On Multifractal Analysis and Local Dimensions of Measures

Master's thesis

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Preface

In the field of fractal geometry one is often interested in studying different notions of dimension of sets in Euclidean, or more general metric spaces. Fractal geometry provides tools for estimating these notions of dimension with methods from the measure theory and other areas of mathematical analysis. Classically, one is usually interested in approximating the fractal dimension of a given set with as high of a precission as possible. In addition to studying the dimensions of sets, it is often useful to assign notions of dimension for measures, the ones of interest of which are usually supported on fractal like sets or otherwise have a fractal nature. In this thesis we introduce a method for finely analysing dimensions of measures, *multifractal analysis*, where we consider not only the dimension of the measure as whole, but also take into account how the intensity of the measure changes on the supporting set.

This thesis is composed of two independent parts. Part I serves as an introduction to multifractal analysis and establishes key concepts in the field. We begin with preliminary measure theory and fractal geometry, and afterwards shift our focus to studying self-similar measures satisfying the strong separation condition in Euclidean spaces. A key concept we introduce in the later sections of Part I is the multifractal formalism, a heuristic principle with origins in physics literature. Loosely speaking, the multifractal formalism states that we may obtain dimensional information of certain sets associated with measures by examining the small scale distribution of the measure and using a simple Legendre transform. This is made more precise in Section 4 of Part I. Our end goal is to establish the multifractal formalism rigorously for self-similar measures satisfying the strong separation condition.

The second part of the thesis is independent of the first part and readers who are familiar with the basics of multifractal analysis may skip Part I if they so wish. In Part II we consider a generalisation of multifractal analysis to metric spaces and introduce a local variant of the theory in the first part. The second part is mainly based on two articles by Käenmäki, Rajala and Suomala [16,17] and the aim is to fix a small inaccuracy found in [16]. Part II of the thesis has also been published as an article in The PUMP Journal of Undergraduate Research [1].

Part I
Introduction to Multifractal Analysis

1 Introduction

Fractal geometry is a field that studies certain irregular subsets of (metric) spaces called *fractals*. Usually a precise definition of a fractal is avoided because different types of sets can be thought of as fractal even though they might not share all of their features. Falconer in [7] loosely defines sets F to be fractal if they have some or all of the following informal properties

1. F has a fine structure (i.e. detail on arbitrarily small scales)
2. F is too irregular to be described in traditional geometrical language, both locally and globally
3. Often F has some form of self-similarity, perhaps approximate or statistical
4. Usually, the *fractal dimension* of F (defined in some appropriate way) is greater than its topological dimension
5. In most cases of interest F is defined in a very simple way, perhaps recursively.

Such sets appear often in many different branches of mathematics and other related sciences such as number theory, dynamical systems, physics and finance so understanding their properties is crucial.

In fractal geometry, the broad objective is to describe the complexity of these fractal sets. This is most often done using different notions of fractal dimension which roughly speaking describe how the amount of detail in the set scales when the scale of observation changes. Often the amount of detail in the set (whatever that means in the given case), follows a power rule with the scaling i.e when measured at scale ε the amount of detail N of the set fractal set scales with $N \sim \varepsilon^{-s}$ for some exponent s. This exponent is, in this naive example, called the fractal dimension of the set. A classical introductory example is the coastline of a country. When measured with a measuring stick of lenght, say 100 km, the

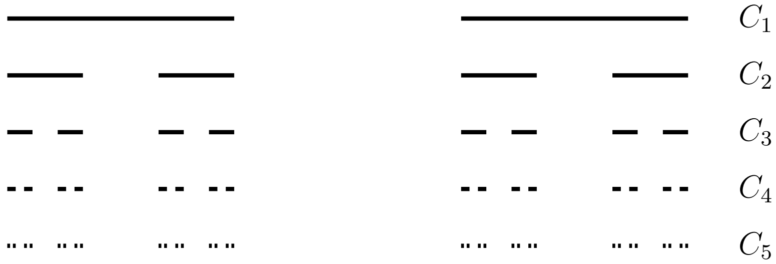


Figure 1.1: The first five steps of the construction of the Cantor $\frac{1}{3}$-set. The Cantor set C is the limit of the construction $C = \bigcap_{i=1}^{\infty} C_i$

approximate lenght of the coastline is smaller than when measured with a stick of lenght 10 cm, due to the irregularities in the coastline. The amount of detail in this case can be quantified, by calculating *how many* sticks of a given lenght are needed for measuring the lenght of the coastline. If the coastline is regular enough, say a square, the number of sticks needed to measure the perimeter would double when the lenght of the stick gets halved, in other words $N \sim \varepsilon^{-1}$, so the fractal dimension s of the coastline would be 1. However, calculations show that in real world cases the coastlines have different values of fractal dimension, for example the coastline of Norway has a fractal dimension approximately equal to 1.52 [9].

In mathematics, perhaps the most classic example of a fractal set is the so called Cantor $\frac{1}{3}$-set (see Figure 1.1) which is constructed by starting with the interval $[0, 1]$ and removing the middle third of it and iteratively removing the middle thirds of the intervals left remaining in the previous step. Different notions of fractal dimension have been developed and extensively studied by mathematicians in the previous decades and with most of the common definitions, the dimension of the Cantor set is equal to $\frac{\log(3)}{\log(2)}$.

Many of the tools in fractal geometry are based on measure theory (see Section 2), and measures are also otherwise closely related to fractals. They can be supported on fractal sets or otherwise have a fractal like nature, which motivates assigning notions of dimension to the measures themselves. These notions of dimension provide insight to the intensity and the distribution of the measures on their supporting sets. For some measures, this intensity varies widely at a small scale, so the level-sets of the intesities may define a spectrum of fractal sets. We call measures of this type *multifractal measures*, and our aim in this thesis is to provide a basic theoretical framework for studying the dimension of these level sets.

This thesis aims to give a mostly self contained look in to the world of multifractals and the multifractal formalism. Part I of the thesis is structured as follows. We begin, in Section 2, by briefly going over some of the necessary measure theory. Section 3 gives an overview of the fractal geometric techniques used in multifractal analysis, and provides a basis for studying dimensions of fractal sets and measures. Finally in Section 4 we provide two different theoretical frameworks for the multifractal analysis of measures in Euclidean spaces, and show that multifractal formalism, a celebrated heuristic principle for estimating the multifractal spectra of measures, holds in the case of strongly separated self-similar measures.

2 Measure Theory

This section gives a brief overview of introductory measure theory. The section is mainly based on [3], [7] and [20] and is aimed to be a quick rundown of the notation used in this study and doesn't aim to provide too much intuitional insight into the concepts.

2.1 Measure spaces

Definition 2.1. A *σ-algebra* of a space X is a collection $\Gamma \subset \mathcal{P}(X)$ that satisfies the following properties

1. $\emptyset \in \Gamma$
2. If $A \in \Gamma$, then $A^c \in \Gamma$
3. If $A_i \in \Gamma$ for all $i = 1, 2, \ldots$, then $\bigcup_{i=1}^{\infty} A_i \in \Gamma$

Remark. It is readily seen from 1. and 2. that $X \in \Gamma$ and from 2. and 3. that $\bigcap_{i=1}^{\infty} A_i \in \Gamma$ if $A_i \in \Gamma$ for all $i \in \mathbb{N}$

Perhaps the most important σ-algebra in our case is the *Borel (σ-)algebra* defined in a topological space (X, τ) as

$$\mathcal{B} = \bigcap_{\Delta \subset \Gamma \subset \mathcal{P}(X) \text{ is a } \sigma-\text{algebra}} \Gamma$$

where Δ is the collection of all open subsets $U \subset X$ i.e. the Borel algebra is the smallest σ-algebra containing every open set. The elements of $\mathcal{B}$ are called *Borel sets.*

Definition 2.2. Let Γ be a σ-algebra in X. A set function $\mu : \Gamma \to \overline{\mathbb{R}}_{\geq 0}$ is a *measure* if

1. $\mu(\emptyset) = 0$
2. $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for all separate $A_i \in \Gamma$, $i \in \mathbb{N}$.

The triple (X, Γ, μ) is then called a *measure space.* The *support* of μ denoted by $\operatorname{spt}(\mu)$ is the set of all points of X such that every open neighbourhood of the point has positive measure. For Euclidean spaces this is the same as the largest set $A \subset \mathbb{R}^d$ such that $\mu(\mathbb{R}^d \setminus A) = 0$.

Next we give a few elementary but essential properties of measures.

Theorem 2.3. *Let (X, Γ, μ) be a measure space and $A, B, A_i \in \Gamma$, $i \in \mathbb{N}$. Then*

1. *If $A \subset B$, then $\mu(A) \leq \mu(B)$.*
2. *If $A \subset B$ and $\mu(A) < \infty$, then $\mu(B \backslash A) = \mu(B) - \mu(A)$.*
3. *$\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$.*
4. *If $A_1 \subset A_2 \subset \ldots$, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n)$*
5. *If $A_1 \supset A_2 \supset \ldots$, and $\mu(A_k) < \infty$ for some k, then*

$$\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n).$$

Going forward, property 1. of the previous theorem is referred to as *monotonicity* of the measure, property 3. is called *(countable) sub-additivity* and properties 4. and 5. are called the *continuity of the measure from below and above* respectively. Proofs of the statements can be found in any introductory book to measure theory e.g. [4].

2.2 Outer measure, and measurable sets and functions

Definition 2.4. Let X be a arbitrary set. A set function $\mu_* : \mathcal{P}(X) \to \overline{\mathbb{R}}_{\geq 0}$ is an *outer measure* if it satisfies the following

1. $\mu_*(\emptyset) = 0$
2. If $A \subset B \subset X$, then $\mu_*(A) \leq \mu_*(B)$
3. If $A \subset \bigcup_{k=1}^{\infty} A_k$, then $\mu_*(A) \leq \sum_{k=1}^{\infty} \mu_*(A_k)$

Notice that the difference between a measure defined as in 2.2 and an outer measure defined above is that the outer measure is well defined for all subsets of the sample space X where as a measure is defined only for a appropriately well behaved subcollection of the subsets. Informally speaking, the fact that the outer measure may behave badly on some of the subsets is why it only satisfies countable *sub*-additivity where as a measure satisfies the stronger property of countable additivity (property 2. in definition 2.2). Next we briefly study the difference between outer measures and measures a bit further.

Definition 2.5. Let μ_* be an outer measure in X. A set $A \subset X$ is said to be μ_*-measurable if

$$\mu_*(E) = \mu_*(E \cap A) + \mu_*(E \backslash A),$$

for all sets $E \subset X$.

As an immediate consequence of the definition we have for μ_*-measurable subsets A and B

$$\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$$

by choosing $E = A \cup B$. It is also clear that to verify the μ_*-measurability of a set one needs only to verify the inequality

$$\mu_*(E) \geq \mu_*(E \cap A) + \mu_*(E \backslash A),$$

since the other inequality is a consequence of the sub-additivity of the outer measure μ_*.

One of the most important elementary results concerning outer measures is that every outer measure μ_* can be restricted to a σ-algebra Γ of μ_*-measurable sets in X and the restriction of μ_* to Γ is a measure.

Theorem 2.6. *Let X be an arbitrary set, μ_* a outer measure in X and $\Gamma = \{A \subset X : A$ is μ_*-measurable$\}$, then the triple (X, Γ, μ) is a measure space, where $\mu(A) = \mu_*(A)$ for all $A \in \Gamma$.*

Again, the proof is available in any introductory book to measure theory e.g. [4]. In the following we sometimes drop the distinction between outer measures and measures, since we may construct a measure from any outer measure by restricting it to a suitable subcollection of $\mathcal{P}(X)$.

Definition 2.7. Let (X, Γ, μ) be a measure space. A function $f : X \to \mathbb{R}$ is a *measurable function* if

$$f^{-1}(B) \in \Gamma,$$

for all Borel sets B.

Often it is convinient to define (outer) measures on some space by using a mapping from one measure space to another. The next proposition formalises this idea.

Definition 2.8. Let (X, Γ_1, μ) be a measure space, and Γ_2 be a σ-algebra on Y (the pair (Y, Γ_2) is often called a *measurable space*). Let $f : X \to Y$ be a measurable function. We call the measure $f_*\mu : \Gamma_2 \to \mathbb{R}$ defined by

$$f_*\mu(A) = \mu \circ f^{-1}(A) = \mu(f^{-1}(A)),$$

the *pushforward of μ under the mapping f.*

It is a simple exercise to show that $f_*\mu$ is indeed a measure on (Y, Γ_2). The definition also works for outer measures with trivial modifications

2.3 Regularity properties of outer measures

Here we briefly state a few regularity properties of outer measures which we refer to in the later parts of the thesis.

Definition 2.9. Let μ_* be an outer measure in a (topological) space X. Then

1. μ_* is *regular* if for every $A \subset X$ there exists a μ_*-measurable $B \subset X$ such that $A \subset B$ and $\mu_*(A) = \mu_*(B)$.

2. μ_* is *finite* if $\mu_*(X) < \infty$.

3. μ_* is *locally finite* if $\mu_*(K) < \infty$ for all compact $K \subset X$.

4. μ_* is a *Borel outer measure* if every Borel set is μ_*-measurable.

5. A Borel outer measure μ_* is *Borel regular* if for every $A \subset X$ there exists a Borel set $B \subset X$ such that $A \subset B$ and $\mu_*(A) = \mu_*(B)$.

6. μ_* is a *Radon outer measure* if it is locally finite and for every open $U \subset X$

$$\mu_*(U) = \sup\{\mu_*(K)\ K \subset U \text{ is compact}\}$$

   and for all $A \subset X$

$$\mu_*(A) = \inf\{\mu_*(U)\ U \text{ is open, } A \subset U\}.$$

2.4 Integration

For completeness of the theory we briefly go over basic definitions for integration on measure spaces. For this section, assume that (X, Γ, μ) is a measure space.

Definition 2.10. A function $f : X \to \mathbb{R}$ is called *simple*, if

$$f = \sum_{i=1}^{k} a_i \chi_{A_i},$$

where $A_i \subset X$ and χ_{A_i} denotes the characteristic function of A_i. The family of non-negative simple functions is denoted by $\mathcal{Y}^+$, and the integral of a simple function $f \in \mathcal{Y}^+$ with respect to the measure μ over a set $E \in \Gamma$, is given by

$$\int_E f \mathrm{d}\mu = \sum_{i=1}^{k} a_i \mu(A_i \cap E).$$

Definition 2.11. Let $f : X \to [0, \infty)$ a measurable function. The integral of f with respect to the measure μ over a set E is given by

$$\int_E f \mathrm{d}\mu = \sup\left\{\int_E g \mathrm{d}\mu \ :\ g \in \mathcal{Y}^+, \quad 0 \le g \le f\right\}.$$

Definition 2.12. Let $f : X \to \mathbb{R}$ a measurable function and set $f^+(x) = \max\{0, f(x)\}$ and $f^-(x) = \max\{0, -f(x)\}$. Notice that $f = f^+ - f^-$. Let $E \in \Gamma$. If

$$\int_E f^+ \mathrm{d}\mu < \infty, \quad \text{or} \quad \int_E f^- \mathrm{d}\mu < \infty,$$

we define the integral of f with respect to the measure μ over E as

$$\int_E f \mathrm{d}\mu = \int_E f^+ \mathrm{d}\mu - \int_E f^- \mathrm{d}\mu.$$

Remark 2.13. We sometimes ommit the subscript from the integral symbol when integrating over the whole space X. That is

$$\int f \mathrm{d}\mu := \int_X f \mathrm{d}\mu.$$

We also remark that integration is linear as well as monotone, meaning that

1. $\int af + bg \,\mathrm{d}\mu = a \int_X f \mathrm{d}\mu + b \int g \mathrm{d}\mu$, for measurable f, g and $a, b \in \mathbb{R}$,
2. $\int_A f \mathrm{d}\mu \leq \int_B f \mathrm{d}\mu$, for $A \subset B \subset X$.

These properties are easily verifiable by passing down to simple functions, and verifying the properties for them using definition 2.10.

Let us state without proof some of the most important properties of the integral. The proofs are easily found in virtually any book on measure theory, e.g. [3].

Theorem 2.14 (Monotone Convergence Theorem)**.** *Let $f_n : A \to [0, \infty)$ be a sequence of measurable functions, such that $f_{n+1} \geq f_n$, for all $n \in \mathbb{N}$. Then*

$$\int_A \lim_{n\to\infty} f_n \mathrm{d}\mu = \lim_{n\to\infty} \int_A f_n \mathrm{d}\mu.$$

The following weaker version of 2.14 is commonly known as Fatou's lemma

Theorem 2.15 (Fatou's lemma)**.** *Let $f_n : A \to [0, \infty)$ be a sequence of measurable functions. Then*

$$\int_A \liminf_{n\to\infty} f_n \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_A f_n \mathrm{d}\mu$$

As a corollary of the monotone convergence theorem, we see that integration induces a measure on the space X.

Corollary 2.16. *If $f : X \to [0, \infty)$ is measurable, then the set function $\nu : \Gamma \to [0, \infty]$ defined as*

$$\nu(E) = \int_E f \mathrm{d}\mu,$$

is a measure.

Proof. We only need to check the σ-additivity, since the other properties easily follow from the definition. Let $E_n \in \Gamma$, $n \in \mathbb{N}$ be disjoint. Define $f_n = \sum_{i=1}^n f\chi_{E_k}$. Clearly $f_n \leq f_{n+1}$ and each of the f_n is measurable. Also, for any $x \in \bigcup_{i=1}^n E_n$, we have

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Thus by the monotone convergence theorem 2.14, we have

$$\nu\left(\bigcup_{i=1}^n E_n\right) = \int_{\bigcup_{i=1}^n E_n} f \mathrm{d}\mu = \lim_{n\to\infty} \int_{\bigcup_{i=1}^n E_n} f_n \mathrm{d}\mu = \lim_{n\to\infty} \int \sum_{i=1}^n f\chi_{E_i} \mathrm{d}\mu =$$
$$\lim_{n\to\infty} \sum_{i=1}^n \int f\chi_{E_i} \mathrm{d}\mu = \sum_{i=1}^\infty \int_{E_i} f \mathrm{d}\mu = \sum_{i=1}^\infty \nu(E_i),$$

and the claim follows. □

This innocent looking result is in fact quite a powerfull statement, since it allows us to utilise the entire machinery of measure theory with integrals.

2.5 Basic results

In this section we state a few useful basic results from the theory of measures, which are later used in Section 4.2. First we state the measure theoretic version of the inequality originated in probability theory, the Markov's inequality

Theorem 2.17 (Markov's inequality). *Let (X, Γ, μ) be a measure space, and f a μ-measurable function. Then for any $\varepsilon > 0$,*

$$\mu(x \in X \,:\, |f| \geq \varepsilon) \leq \frac{1}{\varepsilon} \int |f| \,\mathrm{d}\mu.$$

Proof. Consider the set $A = \{x \in X \,:\, |f| \geq \varepsilon\}$. Now

$$\frac{1}{\varepsilon} \int |f| \,\mathrm{d}\mu \geq \frac{1}{\varepsilon} \int_A |f| \,\mathrm{d}\mu \geq \frac{1}{\varepsilon} \int_A \varepsilon \mathrm{d}\mu = \int_A \mathrm{d}\mu = \mu(A),$$

which completes the proof. □

Next we give the measure theoretic version of the Borel-Cantelli lemma, which also has origins in probability theory.

Theorem 2.18 (Borel-Cantelli lemma). *Let (X, Γ, μ) be a measure space and let $A_n \in \Gamma$, for all $n \in \mathbb{N}$. If*

$$\sum_{n=1}^\infty \mu(A_n) < \infty,$$

then

$$\mu(\limsup_{n\to\infty} A_n) = 0,$$

where $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m$.

Proof. By sub-additivity

$$\mu(\bigcup_{m=n}^{\infty} A_m) \le \sum_{m=n}^{\infty} \mu(A_m),$$

and since $\bigcup_{m=n}^{\infty} A_m \supset \bigcup_{m=n+1}^{\infty} A_m$, for all $n \in \mathbb{N}$, by upper continuity

$$\mu\Big(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m\Big) = \lim_{n\to\infty} \mu\Big(\bigcup_{m=n}^{\infty} A_m\Big) \le \lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(A_m) = 0,$$

where the last equality holds since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. $\square$

The following theorem gives a useful way to give quantitative bounds for measures of subsets of sets with infinite measure.

Theorem 2.19. *Let μ be a measure on X. Assume that every subset of X with infinite measure contains a Borel subset of finite measure. Let $A \subset X$ with $\mu(A) = \infty$. Then for any $0 < c < \infty$ there exists a Borel subset B of A such that $c < \mu(B) < \infty$.*

Proof. We set $c_0 = \sup\{\mu(B) : B \subset A,\ 0 < \mu(B) < \infty\}$. The supremum exists according to the hypothesis and is either positively finite or infinite. Suppose $c_0 < \infty$. Now for each $n \in \mathbb{N}$ we choose $B_n \subset A$ such that

$$\mu(B_n) > c_0 - \frac{1}{n},$$

and $B_n \subset B_{n+1}$. Let $B = \bigcup_{n=1}^{\infty} B_n$. Since $\mu(B) \le c_0$ by construction, from the lower continuity of the measure we get

$$\mu(B) = c_0.$$

Now, since $\mu(A) = \infty$, we have

$$\mu(A) = \mu(A \backslash B) + \mu(B),$$

and since $\mu(B)$ is finite we have $\mu(A\backslash B) = \infty$. By the hypothesis, there exists $C \subset A\backslash B$ such that $0 < \mu(C) < \infty$. By our construction B and C are separate, hence

$$\mu(C \cup B) = \mu(C) + \mu(B) > c_0,$$

and since $C \subset A$, this contradicts the choice of c_0. Thus $\sup\{\mu(B) : B \subset A,\ 0 < \mu(B) < \infty\} = \infty$, which implies that for any $c > 0$ there exists $B \subset A$ such that $c < \mu(B) < \infty$. $\square$

3 Fractal Geometry

Fractal geometry provides tools to help understand the complexity of fractal sets. Usually the geometry of fractal sets is described using different notions of fractal dimension. This section introduces some of the classical definitions of fractal dimensions, most important of which is perhaps the Hausdorff dimension. The theory is then expanded to studying some notions of dimension for measures as well. As an introduction to fractal dimensions, we first take a look at the box-counting dimension.

3.1 Box-counting dimension

The box-counting dimension (also known as Minkowski dimension or simply as box dimension) is one of the most widely used notions of dimension in concrete applications due to the ease of it's numerical estimation. It's major advantages are that is quite easily defined and simple to calculate mathematically or estimate computationally. However, it has some theoretical downsides, for example it lacks countable stability, which means that countable sets can have non-zero box dimension and a countable union of a collection of sets with box dimension 0 can have a positive box dimension.

Box dimension has multiple equivalent definitions, and the one we give works in any metric space. Recall that for a non-empty subset A of a metric space (X, d), the *diameter* of A is defined as

$$\operatorname{diam}(A) = \sup\{d(x, y) \,:\, x, y \in A\}$$

from now on we assume that X is a metric space with a metric d unless stated otherwise.

Definition 3.1. Let $A \subset X$ be non-empty and bounded. We define

$$N_\delta(A) = \min\left\{k \in \mathbb{N} \,:\, A \subset \bigcup_{i=1}^{k} A_i,\ \text{with } A_i \subset X, \operatorname{diam}(A_k) \leq \delta\right\}$$

Now we define the *lower box dimension* of A as

$$\underline{\dim}_B A = \liminf_{\delta \downarrow 0} \frac{\log N_\delta(A)}{-\log \delta}$$

and the *upper box dimension* of A as

$$\overline{\dim}_B A = \limsup_{\delta \downarrow 0} \frac{\log N_\delta(A)}{-\log \delta}.$$

If the upper and lower box dimensions are equal then the common value is refered to as the *box dimension* of A

$$\dim_B A = \lim_{\delta \downarrow 0} \frac{\log N_\delta(A)}{-\log \delta}.$$

Other equivalent definitions are given by defining $N_\delta(A)$ differently [7], for example as

1. the smallest number of open balls of radius δ that cover A
2. in $\mathbb{R}^d$, the smallest number of cubes of side length δ that cover A (hence the name box dimension)
3. in $\mathbb{R}^d$, the number of δ-mesh cubes that intersect A
4. the largest number of disjoint balls of radius δ with centers in A

Example 3.2. To demonstrate the ease of calculating the box dimension, let us consider the Cantor $\frac{1}{3}$-set C (see figure 1.1). Obviously we can cover the set C_k, that is the k^{th} step of the construction of C with 2^k intervals of length 3^{-k}, which gives that $N_{3^{-k}}(C) \leq 2^k$ and thus

$$\overline{\dim}_B C = \limsup_{k \to \infty} \frac{\log N_{3^{-k}}(C)}{-\log 3^{-k}} \leq \limsup_{k \to \infty} \frac{\log 2^k}{-\log 3^{-k}} = \frac{\log 2}{\log 3}.$$

On the other hand, an interval of length δ, where $3^{-(k+1)} \leq \delta < 3^{-k}$ intersects at most one of the intervals in the construction of C of length 3^{-k}. The number of such intervals is again 2^k and so $N_\delta(C) \geq 2^k$, when $3^{-(k+1)} \leq \delta < 3^{-k}$. Then

$$\underline{\dim}_B C = \liminf_{k \to \infty} \frac{\log N_\delta(C)}{-\log 3^{-k}} \geq \liminf_{k \to \infty} \frac{\log 2^k}{-\log 3^{-k}} = \frac{\log 2}{\log 3}.$$

from which it follows that $\dim_B C = \frac{\log 2}{\log 3}$.

To see the downsides of the box dimension, consider the rational numbers. It is clear that the box dimension of a single point is 0, but it is also easy to see that since the rational numbers $\mathbb{Q}$ are dense in the reals, they have a box dimension of 1. This means that taking a countable union of sets with 0 box dimension can result in a set of positive box dimension. Next we introduce the Hausdorff dimension, which provides a more stable basis for theoretical considerations.

3.2 Hausdorff dimension

The Hausdorff dimension is perhaps the most widely used notion of dimension in theoretical considerations. The definition is given via a certain measure called the Hausdorff measure, which allows the dimension to inherit many of the useful properties of measures. The construction of the measure is based on a classical way to construct outer measures proposed by Carathédory (see e.g. [3]). The Hausdorff dimension has the adantage of being defined for any set, and is mathematically convenient, as it is based on measures, which are relatively well behaved and have a rich theory behind them. A major disadvantage however is that in many cases it is difficult to calculate or estimate empirically. In this section, we define the Hausdorff measure and derive some of its most well known properties.

Definition 3.3. Let $F \subset X$ and $\{U_i\}$ be a countable collection of sets of diameter at most $\delta > 0$ that cover F i.e.

$$F \subset \bigcup_{i=1}^{\infty} U_i,$$

with $\mathrm{diam}(U_i) \leq \delta$ for all $i \in \mathbb{N}$. Then $\{U_i\}$ is said to be a *δ-cover* of F.

Definition 3.4 (Hausdorf measure)**.** Let $F \subset X$ and $s \geq 0$. For any $\delta > 0$ we define the *s-dimensional Hausdorff δ-measure* as

$$\mathcal{H}^s_\delta(F) = \inf \Big\{ \sum_{i=1}^{\infty} \mathrm{diam}(A_i)^s \ : \ \{A_i\} \text{ is a } \delta\text{-cover of } F \Big\}$$

Now the *s-dimensional Hausdorff measure* of F is defined as

$$\mathcal{H}^s(F) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(F) = \sup_{\delta > 0} \mathcal{H}^s_\delta(F).$$

Notice that the limit is always defined since $\mathcal{H}^s_\delta(F)$ increases as δ decreases. By the Carathédory construction $\mathcal{H}^s$ is a Borel outer measure and if we restrict the sets of the δ-cover to, say closed sets, then $\mathcal{H}^s$ is Borel regular (it is easy to show that this doesn't affect the value of $\mathcal{H}^s$).

Hausdorff dimension can be tought of as generalisation of the familiar ideas of length, area, volume and higher dimensional equivalents. With a little bit of effort it is possible to show that for a Borel set $F \subset \mathbb{R}^d$

$$\mathcal{H}^d(F) = c_d \lambda(F), \tag{3.1}$$

where $\lambda(F)$ is the d-dimensional Lebesgue measure of F. In other words the d-dimensional Hausdorff measure of a set is within a constant multiple of the Lebesgue measure of the set. For the definition of Hausdorff dimension we note an interesting property of the Hausdorff measure given by the following simple theorem.

Theorem 3.5. *Let $A \subset X$ and $0 < s < t$, then*

1. *If $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^t(A) = 0$.*
2. *If $\mathcal{H}^t(A) > 0$ then $\mathcal{H}^s(A) = \infty$.*

Proof. Clearly 1. and 2. are equivalent, but we have given two statements to give emphasis on the property. To prove statement 1. (and hence 2.) we fix $0 < s < t$, $\delta > 0$ and let $\mathcal{H}^s(A) < \infty$. We take a δ-cover $\{U_i\}$ of the set A such that

$$\sum_{i=1}^{\infty} \mathrm{diam}(U_i)^s \leq \mathcal{H}^s_\delta(A) + 1,$$

which is possible since $\mathcal{H}^s_\delta(A)$ is defined as the infimum of the $\sum_{i=1}^{\infty} \mathrm{diam}(U_i)^s$ over the δ-covers of A. Now since $\{U_i\}$ is a δ-cover of A

$$\begin{aligned}\mathcal{H}^t_\delta(A) &\leq \sum_{i=1}^{\infty} \mathrm{diam}(U_i)^t = \sum_{i=1}^{\infty} \mathrm{diam}(U_i)^{t-s} \mathrm{diam}(U_i)^s \\ &\leq \delta^{t-s} \sum_{i=1}^{\infty} \mathrm{diam}(U_i)^s \leq \delta^{t-s} \underbrace{(\mathcal{H}^s_\delta(A) + 1)}_{<\infty}\end{aligned}$$

and by taking $\delta \to 0$ we get $\mathcal{H}^t(A) = 0$. □

The previous theorem shows that there is an exponent s_0 for which the Hausdorff dimension jumps from 0 to ∞. This critical value is defined to be the Hausdorff dimension of the set A. More formally

Definition 3.6. For a set $A \subset X$ the *Hausdorff dimension* of A is

$$\dim_H(A) = \inf\{s > 0 \,:\, \mathcal{H}^s(A) = 0\} = \sup\{s > 0 \,:\, \mathcal{H}^s(A) = \infty\}$$

An immediate consequence of Theorem 3.5 and the previous definition is that

$$\mathcal{H}^s(A) = \begin{cases} \infty, & \text{if } s < \dim_H(A) \\ 0, & \text{if } s > \dim_H(A), \end{cases}$$

and moreover, if $0 < \mathcal{H}^s(A) < \infty$, then $\dim_H A = s$. The converse however does not hold, that is if $s = \dim_H(A)$ then the s-dimensional Hausdorff measure of A may be zero, finite or even infinite. We call a Borel set that has a finite s-dimensional Hausdorff measure an *s-set*.

The Hausdorff dimension has several advantages over the more crude box dimension. For example it has the desireable properties of monotonicity and stability under countable unions. The following proposition collects some of these properties of the Hausdorff dimension.

Proposition 3.7. *Let X be a metric space. Then*

1. *If $A \subset X$ is finite, then* $\dim_H A = 0$,
2. *If $A \subset B \subset X$, then* $\dim_H A \leq \dim_H B$,

3. If $A_i \subset X$, for all $i \in \mathbb{N}$, then $\dim \bigcup_{i\in\mathbb{N}} A_i = \sup_{i\in N} \dim A_i$.

4. $\dim_H \mathbb{R}^d = d$*, and moreover, for any* $A \subset \mathbb{R}^d$*,* $0 \leq \dim_H A \leq d$*.*

Proof. 1. Let $A = \{a_1, a_2, \dots, a_n\} \subset X$. Clearly $\{B(a_i, \delta)\}_{i=1}^n$ is a δ-cover for A for any $\delta > 0$, and as such

$$\mathcal{H}^0_\delta = \sum_{i=1}^{n} \operatorname{diam}(B(a_i, \delta))^0 = n,$$

so $\mathcal{H}^0(A) = n$, and thus $\dim_H A = 0$.

2. Let $s > \dim_H B$. By Theorem 3.5, we have $\mathcal{H}^s(B) = 0$. This on the other hand implies that $\mathcal{H}^s(A) \leq \mathcal{H}^s(B) = 0$, so $\mathcal{H}^s(A) = 0$. Since this holds for any $s > \dim_H B$, we must have

$$\dim_H A = \inf\{s > 0 \,:\, \mathcal{H}^s(A) = 0\} \leq \dim_H B.$$

3. Let $A_i \subset X$, for all $i \in \mathbb{N}$, and $s_0 = \sup_{i\in N} \dim A_i$. By 2. $\dim_H \bigcup_{i\in\mathbb{N}} A_i \geq \dim_H A_i$, for all $i \in \mathbb{N}$, so

$$\dim_H \bigcup_{i\in\mathbb{N}} A_i \geq s_0.$$

Now take $s > s_0$. Then $\dim_H A_i < s$, for all $i \in \mathbb{N}$ and consequently $\mathcal{H}^s(A_i) = 0$, for all $i \in \mathbb{N}$. Thus

$$\mathcal{H}^s\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \sum_{i\in\mathbb{N}} \mathcal{H}^s(A_i) = 0,$$

so by similar reasoning to 2. we have

$$\dim_H \bigcup_{i\in\mathbb{N}} A_i \leq s_0.$$

4. Let A_i be a countable cover for $\mathbb{R}^d$, s.t. each A_i has positive and finite Lebesgue measure, and thus positive and finite d-Hausdorff measure. For example one can choose A_i to be a mesh of cubes with equal side length. Then $\dim_H A_i = d$, for all $i \in \mathbb{N}$, and by 3.

$$\dim_H \mathbb{R}^d = d.$$

The second claim follows from 2. □

Remark 3.8. In what follows we are mainly interested in the Euclidean setting, that is $X = \mathbb{R}^d$. In this case we note that equation (3.1) together with Theorem 3.5 implies that if $A \subset \mathbb{R}^d$, with $0 < \lambda(A) < \infty$, then $\dim_H(A) = d$, so the only sets of real interest are sets of zero d-dimensional Lebesgue measure, or sets with unknown Lebesgue measure.

It is usually much simpler to obtain upper estimates for the Hausdorff measure, and consequently the Hausdorff dimension. This is because we need only calculate the sum in the definition for one δ-covering to obtain an upper estimate, but obtaining a lower estimate requires taking an infimum over arbitrary coverings. A plethora of tools have been developed over the years for obtaining lower estimates for the Hausdorff dimension, the first of which we introduce being the well known mass distribution principle.

Proposition 3.9. *Let μ be a finite measure, and let $E \subset X$, with $\mu(E) > 0$. Suppose that there are numbers $s \geq 0$, $c > 0$ and $\delta_0 > 0$, such that for all sets $U \subset X$, with $\mathrm{diam}(U) \leq \delta_0$, we have*

$$\mu(U) \leq c \cdot \mathrm{diam}(U)^s.$$

Then $\mathcal{H}^s(E) \geq \frac{\mu(E)}{c}$, and moreover

$$s \leq \dim_H E.$$

Proof. Let $\delta \leq \delta_0$, and $\{U_i\}_i$ be a δ-cover for E. Then by hypothesis

$$\mu(E) \leq \mu\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mu(U_i) \leq c \sum_{i=1}^{\infty} \mathrm{diam}(U)^s,$$

so $\mu(E) \leq c\mathcal{H}^s_\delta$. Taking $\delta \to 0$ gives the estimate

$$\mathcal{H}^s(E) \geq \frac{\mu(E)}{c},$$

and recalling 3.5 gives $\dim_H E \geq s$. □

3.3 Packing dimension

Just as the Hausdorff dimension of a set A is defined in terms of small sets covering A, it is natural to consider small sets packed inside of A and define a dimension in that way. With this motivation we define the packing dimension of A in a similar fashion to the Hausdorff dimension.

Definition 3.10. Let $A \subset X$ and define

$$\mathcal{P}^s_\delta = \sup\left\{ \sum_{i=1}^{\infty} \mathrm{diam}(B_i) : \begin{array}{l} \{B_i\} \text{ is a collection of disjoint} \\ \text{closed balls of radii at most } \delta \\ \text{with centres in } A \end{array} \right\}$$

and similarily to the construction of Hausdorff measure define

$$\mathcal{P}^s_0 = \lim_{\delta \downarrow 0} \mathcal{P}^s_\delta.$$

Unlike in the Hausdorff case, the entity defined above is not a measure, which can be seen by considering countable dense sets (sub-additivity is violated). Hence we define the *s-dimensional packing measure* as

$$\mathcal{P}^s = \inf\left\{\sum_{i=1}^{\infty}\mathcal{P}_0^s(A_i)\ :\ A \subset \bigcup_{i=1}^{\infty} A_i\right\}$$

As in the definition of the Hausdorff dimension, we define the *packing dimension* of A as

$$\dim_P(A) = \inf\{s > 0\ :\ \mathcal{P}^s(A) = 0\} = \sup\{s > 0\ :\ \mathcal{P}^s(A) = \infty\},$$

which can be verified to be well defined similarly to the Hausdorff dimension.

This extra step in the definition of the packing measure obviously leads to some difficulties in its mathematical manipulation. In the following we shall mainly consider results related to Hausdorff dimensions, and will not concern ourselves too much with packing dimensions. Still it is beneficiary to give the definition here so we have the possibility to briefly refer to corresponding results for packing dimensions.

3.4 Dimensions of measures

The next logical step when considering dimensions is to switch our focus from sets to measures. First we consider local properties of a measure and define a local dimension for a measure, which we afterwards use to give estimates of Hausdorff dimensions of corresponding sets. The classical way of defining a local dimension of a measure is to compare the measure of a small ball to its radius. For the rest of this section we consider a locally finite Borel regular (outer) measure μ, defined on the set $\mathcal{P}(X)$, where X is a metric space.

Before we dive in to defining dimensions for measures, we obtain a couple useful results concerning finite measures, which later on enable us to link the dimensions of measures to Hausdorff or packing dimensions of certain sets. First we prove a classical covering lemma which we require in one of the following proofs.

Lemma 3.11. *Let $\mathcal{B}$ be a collection of balls contained in a bounded region of a separable metric space X. Then there exists an at most countable subcollection $\mathcal{C} \subset \mathcal{B}$ of disjoint balls such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in\mathcal{C}} 4B, \tag{3.2}$$

where we use the notation $4B$ for a ball concentric with B and of four times the radius.

Proof. We construct $\mathcal{C}$ by inductively choosing appropriate subcollections of $\mathcal{B}$. Let $r = \sup\{\mathrm{rad}(B)\ :\ B \in \mathcal{B}\}$, where $\mathrm{rad}(B)$ denotes the radius of B. Let $\varepsilon > 0$ and let C_n be the subcollection of $\mathcal{B}$ such that the radii of the balls in C_n lies in the interval $((1+\varepsilon)^{-n-1}r, (1+\varepsilon)^{-n}r]$. We now choose $\mathcal{C}_0$ as the maximal disjoint subcollection of C_0. Note that C_0 is finite since otherwise the collection $\mathcal{B}$ could not be contained in a bounded region. Assume that $\mathcal{C}_0, \ldots, \mathcal{C}_{k-1}$ have been chosen and define

$$A_k = \left\{ B \in C_k \,:\, B \cap C = \emptyset,\ C \in \bigcup_{i=0}^{k-1} \mathcal{C}_i \right\}$$

we now choose $\mathcal{C}_k$ as the maximal disjoint subcollection of A_k. Since every $\mathcal{C}_k$ defined as above is finite we get a countable collection

$$\mathcal{C} = \bigcup_{k=0}^{\infty} \mathcal{C}_k.$$

It remains to show that condition (3.2) holds for this collection. Take $B \in \mathcal{B}$. Now $B \in C_n$ for some $n \in \mathbb{N}$ and so either $B \in \mathcal{C}_n$ or B intersects at least one of the balls in the union $\bigcup_{k=0}^{n} \mathcal{C}_k$. If $B \in \mathcal{C}_n$ then (3.2) is obvious, otherwise let $\overline{B}$ be the ball in $\bigcup_{k=0}^{n-1} \mathcal{C}_k$ that B intersects. Now $\mathrm{rad}(\overline{B}) > (1+\varepsilon)^{-n-1}$ and since $\mathrm{rad}(B) < (1+\varepsilon)^{-n}$ we have

$$\frac{\mathrm{rad}(B)}{\mathrm{rad}(\overline{B})} < 1+\varepsilon$$

hence $B \subset (3+2\varepsilon)\overline{B}$ and by choosing $\varepsilon = \frac{1}{2}$ we obtain the result. □

This lemma allows us to prove the next proposition which provides useful estimates for Hausdorff and packing dimensions.

Proposition 3.12. *Let μ be a finite measure on X, $A \subset X$ be a Borel set and let $0 < c < \infty$ be a constant.*

1. *If $\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} < c$ for all $x \in A$ then $\mathcal{H}^s(A) \geq \mu(A)/c$.*
2. *If $\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} > c$ for all $x \in A$ then $\mathcal{H}^s(A) \leq 8^s \mu(X)/c$.*

Proof. 1. For each $n \in \mathbb{N}$ define a set

$$A_n = \{x \in A \,:\, \mu(B(x,r)) < cr^s,\ \text{for all } 0 < r < \frac{1}{n}\}$$

It is obvious that the sequence $\{A_n\}$ is increasing and since for all $x \in A$

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} < c,$$

then for all $x \in A$ there exists $n \in \mathbb{N}$ such that $\mu(B(x,r))/r^s < c$ for all $0 < r < \frac{1}{n}$, hence

$$A = \bigcup_{n=1}^{\infty} A_n$$

and by the continuity of the measure $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

We now take $0 < \delta < \frac{1}{n}$ and let $\{U_i\}$ be a δ-cover of A and thus A_n. For each U_i containing a point $x \in A_n$, $B(x, \operatorname{diam}(U_i))$ covers that U_i. Then

$$\mu(U_i) \leq \mu(B(x, \operatorname{diam}(U_i))) \leq c \cdot \operatorname{diam}(U_i)^s$$

so by the mass distribution principle 3.9 we have

$$\mathcal{H}^s(A) \geq \frac{\mu(A_n)}{c},$$

and by taking $n \to \infty$ we get the claim.

2. For the proof of the second statement we require Lemma 3.11. First we assume that A is bounded. Fix $\delta > 0$ and define the collection

$$\mathcal{B} = \{B(x, r) : x \in A,\ 0 < r < \delta,\ \text{and } \mu(B(x, r)) > cr^s\}$$

since

$$\limsup_{r\downarrow 0} \frac{\mu(B(x, r))}{r^s} > c,$$

then for all $x \in A$ there exists $r < \delta$ such that $\mu(B(x, r)) > cr^s$ and hence

$$A \subset \bigcup_{B\in\mathcal{B}} B$$

By Lemma 3.11 we have a countable subcollection $\mathcal{C}$ of $\mathcal{B}$ such that the balls in $\mathcal{C}$ are disjoint and

$$A \subset \bigcup_{B\in\mathcal{C}} 4B.$$

Thus $\mathcal{C}$ is an 8δ-cover of A and

$$\begin{aligned}\mathcal{H}^s_{8\delta}(A) &\leq \sum_{B\in\mathcal{C}} (4\operatorname{diam}(B))^s \leq 4^s \sum_{B\in\mathcal{C}} \operatorname{diam}(B)^s \\ &< 8^s c^{-1} \sum_{B\in\mathcal{C}} \mu(B) \leq 8^s c^{-1} \mu(X) < \infty\end{aligned}$$

Letting $\delta \to 0$ we get $\mathcal{H}^s(A) \leq 8^s c^{-1}\mu(X) < \infty$ for all bounded A. If we then assume A to be unbounded such that $\mathcal{H}^s(A) > 8^s c^{-1}\mu(X)$, then it follows that for some large enough bounded set $A' \subset A$, the value of $\mathcal{H}^s(A')$ exceeds $8^s c^{-1}\mu(X)$, which according to above is a contradiction. $\square$

Motivated by the proposition above, we define local dimensions for measures by how the measure of a ball scale compared to the radius.

Definition 3.13. The *lower local dimension* of a measure μ at point $x \in X$ is defined as

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

and the *upper local dimension* similarily

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ is the open ball with center x and radius r. If the upper and lower dimensions coincide at a point x we say that μ has a local dimension at x which is denoted by

$$\dim_{\mathrm{loc}} \mu(x) = \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

The measure μ is said to be *exact lower dimensional* if $\underline{\dim}_{\mathrm{loc}}\mu(x) = s$ for μ-almost all x and *exact upper dimensional* if $\overline{\dim}_{\mathrm{loc}}\mu(x) = s$ for μ-almost all x. If $\dim_{\mathrm{loc}} \mu(x) = s$ for μ-almost all x, then the measure μ is *exact dimensional.*

It is hardly surprising that by Proposition 3.12 we find a connection between the local dimensions of measures and Hausdorff dimensions of sets.

Proposition 3.14. *Let $A \subset X$ be a non-empty Borel set.*

1. *If $\underline{\dim}_{loc}\mu(x) \geq s$ for all $x \in A$ and $\mu(A) > 0$ then $\dim_H(A) \geq s$.*
2. *If $\underline{\dim}_{loc}\mu(x) \leq s$ for all $x \in A$ then $\dim_H(A) \leq s$.*

Proof. 1. By the hypotesis, for every $\varepsilon > 0$ there exists $0 < r_0 \leq 1$ such that for every $r < r_0$

$$\frac{\log \mu B(x,r)}{\log r} > s - \varepsilon \iff \mu(B(x,r)) < r^{s-\varepsilon},$$

for all $x \in A$. This implies that

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^{s-\varepsilon}} < \limsup_{r \downarrow 0} \frac{r^{s-\varepsilon}}{r^{s-\varepsilon}} = 1$$

thus by Proposition 3.12 we have $\mathcal{H}^{s-\varepsilon}(A) \geq \mu(A) > 0$ which by Theorem 3.5 implies that $\dim_H A \geq s - \varepsilon$. Since ε was arbitrary, the claim follows.

2. In contrast to 1 we have for every $\varepsilon > 0$ a decreasing sequence $\{r_n\}$ such that

$$\frac{\log \mu B(x,r_n)}{\log r_n} < s + \varepsilon \iff \mu(B(x,r_n)) > r_n^{s-\varepsilon},$$

for all $x \in A$ and $n \in \mathbb{N}$. Then

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^{s+\varepsilon}} > \lim_{n \to \infty} \frac{r_n^{s+\varepsilon}}{r_n^{s+\varepsilon}} = 1$$

thus by Proposition 3.12 we have $\mathcal{H}^{s+\varepsilon}(A) \leq 8^{s+\varepsilon}\mu(X) < \infty$ which by Theorem 3.5 implies that $\dim_H A \leq s + \varepsilon$ and the claim follows. □

It is noteworthy that it is exactly the *lower* local dimensions which are related to the Hausdorff measure of a set. By a similar argument one may show that the upper local dimensions are in correspondence with the packing measure of a set.

Next we define the Hausdorff dimension for a measure. We give the quite natural definition using Hausdorff dimension of sets associated with the measure. We note however that some authors use the proposition we give afterwards as the definition of the Hausdorff dimension of a set, which is also rather natural given Proposition 3.14.

Definition 3.15. Let μ be a finite Borel measure. The *Hausdorff dimension of* μ is defined as

$$\dim_H \mu = \inf\{\dim_H E : E \text{ is a Borel set with } \mu(E) > 0\}.$$

Recalling Proposition 3.14, one should not be supprised to find that the Hausdorff dimension of a measure may be expressed in terms of the local dimension.

Proposition 3.16. *For a finite Borel measure* μ

$$\dim_H \mu = \sup\{s : \underline{\dim}_{\text{loc}}\mu(x) \geq s \text{ for } \mu\text{-almost all } x\}. \tag{3.3}$$

Proof. Denote by s_0 the right hand side of equation (3.3). First let $s < s_0$, so by definition,

$$\underline{\dim}_{\text{loc}}\mu(x) \geq s,$$

for all $x \in E_0$, where E_0 is a set of full measure. Let E be a Borel set, with $\mu(E) > 0$. The previous implies that

$$\underline{\dim}_{\text{loc}}\mu(x) \geq s,$$

for all $x \in E \cap E_0$, where $\mu(E \cap E_0) = \mu(E) > 0$, since E_0 has full measure. Now by Proposition 3.14, we have

$$\dim_H E \geq \dim_H E \cap E_0 \geq s.$$

Since E and s were arbitrary, this implies that $\dim_H \mu \geq s_0$.

For the other inequality take $s > s_0$, so by definition,

$$\underline{\dim}_{\text{loc}}\mu(x) \leq s,$$

in some Borel set E, with $\mu(E) > 0$. Once again using Proposition 3.14, we get that $\dim_H E \leq s$, so by definition $\dim_H \mu \leq s$. Since this again holds for all $s > s_0$, we arrive at the claim. □

3.5 Self-Similar Sets and Measures

In Section 4 we give a strong emphasis on the well studied class of measures called the self-similar measures. This section is devoted to familiarising the reader with the concept of self-similar sets and corresponding measures. We begin by introducing iterated function systems (IFS), and afterwards move on to defining self-similar measures supported on the attractors of these systems. Since the focus of Section 4 is in the Euclidean case, we will restrict ourselves to $\mathbb{R}^d$ also in this chapter. The theory can however easily be developed in any metric space X with the obvious modifications. The study of self-similar measures has its origins in the 1980s, and much of the theory developed in this section is due to Hutchinson [12].

Recall that if a function $F : \mathbb{R}^d \to \mathbb{R}^m$ satisfies $||F(x) - F(y)|| \leq c\,||x - y||$, for some positive constant $c < 1$, it is called a *contraction* and if it satisfies $||F(x) - F(y)|| = c\,||x - y||$, it is called a *(contractive) similarity*. The constant c is referred to as the *contraction ratio* in the former case and the *similarity ratio* in the latter.

Definition 3.17. A family $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ of contractions from a closed subset of C of $\mathbb{R}^d$ to itself is called an *iterated function system* or an IFS for short. If the functions F_i are similarities, then $\mathcal{F}$ is called a *self-similar IFS*.

Theorem 3.18. *Let $\mathcal{F}$ be an IFS. Then there exists a unique non-empty compact set $E \subset C$ called the* attractor *or* invariant set *of $\mathcal{F}$, which satisfies*

$$E = \bigcup_{i=1}^{n} F_i(E). \tag{3.4}$$

Moreover, if $F : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ is defined as

$$F(A) = \bigcup_{i=1}^{n} F_i(A),$$

then for any non-empty compact set $A \subset C$, with $F_i(A) \subset A$, we have

$$E = \bigcap_{k=1}^{\infty} F^{\circ k}(A),$$

where $F^{\circ k}(A) = \underbrace{F \circ F \circ \ldots \circ F}_{k \text{ times}}(A)$.

This is a well known theorem, and the proof can be found for example in [8]. The second part of the previous theorem motivates the term iterated function system, since we may obtain the set A by iteratively applying the functions F_i to some appropriate set.

Studying the properties of self similar sets is easier with some information on the overlap of the images of each of the functions F_i. For this, a multitude of separation conditions have been developed. Perhaps the most obvious separation condition, and a great first introduction to the topic, is the *strong separation condition*, which the IFS $\mathcal{F}$ is said to satisfy if the union $\bigcup_{i=1}^{n} F_i(A)$ is disjoint, where A is the attractor of the IFS $\mathcal{F}$. A multitude of other separation conditions have been studied in the literature, but we restrict our selves to this simple case.

The invariant sets of iterated function systems are interesting in their own right, but the object of our study is the measures supported by these invariant sets. In fact the following theorem by Hutchinson [12] provides us with a natural way of assigning a probability measure on the attractor of an IFS.

Theorem 3.19. *Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be an IFS and let E be the invariant set of F. Fix probabilities $p_1, \ldots, p_n \in]0, 1[$ such that*

$$\sum_{i=1}^{n} p_i = 1.$$

Then there exists a unique Borel reqular probability measure μ such that

$$\mu(B) = \sum_{i=1}^{n} p_i \mu \circ F_i^{-1}(B),$$

for all $B \subset \mathbb{R}^d$, with spt$(\mu) = E$.

Proof. The standard proof is a clever application of the contraction mapping theorem, but we omit it here. For the proof, see [8, Theorem 2.8] or [12]. □

The measure μ given by the theorem above is called the invariant measure of $\mathcal{F}$, and if $\mathcal{F}$ is a self-similar IFS, then μ is called a *self-similar measure*. There is a natural way of constructing self-similar measures. Let $\mathcal{F}$ be a self-similar IFS satisfying the strong separation condition, with similarity ratios $c_1, c_2, \ldots, c_n$ and fix probabilities $p_1, p_2, \ldots, p_n \in]0, 1[$, such that $\sum_{i=1}^{n} p_i = 1$. Here and afterwards, we use $\Sigma_k = \{(i_1, i_2, \ldots, i_k) : i_j \in \{1, 2, \ldots, n\}\}$ to denote the set of finite sequences of length k of the numbers i_j and $\Sigma = \{(i_1, i_2, \ldots) : i_j \in \{1, 2, \ldots, n\}\}$ to denote such sequences that are infinitely long. A fixed sequence $(i_1, \ldots, i_k) \in I_k$ is abbreviated with $\mathtt{i}$, and for every $\mathtt{i} = (i_1, i_2, \ldots, i_k) \in \Sigma_k$, we let $\mathtt{i}^- = (i_1, i_2, \ldots, i_{k-1})$.

Let $E \subset \mathbb{R}^d$, such that $F_i(E) \subset E$, for all $i = 1, \ldots, n$, for example we may take E to be the attractor of the IFS in question. For simplicity we assume here and hereafter that $\operatorname{diam}(E) = 1$. This assumption is does not restrict us, since we may always uniformly scale the set E to have a diameter of 1, and uniform scaling has no effect on any of the notions of dimension we study [7]. Write

$$E_{\mathtt{i}} = E_{i_1,\ldots,i_k} = F_{i_1} \circ \ldots, \circ F_{i_k}(E),$$

and denote by $\mathcal{E} = \{E_{\mathtt{i}} : \mathtt{i} \in \Sigma_k, \text{ for some } k \in \mathbb{N}\}$. The self-similar measure ν can then be constructed by setting $\nu(E) = 1$ and dividing the mass by $\nu(F_i(E)) = p_i \nu(E)$ and inductively

$$\nu(E_{\mathtt{i}}) = p_{\mathtt{i}} = p_{i_1} p_{i_2} \ldots p_{i_k},$$

for any $E_{\mathtt{i}} \in \mathcal{E}$. A simple calculation shows that by Theorem 3.18 we have $E = \bigcap_{k=1}^{\infty} \bigcup_{\mathtt{i} \in \Sigma_k} E_{\mathtt{i}}$, where E is the attractor of $\mathcal{F}$. Now ν may be extended from the sets $\mathcal{E}$ to the whole space $\mathbb{R}^d$ by setting for all $B \subset \mathbb{R}^d$,

$$\nu(B) = \inf \left\{ \sum_{i=1}^{\infty} \nu(V_i) \; : \; B \cap E \subset \bigcup_{i=1}^{\infty} V_i, \text{ and } V_i \in \mathcal{E} \right\}. \tag{3.5}$$

The following proposition shows that this measure is the measure given by Theorem 3.19. However, first we observe that since the functions F_i are contractive similarities with similarity ratios c_i, we have $||F_i(x) - F_i(y)|| = c_i \, ||x - y||$, for all $x, y \in E$, so taking the infimum over both sides yields

$$\operatorname{diam}(F_i(E)) = c_i \operatorname{diam}(E) = c_i.$$

Moreover, by induction it is easy to see that for any $\mathtt{i} \in \Sigma_k$ we have

$$\operatorname{diam}(E_{\mathtt{i}}) = c_{\mathtt{i}} = c_{i_1} c_{i_2} \dots c_{i_k} \leq \max_{j=1,\dots,k} c_{i_j}^k. \tag{3.6}$$

Proposition 3.20. *Let $\mathcal{F}$ be a self-similar IFS satisying the strong separation condition, $p_1, \dots, p_n \in]0,1[$ be fixed such that $\sum_{i=1}^n p_i = 1$, and ν, μ be the corresponding measures given by equation (3.5) and Theorem 3.19 respectively. Then for all $B \subset \mathbb{R}^d$ we have*

$$\nu(B) = \mu(B).$$

Proof. Let us first show that the equality holds for any $V \in \mathcal{E}$. Recall that by the strong separation condition $F_i(E) \cap F_j(E) = \emptyset$, for all $i \neq j$. Since by definition $V = E_{\mathtt{i}}$, for some $\mathtt{i} \in \Sigma_k$, $k \in \mathbb{N}$, that is

$$V = E_{\mathtt{i}} = F_{i_1} \circ \dots \circ F_{i_k}(E),$$

we have $F_i^{-1}(V) = \emptyset$ if $i \neq i_1$, and

$$\mu(V) = \sum_{i=1}^{n} p_i \mu \circ F_i^{-1}(E_{\mathtt{i}}) = p_{i_1} \mu(E_{(i_2,\dots,i_k)}).$$

Iterating this k times gives $\mu(V) = \mu(E_{\mathtt{i}}) = p_{i_1} p_{i_2} \dots p_{i_k} = \nu(E_{\mathtt{i}}) = \nu(V)$.

Now let $B \subset \mathbb{R}^d$ and let $\varepsilon > 0$. By definition we may choose a collection $\{V_i\}_i$, with $V_i \in \mathcal{E}$ and $B \cap E \subset \bigcup_{i=1}^{\infty} V_i$, such that

$$\sum_{i=1}^{\infty} \nu(V_i) \leq \nu(B) + \varepsilon.$$

Then, since $\operatorname{spt}(\mu) = E$, we have

$$\begin{aligned}\mu(B) = \mu(B \cap E) &\leq \mu\left(\bigcup_{i=1}^{\infty} V_i\right) \leq \sum_{i=1}^{\infty} \mu(V_i) \\ &= \sum_{i=1}^{\infty} \nu(V_i) \leq \nu(B) + \varepsilon.\end{aligned}$$

Taking $\varepsilon \to 0$ gives $\mu(B) \leq \nu(B)$.

For the other inequality we make the observation that for all $\varepsilon > 0$ and for any open set U, with $B \cap E \subset U$, there exists a disjoint collection $\{V_i\}_i$, with $V_i \in \mathcal{E}$, such that $\bigcup_i V_i \subset U$ and

$$\nu(B) = \nu(B \cap E) \leq \nu\left(\bigcup_{i=1}^{\infty} V_i\right) + \varepsilon. \tag{3.7}$$

This can be seen as follows. Let $\mathcal{E}_k = \{E_{\mathtt{i}} : \mathtt{i} \in \Sigma_k\}$. It follows from the strong separation condition that the union $\bigcup_{V \in \mathcal{E}_k} V$ is disjoint, for any $k \in \mathbb{N}$ and Theorem 3.18 implies that

$$E = \bigcap_{k=1}^{\infty} \bigcup_{V \in \mathcal{E}_k} V. \tag{3.8}$$

Now let U be as above and $\varepsilon > 0$. Since U is open, for any $a \in B \cap E$ there exists $r_a > 0$, such that $B(a, r_a) \subset U$. Let $E_k = \left\{a \in B \cap E \,:\, r_a \geq \frac{1}{k}\right\}$. Clearly we have

$$B \cap E = \bigcup_{k=1}^{\infty} E_k,$$

and $E_k \subset E_{k+1}$, so by lower continuity of the measure ν we have

$$\lim_{k \to \infty} \nu(E_k) = \nu(B \cap E) = \nu(B). \tag{3.9}$$

On the other hand, a simple consequence of equation (3.6) is that for each E_k we may choose n_k large enough, such that

$$\max_{V \in \mathcal{E}_{n_k}} \operatorname{diam} V \leq \frac{1}{k}.$$

Then for each $V \in \mathcal{E}_{n_k}$, with $V \cap E_k \neq \emptyset$ we have $V \subset U$, since for any $a \in V \cap E_k$, we have $V \subset B(a, \frac{1}{k}) \subset B(a, r_a) \subset U$. Let us denote $\tilde{\mathcal{E}}_{n_k} = \{V \in \mathcal{E}_{n_k} \,:\, V \cap E_k \neq \emptyset\}$. Then by the previous and equation (3.8), we have

$$E_k \subset \bigcup_{V \in \tilde{\mathcal{E}}_{n_k}} V \subset U. \tag{3.10}$$

Thus by equations (3.9) and (3.10), for any $\varepsilon > 0$ we have

$$\nu(E) \le \nu(E_{k_0}) + \varepsilon \le \nu\Big(\bigcup_{V \in \tilde{\mathcal{E}}_{n_{k_0}}} V \Big) + \varepsilon,$$

for some large enough $k_0 \in \mathbb{N}$. Additionally, the union $\bigcup_{V \in \tilde{\mathcal{E}}_{n_{k_0}}} V \subset U$, is disjoint.

Let us now prove the other inequality. Let $\varepsilon > 0$. Recall that μ is outer regular by Theorem 3.19, so we may choose an open set U, with $B \cap E \subset U$, such that

$$\mu(B \cap U) \ge \mu(U) - \frac{\varepsilon}{2}.$$

Choose a disjoint collection $\{V_i\}_i$, with $V_i \in \mathcal{E}$ such that (3.7) is satisfied with $\frac{\varepsilon}{2}$ in place of ε. Then

$$\begin{aligned}
\mu(B) = \mu(B \cap E) \ge \mu(U) - \frac{\varepsilon}{2} &\ge \mu\left(\bigcup_{i=1}^{\infty} V_i \right) - \frac{\varepsilon}{2} \\
&= \sum_{i=1}^{\infty} \mu(V_i) - \frac{\varepsilon}{2} = \sum_{i=1}^{\infty} \nu(V_i) - \frac{\varepsilon}{2} \\
&= \nu\left(\bigcup_{i=1}^{\infty} V_i \right) - \frac{\varepsilon}{2} \ge \nu(B \cap E) - \varepsilon = \nu(B) - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get the claim. □

The Hausdorff dimension of these strongly separated self-similar IFSs are not too difficult to calculate. In fact, the following theorem provides a way of determining the Hausdorff dimension exactly, for strongly separated IFSs. The proof can quite easily be modified to hold in the case of IFSs satisfying probably the second most well known separation condition, the open set condition, but since we do not concern ourselves with that class of self-similar systems, we leave the proof for the interested reader.

Theorem 3.21. *Let $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ be an IFS on $\mathbb{R}^d$ with similarity ratios $c_1, c_2, \dots, c_n$, which satisfies the strong separation condition and let E be the attractor of $\mathcal{F}$. Then $\dim_H E = s$, where s is the unique real number satisfying*

$$\sum_{i=1}^{n} c_i^s = 1. \tag{3.11}$$

Moreover $0 < \mathcal{H}^s(E) < \infty$, for this value of s.

Proof. Let s satisfy (3.11), and let E be the attractor of $\mathcal{F}$. Using (3.4) repeatedly we have

$$E = \bigcup_{\mathtt{i} \in \Sigma_k} E_{\mathtt{i}}.$$

Let $\delta > 0$. By (3.6) we may choose a large enough $k \in \mathbb{N}$, such that $c_{\mathtt{i}} = \operatorname{diam}(E_{\mathtt{i}}) \leq \delta$, for all $\mathtt{i} \in \Sigma_k$. That is, the sets $E_{\mathtt{i}}$, with $\mathtt{i} \in \Sigma_k$ constitute a δ-cover of E. Moreover by the hypothesis,

$$\sum_{\mathtt{i}\in\Sigma_k} \operatorname{diam}(E_{\mathtt{i}})^s = \sum_{\mathtt{i}\in\Sigma_k} c_{\mathtt{i}}^s = \left(\sum_{i_1=1}^{n} c_{i_1}^s\right) \cdots \left(\sum_{i_k=1}^{n} c_{i_1}^s\right) = 1,$$

so $\mathcal{H}_\delta^s(E) \leq 1$, and since this holds for arbitrarily small $\delta > 0$, $\mathcal{H}^s(E) \leq 1$, and consequently $\dim_H E \leq s$.

As we mentioned previously, the lower bounds are usually a little trickier to obtain. Luckily we have introduced the mass distribution principle 3.9, so the only thing we need to do is concentrate a suitable measure on E to obtain the lower bound. Recall that $\Sigma = \{(i_1, i_2, \ldots) : i_j \in \{1, \ldots, n\}\}$ denotes infinite sequences of the numbers $i_j \in \{1, \ldots, n\}$. In what follows,

$$\Sigma_{\mathtt{i}} = \{(\mathtt{i}, q_{k+1}, q_{k+1}, \ldots) : q_j \in \{1, \ldots, n\}\} \subset \Sigma$$

denotes the so called *cylinder* of infinite sequences with initial terms $\mathtt{i}$. Let us concentrate a measure on Σ, which we may transfer to a measure on E in a natural way. Let μ be a measure on Σ, such that $\mu(\emptyset) = 0$, and

$$\mu(\Sigma_{\mathtt{i}}) = c_{\mathtt{i}}^s,$$

for all $\Sigma_{\mathtt{i}} \subset \Sigma$. We note that by 3.11

$$\mu(\Sigma_{\mathtt{i}}) = c_{\mathtt{i}}^s = \sum_{j=1}^{n} (c_{\mathtt{i}} c_j)^s = \sum_{j=1}^{n} \mu(\Sigma_{(\mathtt{i},j)}),$$

which implies that μ is a measure on Σ, with $\mu(\Sigma) = 1$. Recall that by strong separation, for each $x \in E$, there exists a unique sequence $(i_1, i_2, \ldots) \in \Sigma$, such that

$$x = \bigcap_{k=1}^{\infty} E_{(i_1, i_2, \ldots, i_k)}.$$

See [7]. Define a projection mapping $\pi : \Sigma \to E$, by setting $\pi(i_1, i_2, \ldots) = \bigcap_{k=1}^{\infty} E_{(i_1,i_2,\ldots,i_k)}$, for each $(i_1, i_2, \ldots) \in \Sigma$. Now it is natural to define a measure $\tilde{\mu}$ on E as the pushforward of the measure μ under the mapping π, more concretely

$$\tilde{\mu}(A) = \pi_* \mu(A) = \mu \circ \pi^{-1}(A).$$

It is clear that $\pi^{-1}(E) = \Sigma$, so $\tilde{\mu}(E) = 1$, and also clearly $\pi^{-1}(E_{\mathtt{i}}) = \Sigma_{\mathtt{i}}$, for any $k \in \mathbb{N}$ and $\mathtt{i} \in \Sigma_k$.

Let us now check that $\tilde{\mu}$ satisfies the conditions of the mass distribution principle 3.9. Let $l = \min\{d(E_i, E_j : i \neq j\}$, which is positive and finite, since $\mathcal{F}$ is strongly separated. Let $x \in E$, and $(i_1, i_2, \ldots) = \pi^{-1}(x)$. For $0 < r < l$ denote by k the smallest natural number for which

$$c_{\mathtt{i}_k} l \le r < c_{\mathtt{i}_{k-1}} l, \tag{3.12}$$

where $\mathtt{i}_k$ denotes the first k terms of the infinite sequence $(i_1, i_2, \ldots)$. Notice that for any $\mathtt{j} \in \Sigma_k$, which differs from $\mathtt{i}_k$, the separation of the sets $E_{\mathtt{i}_k}$ and $E_{\mathtt{j}}$ is at least $c_{\mathtt{i}_{k-1}} l > r$, so $E \cap B(x,r) \subset E_{\mathtt{i}_k}$. This together with 3.12 implies that for any $x \in E$ and $0 < r < l$, we have

$$\begin{aligned}\tilde{\mu}(E \cap B(x,r)) \le \tilde{\mu}(E_{\mathtt{i}_k}) = \mu(\pi^{-1}(E_{\mathtt{i}_k})) = \mu(\Sigma_{\mathtt{i}_k})\\ = c_{\mathtt{i}_k}^s \le d^{-s} r^s.\end{aligned} \tag{3.13}$$

Now let $0 < \delta < l$ and $U \subset \mathbb{R}^d$, with $r \coloneqq \operatorname{diam}(U) \le \delta$. Since $\tilde{\mu}$ is supported on E, we may assume without loss of generality that $U \cap E \neq \emptyset$. Choose $x \in U \cap E$. Then for any $1 < c < \frac{l}{\delta}$, $U \subset B(x, cr)$, so by 3.13 we have

$$\tilde{\mu}(U) \le d^{-s} c^s r^s = d^{-s} c^s \operatorname{diam}(U)^s,$$

and taking $c \to 1$ shows that $\tilde{\mu}$ satisfies the assumptions of the mass distribution principle 3.9. Thus $\mathcal{H}^s(E) \ge d^s > 0$, and $\dim_H E \ge s$, finishing the proof.

□

4 Multifractal Analysis

In multifractal analysis, one is interested in obtaining detailed information about the scaling properties of certain measures called *multifractal measures*. As is the case with fractals, a formal definition of a multifractal measure is avoided as not to restrict ourselves too much. The sets under analysis are

$$E_\alpha = \{x \in X \,:\, \dim_{\text{loc}} \mu(x) = \alpha\},$$

i.e. the α-level-sets of the local dimension of some measure μ. The measures for which these sets E_α are non-empty and of fractal nature over a range of α are called *multifractal measures*. A natural object to study is the dimensions of these level sets, which may depend on the variable α. These dimensions $f(\alpha) \equiv \dim(E_\alpha)$ (for some suitable definition of dimension) are called the *multifractal spectrum*, or sometimes the singularity spectrum, of μ.

Generally, self-similar measures provide an example of a multifractal measures. Even strongly separated self-similar measures may exhibit multifractal properties, which may at first sound surprising, since self-similar measures are a prototypical example of exact dimensional measures [8].

The so called *multifractal formalism* is a celebrated heuristic principle, originated in physics literature, which describes the multifractal properties of a given measure. While studying turbulent flow in fluids, the authors of [10] noticed that in some cases the Hausdorff measure of the sets E_α, related to some specific measures correspond to Legendre transforms of certain functions that are easily estimated.

Our aim is that by the end of Part I we have developed this correspondence rigorously and provided examples of multifractal measures for which the multifractal formalism holds.

In addition to the study of turbulence, multifractal analysis has found applications in a plethora of situations. Some examples include studying residence measures on the attractors of dynamical systems [7], economics [13], medical imaging [14] and complex networks [21].

4.1 Fine and coarse multifractal theory

This section is mostly based on [7, Chapter 10] and [8, Chapter 11] and provides basis for two different approaches to multifractal analysis. The *fine theory* considers the geometry of the sets E_α themselves, where as the *coarse theory* is concerned in the irregularities of the distribution of $\mu(B(x,r))$ on small scales. For many basic measures (e.g. self-similar measures on Euclidean spaces with strong enough separation) the multifractal spectra of both of these approaches coincide. The coarse theory is similar to the original ideas of multifractal formalism by physicists and is better suited for estimating multifractal spectra numerically, whereas the fine theory is a little more mathematically convinient since it is based on the Hausdorff dimension.

For the rest of this section μ refers to a finite Borel regular measure on $\mathbb{R}^d$. For an arbitrary $0 < \delta < \infty$ we let $\mathcal{D}_\delta$ denote the cubes of the δ-coordinate mesh (that is cubes of form $[m_1\delta, (m_1+1)\delta] \times \ldots \times [m_n\delta, (m_n+1)\delta]$, where $m_1, \ldots, m_n$ are integers) that intersect the support of μ. We define

$$N_\delta^\mu(\alpha) = \#\{D \in \mathcal{D}_\delta : \mu(D) \geq \delta^\alpha\}, \tag{4.1}$$

where $\#$ denotes the cardinality of the set. We also define the L^q moment sums over the δ-mesh cubes

$$S_\delta^\mu(q) = \sum_{D \in \mathcal{D}_\delta} \mu(D)^q, \tag{4.2}$$

where the sum runs over the delta mesh cubes of positive μ-measure.

Definition 4.1. The *upper and lower L^q-spectrum* of μ are defined as

$$\underline{\tau}_\mu(q) = \liminf_{\delta \to 0} \frac{\log S_\delta^\mu(q)}{\log \delta}$$

and

$$\overline{\tau}_\mu(q) = \limsup_{\delta \to 0} \frac{\log S_\delta^\mu(q)}{\log \delta},$$

respectively. When the limits agree, we call $\tau_\mu(q) = \underline{\tau}_\mu(q) = \overline{\tau}(q)$ the *L^q-spectrum* of μ.

Definition 4.2. Let $0 \le \alpha < \infty$. We define the *lower* and *upper coarse multifractal spectrum* of μ as

$$\underline{f}_C^\mu(\alpha) = \lim_{\varepsilon \to 0} \liminf_{\delta \to 0} \frac{\log(N_\delta^\mu(\alpha+\varepsilon) - N_\delta^\mu(\alpha-\varepsilon))}{-\log \delta}$$

and

$$\overline{f}_C^\mu(\alpha) = \lim_{\varepsilon \to 0} \limsup_{\delta \to 0} \frac{\log(N_\delta^\mu(\alpha+\varepsilon) - N_\delta^\mu(\alpha-\varepsilon))}{-\log \delta}$$

respectively. If the limits agree, we call the following double limit the *coarse multifractal spectrum* of μ

$$f_C^\mu(\alpha) = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \frac{\log(N_\delta^\mu(\alpha+\varepsilon) - N_\delta^\mu(\alpha-\varepsilon))}{-\log \delta}. \tag{4.3}$$

Definition 4.3. Let $E_\alpha^\mu = \{x \in \mathbb{R}^d : \dim_{\mathrm{loc}} \mu(x) = \alpha\}$. We define the *Hausdorff (or fine) multifractal spectrum* of μ as

$$f_H^\mu(\alpha) = \dim_H(E_\alpha)$$

Definition 4.4. For $0 \le \alpha < \infty$, the *lower* and *upper Legendre spectrum* of μ are defined as

$$\underline{f}_L^\mu(\alpha) = \inf_{q \in \mathbb{R}} \{q\alpha - \underline{\tau}_\mu(q)\}$$

and

$$\overline{f}_L^\mu(\alpha) = \inf_{q \in \mathbb{R}} \{q\alpha - \overline{\tau}_\mu(q)\},$$

respectively. When the previous are equal we refer to the function $f_L^\mu(\alpha) = \underline{f}_L^\mu(\alpha) = \overline{f}_L^\mu(\alpha)$ as the *Legendre spectrum* of μ.

For a large class of well behaved measures, all of these definitions give the same spectrum. Our aim is now to establish some basic relationships between the different spectra in a few different cases. Let us first examine the spectra with no further restrictions to the measure μ. For this we give a theorem concerning Hausdorff measure. For a sketch of the proof see e.q. [7, Theorem 4.10].

Theorem 4.5. *Let $A \subset \mathbb{R}^d$ with $\mathcal{H}^s(A) = \infty$. Then there exists a compact set $B \subset A$ such that $0 < \mathcal{H}^s(B) < \infty$.*

Corollary 4.6. *Let $A \subset \mathbb{R}^d$ with $\mathcal{H}^s(A) = \infty$. Then for any $0 < c < \infty$ there exists a Borel set $B \subset A$ such that $c < \mathcal{H}^s(B) < \infty$.*

Proof. Apply Theorems 4.5 and 2.19. □

The next lemma establishes the basic relationship between the different spectra.

Lemma 4.7. *Let μ be a finite Borel regular measure on $\mathbb{R}^d$. Then*

1. $f_H^\mu(\alpha) \le \underline{f}_C^\mu(\alpha) \le \overline{f}_C^\mu(\alpha)$

2. $\overline{f}_C^\mu(\alpha) \le \overline{f}_L^\mu(\alpha)$

3. $\underline{f}_C^\mu(\alpha) \le \underline{f}_L^\mu(\alpha)$

for all $\alpha \ge 0$.

Proof. 1. The right hand side inequality is obvious from the definition. Let us prove the left hand side inequality. Fix $\alpha \ge 0$, and we write $h = f_H^\mu(\alpha) = \dim_H(E_\alpha)$. Assume $h > 0$. Now for any $\varepsilon > 0$ we have $\mathcal{H}^{h-\varepsilon}(E_\alpha) = \infty$. By Corollary 4.6 and the definition of E_α we can find a set $E'_\alpha \subset E_\alpha$, with $\mathcal{H}^{h-\varepsilon}(E'_\alpha) > 1$ such that

$$3^n r^{\alpha+\varepsilon} \le \mu(B(x,r)) \le (2\sqrt{n})^{\varepsilon-\alpha} r^{\alpha-\varepsilon},$$

for all $x \in E'_\alpha$ and all $r < r_0$. By the definition of $\mathcal{H}^s$ we may choose $0 < \delta < \frac{1}{2\sqrt{n}} r_0$ such that

$$\mathcal{H}_\delta^{h-\varepsilon}(E'_\alpha) \ge 1.$$

For $r \le \delta$, consider the r-mesh cubes $\mathcal{Q}_r = \{D \in \mathcal{D}_r \,:\, E'_\alpha \cap D \ne \emptyset\}$. Each of the cubes $D \in \mathcal{Q}_r$ is surrounded by $3^n - 1$ similar cubes, denoted by $\{D_i\}_{i=1}^{3^n-1}$ and thus if $x \in D$ for some r-mesh cube in $\mathcal{Q}_r$, we have

$$B(x,r) \subset D \cup \bigcup_{i=1}^{3^n-1} D_i \subset B(x, 2\sqrt{n}r),$$

thus

$$3^n r^{\alpha+\varepsilon} \le \mu(B(x,r)) \le \mu(\bigcup_{i=1}^{3^n-1} D_i) \le \mu(B(x, 2\sqrt{n}r)) \le r^{\alpha-\varepsilon},$$

such that

$$r^{\alpha+\varepsilon} \le \mu(D') < r^{\alpha-\varepsilon}, \tag{4.4}$$

where D' is the r-mesh cube in $\{D_i\}_{i=1}^{3^n-1}$ which contains x. Since $\mathcal{Q}_r$ is a δ-cover for E'_α, by the definition of $\mathcal{H}_\delta^{h-\varepsilon}(E'_\alpha)$, we have

$$\#\mathcal{Q}_r \ge \frac{\sum_{D\in\mathcal{Q}_r} \mathrm{diam}(D)^{h-\varepsilon}}{r^{h-\varepsilon}} \ge \mathcal{H}_\delta^{h-\varepsilon}(E'_\alpha) r^{\varepsilon-h} \ge r^{\varepsilon-h},$$

thus there are at least $\frac{1}{3^n} r^{\varepsilon-h}$ cubes D' that satisfy (4.4), from which it follows that

$$N_r^\mu(\alpha+\varepsilon) - N_r^\mu(\alpha-\varepsilon) \ge \frac{1}{3^n} r^{\varepsilon-h},$$

hence

$$\underline{f}_C^\mu(\alpha) = \lim_{\varepsilon\to 0}\liminf_{r\to 0}\frac{\log(N_r^\mu(\alpha+\varepsilon)-N_r^\mu(\alpha-\varepsilon))}{-\log r} \geq \lim_{\varepsilon\to 0}\liminf_{r\to 0}\frac{\log(\frac{1}{3^n}r^{\varepsilon-h})}{-\log r}$$
$$\geq \lim_{\varepsilon\to 0}\liminf_{r\to 0}\frac{(\varepsilon-h)\log r+\log\frac{1}{3^n}}{-\log r} \geq \lim_{\varepsilon\to 0} h-\varepsilon = f_H^\mu(\alpha).$$

2. First we prove the inequality for $q \geq 0$. Let $\eta > 0$. By definition of $f_C^\mu(\alpha)$, we have an $0 < \varepsilon < \frac{\eta}{q}$ for which there exists a $\delta_0 > 0$ such that

$$\delta^{-\underline{f}_C^\mu(\alpha)+\eta} \leq N_\delta^\mu(\alpha+\varepsilon) - N_\delta^\mu(\alpha-\varepsilon) \leq \delta^{-\underline{f}_C^\mu(\alpha)-\eta}, \tag{4.5}$$

for all $\delta < \delta_0$. Now obviously

$$S_\delta^\mu = \sum_{D\in\mathcal{D}_\delta}\mu(D)^q \geq N_\delta^\mu(\alpha+\varepsilon)\delta^{q(\alpha+\varepsilon)} \geq \delta^{-\underline{f}_C^\mu(\alpha)+\eta}\delta^{q(\alpha+\varepsilon)} \geq \delta^{-\underline{f}_C^\mu(\alpha)+2\eta+q\alpha}$$

Since $\eta > 0$ was arbitrary, it follows that

$$\tau_\mu(q) \leq -\underline{f}_C^\mu(\alpha) + 2\eta + q\alpha \Rightarrow \underline{f}_C^\mu(\alpha) \leq \inf_{q\geq 0}\{q\alpha - \tau_\mu(q)\}. \tag{4.6}$$

For $q < 0$ we note that by definition

$$S_\delta^\mu(q) \geq \#\{D \in \mathcal{D}_\delta : 0 < \mu(D) \leq \delta^\alpha\}\delta^{q\alpha}. \tag{4.7}$$

For $\eta > 0$ and $0 < \varepsilon \leq \frac{\eta}{q}$ (4.5) holds. By (4.7) we have

$$\begin{aligned} S_\delta^\mu(q) &\geq \#\{D \in \mathcal{D}_\delta : 0 < \mu(D) \leq \delta^{\alpha-\varepsilon}\}\delta^{q(\alpha-\varepsilon)} \\ &\geq (N_\delta^\mu(\alpha+\varepsilon) - N_\delta^\mu(\alpha-\varepsilon))\delta^{q(\alpha-\varepsilon)} \geq \delta^{-\underline{f}_C^\mu(\alpha)+q\alpha}, \end{aligned}$$

and it follows that

$$\tau_\mu(q) \leq -\underline{f}_C^\mu(\alpha) + q\alpha \Rightarrow \underline{f}_C^\mu(\alpha) \leq \inf_{q<0}\{q\alpha - \tau_\mu(q)\}. \tag{4.8}$$

By combining (4.6) and (4.8) we get the result. The proof of 3. is similar to that of 2. □

4.2 Multifractal formalism for self-similar measures

From the physical origins of the multifractal study, for a qiven measure μ the multifractal formalism is usually said to hold [18] if there exists a function $\tau(q)$, such that

1. (a) τ is increasing, concave and smooth
   (b) τ has affine asymptotes as $q \to \pm\infty$

2. There exist real numbers $0 < \alpha_1 < \alpha_2$, such that

$$f_H^\mu(\alpha) = f_C^\mu(\alpha) = \begin{cases} \inf_q\{\alpha q - \tau(q)\}, & \alpha \in [\alpha_1, \alpha_2], \\ 0, & \alpha \notin [\alpha_1, \alpha_2] \end{cases}$$

3. $\tau(q)$ can be calculated with the box counting argument in definition 4.1, in other words $\tau(q) = \tau_\mu(q)$

Note that the combination of 2. and 3. above states that for appropriate values of α, we have

$$f_H^\mu(\alpha) = f_C^\mu(\alpha) = f_L^\mu(\alpha),$$

so all of the definitions of the multifractal spectra given in the previous section agree. As stated, our goal in this section is to establish the multifractal formalism rigorously, for self-similar measures in the sense of Section 3.5. For the rest of the section we let $\mathcal{F}$ be a self similar IFS with similarity ratios $c_1, \ldots, c_n$ and μ be the corresponding self-similar measure, with probabilities $p_1, \ldots, p_n$. First we show that the fine multifractal spectrum of μ can be obtained as the Legendre transform of a function $\tau(q)$ defined as follows.

Definition 4.8. Let μ be a self-similar measure corresponding to an IFS $\mathcal{F}$ with similarity ratios $c_1, \ldots, c_n$ and probabilities $p_1, \ldots, p_n$. The *similarity spectrum* τ of μ is defined for each $q \in \mathbb{R}$ as the unique real number satisfying

$$\sum_{i=1}^{n} p_i^q c_i^{-\tau(q)} = 1. \tag{4.9}$$

The existence and uniqueness as well as the smoothness of the function τ are all direct concequences of the implicit function theorem. The following lemma gives a couple of useful properties of the similarity spectrum.

Lemma 4.9. *The function τ satisfies the following*

1. $\tau(1) = 0$

2. *$\tau(q)$ is increasing and concave, with strict concavity if $\frac{\log p_i}{\log c_i} \neq \frac{\log p_j}{\log c_j}$, for some $i \neq j$.*

3. *The function $\tau(q)$ has the affine asymptotes $\alpha_{\max} q - M$ and $\alpha_{\min} q - m$ corresponding to q approaching $-\infty$ and ∞ respectively, where*

$$\alpha_{\min} = \min_{1=1,\ldots,k} \frac{\log p_i}{\log c_i} \qquad \alpha_{\max} = \max_{1=1,\ldots,k} \frac{\log p_i}{\log c_i},$$

and $m, M \in \mathbb{R}$.

Proof. 1. Obvious from the definition.

2. Differentiating equation (4.9) implicitly once gives

$$\sum_{i=1}^{n} p_i^q c_i^{-\tau(q)} \Big(\log p_i - \tau'(q) \log c_i \Big) = 0, \tag{4.10}$$

which implies that $\tau'(q) > 0$ and thus $\tau(q)$ is increasing. Second differentiation gives us

$$\sum_{i=1}^{n} p_i^q c_i^{-\tau(q)} \Big(-\tau''(q) \log c_i + (\log p_i - \tau'(q) \log c_i)^2 \Big) = 0,$$

which implies that $\tau''(q) \leq 0$ so $\tau(q)$ is concave. The equality holds if and only if

$$\log p_i - \tau'(q) \log c_i = 0, \text{ for all } i.$$

By simple rearrangement, this implies that if $\frac{\log p_i}{\log c_i} \neq \frac{\log p_j}{\log c_j}$, for some $i \neq j$, we have strict concavity. To avoid the degenerate cases where the similarity spectrum might not be strictly concave, we assume from now on that $\frac{\log p_i}{\log c_i}$ are not the same for all values of i.

3. For the second statement, we first define

$$\alpha_i = \frac{\log p_i}{\log c_i}, \qquad \alpha_{\min} = \min_{1=1,\ldots,k} \alpha_i, \text{ and} \qquad \alpha_{\max} = \max_{1=1,\ldots,k} \alpha_i.$$

Let $I_{\max} = \{i = 1, \ldots, k \, : \, \alpha_i = \alpha_{\max}\}$ and $I_{\min} = \{i = 1, \ldots, k \, : \, \alpha_i = \alpha_{\min}\}$. We claim that

$$\lim_{q \to \infty} \alpha_{\min} q - \tau(q) = m,$$

where m is the unique finite real number that satisfies

$$\sum_{i \in I_{\min}} c_i^m = 1. \tag{4.11}$$

This can be seen by first observing that

$$\begin{aligned}
1 = \sum_{i=1}^{n} p_i^q c_i^{-\tau(q)} &= \sum_{i=1}^{n} c_i^{-\alpha_{\min} q} p_i^q c_i^{\alpha_{\min} q - \tau(q)} \\
&= \sum_{i \in I_{\min}} c_i^{-\alpha_{\min} q} p_i^q c_i^{\alpha_{\min} q - \tau(q)} + \sum_{i \notin I_{\min}} c_i^{-\alpha_{\min} q} p_i^q c_i^{\alpha_{\min} q - \tau(q)} \\
&= \sum_{i \in I_{\min}} c_i^{\frac{\log p_i^{-q}}{\log c_i}} p_i^q c_i^{\alpha_{\min} q - \tau(q)} + \sum_{i \notin I_{\min}} c_i^{-\alpha_{\min} q} p_i^q c_i^{\alpha_{\min} q - \tau(q)} \\
&= \sum_{i \in I_{\min}} c_i^{\alpha_{\min} q - \tau(q)} + \sum_{i \notin I_{\min}} c_i^{-\alpha_{\min} q} p_i^q c_i^{\alpha_{\min} q - \tau(q)}.
\end{aligned} \tag{4.12}$$

Next, note that $\alpha_{\min}q - \tau(q)$ is non-increasing, since by (4.10) we have

$$\frac{d}{dq}\alpha_{\min}q - \tau(q) = \alpha_{\min} - \tau'(q) = \alpha_{\min} - \frac{\sum_{i=1}^n p_i^q c_i^{-\tau(q)} \log p_i}{\sum_{i=1}^n p_i^q c_i^{-\tau(q)} \log c_i}$$
$$= \frac{\sum_{i=1}^n p_i^q c_i^{-\tau(q)} (\alpha_{\min} - \alpha_i) \log c_i}{\sum_{i=1}^n p_i^q c_i^{-\tau(q)} \log c_i} \leq 0,$$

so the limit $\lim_{q\to\infty} \alpha_{\min}q - \tau(q)$ exists and is either $-\infty$ or m for some finite real number m. Assuming $\lim_{q\to\infty} \alpha_{\min}q - \tau(q) = -\infty$, taking $q \to \infty$ in equation (4.12) would yield $1 = \infty$, which is an obvious contradiction, so we must have $\lim_{q\to\infty} \alpha_{\min}q - \tau(q) = m$, for some finite m. Note that when $i \notin I_{\min}$, we have $c_i^{-\alpha_{\min}} p_i < 1$, so taking the limits in this case yields the equation (4.11). This implies that the affine function $q \mapsto \alpha_{\max}q - m$ is an asymptote to the function $\tau(q)$ as $q \to \infty$.

A similar argument shows that when $q \to -\infty$, $\tau(q)$ has the asymptote

$$\alpha_{\max}q - M,$$

where M is given by the solution to the equation

$$\sum_{i \in I_{\max}} c_i^M = 1.$$

Notice also that if the sets $I_{\max}$ and $I_{\min}$ contain only a single index, then we have $\lim_{q\to\infty} \alpha_{\max}q - \tau(q) = \lim_{q\to-\infty} \alpha_{\min}q - \tau(q) = 0$. □

The previous lemma already shows that the similarity spectrum satisfies the first of the heuristic properties, for the multifractal formalism. Next we begin the study of the Legendre transform of this singularity spectrum and show that we may obtain both the coarse and the fine multifractal spectra from the Legendre transform of the similarity spectrum and that the singularity spectrum and the L^q-spectrum of the self-similar measure agree.

Recall that the Legendre transform f of the function τ is given by

$$f(\alpha) = \inf_q \alpha q - \tau(q).$$

Since $\tau(q)$ is strictly concave, for a given α the infimum is attained at a unique q. By differentiating, this occurs when

$$\alpha = \tau'(q), \tag{4.13}$$

so the Legenrdre transform of τ is given by

$$f(\alpha) = \alpha q - \tau(q) = \tau'(q)q - \tau(q),$$

where $q = q(\alpha)$. Note that if α is given, the values of τ and q may be determined by equations (4.9) and (4.13). Notice also that the correspondence is one-to-one, so in what follows, we may talk about $q = q(\alpha)$ as a function of α, or $\alpha = \alpha(q) = \tau'(q)$ as a function of q.

Lemma 4.10. *The function α satisfies*

$$\lim_{q\to\infty} \alpha(q) = \alpha_{min},$$
$$\lim_{q\to-\infty} \alpha(q) = \alpha_{max}.$$

Moreover, the Legendre transform $f : [\alpha_{min}, \alpha_{max}] \to \mathbb{R}$ is continuous and concave, $f(\alpha_{min}) = m$ and $f(\alpha_{max}) = M$, where m and M are as in Lemma 4.9.

Proof. Using the notation of Lemma 4.9, we have by simple manipulation of equation (4.10),

$$\begin{aligned}\alpha(q) = \tau'(q) &= \frac{\sum_{i=1}^n p_i^q c_i^{-\tau(q)} \log p_i}{\sum_{i=1}^n p_i^q c_i^{-\tau(q)} \log c_i} \\ &= \frac{\sum_{i\in I_{\max}}^n c_i^{\alpha_{\min} q-\tau(q)} \log p_i + \sum_{i\notin I_{\min}}^n c_i^{\alpha_i q-\tau(q)} \log p_i}{\sum_{i\in I_{\min}}^n c_i^{\alpha_{\min} q-\tau(q)} \log c_i + \sum_{i\notin I_{\min}}^n c_i^{\alpha_i q-\tau(q)} \log c_i}.\end{aligned}$$

Since $\alpha_i > \alpha_{\min}$, when $q \to \infty$ we have $\alpha_i q - \tau(q) \to \infty$, so taking $q \to \infty$ in the equation above yields

$$\lim_{q\to\infty} \alpha(q) = \max_{i=1,\dots,n} \frac{\log p_i}{\log c_i} = \alpha_{\min}.$$

The limit as $q \to -\infty$ is proved similarily. The continuity of the function f follows easily from the smoothness of τ. By Lemma 4.9, it is clear that $f(\alpha_{\min}) = m$ and $f(\alpha_{\max}) = M$.

Finally, by differentiating we get

$$f'(\alpha) = q + \alpha q'(\alpha) - \tau'(q)q'(\alpha) = q, \tag{4.14}$$

and since q decreases, when α increases, the function f is concave. □

Now we define an auxillary function $\Phi : \mathbb{R}^2 \to \mathbb{R}_+$,

$$\Phi(q, \tau) = \sum_{i=1}^n p_i^q c_i^{-\tau}. \tag{4.15}$$

Notice that $\tau(q)$ is defined by the equation $\Phi(q, \tau(q)) = 1$. The following technical lemma proves useful in the future.

Lemma 4.11. *For all $\varepsilon > 0$, there exists $\delta_0 > 0$ such that*

$$\Phi(q + \delta, \tau(q) + (\alpha - \varepsilon)\delta) < 1$$

and

$$\Phi(q-\delta,\tau(q)-(\alpha+\varepsilon)\delta)<1$$

for all $\delta < \delta_0$

Proof. We prove the second inequality, and the first one is then proved similarily. Let $\varepsilon > 0$. Recall that $\alpha = \tau'(q)$, which is equivalent with the statement that for all $y \in \mathbb{R}$

$$\tau(y)=\tau(x)+\alpha(y-x)+(y-x)\varepsilon(y-x),$$

where $\lim_{z\to 0}\varepsilon(z)=0$. Let $\delta > 0$ and choose $y = q-\delta$ and $x = q$, such that

$$\tau(q-\delta)=\tau(q)-\alpha\delta-\delta\varepsilon(\delta).$$

Since $\lim_{z\to 0}\varepsilon(z)=0$, it follows that there exists $\delta_0 > 0$ such that

$$\varepsilon(\delta)<\varepsilon,$$

for all $\delta < \delta_0$, hence

$$\tau(q-\delta)>\tau(q)-(\alpha+\varepsilon)\delta,$$

for all $\delta < \delta_0$. Notice that since $c_i < 1$, it follows that $\Phi(q,\tau)$ decreases with τ, so

$$1=\Phi(q-\delta,\tau(q-\delta))>\Phi(q-\delta,\tau(q)-(\alpha+\varepsilon)\delta).$$

□

Recall that our aim is to find the Hausdorff dimension of E_α. Our procedure is to concentrate a measure ν on E_α and use Proposition 3.14 to find estimates for the dimension. Let us define the measure ν with a similar motivation as the construction of the measure $\tilde{\mu}$ the proof of 3.21. We skip the abstraction of pushing the measure forward from the set Σ, but keep in mind that it may be shown similarily to the proof of Theorem 3.21, that ν is in fact a probability measure concentrated on $\mathrm{spt}(\mu)$.

For a fixed q, denote $\tau = \tau(q)$. Then for every $\mathtt{i} \in \Sigma_k$, set

$$\nu(E_{\mathtt{i}})=p_{\mathtt{i}}^q c_{\mathtt{i}}^{-\tau}=(p_{i_1}\dots p_{i_k})^q(c_{i_1}\dots c_{i_k})^{-\tau},$$

and extend the measure to the whole space as in Section 3.5. In the following, we write $E_k(x)$ for the unique set $E_{(i_1,\dots,i_k)}$ that contains x.

Lemma 4.12. *For a self-similar measure satisfying the strong separation condition, the following holds*

$$\lim_{r\downarrow 0}\frac{\log\mu(B(x,r))}{\log r}=\alpha \iff \lim_{k\to\infty}\frac{\log\mu(E_k(x))}{\log\operatorname{diam}(E_k(x))}=\alpha,$$

for any $\alpha \geq 0$

Proof. "$\Rightarrow$" Let $x \in \mathbb{R}^d$ such that

$$\lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha.$$

First we note that $\mu(B(x,r)) = \mu(B(x,r)) \cap \operatorname{spt}(\mu)$, where

$$\operatorname{spt}(\mu) = \bigcap_{k=1}^{\infty} \bigcup_{\mathtt{i} \in \Sigma_k} E_{\mathtt{i}}.$$

We can assume that $x \in \operatorname{spt}(\mu)$, since otherwise the result holds trivially.

Observe that $E_k(x) \subset B(x, \operatorname{diam}(E_k(x)))$, so

$$\begin{aligned}\lim_{k \to \infty} \frac{\log \mu(E_k(x))}{\log \operatorname{diam}(E_k(x))} &\geq \lim_{k \to \infty} \frac{\log \mu(B(x, \operatorname{diam}(E_k(x))))}{\log \operatorname{diam}(E_k(x))} \\ &= \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha.\end{aligned}$$

For the opposite inequality we note that since μ satisfies the strong separation condition and $E_{\mathtt{i}}$ is compact for every $\mathtt{i} \in \Sigma_k$, we have

$$d(E_{\mathtt{i}}, E_{\mathtt{j}}) > 0,$$

for all $\mathtt{j}, \mathtt{i} \in \Sigma_k$, with $\mathtt{j} \neq \mathtt{i}$. In particular for all $k \in \mathbb{N}$

$$d(E_k(x), E_{\mathtt{j}}) = \delta_k > 0,$$

for all $E_{\mathtt{j}} \neq E_k(x)$ with $\mathtt{j} \in \Sigma_k$. Now we may choose $r_k \leq \min\{\delta_k, 2^{-1}\operatorname{diam}(E_k(x))\}$, so

$$B(x, r_k) \cap \operatorname{spt}(\mu) \subset E_k(x).$$

Since $\log r_k < 0$ it follows that

$$\lim_{k \to \infty} \frac{\log \mu(E_k(x))}{\log \operatorname{diam}(E_k(x))} \leq \lim_{k \to \infty} \frac{\log \mu(B(x, r_k))}{\log r_k} = \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha.$$

The second direction is proved with similar estimates.

□

The following proposition is the main tool we use in proving that $f(\alpha) = f_H^{\mu}(\alpha)$.

Proposition 4.13. *With α, q, τ, f and ν defined as above, we have*

1. $\nu(E_\alpha) = 1$
2. $\lim_{r \downarrow 0} \frac{\log \nu(B(x,r))}{\log r} = f(\alpha)$, *for all $x \in E_\alpha$.*

Proof. Let $\varepsilon > 0$. Then for all $\delta > 0$

$$\begin{aligned}
\nu(x : \mu(E_k(x)) \geq \operatorname{diam}(E_k)^{\alpha-\varepsilon}) &= \nu(x : \mu(E_k(x))^{\delta}\operatorname{diam}(E_k(x))^{(\varepsilon-\alpha)\delta} \geq 1) \\
&\leq \int \mu(E_k(x))^{\delta}\operatorname{diam}(E_k(x))^{(\varepsilon-\alpha)\delta}\,\mathrm{d}\nu(x) \\
&= \sum_{\mathtt{i}\in\Sigma_k} \mu(E_{\mathtt{i}})^{\delta}\operatorname{diam}(E_{\mathtt{i}})^{(\varepsilon-\alpha)\delta}\nu(E_{\mathtt{i}}) \\
&= \sum_{\mathtt{i}\in\Sigma_k} p_{\mathtt{i}}^{\delta}c_{\mathtt{i}}^{(\varepsilon-\alpha)\delta}p_{\mathtt{i}}^{q}c_{\mathtt{i}}^{-\tau} \\
&= \left(\sum_{i=1}^{n} p_i^{q+\delta}c_i^{-(\tau+(\alpha-\varepsilon))\delta}\right)^{k} \\
&= [\Phi(q+\delta, \tau+(\alpha-\varepsilon)\delta)]^{k}.
\end{aligned}$$

using Markov's inequality 2.17 and a multinomial expansion. By Lemma 4.11 we may choose a small enough $\delta > 0$ such that

$$\nu(x : \mu(E_k(x)) \geq \operatorname{diam}(E_k)^{\alpha-\varepsilon}) \leq \gamma^{k},$$

for some $\gamma < 1$, which does not depend on k. It follows that

$$\sum_{k=1}^{\infty} \nu(x : \mu(E_k(x)) \geq \operatorname{diam}(E_k)^{\alpha-\varepsilon}) \leq \sum_{k=1}^{\infty} \gamma^{k} \leq \frac{\gamma}{1-\gamma} < \infty.$$

Applying the Borel-Cantelli lemma we get

$$\begin{aligned}
&\nu(\limsup_{k\to\infty}\{x : \mu(E_k(x)) \geq \operatorname{diam}(E_k)^{\alpha-\varepsilon}\}) = 0 \\
&\iff \nu(\liminf_{k\to\infty}\{x : \mu(E_k(x)) < \operatorname{diam}(E_k)^{\alpha-\varepsilon}\}) = 1
\end{aligned}$$

and thus for ν-almost all x we have

$$\liminf_{k\to\infty} \frac{\log \mu(E_k(x))}{\log \operatorname{diam}(E_k(x))} > \alpha - \varepsilon.$$

Since this holds for all $\varepsilon > 0$ we have

$$\liminf_{k\to\infty} \frac{\log \mu(E_k(x))}{\log \operatorname{diam}(E_k(x))} \geq \alpha,$$

for ν-almost all x. On the other hand we can estimate $\nu(x : \mu(E_k(x)) \leq \operatorname{diam}(E_k(x))^{\alpha+\varepsilon})$ in a similar fashion to obtain

$$\begin{aligned}
\nu(x : \mu(E_k(x)) \leq \operatorname{diam}(E_k)^{\alpha+\varepsilon}) &= \nu(x : \mu(E_k(x))^{-\delta}\operatorname{diam}(E_k(x))^{(\varepsilon+\alpha)\delta} \geq 1) \\
&\leq [\Phi(q-\delta, \tau-(\alpha+\varepsilon)\delta)]^{k},
\end{aligned}$$

and by Lemma 4.11 we again get that

$$\nu(x : \mu(E_k(x)) \leq \mathrm{diam}(E_k)^{\alpha+\varepsilon}) \leq \gamma^k,$$

for some $\gamma < 1$ independent of k. By similar application of Borel-Cantelli we get

$$\begin{aligned}
&\nu(\limsup_{k\to\infty}\{x : \mu(E_k(x)) \leq \mathrm{diam}(E_k)^{\alpha+\varepsilon}\}) = 0 \\
&\Rightarrow \nu(\liminf_{k\to\infty}\{x : \mu(E_k(x)) \leq \mathrm{diam}(E_k)^{\alpha+\varepsilon}\}) = 0 \\
&\Rightarrow \nu(\limsup_{k\to\infty}\{x : \mu(E_k(x)) > \mathrm{diam}(E_k)^{\alpha+\varepsilon}\}) = 1
\end{aligned}$$

and thus for ν-almost all x

$$\begin{aligned}
\alpha &\leq \liminf_{k\to\infty} \frac{\log \mu(E_k(x))}{\log \mathrm{diam}(E_k(x))} \\
&\leq \limsup_{k\to\infty} \frac{\log \mu(E_k(x))}{\log \mathrm{diam}(E_k(x))} \leq \alpha.
\end{aligned}$$

By Lemma 4.12 we see that

$$\dim_{\mathrm{loc}} \mu(x) = \lim_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha,$$

for ν-almost all x and thus $\nu(E_\alpha) = 1$ proving claim 1.

For 2. note that

$$\nu(E_k(x)) = \mu(E_k(x))^q \mathrm{diam}(E_k(x))^{-\tau(q)}.$$

Then

$$\frac{\log \nu(E_k(x))}{\log \mathrm{diam}(E_k(x))} = q\frac{\log \mu(E_k(x))}{\log \mathrm{diam}(E_k(x))} - \tau(q)\frac{\log \mathrm{diam}(E_k(x))}{\log \mathrm{diam}(E_k(x))}.$$

By letting $k \to \infty$ we get

$$\frac{\log \nu(E_k(x))}{\log \mathrm{diam}(E_k(x))} \to q\alpha - \tau(q)$$

and by applying Lemma 4.12 we obtain 2. □

Finally it is time to establish the multifractal formalism for self-similar measures rigorously. Unfortunately our definitions are not strong enough to establish the formalism for all values of α, but we do get the results for values of $\alpha(q)$ corresponding to $q \geq 0$. It is possible to alter the definitions slightly to obtain similar results for all values of q [19] , but this complicates matters further so we are satisfied with the results for $q \geq 0$.

Theorem 4.14. *Let μ be a self similar measure as defined above, and let $\tau(q)$ be the similarity spectrum of μ. Then*

$$\tau(q) = \tau_\mu(q),$$

for all $q \geq 0$.

Proof. Let μ be a self-similar measure in $\mathbb{R}^d$. Fix $q \geq 0$ and let $\tau = \tau(q)$. Let $l = \min\{d(E_i, E_j) : i \neq j\}$, $a = \frac{2\sqrt{d}}{l}$ and denote by $c_{\min} = \min\{c_i : i = 1, \ldots, k\}$. For every $\delta \leq \frac{\operatorname{diam}(E)}{a}$, let $\mathcal{D}_\delta$ be the family δ-mesh cubes with positive measure. Recall that for every $\mathtt{i} = (i_1, i_2, \ldots, i_k) \in \Sigma_k$, we let $\mathtt{i}^- = (i_1, i_2, \ldots, i_{k-1})$ and define

$$J = \{\mathtt{i} \in \Sigma_k : \operatorname{diam}(E_\mathtt{i}) \leq a\delta, \text{ but } \operatorname{diam}(E_{\mathtt{i}^-}) > a\delta, \, k \in \mathbb{N}\}.$$

Notice that each point of $E = \operatorname{spt}(\mu)$ lies in exactly one of the sets $E_\mathtt{i}$ with $\mathtt{i}$ in J. Now we have

$$c_{\min} a\delta < \operatorname{diam}(E_\mathtt{i}) = c_\mathtt{i} \leq a\delta, \tag{4.16}$$

and for any $\mathtt{i} \neq \mathtt{j} \in J$, the separation of the sets $E_\mathtt{i}$ and $E_\mathtt{j}$ is at least

$$\min_{\mathtt{i} \in J}\{\operatorname{diam}(E_{\mathtt{i}^-})\} l > a\delta l = 2\sqrt{d}\delta,$$

and since $\operatorname{diam}(D) = \sqrt{d}\delta$, for any $D \in \mathcal{D}_\delta$, each of the δ-mesh cubes which intersects the support of μ, intersects exactly one of the sets $E_\mathtt{i}$, with $\mathtt{i} \in J$. For each $\mathtt{i} \in J$, choose $D_\mathtt{i} \in \mathcal{D}_\delta$, such that $D_\mathtt{i} \cap E_\mathtt{i} \neq \emptyset$. Let $c_0 = \max\{c_{\min}^\tau a^\tau, a^\tau\}$, so that

$$\begin{aligned} S_\delta^\mu(q) &= \sum_{D \in \mathcal{D}_\delta} \mu(D)^q \leq \sum_{\mathtt{i} \in J} \mu(D_\mathtt{i})^q \leq \sum_{\mathtt{i} \in J} \mu(E_\mathtt{i})^q c_\mathtt{i}^{-\tau} c_\mathtt{i}^\tau \\ &\leq c_0 \delta^\tau \sum_{\mathtt{i} \in J} p_\mathtt{i}^q c_\mathtt{i}^{-\tau} = c_0 \delta^\tau. \end{aligned}$$

The last equality follows by noticing that

$$\sum_{i=1}^m p_{\mathtt{i},i}^q c_{\mathtt{i},i}^{-\tau} = p_\mathtt{i}^q c_\mathtt{i}^{-\tau} \sum_{i=1}^m p_i^q c_i^{-\tau} = p_\mathtt{i}^q c_\mathtt{i}^{-\tau},$$

where $\mathtt{i}, i = i_1, \ldots, i_k, i$, so by applying the formula repeatedly, we get the identity

$$\sum_{\mathtt{i} \in J} p_\mathtt{i}^q c_\mathtt{i}^{-\tau} = 1, \tag{4.17}$$

Taking logarithms and rearranging gives us the estimate

$$\frac{\log S_\delta^\mu(q)}{\log \delta} \geq \tau + \frac{\log c_0}{\log \delta},$$

and taking $\delta \to 0$ gives us

$$\tau_\mu(q) = \lim_{\delta\to 0} \frac{\log S_\delta^\mu(q)}{\log \delta} \geq \tau.$$

For the opposite inequality, for each $\mathtt{i} \in J$ let $\mathcal{D}_\mathtt{i} = \{D \in \mathcal{D}_\delta : D \cap E_\mathtt{i} \neq \emptyset\}$. By (4.16), we have $\#\mathcal{D}_\mathtt{i} \leq c_{\min}^{-1}$. Notice also that $E_\mathtt{i} \subset \bigcup_{D\in\mathcal{D}_\mathtt{i}} D$, so we have

$$\mu(E_\mathtt{i})^q \leq \left(\sum_{D\in\mathcal{D}_\mathtt{i}} \mu(D)\right)^q \leq (\#\mathcal{D}_\mathtt{i})^q \sum_{D\in\mathcal{D}_\mathtt{i}} \mu(D)^q \leq c_{\min}^{-q} \sum_{D\in\mathcal{D}_\mathtt{i}} \mu(D)^q.$$

On the other hand, recall that each $D \in \mathcal{D}_\delta$ intersects exactly one of the sets $E_\mathtt{i}$, with $\mathtt{i} \in J$, so letting $c_1 = \min\{c_{\min}^\tau a^\tau, a^\tau\}$ and using (4.16) and (4.17) we may estimate

$$\begin{aligned} S_\delta^\mu(q) &= \sum_{D\in\mathcal{D}_\delta} \mu(D)^q = \sum_{\mathtt{i}\in J_n} \sum_{D\in\mathcal{D}_\mathtt{i}} \mu(D)^q \geq c_{\min}^q \sum_{\mathtt{i}\in J_n} \mu(E_\mathtt{i})^q c_\mathtt{i}^{-\tau} c_\mathtt{i}^\tau \\ &\geq c_{\min}^q c_1 \sum_{\mathtt{i}\in J_n} p_\mathtt{i}^q c_\mathtt{i}^{-\tau} = c_{\min}^q c_1 \end{aligned}$$

Again taking logarithms and rearranging leads to

$$\frac{\log S_\delta^\mu(q)}{\log \delta} \leq \tau + \frac{\log c_{\min}^q c_1}{\log \delta},$$

and taking $\delta \to 0$ yields the second inequality

$$\tau_\mu(q) = \lim_{\delta\to 0} \frac{\log S_\delta^\mu(q)}{\log \delta} \leq \tau.$$

□

Theorem 4.15. *Let μ be a self similar measure as defined above, and let*

$$E_\alpha = \{x \in \mathbb{R}^d : \dim_{\mathrm{loc}} \mu(x) = \alpha\},$$

as previously. Then if $\alpha \notin [\alpha_{min}, \alpha_{max}]$, we have $E_\alpha = \emptyset$ and if $\alpha \in [\alpha_{min}, \alpha_{max}]$, we have

$$f_H^\mu(\alpha) = f(\alpha), \tag{4.18}$$

with f defined as before. In addition

$$f_C^\mu(\alpha) = f(\alpha), \tag{4.19}$$

for all $\alpha = \alpha(q)$, where $q \geq 0$.

Proof. First we note that for all $\mathtt{i} \in \Sigma_k$ we have

$$\frac{\log \mu(E_\mathtt{i})}{\log \mathrm{diam}(E_\mathtt{i})} = \frac{\sum_{i=1}^k \log p_i}{\sum_{i=1}^k \log c_i} \in [\alpha_{\min}, \alpha_{\max}]$$

and since the interval $[\alpha_{\min}, \alpha_{\max}]$ is closed we have

$$\lim_{k\to\infty} \frac{\log \mu(E_k(x))}{\log \operatorname{diam}(E_k(x))} \in [\alpha_{\min}, \alpha_{\max}],$$

for all $x \in \mathbb{R}^d$. By Lemma 4.12 it follows that $E_\alpha = \emptyset$ if $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$. If we then take $\alpha \in]\alpha_{\min}, \alpha_{\max}[$, by Proposition 4.13 there exists a measure ν concentrated on E_α such that

$$\dim_{\mathrm{loc}} \nu(x) = \lim_{r\downarrow 0} \frac{\log \nu(B(x,r))}{\log r} = f(\alpha),$$

for all $x \in E_\alpha$. By Proposition 3.14 it follows that $f_H^\mu(\alpha) \coloneqq \dim_H(E_\alpha) = f(\alpha)$. The equality also holds at $\alpha_{\min}$ and $\alpha_{\max}$, but we do not give the proof here.

To prove (4.19) we first note that the inequality $\underline{f}_C^\mu(\alpha) \geq f(\alpha) = f_H^\mu(\alpha)$ follows from Lemma 4.7.

For the inequality $\overline{f}_C^\mu(\alpha) \leq f(\alpha)$ we use a similar approach as the proof of Theorem 4.14. Let

$$l = \min\{d(E_i, E_j) \,:\, i \neq j\},$$

which is positive, since $\mathcal{F}$ satisfies the strong separation condition. Write $a = \frac{2\sqrt{n}}{l}$, and for a given $\delta < \frac{\operatorname{diam}(E)}{a}$ let J be as in proof of 4.14. Recall that we have

$$ac_{\min}\delta < \operatorname{diam}(E_{\mathtt{i}}) = c_{\mathtt{i}} \leq a\delta,$$

where $c_{\min}$ is the minimal similarity constant of $\mathcal{F}$ and that the separation of the sets $E_{\mathtt{i}}$ and $E_{\mathtt{j}}$ is at least $a\delta l = 2\sqrt{n}\delta$, for $\mathtt{i}, \mathtt{j} \in J$, $\mathtt{i} \neq \mathtt{j}$.

Assume $q \geq 0$ and let $\tau = \tau(q)$, α and f be defined as above. By aplying Markov's inequality 2.17 to the counting measure and using (4.17) and (4.16) we get

$$\begin{aligned}
\#\{\mathtt{i} \in J \,:\, \mu(E_{\mathtt{i}}) \geq a^{-\alpha}\operatorname{diam}(E_{\mathtt{i}})^\alpha\} &= \#\{\mathtt{i} \in J \,:\, a^{\alpha q} p_{\mathtt{i}}^q c_{\mathtt{i}}^{-\alpha q} \geq 1\} \\
&\leq \sum_{\mathtt{i}\in J} a^{\alpha q} p_{\mathtt{i}}^q c_{\mathtt{i}}^{-\alpha q} \\
&= a^{\alpha q} \sum_{\mathtt{i}\in J} p_{\mathtt{i}}^q c_{\mathtt{i}}^{-\tau} c_{\mathtt{i}}^{\tau-\alpha q} \\
&\leq a^{\alpha q}(ac_{\min})^{-f(\alpha)}\delta^{-f(\alpha)}.
\end{aligned}$$

Note that the first equality here is where we use the assumption $q \geq 0$. Consider the δ-mesh cubes $\mathcal{D}_\delta$. Recall also from the proof of 4.14 that each $D \in \mathcal{D}_\delta$ intersects at most one of the sets $E_{\mathtt{i}}$, where $\mathtt{i} \in J$. We may estimate $N_\delta^\mu(\alpha)$ as follows

$$\begin{aligned}
N_\delta^\mu(\alpha) &= \#\{D \in \mathcal{D}_\delta \,:\, \mu(D) \geq \delta^\alpha\} \\
&\leq \#\{\mathtt{i} \in J \,:\, \mu(E_{\mathtt{i}}) \geq a^{-\alpha}\operatorname{diam}(E_{\mathtt{i}})^\alpha\} \\
&\leq a^{\alpha q}(ac_{\min})^{-f(\alpha)}\delta^{-f(\alpha)}.
\end{aligned}$$

Now we estimate $N^\mu_\delta(\alpha+\varepsilon) - N^\mu_\delta(\alpha-\varepsilon)$. By choosing ε and δ small enough we get that

$$\begin{aligned} N^\mu_\delta(\alpha+\varepsilon) - N^\mu_{\delta_1}(\alpha-\varepsilon) &\le N^\mu_\delta(\alpha+\varepsilon) \\ &\le a^{(\alpha+\varepsilon)q}(ac_{\min})^{-f(\alpha+\varepsilon)}\delta^{-f(\alpha+\varepsilon)} \\ &\le a^{\alpha q+1}\delta^{-f(\alpha+\varepsilon)}. \end{aligned}$$

Now

$$\begin{aligned} \overline{f}^\mu_C(\alpha) &= \lim_{\varepsilon\to 0}\limsup_{\delta\to 0}\frac{\log(N^\mu_\delta(\alpha+\varepsilon)-N^\mu_\delta(\alpha-\varepsilon))}{-\log\delta} \\ &\le \lim_{\varepsilon\to 0}\limsup_{\delta\to 0}\frac{\log(a^{\alpha q+1})+\log(\delta^{-f(\alpha+\varepsilon)})}{-\log\delta} \\ &= \lim_{\varepsilon\to 0} f(\alpha+\varepsilon) = f(\alpha), \end{aligned}$$

where the last equality is a consequence of the continuity of f. Thus $\overline{f}^\mu_C(\alpha) \le f(\alpha)$, when $q \ge 0$ and we arrive at the claim. $\square$

Remark 4.16. The theorem fails for values of $q < 0$, since we would have to estimate the number of δ-mesh cubes D with $0 < \mu(D) < \delta^\alpha$, which may differ greatly from the number of sets $E_{\mathtt{i}}$, with $\mu(E_{\mathtt{i}}) \le \operatorname{diam}(E_{\mathtt{i}})^\alpha$. There are a number of ways to work around this obstacle by redefining the L^q-spectrum and coarse spectrum using for example packings of the space instead of partitions with cubes. Using those definitions, it is possible to extend the arguments to all values of α.

4.3 Further properties and examples

In addition to encoding the scaling properties of the level-sets E_α, the multifractal spectrum $f(\alpha)$ as defined above, includes some intereseting information about the properties of the measure μ, as seen by the following proposition.

Proposition 4.17. *Let μ be a self-similar measure as above, with f and $\alpha(q)$ defined as previously, then*

1. *The function f attains it maximum at $\alpha = \alpha(0)$. Moreover $f(\alpha(0)) = \dim_H \operatorname{spt}(\mu)$.*
2. *$f(\alpha(1)) = \alpha(1) = \dim_H \mu$*

Proof. Since f is convex, the maximum is attained at the point α where $f'(\alpha) = 0$. By considering $q = q(\alpha)$ and recalling that

$$f'(\alpha) = q + \alpha q'(\alpha) - \tau'(q)q'(\alpha) = q,$$

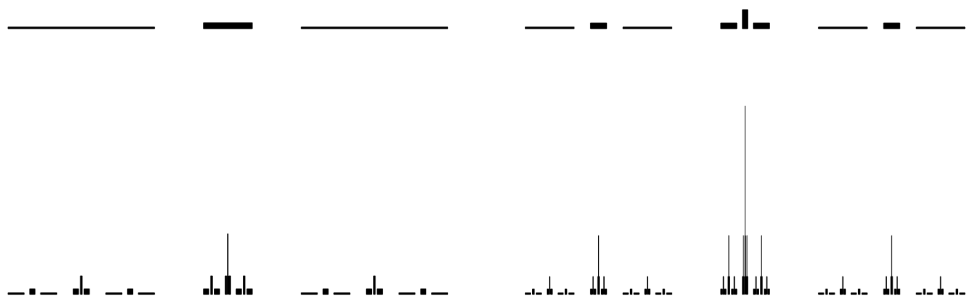


Figure 4.1: The first four steps of the construction of the measure, with $p = \frac{1}{4}$. The area of the rectangle represents the measure of that subinterval.

as in (4.14), we find that f attains its maximum when $q = 0$. For the second part of 1. we note that $\operatorname{spt}(\mu)$ is exactly the attractor of the associated IFS. Now $f(\alpha(0)) = -\tau(0)$, where $-\tau(0)$ by definition satisfies

$$\sum_{i=1}^{n} c_i^{-\tau(0)} = 1.$$

Then by Theorem 3.21, we have $f(\alpha(0)) = -\tau(0) = \dim_H \operatorname{spt}(\mu)$.

For 2. we note that when $q = 1$ we have $\tau(1) = 0$, so $f(\alpha) = \alpha$. Since $\tau(1) = 0$ we also have that the measures ν and μ are identical, whereby it follows from Proposition 4.13 that $\mu(E_{\alpha(1)}) = 1$ and that $\dim_{\text{loc}} \mu(x) = f(\alpha(1)) = \alpha(1)$, for μ-almost all x, so claim 2. follows from Proposition 3.16. □

Let us now see an example of how to calculate the multifractal spectrum of a concrete self-similar measure using the theory developed in this section.

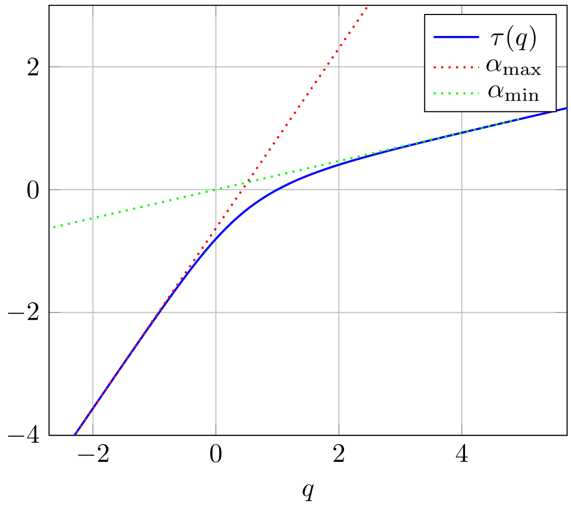
Example 4.18. Choose $0 < p < \frac{1}{3}$. As an example we give the probabilistic IFS defined by the probabilities $p, 1-2p, p$ associated respectively with the iterated function system $\mathcal{F} = \{F_1(x), F_2(x), F_3(x)\}$, where $F_i : [0,1] \to [0,1]$ are defined as

$$\begin{aligned} F_1(x) &= \frac{1}{3}x, \\ F_2(x) &= \frac{1}{9}x + \frac{4}{9} \\ F_3(x) &= \frac{1}{3}x + \frac{2}{3}. \end{aligned}$$

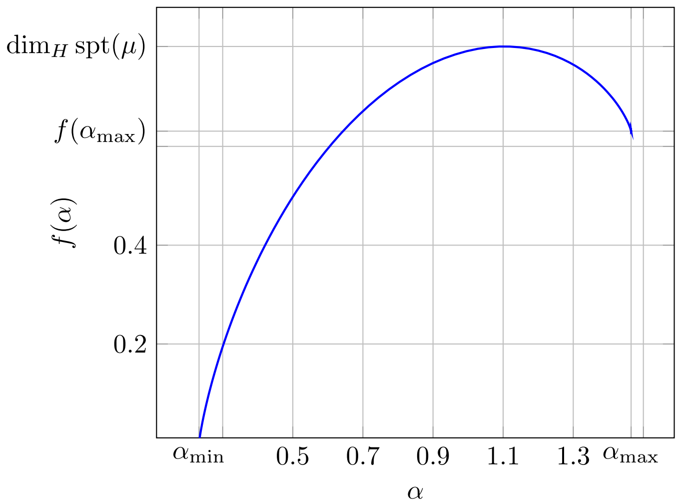
Figure 4.1 illustrates the measure generated by this construction. Let us now obtain the multifractal spectrum $f(\alpha)$ of this self-similar measure using the theory we developed in this section. From equation (4.9) we get

$$2p^q \cdot 3^{-\tau(q)} + (1-2p)^q \cdot 9^{-\tau(q)} = 1,$$

and by solving for τ we get



(a) The similarity spectrum $\tau(q)$ of the measure in example 4.18



(b) The multifractal spectrum $f(\alpha)$ of the measure in example 4.18

Figure 4.2: In (a) $\alpha_{\min}$ and $\alpha_{\max}$ are the slopes of the asymptotes of the similarity spectrum. The general shapes in the figures are typical for the similarity and multifractal spectra of strongly separated self-similar measures.

$$\tau(q) = -\frac{\log(p^q + \sqrt{p^{2q} + (1-2p)^q})}{\log 3}.$$

Differentiating τ gives us the parametrization of α with respect to q

$$\alpha(q) = -\frac{2p^q(\sqrt{p^{2q} + (1-2p)^q} + p^q)\log p + (1-2p)^q \log(1-2p)}{\log 9(p^{2q} + p^q\sqrt{p^{2q} + (1-2p)^q} + (1-2p)^q)}.$$

Now at $\alpha(q)$ the multifractal spectrum of μ is given simply as

$$f(\alpha(q)) = \alpha(q)q - \tau(q),$$

which by Theorem 4.15 corresponds with the fine spectrum $f_H^\mu(\alpha)$, for all values of $q \in \mathbb{R}$, and the coarse spectrum $f_C^\mu(\alpha)$ for values of $q \geq 0$. Figure 4.2 shows the function $\tau(q)$ against the values of q and the values of $f(\alpha)$ against α.

Interesting properties to note from the multifractal spectrum of μ are that from Proposition 4.17 we get

$$\dim_H \operatorname{spt}(\mu) = f(\alpha(0)) = -\tau(0) = \frac{\log(1+\sqrt{2})}{\log 3}$$

and

$$\dim_H \mu = \frac{(p - \sqrt{(p-1)^2}) \log(1-2p) - 2p\log(p)}{\sqrt{(p-1)^2}\log 9}.$$

It is worth to note that the value of the dimension of the support is not dependent on the value of p, which is to be expected, since the support is the attractor of the IFS $\mathcal{F}$, which is not dependent on p.

5 Discussion

We have established the multifractal formalism in a simple case of strongly separated self-similar measures for positive q. This case is obviously quite restrictive and does not provide too many useful tools for considering more applied examples of multifractals. Multifractal formalism has however been established in cases with much less restrictive assumptions. For example the case of negative q has been taken care of in [19] and the multifractal formalism is also known to hold for self-similar measures satisfying the open set condition [5] and for Gibbs measures on cookie-cutter sets [8], to name a few examples. Many other partial results have also been studied and the formalism has been established in some cases of overlapping IFSs for some values of q [6].

Quite recently, a concept of *local* multifractal analysis has been studied for Moran measures in doubling metric spaces [16] and for functions, measures and distributions in Euclidean spaces [2]. The study of multifractals is ongoing in the field of fractal geometry and geometric measure theory and the applications seem plentifull so it is to be expected that multifractal analysis will be carried out for many more objects with multifractal nature in the following years.

Part II
Local entropy and L^q-dimensions of measures in doubling metric spaces

Background

The second part of the thesis arises from the authors work as a university trainee in the Fractal Geometry Research Group at the Department of Mathematics in the summer of 2020. This part was also published as an article in The PUMP Journal of Undergraduate Research [1]. The aim of this part is to correct a slight inaccuracy found in [16], and in doing so provide a basis for local multifractal analysis in doubling metric spaces. The second part has some overlap with Part I of the thesis, especially in the introduction and definitions, to keep the parts independent.

1 Introduction

In multifractal analysis, one is interested in the behaviour of the local dimension map

$$x \mapsto \dim_{\mathrm{loc}}(\mu, x) = \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

where μ is often a fractal type measure, and the level sets of the local dimension map exhibit fractal scaling according to a spectrum of dimensions. These types of measures are often called *multifractal measures* although a precise definition is avoided. In particular one is interested in properties of the level sets

$$E_\alpha = \{x : \dim_{\mathrm{loc}}(\mu, x) = \alpha\},$$

mainly the Hausdorff and packing dimensions of said sets. Multifractal formalism is a heuristic principle with origins in physics literature [10] which states that the Hausdorff and packing dimensions of these levels sets are given by the Legendre transform of the L^q-spectrum $\tau_q(\mu)$ of the measure (see Section 2.2 for the precise definitions), that is

$$\dim_H E_\alpha = \dim_P E_\alpha = \inf_{q\in\mathbb{R}}\{q\alpha - \tau_q(\mu)\}.$$

Recently, multifractal analysis has received quite a bit of interest due to it's many applications in different fields adjacent to mathematics. Multifractals have proved to be a useful tool in understanding turbulence in fluids [10], complex networks [21], economics [13] and medical imaging [14] just to name a few examples. Mathematically the problem is also interesting since in applications the definitions are quite non-standard and often establishing the multifractal formalism in a mathematically rigorous way requires careful investigation of the measures in question.

The most simple and classical example of a class of measures for which the multifractal formalism is known to hold are self-similar measures under the strong separation condition (see e.g. [7], [8]). The results presented by Käenmäki, Rajala and Suomala in [16] and [17] provide a generalisation of this classical situation into doubling metric spaces. In addition, the authors provide a local variant of multifractal analysis, which is also useful in the Euclidean case.

This part of the thesis aims to correct a small inaccuracy in [16]. The authors claim in [16, Proposition 3.2] that the global L^q-spectrum can be calculated using partitions of the doubling metric space X instead of packings (see Section 2.1 for definitions), which are used in the definition. However, the counterexample in Section 2 shows that this is indeed not the case. Our aim is to provide an alternate version of the statement using the restricted L^q-spectrum, which gives correct proofs for the results in [16] that make use of the incorrect proposition.

This part is organised as follows: Section 2 gives a brief recap of the concepts we are working with and introduces some notation, as well as gives a counterexample to [16, Proposition 3.2]. In Section 3 we provide an alternate version of the erroneous proposition and give a proof for the statement. We conclude the paper by stating the main results in Section 4 and by giving a remark concerning the application of the theory in Section 5. For an introduction to multifractal analysis we refer to the first part of the thesis.

2 Preliminaries

2.1 Notation

For constants, we use the notation $c = c(\ldots)$, meaning that the constant depends on the parameters listed inside the parentheses. Closed balls, with center $x \in X$ and radius r are denoted by $B(x,r) = \{y \in X : d(x,y) \le r\}$. For $M > 0$ and a ball $B = B(x,r)$ we use the abbreviation $MB = B(x, Mr)$, when the radius and center of the ball B are fixed.

For the rest of the thesis we always work in *doubling metric spaces* (X, d), which means that there exists a constant $N = N(X)$ called the *doubling constant* of X, such that any closed ball in X, with centre x and radius $r > 0$ can be covered with N balls of radius $r/2$. The distance function d is fixed for the space so we refer to (X, d) simply as X.

Any countable collection $\mathcal{B}$ of pairwise disjoint closed balls is called a *packing* and if the centres of the balls are in a subset $A \subset X$ it is called a packing of A. For $\delta > 0$ the packing $\mathcal{B}$ is called a δ-packing if each of the balls in $\mathcal{B}$ has a radius of δ. A collection $\mathcal{Q}$ of non-empty subsets of X, such that every element of X is contained in exactly one $Q \in \mathcal{Q}$ is called a *partition (of X)*.

Let $1 \leq \Lambda < \infty$. For $\delta > 0$, a countable partition $\mathcal{Q}$ of X is called a (δ, Λ)-*partition* if all of the sets $Q \in \mathcal{Q}$ are Borel sets and for each $Q \in \mathcal{Q}$ there exists a ball B_Q such that $Q \subset \Lambda B_Q$ and the collection $\{B_Q : Q \in \mathcal{Q}\}$ is a δ-packing. Usually we assume that Λ has been fixed and only talk about δ-partitions, since the choice of Λ is often irrelevant.

Let $(\delta_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers so that there is $0 < c < 1$ for which

$$\delta_n < c^n \tag{2.1}$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{\log \delta_n}{\log \delta_{n+1}} = 1. \tag{2.2}$$

Here and hereafter we assume that for each $n \in \mathbb{N}$ we have fixed a δ_n-partition that satisfies the previous assumptions and denote it by $\mathcal{Q}_n$. For $x \in X$, we denote the unique element of $\mathcal{Q}_n$ containing x by $\mathcal{Q}_n(x)$. For $A \subset X$ we set $\mathcal{Q}_n(A) = \{Q \in \mathcal{Q}_n : A \cap Q \neq \emptyset\}$. For a fixed δ_n-partition we silently assume that Λ is the same for all δ_n.

From now on, a *measure* always refers to a locally finite Borel regular (outer) measure defined on all subsets of X. The *support* of a measure μ is the smallest closed subset of X with full μ-measure and is denoted by $\operatorname{spt}(\mu)$.

Next we present a lemma which shows that it is possible to state the doubling property of the metric space X in multiple equivalent ways. The proof of the lemma is a simple exercise (see e.g. [11])

Lemma 2.1. *For a metric space X, the following statements are equivalent*

1. *X is doubling*

2. *There are $s > 0$ and $c > 0$ such that for all $R > r > 0$ any ball of radius R can be covered by $c(r/R)^{-s}$ balls of radius r.*

3. *There are $s > 0$ and $c > 0$ such that if $R > r > 0$ and $\mathcal{B}$ is an r-packing of a closed ball of radius R, then the cardinality of $\mathcal{B}$ is at most $c(r/R)^{-s}$*

4. *For every $0 < \lambda < 1$ there is a constant $M = M(X, \lambda) \in \mathbb{N}$ satisfying the following: if $\mathcal{B}$ is a collection of closed balls of radius $\delta > 0$ so that $\lambda\mathcal{B}$ is pairwise disjoint, then there are δ-packings $\{\mathcal{B}_1, \ldots, \mathcal{B}_M\}$ so that $\mathcal{B} = \bigcup_{i=1}^{M} \mathcal{B}_i$.*

5. *There is $M = M(X) \in \mathbb{N}$ such that if $A \subset X$ and $\delta > 0$, then there are δ-packings of A, $\mathcal{B}_1, \ldots, \mathcal{B}_M$ whose union covers A.*

One final property referenced in this paper is the *density point property* for a measure μ, which is said to hold if

$$\lim_{r \downarrow 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} = 1,$$

for μ-almost all $x \in A$, whenever $A \subset X$ is μ-measurable. We note that the property holds for every finite Borel measure in Euclidean spaces, but not necessarily in arbitrary doubling metric spaces.

2.2 Dimensions of measures

The *upper and lower local dimensions* of a measure μ at the point x are given by

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) = \limsup_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r},$$
$$\underline{\dim}_{\mathrm{loc}}(\mu, x) = \liminf_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r},$$

respectively. If the upper and lower dimensions agree, their common value is referred to as the *local dimension* of μ at x and we write $\dim_{\mathrm{loc}}(\mu, x) = \overline{\dim}_{\mathrm{loc}}(\mu, x) = \underline{\dim}_{\mathrm{loc}}(\mu, x)$

For a bounded subset $A \subset X$, the *L^q-moment sum of μ on A at scale δ* is defined by

$$S_q(\mu, A, \delta) = \sup\{\sum_{B \in \mathcal{B}} \mu(B)^q : \mathcal{B} \text{ is a } \delta\text{-packing of } A \cap \operatorname{spt}(\mu)\}. \tag{2.3}$$

We then define the *(global) L^q-spectrum of μ on A* as

$$\tau_q(\mu, A) = \liminf_{\delta \downarrow 0} \frac{\log S_q(\mu, A, \delta)}{\log \delta}.$$

The definition given here is the same that is used in [16]. Notice that for $q \geq 0$ the definition remains unchanged if $A \cap \operatorname{spt}(\mu)$ is replaced with A in (2.3). For $q \neq 1$ we define the *L^q-dimension of μ on A* by

$$\dim_q(\mu, A) = \tau_q(\mu, A)/(q - 1).$$

If X is bounded, we use the shorthand notation $\tau_q(\mu) = \tau_q(\mu, X)$ and $\dim_q(\mu) = \dim_q(\mu, X)$. For any $A \subset X$, with $\mu(A) > 0$ we define the *(global) upper and lower entropy dimensions of μ on A* as

$$\overline{\dim}_1(\mu, A) = \limsup_{\delta\downarrow 0} \fint_A \frac{\log \mu(B(y,\delta))}{\log \delta} \mathrm{d}\mu(y),$$
$$\underline{\dim}_1(\mu, A) = \liminf_{\delta\downarrow 0} \fint_A \frac{\log \mu(B(y,\delta))}{\log \delta} \mathrm{d}\mu(y)$$

respectively. Here and hereafter we use the notation

$$\fint_A f(x)\mathrm{d}\mu(x) = \mu(A)^{-1} \int_A f(x)\mathrm{d}\mu(x).$$

If the values of the dimensions agree then the common value is referred to as the *(global) entropy dimension of μ on A* and is denoted by $\dim_1(\mu, A)$.

From the above definitions we derive the local variants. The *local L^q-spectrum of μ at $x \in spt(\mu)$* is defined as

$$\tau_q(\mu, x) = \lim_{r\downarrow 0} \tau_q(\mu, B(x, r)),$$

and the *local L^q-dimension of μ at x* as

$$\dim_q(\mu, x) = \tau_q(\mu, x)/(q - 1).$$

Correspondingly we define the *local upper and lower entropy dimensions of μ at $x \in spt(\mu)$* as

$$\overline{\dim}_1(\mu, x) = \limsup_{r\downarrow 0} \overline{\dim}_1(\mu, B(x, r)),$$
$$\underline{\dim}_1(\mu, x) = \liminf_{r\downarrow 0} \underline{\dim}_1(\mu, B(x, r)).$$

The following theorem explains the choice of notation for the entropy dimensions.

Theorem 2.2. [16, Theorem 2.2] *If μ is a measure on a doubling metric space X, then*

$$\lim_{q\downarrow 1} \dim_q(\mu, x) \leq \underline{\dim}_{\mathrm{loc}}(\mu, x) \leq \overline{\dim}_{\mathrm{loc}}(\mu, x) \leq \lim_{q\uparrow 1} \dim_q(\mu, x), \tag{2.4}$$

for μ-almost all $x \in X$ and

$$\lim_{q\downarrow 1} \dim_q(\mu, x) \leq \underline{\dim}_1(\mu, x) \leq \overline{\dim}_1(\mu, x) \leq \lim_{q\uparrow 1} \dim_q(\mu, x), \tag{2.5}$$

for every $x \in \mathrm{spt}(\mu)$.

Moreover, if the measure has the density point property, then

$$\underline{\dim}_{\mathrm{loc}}(\mu, x) \leq \underline{\dim}_1(\mu, x) \leq \overline{\dim}_1(\mu, x) \leq \overline{\dim}_{\mathrm{loc}}(\mu, x), \tag{2.6}$$

for μ-almost all $x \in X$.

The proofs of (2.4) and (2.6) can be found in [17]. However the proof of claim (2.5) is erroneous as it makes use of the incorrect [16, Proposition 3.2]. Our main goal in this article is to give a correct proof for this theorem. A correct proof for claim (2.5) will be given at the end of Section 4.

2.3 Counterexample

In [16] the authors aim to formulate an alternate way of defining the L^q-spectrum using partitions of the space X, which is a little easier to work with than the definition using packings. In this section we introduce the original Proposition of [16] which claims that the L^q-spectrum of a measure on a subset $A \subset X$ can be calculated using partitions of the space X instead of packings, and provide a counterexample for the proposition.

Claim. [16, Proposition 3.2] *If μ is a measure on a doubling metric space X, $A \subset X$ is bounded with $\mu(A) > 0$ and $q \geq 0$, then*

$$\tau_q(\mu, A) = \liminf_{n\to\infty} \frac{\log \sum_{Q\in\mathcal{Q}_n(A)} \mu(Q)^q}{\log \delta_n}.$$

Intuitively, the claim feels plausible and indeed it holds when $A = X$. However there is a problem; in the δ-packing definition of the L^q-spectrum we only require the center of the ball to be in the set A and thus the balls of the packing may intersect the complement of A even with small δ and if the measure in question is concentrated near the set A, the original definition may give a much smaller value for the spectrum than the claim above. The following simple counterexample originally proposed by Laurent Dufloux shows that if A is a proper subset of X, the proposition above does not necessarily hold.

Counterexample 2.3. To simplify the notation we set

$$T_q(\mu, A) = \liminf_{n\to\infty} \frac{\log \sum_{Q\in\mathcal{Q}_n(A)} \mu(Q)^q}{\log \delta_n}.$$

Let $X = [-1, 1]$ with the usual Euclidean metric in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x \in [-1, 0], \\ x^{-\frac{1}{2}}, & \text{if } x \in (0, 1]. \end{cases}$$

We define the measure $\mu : \mathcal{L}(X) \to \mathbb{R}^+$ as

$$\mu(A) = \int_A f(x)\mathrm{d}x,$$

where $\mathcal{L}(X)$ is the Lebesgue σ-algebra on X. We show that for $A = [-1, 0]$,

$$\tau_q(\mu, A) < T_q(\mu, A),$$

and thus [16, Proposition 3.2] does not hold.

First we calculate an upper bound for $\tau_q(\mu, A)$. Since $0 \in A$, $\{B(0,\delta)\}$ is a δ-packing of A and

$$S_q(\mu, A, \delta) \geq \mu(B(0,\delta))^q = \left(\int_{-\delta}^{\delta} f(x)\mathrm{d}x\right)^q \geq \left(\int_0^{\delta} x^{-\frac{1}{2}}\mathrm{d}x\right)^q = 2^q\delta^{\frac{1}{2}q}.$$

Hence

$$\tau_q(\mu, A) \leq \liminf_{\delta\downarrow 0} \frac{\log 2^q\delta^{\frac{1}{2}q}}{\log\delta} = \frac{q}{2}.$$

Next we calculate $T_q(\mu, A)$. For each $n \in \mathbb{N}$ we divide the space X in to the dyadic intervals $(m2^{-n}, (m+1)2^{-n}]$ of length 2^{-n} and obtain a 2^{-n}-partition of X denoted by $\mathcal{Q}_n$ (we include the point -1 in the appropriate interval). Notice that (2.1) and (2.2) hold for the dyadic partition. Obviously

$$\sum_{Q\in\mathcal{Q}_n(A)} \mu(Q)^q = 2^n 2^{-qn},$$

since the number of dyadic intervals intersecting A is 2^n. Hence

$$T_q(\mu, A) = \liminf_{n\to\infty} \frac{\log 2^n 2^{-qn}}{\log 2^{-n}} = q - 1 > \tau_q(\mu, A),$$

when $q > 2$.

We note that the mistake in the proof of [16, Proposition 3.2] is in the statement "Since C_B is a cover for B" which does not necessarily hold if A is a proper subset of X.

3 Restricted L^q-spectrum and entropy dimension using partitions

3.1 Restricted L^q-spectrum

As the calculation of $T_q(\mu, A)$ in Counterexample 2.3 shows, the property of [16, Proposition 3.2] is rather desirable, since it would greatly simplify the calculation of the L^q-spectrum in some cases. Thus it is in our best interest to try and provide a variant of [16, Proposition 3.2] which holds and allows us to calculate at least the local dimensions, which are the main object of study in this paper, using partitions.

For the rest of the paper we use the notation μ_A for the restriction of the measure μ on the set $A \subset X$, i.e. for all $B \subset X$ we set $\mu_A(B) = \mu(B \cap A)$. We call the measure μ_A the *restricted measure* and the L^q-spectrum of the restricted measure the *restricted L^q-spectrum*. Similarly we call the entropy and L^q-dimensions of the restricted measure the *restricted entropy and L^q-dimensions*. Our aim is to provide a formulation of [16, Proposition 3.2] using the restricted L^q-spectrum and relate that with the L^q-spectrum of the measure on the whole space. First we note that since it is clear that $\mu_X = \mu$, then if X is bounded we have

$$\tau_q(\mu) = \tau_q(\mu_X), \tag{3.1}$$

for all $q \in \mathbb{R}$.

Using the restricted measure we define the *restricted local L^q-spectrum of μ at $x \in spt(\mu)$* as

$$\tau_q^*(\mu, x) = \lim_{r \downarrow 0} \tau_q(\mu_{B(x,r)}, B(x, r)).$$

Similarly we define the *restricted local L^q-dimension of μ at $x \in spt(\mu)$* as

$$\dim_q^*(\mu, x) = \tau_q^*(\mu, x)/(q - 1).$$

The *restricted local upper and lower entropy dimensions of μ at $x \in spt(\mu)$* are defined as

$$\overline{\dim}_1^*(\mu, x) = \limsup_{r \downarrow 0} \overline{\dim}_1(\mu_{B(x,r)}, B(x, r)),$$
$$\underline{\dim}_1^*(\mu, x) = \liminf_{r \downarrow 0} \underline{\dim}_1(\mu_{B(x,r)}, B(x, r)),$$

respectively.

3.2 Properties of the L^q-spectra and L^q-dimensions

First we state a lemma which provides some basic properties for the L^q-spectrum and dimension (see [17, Lemma 2.7], for a proof).

Lemma 3.1. *If μ is a measure on a doubling metric space X, the set $A \subset X$ is bounded, with $\mu(A) > 0$, setting $q_0 = \inf\{q \in \mathbb{R} : \tau_q(\mu, A) > -\infty\}$, and $s > 0$ as in Lemma 2.1(2 and 3), then*

1. $\tau_1(\mu, A) = 0$,
2. $\min\{0, (q - 1)s\} \leq \tau_q(\mu, A) \leq \max\{0, (q - 1)s\}$ *for all* $0 \leq q < \infty$,
3. $0 \leq \dim_q(\mu, A) \leq s$ *for all* $0 \leq q < \infty$ *with* $q \neq 1$,
4. *the mapping* $q \mapsto \tau_q(\mu, A)$ *is concave on* (q_0, ∞),
5. *the mapping* $q \mapsto \dim_q(\mu, A)$ *is continuous and decreasing on both* $(q_0, 1)$ *and* $(1, \infty)$.

Furthermore if $x \in \operatorname{spt}(\mu)$, then all the claims remain true if $\tau_q(\mu, A)$ is replaced by $\tau_q(\mu, x)$ and $\dim_q(\mu, A)$ by $\dim_q(\mu, x)$.

The following lemma states the basic relationship of the restricted L^q-spectrum and the L^q-spectrum of the measure on the whole space.

Lemma 3.2. *If μ is a measure on a doubling metric space X and $A \subset X$ is bounded, then*

$$\tau_q(\mu_A, A) \geq \tau_q(\mu, A),$$

if $q \geq 0$.

Proof. Let $q \geq 0$. Any δ-packing $\mathcal{B}$ of $A \cap \operatorname{spt}(\mu_A)$ is also a δ-packing of $A \cap \operatorname{spt}(\mu)$ and since $B \cap A \subset B$, we have

$$\sum_{B \in \mathcal{B}} \mu_A(B)^q \leq \sum_{B \in \mathcal{B}} \mu(B)^q.$$

Therefore

$$S_q(\mu_A, A, \delta) \leq S_q(\mu, A, \delta),$$

and claim follows by taking logarithms, dividing by $\log \delta$ and taking limits. □

The following proposition is the main tool of this paper and relates the restricted local L^q-spectrum with the local L^q-spectrum of the measure on the whole space and provides us with the useful fact that the local spectra are indeed equal with positive values of q.

Proposition 3.3. *If μ is a measure on a doubling metric space X and $q \geq 0$, then*

$$\tau_q^*(\mu, x) = \tau_q(\mu, x), \tag{3.2}$$

for every $x \in \operatorname{spt}(\mu)$.

Proof. Let $q \geq 0$. By Lemma 3.2 we need only prove that $\tau_q^*(\mu, x) \leq \tau_q(\mu, x)$. Let $x \in \operatorname{spt}(\mu)$, $r > 0$ and $0 < \delta < r$. Let $\mathcal{B}$ be a δ-packing of $B(x, r)$. Then

$$\begin{aligned}\sum_{B \in \mathcal{B}} \mu(B)^q &= \sum_{B \in \mathcal{B}} \mu(B \cap B(x, r + \delta))^q \\ &\leq \sum_{B \in \mathcal{B}} \mu(B \cap B(x, 2r))^q = \sum_{B \in \mathcal{B}} \mu_{B(x,2r)}(B)^q,\end{aligned}$$

and since $\mathcal{B}$ is also a δ-packing of $B(x, 2r)$, we have

$$S_q(\mu, B(x, r), \delta) \leq S_q(\mu_{B(x,2r)}, B(x, 2r), \delta),$$

and the claim then follows by taking logarithms dividing by $\log \delta$ and then taking first $\delta \to 0$ and then $r \to 0$. □

As an immediate consequence we get the following corollary

Corollary 3.4. *If μ is a measure on a doubling metric space X and $1 \neq q \geq 0$, then*

$$\dim_q^*(\mu, x) = \dim_q(\mu, x),$$

for every $x \in \operatorname{spt}(\mu)$.

3.3 Restricted L^q-spectrum using partitions

Next we reformulate [16, Proposition 3.2] using the restricted measure and provide a proof for the statement. Recall that for all $n \in \mathbb{N}$, $\mathcal{Q}_n$ is a δ_n-partition of X for a fixed sequence $(\delta_n)_{n\in\mathbb{N}}$ which satisfies (2.1) and (2.2).

Proposition 3.5. *If μ is a measure on a doubling metric space X, $A \subset X$ is bounded with $\mu(A) > 0$ and $q \geq 0$, then*

$$\tau_q(\mu_A, A) = \liminf_{n\to\infty} \frac{\log \sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)^q}{\log \delta_n}.$$

Proof. The proof closely follows the ideas in proof of [16, Proposition 3.2]. Let $0 < \delta < \delta_1$ and $n \in \mathbb{N}$ so that $\delta_{n+1} < \delta < \delta_n$. First we show that for a constant $c_1(N, \Lambda, q) > 0$, we have

$$S_q(\mu_A, A, \delta) \leq c_1 \left(\frac{\delta_n}{\delta}\right)^s \sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)^q, \tag{3.3}$$

where $s = s(N) > 0$ is the constant given by Lemma 2.1(3), $N = N(X)$ is the doubling constant of the metric space X and Λ is the constant used in defining the partitions $\mathcal{Q}_n$, which is not dependent on n.

Let us fix a δ-packing $\mathcal{B}$ of A and set

$$C_B = \{Q : Q \in \mathcal{Q}_n(A),\, Q \cap B \cap A \neq \emptyset\}$$

for all $B \in \mathcal{B}$. Clearly $\{Q \cap A : Q \in C_B\}$ is a cover for $B \cap A$ and we have

$$\mu_A(B)^q \leq \left(\sum_{Q\in C_B} \mu_A(Q)\right)^q \leq (\#C_B)^q \sum_{Q\in C_B} \mu_A(Q)^q,$$

when $q \geq 0$. Here $\#C_B$ denotes the cardinality of the set C_B. By the definition of $\mathcal{Q}_n$, all the sets of C_B are contained in a ball of radius $(1 + 2\Lambda)\delta_n$, which also by definition has a δ_n-packing of cardinality $\#C_B$. By Lemma 2.1(3) there exists a constant $c_2 = c_2(N, \Lambda) > 0$, such that $\#C_B \leq c_2$ for all $B \in \mathcal{B}$. Hence

$$\sum_{B\in\mathcal{B}} \mu_A(B)^q \leq c_2^q \sum_{B\in\mathcal{B}} \sum_{Q\in C_B} \mu_A(Q)^q.$$

Furthermore, by Lemma 2.1(3) there exists a constant $c_3 = c_3(N, \Lambda) > 0$ so that $\#\{B \in \mathcal{B} : B \cap Q \cap A\} \le c_3\left(\frac{\delta_n}{\delta}\right)^s$, for all $Q \in Q_n$. Claim (3.3) follows then with $c_1 = c_2^q c_3$.

Finding the estimate for the other direction also requires only a minor alteration to the proof of [16, Proposition 3.2]. For each $Q \in \mathcal{Q}_n(A)$ we choose a point $x_Q \in Q \cap A$ and a ball B_Q such that $Q \subset \Lambda B_Q$ and the collection $\{B_Q : Q \in \mathcal{Q}_n(A)\}$ is a δ_n-packing. Obviously $Q \subset B(x_Q, 2\Lambda\delta_n) \subset 3\Lambda B_Q$, for all $Q \in \mathcal{Q}_n(A)$. Lemma 2.1(4) provides us with a constant $M = M(N, \Lambda) \in \mathbb{N}$ and sets $\mathcal{Q}_1, \dots, \mathcal{Q}_M$ so that $\mathcal{Q}_n(A) = \bigcup_{i=1}^M \mathcal{Q}_i$ and $\{3\Lambda B_Q : Q \in \mathcal{Q}_i\}$ is a $3\Lambda\delta_n$-packing for all $i \in \{1, \dots, M\}$ and as a result $\{B(x_Q, 2\Lambda\delta_n) : Q \in \mathcal{Q}_i\}$ is a $2\Lambda\delta_n$-packing with centres in A. Clearly we have

$$\bigcup_{Q \in \mathcal{Q}_n(A)} Q \subset \bigcup_{i=1}^{M} \bigcup_{Q \in \mathcal{Q}_i} B(x_Q, 2\Lambda\delta_n).$$

Moreover since $A \cap Q \cap B(x_Q, 2\Lambda\delta_n) \neq \emptyset$ for all $Q \in \mathcal{Q}_n(A)$, we have

$$\bigcup_{Q \in \mathcal{Q}_n(A)} Q \cap A \subset \bigcup_{i=1}^{M} \bigcup_{Q \in \mathcal{Q}_i} B(x_Q, 2\Lambda\delta_n) \cap A,$$

and we may choose $i \in \{1, \dots, M\}$ so that

$$\sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q)^q \le M \sum_{Q \in \mathcal{Q}_i} \mu_A(B(x_Q, 2\Lambda\delta_n))^q \le M S_q(\mu_A, A, 2\Lambda\delta_n). \tag{3.4}$$

The claim then follows by combining estimates (3.3) and (3.4) and taking logarithms and limits. □

3.4 Entropy dimension using partitions

For convenience we present the formulation of $\dim_1(\mu, A)$ for compact sets A using partitions. The proof of the proposition can be found in [16, Proposition 3.4].

Proposition 3.6. *If μ is a measure on a doubling metric space X and $A \subset X$ is compact with $\mu(A) > 0$, then*

$$\overline{\dim}_1(\mu, A) = \limsup_{n \to \infty} \frac{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q)}{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \delta_n},$$
$$\underline{\dim}_1(\mu, A) = \liminf_{n \to \infty} \frac{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q)}{\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \delta_n}$$

For the restricted entropy dimensions we obtain a definition using partitions which does not require the set in question to be compact.

Proposition 3.7. *If μ is a measure on a doubling metric space X and $A \subset X$ is bounded with $\mu(A) > 0$, then*

$$\overline{\dim}_1(\mu_A, A) = \limsup_{n\to\infty} \frac{\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q)}{\mu(A)\log\delta_n},$$
$$\underline{\dim}_1(\mu_A, A) = \liminf_{n\to\infty} \frac{\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q)}{\mu(A)\log\delta_n}$$

Proof. The proof is a simpler version of the proof for [16, Proposition 3.4]. For each $Q \in \mathcal{Q}_n(A)$ we choose a ball B_Q such that $Q \subset \Lambda B_Q$ and $\{B_Q : Q \in \mathcal{Q}_n(A)\}$ is a δ_n-packing. If $Q \in \mathcal{Q}_n(A)$, then for every $y \in Q \cap A$ we have

$$Q \cap A \subset B(y, 2\Lambda\delta_n) \cap A \subset 3\Lambda B_Q \cap A \subset \bigcup_{Q'\in C_Q} Q' \cap A,$$

where $C_Q = \{Q' \in \mathcal{Q}_n(A) : Q' \cap 3\Lambda B_Q \cap A \neq \emptyset\}$. Notice also that

$$A = \bigcup_{Q\in\mathcal{Q}_n(A)} Q \cap A.$$

We then have

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q) &\leq \sum_{Q\in\mathcal{Q}_n(A)} \int_{Q\cap A} \log\mu_A(B(y, 2\Lambda\delta_n))\mathrm{d}\mu_A(y)\\ &\leq \int_A \log\mu_A(B(y, 2\Lambda\delta_n))\mathrm{d}\mu_A(y)\\ &\leq \sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\sum_{Q'\in C_Q}\mu_A(Q').\end{aligned}$$

By Lemma 2.1(3), there is a constant $c = c(N) < \infty$, such that each Q' is contained in at most $c(3\Lambda)^s$ collections of C_Q. Therefore we have

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q) - &\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\sum_{Q'\in C_Q}\mu_A(Q')\\ &= \sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\Big(1 + \frac{\sum_{Q'\in C_Q\setminus\{Q\}}\mu_A(Q')}{\mu_A(Q)}\Big)\\ &\leq \sum_{Q\in\mathcal{Q}_n(A)}\sum_{Q'\in C_Q\setminus\{Q\}} \mu_A(Q') \leq c(3\Lambda)^s\mu(A).\end{aligned}$$

Combining the previous estimates gives us

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q) &\leq \int_A \log\mu_A(B(y,\delta))\mathrm{d}\mu_A(y)\\ &\leq \sum_{Q\in\mathcal{Q}_{n-1}(A)} \mu_A(Q)\log\mu_A(Q) + c(3\Lambda)^s\mu(A),\end{aligned}$$

for all $2\Lambda\delta_n \le \delta \le 2\Lambda\delta_{n-1}$. From this and (2.2) the claim follows easily. $\square$

4 Relating the dimensions

In this section our aim is to provide relationships between the different notions of dimension discussed in this paper. First we remind ourselves of a small technical lemma introduced in [16].

Lemma 4.1. *Suppose μ is a measure on a doubling metric space X and $A \subset X$ is bounded. Let $s > 0$ and $c > 0$ be as in 2.1(3). Then*

$$\int_A \log \mu(B(y,\delta))\mathrm{d}\mu(y) \ge -\frac{1}{e} - \mu(A)\Big(\log c + s \log \frac{4\mathrm{diam}(A)}{\delta}\Big),$$

for all $\delta > 0$.

Proof. See [16, Lemma 3.3]. $\square$

Using Propositions 3.6 and 3.7 we get the following result:

Proposition 4.2. *If μ is a measure on a doubling metric space X and $A \subset X$ is compact with $\mu(A) > 0$, then*

$$\overline{\dim}_1(\mu_A, A) = \overline{\dim}_1(\mu, A),$$
$$\underline{\dim}_1(\mu_A, A) = \underline{\dim}_1(\mu, A).$$

Proof. First note that for any $y \in A$ and $\delta > 0$ we have $\mu_A(B(y,\delta)) \le \mu(B(y,\delta))$ and hence clearly

$$\begin{aligned}\overline{\dim}_1(\mu_A, A) &= \limsup_{\delta\downarrow 0} \fint_A \frac{\log \mu_A(B(y,\delta))}{\log \delta}\mathrm{d}\mu_A(y)\\ &\ge \limsup_{\delta\downarrow 0} \fint_A \frac{\log \mu(B(y,\delta))}{\log \delta}\mathrm{d}\mu(y) = \overline{\dim}_1(\mu, A),\end{aligned}$$

and the corresponding inequality holds for the lower dimensions as well.

For the other inequality we choose for each $Q \in \mathcal{Q}_n(A)$ a ball B_Q such that $Q \subset \Lambda B_q$ and $\{B_Q : Q \in \mathcal{Q}_n(A)\}$ is a δ_n-packing. If $Q \in \mathcal{Q}_n(A)$, then for every $y \in Q$ we have

$$Q \subset B(y, 2\Lambda\delta_n) \subset 3\Lambda B_Q \subset \bigcup_{Q' \in C_Q} Q',$$

where $C_Q = \{Q' \in \mathcal{Q}_n(A) : Q' \cap 3\Lambda B_Q \ne \emptyset\}$. We set $A_n = \bigcup_{Q \in \mathcal{Q}_n(A)} Q$. Thus

$$\sum_{Q\in\mathcal{Q}_n(A)}\int_Q \log\mu(B(y,2\Lambda\delta_n))\mathrm{d}\mu(y) = \int_{A_n}\log\mu(B(y,2\Lambda\delta_n))\mathrm{d}\mu(y) \tag{4.1}$$

$$\leq \sum_{Q\in\mathcal{Q}_n(A)}\mu(Q)\log\sum_{Q'\in C_Q}\mu(Q'). \tag{4.2}$$

Since by Lemma 2.1(3) there exists a constant $c_1 = c_1(N) < \infty$, such that $Q' \in \mathcal{Q}_n(A)$ is contained in at most $c_1(3\Lambda)^s$ collections C_Q, we have

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}_n(A)}\mu(Q)\log\sum_{Q'\in C_Q}\mu(Q') - \sum_{Q\in\mathcal{Q}_n(A)}\mu(Q)\log\mu(Q)\\
&\qquad = \sum_{Q\in\mathcal{Q}_n(A)}\mu(Q)\log\left(1+\frac{\sum_{Q'\in C_Q\setminus\{Q\}}\mu(Q')}{\mu(Q)}\right)\\
&\qquad \leq \sum_{Q\in\mathcal{Q}_n(A)}\sum_{Q'\in C_Q\setminus\{Q\}}\mu(Q') \leq c_1(3\Lambda)^s\mu(B_0),
\end{aligned}$$

where B_0 is a ball centered at A with radius $\mathrm{diam}(A)+2\Lambda\delta_n$. Combining this with (4.1) we get

$$\int_{A_n}\log\mu(B(y,2\Lambda\delta_n))\mathrm{d}\mu(y) \leq \sum_{Q\in\mathcal{Q}_n(A)}\mu(Q)\log\mu(Q) + c_1(3\Lambda)^s\mu(B_0). \tag{4.3}$$

Next we note that $\mu(Q) = \mu_{A_n}(Q)$, since $Q \subset A_n$ for any $Q \in \mathcal{Q}_n(A)$ and therefore $\mu(Q) = \mu_A(Q) + \mu_{A_n\setminus A}(Q)$ for any $Q \in \mathcal{Q}_n(A)$. Thus

$$\begin{aligned}
\sum_{Q\in\mathcal{Q}_n(A)}\mu(Q)\log\mu(Q) &= \sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\mu(Q) + \sum_{Q\in\mathcal{Q}_n(A)}\mu_{A_n\setminus A}(Q)\log\mu(Q)\\
&\leq \sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\mu(Q) + \log\mu(A_n)\sum_{Q\in\mathcal{Q}_n(A)}\mu_{A_n\setminus A}(Q)\\
&= \sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\mu(Q) + \log\mu(A_n)\mu(A_n\setminus A). \qquad (4.4)
\end{aligned}$$

We also have

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\mu(Q) - \sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\mu_A(Q)\\
&= \sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\frac{\mu(Q)}{\mu_A(Q)} = \sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\frac{\mu_A(Q)+\mu_{A_n\setminus A}(Q)}{\mu_A(Q)}\\
&= \sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\left(1+\frac{\mu_{A_n\setminus A}(Q)}{\mu_A(Q)}\right) \leq \sum_{\mathcal{Q}_n(A)}\mu_{A_n\setminus A}(Q) = \mu(A_n\setminus A) \qquad (4.5)
\end{aligned}$$

and by combining (4.3), (4.4) and (4.5) we get

$$\int_{A_n} \log \mu(B(y,\delta))\mathrm{d}\mu(y) \le \sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q) \qquad (4.6)$$
$$+ \mu(A_n\backslash A)(1+\log\mu(A_n)) + c_1(3\Lambda)^s\mu(B_0),$$

for all $0<\delta\le 2\Lambda\delta_n$. Since A is compact we have $\lim_{n\to\infty}\mu(A_n\backslash A)=0$ and by Lemma 4.1,

$$\lim_{n\to\infty}\frac{1}{\log\delta_n}\int_{A_n\backslash A}\log\mu(B(y,2\Lambda\delta_n))\mathrm{d}\mu(y)=0.$$

From this, (4.6) and Proposition 3.7 the claim follows. □

Corollary 4.3. *If μ is a measure on a doubling metric space X, then*

$$\overline{\dim}_1^*(\mu,x)=\overline{\dim}_1(\mu,x),$$
$$\underline{\dim}_1^*(\mu,x)=\underline{\dim}_1(\mu,x).$$

Proof. Apply Proposition 4.2 to the compact balls $B(x,r)$ and take the limit. □

Next we show that the definitions of restricted entropy dimensions and L^q-dimensions are consistent with the monotonicity of the L^q-dimensions. The result resembles [16, Proposition 3.7], the proof of which is slightly incorrect since it makes use of [16, Proposition 3.2]. Our proof follows the ideas of the proof of [16, Proposition 3.7], but we use Proposition 3.5 instead of [16, Proposition 3.2].

Proposition 4.4. *If μ is a measure on a doubling metric space X and $A\subset X$ is bounded with $\mu(A)>0$, then*

$$\lim_{q\downarrow 1}\dim_q(\mu_A,A)\le\underline{\dim}_1(\mu_A,A)\le\overline{\dim}_1(\mu_A,A)\le\lim_{q\uparrow 1}\dim_q(\mu_A,A).$$

Proof. First notice that the existence of the limits follows from Lemma 3.1(5). The claim then follows if we can show that

$$\tau_q(\mu_A,A)/(q-1)\ge\overline{\dim}_1(\mu_A,A)\ge\underline{\dim}_1(\mu_A,A)\ge\tau_p(\mu_A,A)/(p-1),$$

for $0<q<1<p$. We define a function $h_n(q)=\log\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)^q$, for all $q\ge 0$. First we show that $h_n(q)$ is convex. Take $q,p\in\mathbb{R}$ and $t\in[0,1]$. Now

$$\begin{aligned} h_n(tq+(1-t)p) &= \log\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)^{tq}\mu_A(Q)^{(1-t)p} \\ &\le \log\Big(\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)^q\Big)^t\Big(\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)^p\Big)^{1-t} \\ &= th_n(q)+(1-t)h_n(p). \end{aligned}$$

by Hölder's inequality. Note that $Q_n(A)$ has only a finite number of elements, which implies that $h_n(q)$ is differentiable, and differentiating gives us

$$\begin{aligned} h_n'(1) &= \frac{\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q)}{\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)} \\ &= \frac{\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q)}{\mu(A)}. \end{aligned}$$

By the convexity of $h_n(q)$ we then have

$$\frac{h_n(q)-h_n(1)}{q-1} \le h_n'(1) \le \frac{h_n(p)-h_n(1)}{p-1}.$$

Using the above estimates we calculate

$$\begin{aligned} \frac{1}{q-1}\liminf_{n\to\infty} \frac{\log\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)^q}{\log\delta_n} &= \limsup_{n\to\infty} \frac{h_n(q)-h_n(1)}{(q-1)\log\delta_n} \\ &\ge \limsup_{n\to\infty} \frac{\log\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q)}{\mu(A)\log\delta_n} \\ &\ge \liminf_{n\to\infty} \frac{\log\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q)}{\mu(A)\log\delta_n} \\ &\ge \liminf_{n\to\infty} \frac{h_n(p)-h_n(1)}{(p-1)\log\delta_n} = \frac{1}{p-1}\liminf_{n\to\infty} \frac{\log\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)^p}{\log\delta_n}, \end{aligned}$$

where the first and last equalities hold since $h_n(1)$ does not depend on n. Now the result follows from Propositions 3.5 and 3.7. □

Remark 4.5. Proposition 4.4 does not provide an immediate proof for [16, Proposition 3.7], since we only have $\lim_{q\downarrow 1}\dim_q(\mu_A, A) \le \lim_{q\downarrow 1}\dim_q(\mu, A)$ and $\lim_{q\uparrow 1}\dim_q(\mu_A, A) \ge \lim_{q\uparrow 1}\dim_q(\mu, A)$, by Lemma 3.2. The statement of Proposition 4.4 is enough to show that the definitions of local entropy dimensions are consistent with the monotonicity of the local L^q dimensions, and since our focus is in the local case, [16, Proposition 3.7] remains open. Notice that our proposition does provide the estimates for the whole space i.e. Proposition 4.4 together with (3.1) implies that $\lim_{q\downarrow 1}\dim_q(\mu) \le \underline{\dim}_1(\mu) \le \overline{\dim}_1(\mu) \le \lim_{q\uparrow 1}\dim_q(\mu)$.

As the main result of this paper we provide a correct proof for [16, Theorem 2.2]. A different proof which does not utilize the same global methods used here can be found in [15], which is an earlier arXiv preprint of [16, 17].

Proof. **(Of claim (2.5) in the statement of Theorem 2.2.)**
By Lemma 3.1(5) the limits exist and by Proposition 4.4 we have

$$\begin{aligned} &\dim_p(\mu_{B(x,r)}, B(x,r)) \le \underline{\dim}_1(\mu_{B(x,r)}, B(x,r)) \\ &\le \overline{\dim}_1(\mu_{B(x,r)}, B(x,r)) \le \dim_q(\mu_{B(x,r)}, B(x,r)), \end{aligned}$$

for $0 < q < 1 < p$, and for every $x \in \mathrm{spt}(\mu)$ and $r > 0$. By taking $r \downarrow 0$ we get

$$\dim_p^*(\mu, x) \leq \underline{\dim}_1^*(\mu, x) \leq \overline{\dim}_1^*(\mu, x) \leq \dim_q^*(\mu, x),$$

and the claim follows from Corollaries 3.4 and 4.3. □

5 Discussion

The focus of [16] is in establishing the theory for local multifractal analysis and this paper shows that the local theory in [16] is correct despite the fact that multiple proofs make use of a slightly incorrect result. Although we fail to provide similar results to the second claim of Theorem 2.2 for the case where $A \subsetneq X$ (which was attempted in [16]), our methods do establish the results when considering the two arguably most important cases, the whole space X and the local case. To complete the paper we give a remark considering the applications of the local properties discussed in the paper.

As application of the theory in [16] the authors provide a local multifractal formalism for Moran constructions in doubling metric spaces and use their faulty Proposition 3.2 in the proof of Theorem 4.2. We note here that with only trivial modifications to their proof one can use our Proposition 3.5 in finding the value of $\tau_q^*(\mu, x)$ and then use Proposition 3.3 to obtain a correct proof for the theorem.

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