathcal{F}|/2^k.
\end{aligned}
$$

Hence it follows that $|((H \backslash D)+v) \cap E| \cdot |\mathcal{F}|/2^k \geq t \cdot |\mathcal{F}|$. Therefore, $|(H+v) \cap E| \geq t \cdot 2^k$. Adding the fixed element v, we obtain that $|E \backslash H| > t \cdot 2^k$. □

We now combine Lemmas 6.4.4 and 6.4.6

Lemma 6.4.8. *Let $M = (E, G)$ be a full-rank, r-dimensional, I_5-free, C_5-free and triangle-free matroid. Let $k+3$ be the dimension of a maximum affine geometry restriction contained in M where $k \geq 0$, and suppose that M contains an induced $D^k(M_K)$-restriction. Then $|E| > 2^{\lfloor r/2 \rfloor - 1} + 2^{\lceil r/2 \rceil - 1}$.*

Proof. Let F be a flat for which $M|F \cong D^k(M_K)$; note that the matroid $D^k(M_K)$ contains an induced $D^{k+1}(I_3)$-restriction. Let $M' = (E', F')$ be the contraction of M by F. By Lemma 6.4.4, M' is I_3-free. Also, M' is full-rank, as otherwise M is rank-deficient. By Theorem 6.1.1, $|E'| \geq 2^{\lfloor (r-k-6)/2 \rfloor} + 2^{\lceil (r-k-6)/2 \rceil} - 2$.

Note that $M|F$ contains an induced $D^k(C_6)$-restriction. By Lemma 6.4.6, the cosets of F corresponding to the elements of E' contain at least $3 \cdot 2^k + 1$ elements of E. Note

$3 \cdot 2^k + 1 \geq 4 \cdot (3/2)^k$ for $k \geq 0$. Hence

$$\begin{aligned}
|E| &\geq |E \cap F| + |E'| \cdot 4 \cdot (3/2)^k \\
&\geq 10 \cdot 2^k + (2^{\lfloor (r-k-6)/2 \rfloor} + 2^{\lceil (r-k-6)/2 \rceil} - 2) \cdot 4 \cdot (3/2)^k \\
&= 10 \cdot 2^k - 8 \cdot (3/2)^k + \frac{1}{2} \cdot (3/2)^k \cdot (2^{\lfloor (r-k)/2 \rfloor} + 2^{\lceil (r-k)/2 \rceil}).
\end{aligned}$$

When $k = 0$, then substituting this into the above implies that $|E| > 2^{\lfloor r/2 \rfloor - 1} + 2^{\lceil r/2 \rceil - 1}$. When $k \geq 1$, we can check that

$$\begin{aligned}
(3/2)^k \cdot (2^{\lfloor (r-k)/2 \rfloor} + 2^{\lceil (r-k)/2 \rceil}) &\geq (3/2)^k \cdot 2 \cdot 2^{(r-k)/2} \\
&\geq \frac{3}{\sqrt{(2)}} \cdot 2^{r/2} \\
&\geq 2^{\lfloor r/2 \rfloor} + 2^{\lceil r/2 \rceil}.
\end{aligned}$$

Hence $|E| > 2^{\lfloor r/2 \rfloor - 1} + 2^{\lceil r/2 \rceil - 1}$ when $k \geq 1$ as well.

□

6.4.2 $2T$-freeness

Given an I_5-free, triangle-free and C_5-free matroid $M = (E, G)$, with $M|F$ being a maximum affine geometry restriction contained in M, we will now further restrict the number of elements that are allowed on M/F by finding another excluded induced restriction. The matroid $2T$ is the 4-dimensional matroid whose ground set consists of two disjoint triangles. We now show that M/F is $2T$-free.

We now state and prove the main lemma of this subsection. Note that we start with a *maximal* affine geometry restriction $M|F$, not a maximum affine geometry restriction, in the following lemma, meaning that there is no $x \in G \backslash F$ for which $M|\operatorname{cl}(F \cup \{x\})$ is an affine geometry restriction.

Lemma 6.4.9. *Let $M|F$ be a maximal affine geometry restriction of an I_5-free, triangle-free and C_5-free matroid, where $\dim(F) \geq 3$. Then M/F is $2T$-free, or M contains an affine geometry restriction of dimension $\dim(F) + 1$.*

Proof. Let W be the hyperplane of F for which $W \cap E = \varnothing$.

Suppose for a contradiction that M/F contains an induced $2T$-restriction. Therefore, there exist 6 distinct intersecting cosets of F with respect to M, $A_1, A_2, A_3, B_1, B_2, B_3$, such that $A_1 + A_2 = A_3$, $B_1 + B_2 = B_3$, and $(A_i + B_j) \cap E = \varnothing$ for every $i, j \in \{1, 2, 3\}$.

We now make the following sequence of claims to uncover the structure of M.

6.4.9.1. *For any two distinct $i, j \in \{1, 2, 3\}$, if $a_i \in A_i \cap E$ and $a_j \in A_j \cap E$, then $(a_i + a_j + [W]) \cap E = \varnothing$.*

Subproof: Since M is triangle-free, $a_i + a_j \notin E$. If there exists $a_k \in (a_i + a_j + W) \cap E$, then $\{a_i, a_j, a_k, z, a_i + a_j + a_k + z\}$ is a C_5-restriction for any $z \in F \cap E$, a contradiction. ■

6.4.9.2. *There exists $y \in F \cap E$ such that for any $\{i, j, k\} = \{1, 2, 3\}$, $A_i \cap E + A_j \cap E + y \subseteq A_k \cap E$.*

Subproof: Pick $b_1 \in B_1 \cap E$ and $b_2 \in B_2 \cap E$. As M is triangle-free, $b_1 + b_2 \notin B_3 \cap E$. Due to maximality, we may pick $y \in F \cap E$ such that $b_1 + b_2 + y \notin E$; if not, then $M|\operatorname{cl}(F \cup \{b_1 + b_2\})$ is a strictly larger affine geometry restriction containing $M|F$.

Let $a_i \in A_i \cap E$ and $a_j \in A_j \cap E$. Then $y + a_i + a_j \in A_k \cap E$; otherwise $\{y, a_i, a_j, b_1, b_2\}$ is an induced I_5-restriction. ■

6.4.9.3. *For distinct $i, j \in \{1, 2, 3\}$, $|(A_i \cap E) + (A_j \cap E)| = |A_i \cap E|$.*

Subproof: Let $k \in \{1, 2, 3\} \backslash \{i, j\}$. By 6.4.9.2, we have that $|A_i \cap E + A_j \cap E| \leq |A_k \cap E|$. Also, $|A_k \cap E| \leq |A_k \cap E + A_j \cap E|$. By applying 6.4.9.2 again, we have that $|A_k \cap E + A_j \cap E| \leq |A_i \cap E|$. Hence $|A_i \cap E + A_j \cap E| \leq |A_i \cap E|$. The claim follows. ■

Note that 6.4.9.3 implies in particular that $|A_1 \cap E| = |A_2 \cap E| = |A_3 \cap E|$.

Let $\operatorname{Stab}(A \cap E) = \{w \in [W] \,|\, w + A \cap E = A \cap E\}$. $\operatorname{Stab}(A \cap E)$ is a subspace. Note that trivially $|A \cap E| \geq |\operatorname{Stab}(A \cap E)|$ when $|A \cap E| > 0$. We now claim the following.

6.4.9.4.

- *For $i = 1, 2, 3$, $|A_i \cap E| = |\operatorname{Stab}(A_i \cap E)|$,*
- $\operatorname{Stab}(A_1 \cap E) = \operatorname{Stab}(A_2 \cap E) = \operatorname{Stab}(A_3 \cap E)$.

Subproof: Fix $i \in \{1,2,3\}$, and take some other $j \neq i$, $j \in \{1,2,3\}$. By 6.4.9.3, $|A_i \cap E + A_j \cap E| = |A_i \cap E|$. Let $a \in A_j \cap E$. Then

$$|A_i \cap E| = |A_i \cap E + A_j \cap E| \geq |A_i \cap E + a| = |A_i \cap E|.$$

This implies that $A_i \cap E + A_j \cap E = A_i \cap E + a$. Hence $A_j \cap E \subseteq \operatorname{Stab}(A_i \cap E) + a$. Therefore

$$|\operatorname{Stab}(A_i \cap E)| \geq |A_j \cap E| = |A_i \cap E| \geq |\operatorname{Stab}(A_i \cap E)|.$$

Hence, it follows that $|A_i \cap E| = |\operatorname{Stab}(A_i \cap E)|$. Moreover, $A_j \cap E = \operatorname{Stab}(A_i \cap E) + a$. Taking the stabiliser on both sides, it follows that $\operatorname{Stab}(A_i \cap E) = \operatorname{Stab}(A_j \cap E)$. ∎

6.4.9.4 implies that the three stabilisers coincide on a subspace, and that for each $i = 1,2,3$, $A_i \cap E$ is a coset of that flat.

Now, we may run the same argument, with the role of the A cosets swapped with that of the B cosets. This gives the following.

6.4.9.5.

- $\operatorname{Stab}(A_1 \cap E) = \operatorname{Stab}(A_2 \cap E) = \operatorname{Stab}(A_3 \cap E)$.
- $\operatorname{Stab}(B_1 \cap E) = \operatorname{Stab}(B_2 \cap E) = \operatorname{Stab}(B_3 \cap E)$.
- *For any* $X \in \{A_1, A_2, A_3, B_1, B_2, B_3\}$, $|X \cap E| = |\operatorname{Stab}(X \cap E)|$.

Let F_A be the flat of W for which $F_A = \operatorname{Stab}(A_1 \cap E) \backslash \{0\}$, and F_B another flat of W for which $F_B = \operatorname{Stab}(B_1 \cap E) \backslash \{0\}$.

Fix any two elements $a_1 \in A_1 \cap E$ and $a_2 \in A_2 \cap E$, and similarly $b_1 \in B_1 \cap E$ and $b_2 \in B_2 \cap E$. Then by 6.4.9.1 and 6.4.9.5, we see that $E \cap A_i = [F_A] + a_i$ for $i = 1,2$, and $E \cap A_3$ equals the set $a_1 + a_2 + A_3'$ where $A_3' = [F_A] + y_A$ for some $y_A \in F \cap E$. Similarly, $E \cap B_i = [F_B] + b_i$ for $i = 1,2$, and $E \cap B_3$ equals the set $b_1 + b_2 + B_3'$ where $B_3' = [F_B] + y_B$ for some $y_B \in F \cap E$.

6.4.9.6. $A_3' \cup B_3' = F \cap E$.

Subproof: Suppose not. Pick $z \in (F \cap E) \backslash (A_3' \cup B_3')$. Then $\{a_1, a_2, b_1, b_2, z\}$ is an induced I_5-restriction, a contradiction. ∎

6.4.9.7. *F_A and F_B are hyperplanes of W.*

Subproof: First note that $F_A, F_B \neq W$; otherwise $M|\operatorname{cl}(F \cup A_1)$ or $M|\operatorname{cl}(F \cup B_1)$ is a strictly larger affine geometry restriction containing $M|F$.

Now, suppose for a contradiction that at least one of F_A or F_B, say F_A without loss of generality, has codimension at least 2. But then $|(F \cap E)\backslash(A_3' \cup B_3')| \geq 2^{\dim(F)-1} - 2^{\dim(F_A)} - 2^{\dim(F_B)} \geq 2^{\dim(F)-1} - 2^{\dim(F)-3} - 2^{\dim(F)-2} = 2^{\dim(F)-3} > 0$. This contradicts 6.4.9.6. Therefore both F_A and F_B are hyperplanes of W. ■

By 6.4.9.7, F_A is a hyperplane. Note that $E \cap \operatorname{cl}(F_A \cup \{y_A, a_1, a_2\}) = [F_A] + \{y_A, a_1, a_2, y_A + a_1 + a_2\}$. Hence $\operatorname{cl}(F_A \cup \{y_A, a_1, a_2\}) \backslash E = \operatorname{cl}(F_A \cup \{y_A + a_1, y_A + a_2\})$, which is a flat of dimension $\dim(F_A) + 2 = \dim(W) + 1$. Therefore it follows that $M|\operatorname{cl}(F_A \cup \{y_A, a_1, a_2\})$ is an affine geometry restriction of dimension $\dim(F) + 1$. □

6.4.3 I_3-freeness and $2T$-freeness

In this subsection, we determine the smallest I_3-free, $2T$-free matroids. The proof uses Lemma 2.1.1.

We remark that it is possible to use the structural theorem for claw-free matroids to give an alternative proof; we opt for a more direct approach involving less machinery. We write $\mathrm{PGS}(t_1, t_2)$ for the $(t_1 + t_2)$-dimensional matroid whose ground set consists of two disjoint flats of dimensions t_1 and t_2.

Lemma 6.4.10. *Let $M = (E, G)$ be an r-dimensional, full-rank matroid that is I_3-free and $2T$-free. Then $|E| \geq 2^{r-1}$. Moreover, equality holds if and only if $M \cong \mathrm{AG}(r-1, 2)$ or $M \cong D^k(\mathrm{PGS}(1, t))$ for some $t \leq r$ and $k \geq 0$*

Proof. The result is trivially true when $r = 1$. Let M be a minimum counterexample, on $\dim(M)$.

We first claim that for every proper flat $F \subseteq G$, there exists a mixed coset A of F with respect to M, meaning $0 < |A \cap E| < |A|$. If not, then every coset A of F with respect to M is unmixed. Note first that M/F is full-rank; otherwise M is rank-deficient. Moreover, since F has no mixed cosets with respect to M, for any flat $\bar{F} \subseteq G$ for which $F \cap \bar{F} = \varnothing$, it follows that $M/F \cong M|\bar{F}$. Since $M|\bar{F}$ is an induced restriction of M, $M/F \cong M|\bar{F}$ is I_3-free and $2T$-free. Since $\dim(M/F) < \dim(M)$, it follows that $|M/F| \geq 2^{r-\dim(F)-1}$. Each intersecting coset of F with respect to M contains $2^{\dim(F)}$ elements of E, so we have that

$|E| \geq |E \cap F| + 2^{r-\dim(F)-1} \cdot 2^{\dim(F)} \geq 2^{r-1}$. Moreover, $|E| = 2^{r-1}$ if and only if $|F \cap E| = 0$ and the bound is attained for M/F. Hence, it follows that $M \cong D^{\dim(F)}(M/F)$, and that M/F is either an affine geometry or $M/F \cong D^k(\mathrm{PGS}(1,t))$ for some k. Hence M is an affine geometry, or $M \cong D^{\dim(F)+k}(\mathrm{PGS}(1,t))$, which contradicts minimality. Hence we may assume that, for every proper flat $F \subseteq G$, there exists some mixed coset of F with respect to M. We will now argue that $M \cong \mathrm{PGS}(1, r-1)$, which will give a contradiction.

6.4.10.1. *There exists a hyperplane H of G for which $M|H$ is rank-deficient.*

Subproof: Suppose not, so that $M|H$ is full-rank for every hyperplane H. By minimality, $|E \cap H| \geq 2^{r-2}$. Let $\mathcal{H}$ be the set of hyperplanes of G. Note that any given element in G is contained in precisely $\binom{r-1}{r-2}_2$ hyperplanes of G. By a standard double counting argument, we have the following. Note that the quantity $\binom{p}{q}_2$ is a Gaussian binomial coefficient and equals $\frac{(1-2^p)(1-2^{p-1})\cdots(1-2^{p-q+1})}{(1-2)(1-2^2)\cdots(1-2^q)}$.

$$
\begin{aligned}
|E| = \frac{1}{\binom{r-1}{r-2}_2} \sum_{H \in \mathcal{H}} |E \cap H| &\geq \frac{\binom{r}{r-1}_2}{\binom{r-1}{r-2}_2} \cdot 2^{r-2} \\
&> 2 \cdot 2^{r-2} \\
&= 2^{r-1}
\end{aligned}
$$

This contradicts the minimality of M. ■

Let H be a hyperplane of G for which $M|H$ is rank-deficient. Let F be a hyperplane of H such that $E \cap H \subseteq F$. Let A_1 and A_2 be the remaining two cosets of F. We may assume that $A_i \cap E \neq \varnothing$ for $i = 1, 2$; otherwise M is rank-deficient.

6.4.10.2. *If $v \in ((A_i \cap E) + (A_i \cap E)) \backslash E$ for any $i \in \{1, 2\}$, then $v + w \in E$ for all $w \in (E \cap A_1) \cup (E \cap A_2)$.*

Subproof: Observe that if $v \in ((A_i \cap E) + (A_i \cap E)) \backslash E$, then for any $b \in E \cap A_{3-i}$, $v + b \in E$; write $v = a_1 + a_2$ where $a_1, a_2 \in A_i \cap E$, then otherwise $\{a_1, a_2, b\}$ is a claw. Furthermore, since $|E \cap A_{3-i}| > 0$, there exists some $b \in E \cap A_{3-i}$, and by this observation, $v + b \in E \cap A_{3-i}$, so $v \in ((A_{3-i} \cap E) + (A_{3-i} \cap E)) \backslash E$. Applying the same observation, we conclude that if $v \in ((A_i \cap E) + (A_i \cap E)) \backslash E$ for some $i \in \{1, 2\}$, then in fact $v + w \in E$ for all $w \in (E \cap A_1) \cup (E \cap A_2)$. ■

6.4.10.3. $((A_1 \cap E) + (A_1 \cap E)) \backslash E = ((A_2 \cap E) + (A_2 \cap E)) \backslash E$.

Subproof: Let $v \in ((A_i \cap E) + (A_i \cap E))\backslash E$. Since $|E \cap A_{3-i}| > 0$, take $b \in E \cap A_{3-i}$. By 6.4.10.2, $v + b \in E \cap A_{3-i}$. ∎

6.4.10.4. $((A_1 \cap E) + (A_1 \cap E)) \subseteq E$.

Subproof: Let $P = ((A_1 \cap E) + (A_1 \cap E))\backslash E$, $Q = F \cap E$, $R = F\backslash(P \cup Q)$. We check that there are no triangles T in F for which $|T \cap P| \geq 1$ and $|T \cap R| = 1$. Suppose for a contradiction that such a triangle $T = \{v, w, v+w\}$ exists, with $v \in P$ and $w \in R$. Fix an element $a_1 \in E \cap A_1$. Since $v \in P$ it follows from 6.4.10.2 that $v + a_1 \in E$. Then, since $w \in R$, $a_1+w \notin E$ and $a_1+v+w \notin E$. Now, if $v+w \in E$, then $\{a_1, a_1+v, v+w\}$ is a claw, a contradiction, so $v+w \notin E$. If $v+w \in P$, then 6.4.10.2 would imply that $a_1+v+w \in E$, a contradiction. So $v + w \in R$. By Lemma 2.1.1, it follows that $\mathrm{cl}(P) \subseteq P \cup Q$, and the cosets of $\mathrm{cl}(P)$ in F are contained in Q or R. We now claim that the flat $F' = \mathrm{cl}(P)$ has no mixed cosets.

First, take a coset B of F' in F. Then either $B \subseteq Q$, in which case $B \subseteq E$, and if $B \subseteq R$, then $B \cap E = \varnothing$. It remains to show that the cosets of the form $a + [F']$ where $a \in A_1 \cup A_2$ are not mixed. Take $a \in E \cap A_i$ for some $i \in \{1, 2\}$ and we show that $a + [F'] \subseteq E$. Let $b \in E \cap A_{3-i}$.

Let $1P = P$, and let $kP = P + (k-1)P$ for $k \geq 2$. Then $\mathrm{cl}(P) = \cup_{k\geq 1}(kP)$. We prove by induction that for any $k \geq 1$, any $v \in kP$ satisfies $a+v, b+v \in E$. The base case $k = 1$ follows from 6.4.10.2. Now suppose that the statement is true for $(k-1)P$. Let $v \in kP$ so that there exists $w \in P$ such that $w + v \in (k-1)P$. Note that $v + w + a, v + w + b \in E$ by induction. Now we may apply 6.4.10.2 to conclude that $w + (v + w + a) = v + a$ and $w + (v + w + b) = v + b$ belong to E. By induction, $a + [F'] \subseteq E$, and there are no mixed cosets of F' in G.

By the remark from the beginning of the proof, every proper flat has to have at least one mixed coset of itself with respect to M, so this means that $F' = \mathrm{cl}(P)$ is empty. Hence $((A_i \cap E) + (A_i \cap E)) \subseteq E$ for $i = 1, 2$. ∎

6.4.10.5. $F \subseteq E$.

Subproof: We will first show that $\mathrm{cl}(F \cap E) \subseteq E$. Take $v, w \in F \cap E$, and fix $a_i \in E \cap A_i$, $i = 1, 2$. We claim that $v+w \in E$. So suppose not. By 6.4.10.4, $a_i+v+w \notin E$ for $i = 1, 2$. Note that for each $i = 1, 2$, precisely one of $a_i + v \in E$ and $a_i + w \in E$ must hold; if none holds, then $\{a_i, v, w\}$ is a claw, and if both hold, then the triangle $\{a_i + v, a_i + w, v + w\}$ violates 6.4.10.4. Assume without loss of generality that $a_1+v \in E$ (and $a_1+w \notin E$). Now, if $a_2+w \in E$ (and hence $a_2+v \notin E$), then $M|\,\mathrm{cl}(\{a_1, v, a_2, w\})$ is an induced $2T$-restriction.

Hence $a_2 + v \in E$ and $a_2 + w \notin E$. But then $\{a_1, a_2, w\}$ is a claw, a contradiction. So $v + w \in E$. We therefore conclude that $\mathrm{cl}(F \cap E) \subseteq E$.

It remains to show that $\mathrm{cl}(F \cap E) = F$. If $\mathrm{cl}(F \cap E)$ is a proper subset of F, then by picking $a_i \in E \cap A_i$ for $i = 1, 2$ and observing 6.4.10.4, we see that $E \subseteq \mathrm{cl}((F \cap E) \cup \{a_1, a_2\})$, and so M is rank-deficient, a contradiction. Therefore $\mathrm{cl}(F \cap E) = F$ and hence $F \subseteq E$. ■

Finally, we uniquely determine the matroid M. We claim that $M \cong \mathrm{PGS}(1, r-1)$. We fix $a_i \in A_i \cap E$, $i = 1, 2$. Take $v \in F$. Then $|E \cap \{v + a_1, v + a_2\}| \geq 1$; otherwise $\{v, a_1, a_2\}$ is a claw. Hence $|E| \geq 2 + 2|F| = 2^{r-1}$. If $|E| > 2^{r-1}$, then M satisfies the theorem, so we must have that $|E| = 2^{r-1}$. This equality occurs only if for every $v \in F$, $|E \cap \{v + a_1, v + a_2\}| = 1$. Under this assumption that each $v \in F$ satisfies precisely one of $v + a_1 \in E$ or $v + a_2 \in E$, we now show that there exists $i \in \{1, 2\}$ such that $v + a_i \in E$ for all $v \in F$. Suppose not, so there exist $v, w \in F$ for which $v + a_1, w + a_2 \in E$. Without loss of generality, suppose that $v + w + a_1 \in E$. But then $\{a_2, v, v + w + a_1\}$ is a claw, a contradiction.

This implies that $E = \mathrm{cl}(F \cup \{a_i\}) \cup \{a_{3-i}\}$, so $M \cong \mathrm{PGS}(1, r-1)$. □

6.5 The Main Theorem

We can now combine the results in the prior sections to give our main result, restated below.

Theorem 6.5.1. *Let $M = (E, G)$ be an r-dimensional, full-rank, I_5-free, and triangle-free matroid. Then $|E| \geq 2^{\lfloor r/2 \rfloor - 1} + 2^{\lceil r/2 \rceil - 1}$. Moreover, when $r \geq 6$, equality holds if and only if $M \cong M_1 \oplus M_2$ where M_1 and M_2 are affine geometries of dimension $\lfloor r/2 \rfloor$ and $\lceil r/2 \rceil$ respectively.*

We remark that the condition $r \geq 6$ in the above theorem is necessary. For example, the matroid C_6 is I_5-free and triangle-free with 6 elements, but it is not isomorphic to $\mathrm{AG}(1, 2) \oplus \mathrm{AG}(2, 2)$.

Proof. By Lemma 6.3.3, if M contained a C_5-restriction, then the bound holds, and when $r \geq 6$, no such matroids attain the bound. Hence we may assume that M is C_5-free, hence affine.

Suppose first that M is C_4-free. We first claim that M can have dimension at most 5. For a contradiction, take an independent set of six elements $x_i \in E$, $i = 1, \ldots, 6$.

By triangle-freeness, C_4-freeness and C_5-freeness, it follows that $\sum_{i\in I} x_i \notin E$ for any $I \subseteq \{1,2,3,4,5,6\}$ for which $|I| = 2,3,4$, and therefore, because of I_5-freeness, $\sum_{i\in I} x_i \in E$ for all $|I| = 5$. But then $\{x_1+x_2+x_3+x_4+x_5, x_2+x_3+x_4+x_5+x_6, x_1, x_6\}$ is an induced C_4-restriction, a contradiction (note that $x_1+x_2+x_3+x_4+x_5+x_6 \notin E$ because of triangle-freeness). It is then easy to check that the only possible full-rank matroids that are also C_4-free are I_1, I_2, I_3, I_4 and C_6, all of which satisfy the theorem.

Suppose that $M|F$ is a maximum affine geometry restriction; note that $\dim(F) \geq 3$ since M has an induced C_4-restriction. Let $M' = (E', F')$ denote M/F. Note that M' is full-rank. By Lemma 6.4.2 and maximality, either M' is I_3-free, or M contains an induced $D^k(M_K)$-restriction for some $k \geq 0$ (so that the dimension of F is $k+3$). Then by Lemma 6.4.8, $|E|$ does not attain the bound. Hence we may assume that M' is I_3-free. By Lemma 6.4.9 and maximality, M' is $2T$-free. By Lemma 6.4.10, it follows that $|E'| \geq 2^{r-\dim(F)-1}$. If we let $\mathcal{C}$ be the set of cosets of F, then

$$\begin{aligned}
|E| &= |E\cap F| + \sum_{A\in\mathcal{C}} |E\cap A| \\
&\geq |E\cap F| + |E'| \\
&\geq 2^{\dim(F)-1} + 2^{r-\dim(F)-1} \\
&\geq 2^{\lfloor r/2\rfloor -1} + 2^{\lceil r/2\rceil -1}
\end{aligned}$$

This proves the bound. We now determine the extremal examples. First, note that if $M = (E,G) \cong M_1 \oplus M_2$ where M_1 and M_2 are affine geometries of dimension $\lfloor r/2\rfloor$ and $\lceil r/2\rceil$ respectively, then M is both I_5-free and triangle-free. The matroid M is I_5-free because if we take a five-element subset $I \subseteq E(M_1\oplus M_2)$, then there exist three elements $x,y,z \in I$ that belong to $E(M_i)$ for some $i \in \{1,2\}$. Since both M_1 and M_2 are affine geometries, this implies that $x+y+z \in E(M_i) \subseteq E(M_1\oplus M_2)$. Hence $M|\operatorname{cl}(I) \ncong I_5$.

In order for equality to hold, $\dim(F) = \lfloor r/2\rfloor$ or $\dim(F) = \lceil r/2\rceil$, each intersecting coset of F contains precisely one element of E, and $|E'| = 2^{r-\dim(F)-1}$. Moreover, by Lemma 6.4.10, $M' \cong \mathrm{AG}(r-\dim(F)-1, 2)$ or $M' \cong D^k(\mathrm{PGS}(1,t))$ for some t and k.

Case 1: $M' \cong D^k(\mathrm{PGS}(1,t))$ for some $t,k \geq 0$.

Note that if $t = 0,1$, then $D(\mathrm{PGS}(1,t))$ is an affine geometry, which will be handled in Case 2, so suppose that $t > 1$. In particular, this means that M' contains an induced $\mathrm{PGS}(1,2)$-restriction, corresponding to four intersecting cosets A_1, A_2, A_3 and A_4 of F with respect to M for which $A_1 + A_2 = A_4$, and $(A_3 + B)\cap E = \varnothing$ for $B \in \{A_1, A_2, A_4\}$.

Fix the (unique) elements $x_i \in A_i \cap E$ for $i = 1,2,3$. Since M is triangle-free and C_5-free, it follows that the (unique) element $x_4 \in A_4 \cap E$ satisfies $x_4 = x_1 + x_2 + y$ for some $y \in F \cap E$ (if $y \in F \backslash E$, then $\{x_1, x_2, x_1 + x_2 + y, z, y + z\}$ is a C_5-restriction for any choice of $z \in F \cap E$). Pick $z_1, z_2 \in (F \cap E) \backslash \{y\}$. Then $\{x_1, x_2, x_3, z_1, z_2\}$ is an induced I_5-restriction, a contradiction. So Case 1 does not arise.

Case 2: $M' \cong \mathrm{AG}(r - \dim(F) - 1, 2)$.

We may pick cosets of F, $A_1, \ldots, A_l$ where $l = r - \dim(F)$, that correspond to a basis of E'. From each of these cosets, take its (unique) element $x_i \in E \cap A_i$. Let $F' = \mathrm{cl}(x_1, \ldots, x_l)$; note that $\dim(\mathrm{cl}(x_1, \ldots, x_l)) = l$.

We claim that $M|F' \cong \mathrm{AG}(r - \dim(F) - 1, 2)$. First note that $M|F'$ is triangle-free since M' is triangle-free. Now suppose for a contradiction that there exist $y_1, y_2, y_3 \in F' \cap E$ such that $y_1 + y_2 + y_3 \notin E$. Since $\dim(F) > 2$, we may pick two elements $z_1, z_2 \in E \cap F$ for which $y_1 + y_2 + y_3 + z_1 + z_2 \notin E$; note that $y_1 + y_2 + y_3 + z_i \notin E$ for $i = 1, 2$, as otherwise $\{y_1, y_2, y_3, z_i, y_1 + y_2 + y_3 + z_i\}$ is an induced C_5-restriction. But then $\{y_1, y_2, y_3, z_1, z_2\}$ is an induced I_5-restriction, a contradiction. So $M|F'$ is I_3-free. By Lemma 1.5.6, it follows that $M|F' \cong \mathrm{AG}(r - \dim(F) - 1, 2)$.

Combining these conditions, we obtain that $|E| = 2^{\lfloor r/2 \rfloor - 1} + 2^{\lceil r/2 \rceil - 1}$ if and only if $M \cong M_1 \oplus M_2$ where M_1 and M_2 are affine geometries of dimension $\lfloor r/2 \rfloor$ and $\lceil r/2 \rceil$ respectively. $\square$

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