# Superprojective Banach spaces

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**Abstract**

We study superprojective Banach spaces. We show that they cannot contain copies of $\ell_1$, which restricts the search for non-reflexive examples of these spaces. We also show that the class of superprojective spaces is stable under finite products, certain unconditional sums, certain tensor products, and other operations, providing new examples.

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## 1. Introduction

A Banach space $X$ is called *subprojective* if every (closed) infinite-dimensional subspace of $X$ contains an infinite-dimensional subspace complemented in $X$, and $X$ is called *superprojective* if every infinite-codimensional subspace of $X$ is contained in an infinite-codimensional subspace complemented in $X$. These two classes of Banach spaces were introduced by Whitley [19] in order to find conditions for the conjugate of an operator to be strictly singular or strictly cosingular. More recently, they have been used to obtain some positive solutions to the perturbation classes problem for semi-Fredholm operators. This problem has a negative solution in general [9], but there are some positive answers when one of the spaces is subprojective or superprojective [10].

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There are many examples of subprojective spaces, like $\ell_p$ for $1 \leq p < \infty$, $L_p(0,1)$ for $2 \leq p < \infty$, $C(K)$ with $K$ a scattered compact and some Lorentz and Orlicz spaces. We refer to [10, Proposition 2.4] for references and further examples. It is not difficult to show that subspaces of subprojective spaces are subprojective [19, Lemma 3.1], and quotients of superprojective spaces are superprojective (Proposition 2.3) and, as a consequence of the duality relations between subspaces and quotients, a reflexive space is subprojective (superprojective) if and only if its dual space is superprojective (subprojective), which provides many examples of reflexive superprojective spaces [10, Proposition 3.4]. However, the only examples of non-reflexive superprojective spaces previously known are the $C(K)$ spaces with $K$ a scattered compact and their infinite-dimensional quotients [10, Proposition 3.4].

Some of the duality relations between subprojective and superprojective spaces are known to fail in general:

(a) $X$ being subprojective does not imply that $X^*$ is superprojective, for instance for $X = c_0$ and $X^* = \ell_1$.

(b) $X^*$ being subprojective does not imply that $X$ is superprojective, for instance for the hereditarily indecomposable space obtained in [2] whose dual is isomorphic to $\ell_1$.

However we do not know if the remaining relations are valid:

(a') Does $X$ being superprojective imply that $X^*$ is subprojective?

(b') Does $X^*$ being superprojective imply that $X$ is subprojective?

The answer to these two questions is likely negative, but we know of few examples of non-reflexive superprojective spaces to check, and none of them is a dual space.

Recently Oikhberg and Spinu [14] have studied the stability properties of subprojective spaces under vector sums, tensor products and other operations, obtaining plenty of new examples of subprojective spaces.

We will begin with some auxiliary results in Section 2. Section 3 shows some properties of subprojective and superprojective spaces, such as the fact that superprojective spaces cannot contain copies of $\ell_1$, which restricts the search for non-reflexive examples of these spaces, and we also characterise the superprojectivity of some projective tensor products. In Section 4, following the scheme of [14], we prove several stability results for the class of superprojective spaces under finite products, certain unconditional sums and certain tensor products, and we provide new examples of superprojective spaces. We finish with a collection of open problems in the last section.

We will use standard notation. The dual space of a Banach space $X$ is $X^*$, and the action of $x^* \in X^*$ on $x \in X$ is written as $\langle x^*, x\rangle$. Given a subset $M$ of a Banach space $X$, its annihilator in $X^*$ will be denoted by $M^\perp$; if $M$ is a subset of $X^*$, its annihilator in $X$ will be denoted by $M_\perp$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $X$, then $[x_n : n \in \mathbb{N}]$ will denote the closed linear span of $(x_n)_{n\in\mathbb{N}}$ in $X$. The injective and projective tensor products of $X$ and $Y$ are respectively denoted by $X \hat{\otimes}_\varepsilon Y$ and $X \hat{\otimes}_\pi Y$.

Operators will always be bounded. The identity operator on $X$ is denoted by $I_X$. Given an operator $T: X \longrightarrow Y$, $N(T)$ and $R(T)$ denote the kernel and the range of $T$, and $T^*: Y^* \longrightarrow X^*$ denotes its conjugate operator. An operator $T: X \longrightarrow Y$ is strictly singular if $T|_M$ is an isomorphism only if $M$ is finite-dimensional; and $T$ is strictly cosingular if there is no operator $Q: Y \longrightarrow Z$ with $Z$ infinite-dimensional such that $QT$ is surjective or, equivalently, if there is no infinite-codimensional (closed) subspace $N$ of $Y$ such that $R(T) + N = Y$.

## 2. Preliminaries

The way that superprojective Banach spaces are defined means that we will be dealing with infinite-codimensional subspaces and their induced quotients often, so we will adopt the following definition.

**Definition.** We will say that an operator $T: X \longrightarrow Y$ is a surjection if $T$ is surjective and $Y$ is infinite-dimensional.

The following results will be useful when dealing with complemented subspaces, surjections and superprojective spaces. Similar results were given in [1, Section 2] to study improjective operators.

**Proposition 2.1.** *For a Banach space $X$, the following are equivalent:*

(i) *$X$ is superprojective;*
(ii) *for any surjection $T: X \longrightarrow Y$, there exists another surjection $S: Y \longrightarrow Z$ such that $N(ST)$ is complemented in $X$.*

Proof. For the direct implication, let $T\colon X \longrightarrow Y$ be a surjection, so that $N(T)$ is infinite-codimensional in $X$. By the superprojectivity of $X$, $N(T)$ is contained in a complemented, infinite-codimensional subspace $M$ of $X$, and clearly $T(M)$ is closed in $Y$. Thus the quotient map $Q$ from $Y$ onto $Y/T(M)$ is a surjection such that $N(QT) = M$ is complemented in $X$.

For the converse implication, let $M$ be an infinite-codimensional subspace of $X$, so that $Q_M\colon X \longrightarrow X/M$ is a surjection. Then there exists another surjection $S\colon X/M \longrightarrow Z$ such that $N(SQ_M)$ is infinite-codimensional and complemented in $X$, and contains $M$. $\square$

The next result allows to push the complementation of a subspace through an operator under certain conditions.

**Proposition 2.2.** *Let $X$, $Y$ and $Z$ be Banach spaces and let $T\colon X \longrightarrow Y$ and $S\colon Y \longrightarrow Z$ be operators such that $ST$ is a surjection and $N(ST)$ is complemented in $X$. Then $N(S)$ is complemented in $Y$.*

Proof. Let $H$ be a subspace of $X$ such that $X = N(ST) \oplus H$. Since $ST\colon X \longrightarrow Z$ is a surjection, $ST|_H$ must be an isomorphism onto $Z$; in particular, $T|_H$ is an isomorphism and $Y = N(S) \oplus T(H)$, as proved by the projection $T(ST|_H)^{-1}S\colon Y \longrightarrow Y$. $\square$

A simple consequence of Propositions 2.1 and 2.2 is the fact that the class of superprojective spaces is stable under quotients.

**Proposition 2.3.** *Let $X$ be a superprojective Banach space and let $T\colon X \longrightarrow Y$ be a surjection. Then $Y$ is superprojective.*

Proof. Let $S\colon Y \longrightarrow Z$ be a surjection; then $ST$ is a surjection and, by Proposition 2.1, there exists another surjection $R\colon Z \longrightarrow W$ such that $N(RST)$ is complemented in $X$. By Proposition 2.2, $N(RS)$ is complemented in $Y$, which means, again by Proposition 2.1, that $Y$ is superprojective. $\square$

Finally, we will state a technical observation on the behaviour of surjections on spaces that have a complemented superprojective subspace.

**Proposition 2.4.** *Let $X$ be a Banach space, let $P\colon X \longrightarrow X$ be a projection with $P(X)$ superprojective and let $S\colon X \longrightarrow Y$ be a surjection such that $SP$ is not strictly cosingular. Then there exists another surjection $R\colon Y \longrightarrow Z$ such that $N(RS)$ is complemented in $X$.*

Proof. Let $J: P(X) \longrightarrow X$ be the natural inclusion; then $SP = SJP$ is not strictly cosingular, so neither is $SJ: P(X) \longrightarrow Y$. Therefore, there exists a quotient map $Q: Y \longrightarrow W$ such that $QSJ$ is a surjection, and Proposition 2.1 provides another surjection $R: W \longrightarrow Z$ such that $N(RQSJ)$ is complemented in $P(X)$; by Proposition 2.2, $N(RQS)$ is complemented in $X$, where $RQ: Y \longrightarrow Z$ is a surjection. □

## 3. Some properties of superprojective spaces

The following result gives some simple but useful necessary conditions for a Banach space $X$ to be subprojective or superprojective.

**Proposition 3.1.** *Let $X$ and $Z$ be infinite-dimensional Banach spaces.*

(1) *If $J: Z \longrightarrow X$ is a strictly cosingular embedding, then $X$ is not subprojective.*
(2) *If $Q: X \longrightarrow Z$ is a strictly singular surjection, then $X$ is not superprojective.*

Proof. (1) If $X = M \oplus H$ with $M \subseteq J(Z)$, then $Q_H J$ is surjective. Since $J$ is strictly cosingular, $H$ is finite-codimensional and $M$ is finite-dimensional.

(2) If $X = M \oplus H$ with $N(Q) \subseteq M$, then $Q|_H$ is an embedding. Since $Q$ is strictly singular, $H$ is finite-dimensional. □

In spite of its simplicity, Proposition 3.1 has several straightforward consequences. Proposition 3.2 was proved in [14] for subprojective spaces with the same example but a different argument. Here we extend it to superprojective spaces. Recall that a class $\mathcal{C}$ of Banach spaces satisfies the *three-space property* if a Banach space $X$ belongs to $\mathcal{C}$ whenever $M$ and $X/M$ belong to $\mathcal{C}$ for some subspace $M$ of $X$. We refer to [5] for other equivalent formulations and additional information.

**Proposition 3.2.** *The classes of subprojective and superprojective spaces do not satisfy the three-space property.*

Proof. Let $1 < p < \infty$ and recall that $\ell_p$ is both subprojective and superprojective. Let $Z_p$ be the Kalton-Peck space introduced in [12]. Then there exists an exact sequence

$$0 \to \ell_p \xrightarrow{i} Z_p \xrightarrow{q} \ell_p \to 0$$

in which $i$ is strictly cosingular and $q$ is strictly singular [12, Theorem 6.4]. By Proposition 3.1, $Z_p$ is neither subprojective nor superprojective. $\square$

**Remark.** As for subprojective spaces [14], Proposition 3.2 implies that being superprojective is not open with respect to $\Theta_X(\cdot,\cdot)$, the gap metric between subspaces of a Banach space $X$. In other words, there exists a Banach space $X$ with a superprojective subspace $Y$ so that for every $\varepsilon > 0$ we can find a non-superprojective subspace $Y_\varepsilon$ of $X$ such that $\Theta_X(Y, Y_\varepsilon) < \varepsilon$.

Since $\ell_1$ is subprojective, the following result suggests that the class of non-reflexive superprojective spaces is smaller than that of non-reflexive subprojective spaces.

**Proposition 3.3.** *Let $X$ be a Banach space containing a subspace isomorphic to $\ell_1$. Then $X$ is not superprojective and $X^*$ is not subprojective.*

Proof. If $X$ contains a subspace isomorphic to $\ell_1$, then there exists a surjective operator $Q\colon X \longrightarrow \ell_2$ which is 2-summing [6, Corollary 4.16], therefore weakly compact and completely continuous [6, Theorem 2.17], therefore strictly singular: Indeed, if $Q|_M$ is an isomorphism, then $M$ is reflexive and weakly convergent sequences in $M$ are convergent, so $M$ is finite-dimensional. By Proposition 3.1, $X$ is not superprojective.

For the second part, observe that $Q^{**}\colon X^{**} \longrightarrow \ell_2$ is also 2-summing [6, Proposition 2.19]. Then $Q^{**}$ is strictly singular, hence $Q^*\colon \ell_2 \longrightarrow X^*$ is a strictly cosingular embedding. $\square$

Proposition 3.3 allows to fully characterise the superprojectivity of $C(K)$ spaces. Recall that a compact space is called *scattered* if each of its non-empty subsets has an isolated point.

**Corollary 3.4.** *Let $K$ be a compact set. Then $C(K)$ is superprojective if and only if $K$ is scattered.*

Proof. If $K$ is scattered, then $C(K)$ is superprojective [10, Proposition 3.4]. On the other hand, if $K$ is not scattered, then $C(K)$ contains a copy of $\ell_1$ [15] and cannot be superprojective by Proposition 3.3. $\square$

It also follows immediately that certain tensor products cannot be superprojective.

**Corollary 3.5.** *Let $X$ and $Y$ be Banach spaces and suppose that $X$ admits an unconditional finite-dimensional decomposition and $L(X, Y^*) \neq K(X, Y^*)$. Then $X \hat{\otimes}_\pi Y$ is not superprojective.*

Proof. Note that $(X \hat{\otimes}_\pi Y)^* \equiv L(X, Y^*)$. Since $L(X, Y^*) \neq K(X, Y^*)$, we have that $L(X, Y^*)$ contains $\ell_\infty$ [11, Theorem 6], hence $X \hat{\otimes}_\pi Y$ contains a (complemented) copy of $\ell_1$. □

Since the spaces $\ell_p$ have an unconditional basis and are subprojective and superprojective for $1 < p < \infty$, we can now characterise the superprojectivity of the tensor products $\ell_p \hat{\otimes}_\pi \ell_q$.

**Corollary 3.6.** *Let $1 < p, q < \infty$. Then the following are equivalent:*

(i) *$\ell_p \hat{\otimes}_\pi \ell_q$ is superprojective;*
(ii) *$\ell_p \hat{\otimes}_\pi \ell_q$ is reflexive;*
(iii) *$L(\ell_p, \ell_q^*) = K(\ell_p, \ell_q^*)$;*
(iv) *$p > q/(q-1)$.*

Proof. We have that $\ell_p \hat{\otimes}_\pi \ell_q$ is reflexive if and only if $L(\ell_p, \ell_q^*) = K(\ell_p, \ell_q^*)$ [16, Theorem 4.21] if and only if $p > q/(q-1)$ [13, Proposition 2.c.3]. If $L(\ell_p, \ell_q^*) \neq K(\ell_p, \ell_q^*)$, then $\ell_p \hat{\otimes}_\pi \ell_q$ is not superprojective by Corollary 3.5; otherwise, $\ell_p \hat{\otimes}_\pi \ell_q$ is reflexive and $\ell_p \hat{\otimes}_\pi \ell_q = (\ell_p^* \hat{\otimes}_\varepsilon \ell_q^*)^*$ [16, Theorem 5.33], so $\ell_p^* \hat{\otimes}_\varepsilon \ell_q^*$ is reflexive and subprojective [14, Corollary 3.3] and $\ell_p \hat{\otimes}_\pi \ell_q$ is superprojective. □

**Corollary 3.7.** *$L_p \hat{\otimes}_\pi L_q$ is not superprojective for any $1 \leq p, q \leq \infty$.*

Proof. If $p$ is either 1 or strictly greater than 2, then $L_p$ itself is not superprojective, so neither is $L_p \hat{\otimes}_\pi L_q$, and similarly for $q$. Thus, we are only concerned with the case $1 < p, q \leq 2$, but then both $L_p$ and $L_q^*$ contain complemented copies of $\ell_2$, so $L(L_p, L_q^*) \neq K(L_p, L_q^*)$ and $L_p \hat{\otimes}_\pi L_q$ is not superprojective by Corollary 3.5. □

**Remark.** Observe that $\ell_p \hat{\otimes}_\pi \ell_q$ and $\ell_p \hat{\otimes}_\varepsilon \ell_q$ are subprojective for all $1 \leq p, q < \infty$ [14, Corollary 3.3] and $L_p \hat{\otimes}_\varepsilon L_q$ is subprojective for all $2 \leq p, q < \infty$ [14, Corollary 3.5]. This strengthens the idea that the class of non-reflexive superprojective spaces seems to be smaller than that of non-reflexive subprojective spaces.

## 4. Stability results for superprojective spaces

We will now turn our attention to some stability results for the class of superprojective spaces. Our first result here, and key to subsequent ones, proves that the direct sum of two superprojective Banach spaces is again superprojective.

**Proposition 4.1.** *Let $X$ and $Y$ be Banach spaces. Then $X \oplus Y$ is superprojective if and only if both $X$ and $Y$ are superprojective.*

PROOF. $X$ and $Y$ are quotients of $X \oplus Y$; if $X \oplus Y$ is superprojective, then so are $X$ and $Y$ by Proposition 2.3.

Conversely, assume that $X$ are $Y$ are both superprojective, and define the projections $P_X : X \oplus Y \longrightarrow X \oplus Y$, with range $X$ and kernel $Y$, and $P_Y : X \oplus Y \longrightarrow X \oplus Y$, with range $Y$ and kernel $X$. Take any surjection $S : X \oplus Y \longrightarrow Z$. Then $S = SP_X + SP_Y$ is not strictly cosingular, so either $SP_X$ or $SP_Y$ is not strictly cosingular; without loss of generality, we will assume that it is $SP_X$. By Proposition 2.4, there exists another surjection $R : Z \longrightarrow W$ such that $N(RS)$ is complemented in $X \oplus Y$, which finishes the proof by Proposition 2.1. □

We will now state the main result in this section, which proves that a space is superprojective if it admits a suitable decomposition into superprojective parts. Recall that an operator $T : X \longrightarrow Y$ is *upper semi-Fredholm* if $N(T)$ is finite-dimensional and $R(T)$ is closed, and $T$ is *lower semi-Fredholm* if $R(T)$ is finite-codimensional (hence closed). Note that $T$ is lower semi-Fredholm if and only if $T^*$ is upper semi-Fredholm.

**Theorem 4.2.** *Let $X$ be a Banach space, let $\Lambda$ be a well-ordered set and let $(P_\lambda)_{\lambda \in \Lambda}$ and $(Q_\lambda)_{\lambda \in \Lambda}$ be bounded families of projections on $X$ such that:*

(i) $P_\lambda^* x^* \underset{\lambda}{\longrightarrow} x^*$ *for every* $x^* \in X^*$*;*

(ii) $P_\mu P_\nu = P_{\min\{\mu,\nu\}}$ *and* $Q_\mu Q_\nu = Q_{\min\{\mu,\nu\}}$ *for every* $\mu$, $\nu \in \Lambda$*;*

(iii) $Q_\mu P_\nu = P_\nu Q_\mu$ *for every* $\mu$, $\nu \in \Lambda$, *and* $Q_\mu P_\nu = P_\nu$ *if* $\mu \geq \nu$*;*

(iv) $Q_\lambda(X)$ *is superprojective for every* $\lambda \in \Lambda$*;*

(v) *for every unbounded strictly increasing sequence $(\lambda_k)_{k \in \mathbb{N}}$ of elements in $\Lambda$ and every sequence $(x_k^*)_{k \in \mathbb{N}}$ of non-null elements in $X^*$ such that $x_1^* \in R(P_{\lambda_1}^*)$ and $x_k^* \in R(P_{\lambda_k}^*(I - Q_{\lambda_{k-1}}^*))$ for $k > 1$, the subspace $[x_k^* : k \in \mathbb{N}]_\perp$ is contained in a complemented infinite-codimensional subspace of $X$.*

*Then $X$ is superprojective.*

Here, an unbounded sequence in $\Lambda$ is one that does not have an upper bound within $\Lambda$. Also, this result is only really interesting if $\Lambda$ does not have a maximum element; otherwise, if $\lambda$ is the maximum of $\Lambda$, then $P_\lambda = I_X$ by condition (i) and $Q_\lambda = Q_\lambda P_\lambda = P_\lambda = I_X$ by condition (iii), so $X = Q_\lambda(X)$ is already superprojective by condition (iv).

PROOF. Let $M$ be an infinite-codimensional subspace of $X$ and let us denote its natural quotient map by $S\colon X \longrightarrow X/M$. If there exists $\lambda \in \Lambda$ such that $SQ_\lambda$ is not strictly cosingular, then Proposition 2.4 provides another surjection $R\colon X/M \longrightarrow Z$ such that $N(RS)$ is complemented in $X$. Since $N(RS)$ is infinite-codimensional and contains $M$ we are done.

Otherwise, assume that $SQ_\lambda$ is strictly cosingular for every $\lambda \in \Lambda$. Let $C \geq 1$ be such that $\|P_\lambda\| \leq C$ and $\|Q_\lambda\| \leq C$ for every $\lambda \in \Lambda$, and let $\varepsilon = 1/8C^3 > 0$. We will construct a strictly increasing sequence $\lambda_1 < \lambda_2 < \dots$ of elements in $\Lambda$ and a sequence $(x_n^*)_{n\in\mathbb{N}}$ of norm-one elements in $M^\perp \subseteq X^*$ such that $\|Q_{\lambda_{k-1}}^* x_k^*\| < 2^{-k}\varepsilon$ and $\|P_{\lambda_k}^* x_k^* - x_k^*\| < 2^{-k}\varepsilon$ for every $k \in \mathbb{N}$, where we write $Q_{\lambda_0} = 0$ for convenience. To this end, let $k \in \mathbb{N}$, and assume that $\lambda_{k-1}$ has already been obtained. By hypothesis, $Q_{\lambda_{k-1}}^* S^* = (SQ_{\lambda_{k-1}})^*$ is not an isomorphism, where $S^*\colon (X/M)^* \longrightarrow X^*$ is an isometric embedding with range $M^\perp$, so there exists $x_k^* \in M^\perp$ such that $\|x_k^*\| = 1$ and $\|Q_{\lambda_{k-1}}^* x_k^*\| < 2^{-k}\varepsilon$, and then there is $\lambda_k > \lambda_{k-1}$ such that $\|P_{\lambda_k}^* x_k^* - x_k^*\| < 2^{-k}\varepsilon$ by condition (i), which finishes the inductive construction process. Let $H = [x_k^* : k \in \mathbb{N}] \subseteq X^*$; then $H_\perp$ is infinite-codimensional and contains $M$.

It is easy to check that the operators $T_k := (I - Q_{\lambda_{k-1}})P_{\lambda_k}$ are projections with norm $\|T_k\| \leq (1 + C)C \leq 2C^2$, and that $T_iT_j = 0$ if $i \neq j$.

Let now $z_k^* = T_k^*(x_k^*) = P_{\lambda_k}^*(I - Q_{\lambda_{k-1}}^*)x_k^*$ for each $k \in \mathbb{N}$; then

$$\|z_k^* - x_k^*\| \leq \|P_{\lambda_k}^* x_k^* - x_k^*\| + \|P_{\lambda_k}^* Q_{\lambda_{k-1}}^* x_k^*\| < 2^{-k}\varepsilon + 2^{-k}\varepsilon C \leq 2^{1-k}\varepsilon C < 1/2,$$

so $1/2 < \|z_k^*\| < 3/2$ for every $k \in \mathbb{N}$. If we take $x_k \in X$ such that $\|x_k\| < 2$ and $\langle z_k^*, x_k\rangle = 1$ for each $k \in \mathbb{N}$, and define $z_k = T_k x_k$, it follows that

$$\langle z_k^*, z_k\rangle = \langle z_k^*, T_k x_k\rangle = \langle T_k^* z_k^*, x_k\rangle = \langle z_k^*, x_k\rangle = 1,$$

for every $k \in \mathbb{N}$ and

$$\langle z_i^*, z_j\rangle = \langle T_i^* z_i^*, T_j z_j\rangle = \langle z_i^*, T_i T_j z_j\rangle = 0$$

if $i \neq j$, which makes $(z_k^*, z_k)_{n\in\mathbb{N}}$ a biorthogonal sequence in $(X^*, X)$. In the spirit of the principle of small perturbations [4], let $K\colon X \longrightarrow X$ be the operator defined as $K(x) = \sum_{n=1}^{\infty} \langle x_n^* - z_n^*, x\rangle z_n$; then

$$\sum_{n=1}^{\infty} \|x_n^* - z_n^*\| \, \|z_n\| < \sum_{n=1}^{\infty} (2^{1-n}\varepsilon C)(4C^2) = \sum_{n=1}^{\infty} 2^{-n} = 1,$$

so $K$ is well defined and $U = I + K$ is an isomorphism on $X$. Moreover, $K^*\colon X^* \longrightarrow X^*$ is defined as $K^*(x^*) = \sum_{n=1}^{\infty} \langle x^*, z_n\rangle (x_n^* - z_n^*)$, so $K^*(z_k^*) = x_k^* - z_k^*$ and $U^*(z_k^*) = x_k^*$ for every $k \in \mathbb{N}$. Let $Z = [z_k^* : k \in \mathbb{N}]$; then $U^*(Z) = H$ and $U(H_\perp) = Z_\perp$.

Next we will show that $Z$ is weak* closed in $X^*$. Note first that $T_j P_{\lambda_i} = (I - Q_{\lambda_{j-1}}) P_{\lambda_j} P_{\lambda_i} = (I - Q_{\lambda_{j-1}}) P_{\lambda_j} = T_j$ if $i \geq j$, and $T_j P_{\lambda_i} = (I - Q_{\lambda_{j-1}}) P_{\lambda_j} P_{\lambda_i} = (I - Q_{\lambda_{j-1}}) P_{\lambda_i} = 0$ otherwise. Given that $z_k^* \in R(T_k^*)$ for every $k \in \mathbb{N}$, this means that $P_{\lambda_i}^* z_j^* = z_j^*$ if $i \geq j$ and $P_{\lambda_i}^* z_j^* = 0$ otherwise, so $P_{\lambda_k}^*(Z) = [z_1^*, \ldots, z_k^*]$, which is finite-dimensional, for every $k \in \mathbb{N}$. Let $x^*$ be a weak* cluster point of $Z$; then $P_{\lambda_k}^* x^* \in P_{\lambda_k}^*(Z) \subseteq Z$ and $P_{\lambda_k}^* x^* \underset{k}{\longrightarrow} x^*$ by condition (i), so $x^* \in Z$ and $Z$ is indeed weak* closed. The fact that $H = U^*(Z)$ implies that $H$ is weak* closed, as well.

This means, in turn, that no $Q_\lambda^*$ can be an isomorphism on $H$ for any $\lambda \in \Lambda$. To see this, consider the natural quotient $Q_{H_\perp}\colon X \longrightarrow X/H_\perp$, where $X/H_\perp$ is infinite-dimensional. Since $M \subseteq H_\perp$, the operator $Q_{H_\perp}$ factors through $S = Q_M\colon X \longrightarrow X/M$ and, since $SQ_\lambda$ is strictly cosingular for every $\lambda \in \Lambda$ by our initial hypothesis, it follows that $Q_{H_\perp} Q_\lambda$ cannot be surjective for any $\lambda \in \Lambda$, or even lower semi-Fredholm; equivalently, $Q_\lambda^*$ cannot be upper semi-Fredholm on ${H_\perp}^\perp$ for any $\lambda \in \Lambda$, where ${H_\perp}^\perp = H$ because $H$ is weak* closed.

Finally, we will check that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is unbounded. Assume, to the contrary, that there existed some $\lambda \in \Lambda$ such that $\lambda_k \leq \lambda$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, we would have $T_k Q_\lambda = (I - Q_{\lambda_{k-1}}) P_{\lambda_k} Q_\lambda = (I - Q_{\lambda_{k-1}}) P_{\lambda_k} = T_k$, so $Q_\lambda^* z_k^* = z_k^*$ and $Q_\lambda^*$ would be an isomorphism on $Z$. But then $Q_\lambda^* U^{-1*}$ would be an isomorphism on $H$, where $U^{-1} = I - U^{-1}K$ is a compact perturbation of the identity, so $Q_\lambda^*$ would be upper semi-Fredholm on $H$, a contradiction.

Now that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is known to be unbounded, condition (v) states that $Z_\perp$ is contained in a complemented infinite-codimensional subspace of $X$, and then so is $H_\perp = U^{-1}(Z_\perp)$. □

Note that any sequence $(P_n)_{n\in\mathbb{N}}$ of projections in $X$ satisfying the conditions of Theorem 4.2 effectively defines a Schauder decomposition for $X$, where the components are the ranges of each operator $P_n(I - P_{n-1}) = P_n - P_{n-1}$; equivalently, each $P_n$ is the projection onto the sum of the first $n$ components. For the purposes of Theorem 4.2, these components need not be finite-dimensional.

Regarding condition (v), a further remark is in order. It may very well be the case that there are no unbounded strictly increasing sequences in $\Lambda$, for instance if $\Lambda = [0, \omega_1)$, where $\omega_1$ is the first uncountable ordinal, in which case condition (v) is trivially satisfied and does not impose any additional restriction on $X$ or the projections. In terms of the proof of Theorem 4.2, this means that $SQ_\lambda$ must be eventually not strictly cosingular for some $\lambda \in \Lambda$, and this is so because $Q^*_\lambda$ is an isomorphism on $Z$ for any $\lambda$ greater than the supremum of $(\lambda_k)_{k\in\mathbb{N}}$, so $Q^*_\lambda$ is upper semi-Fredholm on $H$ and $SQ_\lambda$ is not strictly cosingular, as per the last paragraphs of the proof of Theorem 4.2.

Most of the time, we will not need the full strength of Theorem 4.2, and the following, simpler version will suffice, where only a single family of projections $(P_\lambda)_{\lambda\in\Lambda} = (Q_\lambda)_{\lambda\in\Lambda}$ is involved.

**Theorem 4.3.** *Let $X$ be a Banach space, let $\Lambda$ be a well-ordered set and let $(P_\lambda)_{\lambda\in\Lambda}$ be a bounded family of projections on $X$ such that:*

(i) $P^*_\lambda x^* \underset{\lambda}{\longrightarrow} x^*$ *for every* $x^* \in X^*$*;*

(ii) $P_\mu P_\nu = P_{\min\{\mu,\nu\}}$ *for every* $\mu$, $\nu \in \Lambda$*;*

(iii) $P_\lambda(X)$ *is superprojective for every* $\lambda \in \Lambda$*;*

(iv) *for every unbounded strictly increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of elements in $\Lambda$ and every sequence $(x^*_k)_{k\in\mathbb{N}}$ of non-null elements in $X^*$ such that $x^*_1 \in R(P^*_{\lambda_1})$ and $x^*_k \in R(P^*_{\lambda_k} - P^*_{\lambda_{k-1}})$ for $k > 1$, the subspace $[x^*_k : k \in \mathbb{N}]_\perp$ is contained in a complemented infinite-codimensional subspace of $X$.*

*Then $X$ is superprojective.*

Our first use of Theorems 4.2 and 4.3 will be to prove that the (infinite) sum of superprojective spaces, such as $\ell_p(X_n)$ or $c_0(X_n)$, is also superprojective, if the sum is done in a "superprojective" way.

**Definition.** We will say that a Banach space $E \subseteq \mathbb{R}^\mathbb{N}$ is a solid sequence space if, for every $(\alpha_n)_{n\in\mathbb{N}} \in E$ and $(\beta_n)_{n\in\mathbb{N}} \in \mathbb{R}^\mathbb{N}$ with $|\beta_n| \leq |\alpha_n|$ for every $n \in \mathbb{N}$, it holds that $(\beta_n)_{n\in\mathbb{N}} \in E$ and $\|(\beta_n)_{n\in\mathbb{N}}\| \leq \|(\alpha_n)_{n\in\mathbb{N}}\|$.

We will say that $E$ is an unconditional sequence space if it is a solid sequence space and the sequence of canonical vectors $(e_i)_{i\in\mathbb{N}}$ is a normalised basis for $E$, where $e_i = (\delta_{ij})_{j\in\mathbb{N}}$.

If $E$ is an unconditional sequence space, then its canonical basis $(e_n)_{n\in\mathbb{N}}$ is actually 1-unconditional, and its conjugate $E^*$ can be identified with a solid sequence space itself in the usual way, where the action of $\beta = (\beta_n)_{n\in\mathbb{N}} \in E^*$ on $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in E$ is $\langle \beta, \alpha\rangle = \sum_{n=1}^{\infty} \beta_n \alpha_n$. If the canonical basis $(e_n)_{n\in\mathbb{N}}$ is shrinking in $E$, then $E^*$ is additionally unconditional (the coordinate functionals are a basis for $E^*$).

Solid sequence spaces will play a central role in some of our results because of the following construction.

**Definition.** Let $E$ be a solid sequence space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. We will write $E(X_n)$ for the Banach space of all sequences $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ for which $(\|x_n\|)_{n\in\mathbb{N}} \in E$, with the norm $\|(x_n)_{n\in\mathbb{N}}\| = \|(\|x_n\|)_{n\in\mathbb{N}}\|_E$.

The identification of the dual of an unconditional sequence space with another solid sequence space can be carried up to the sum of spaces.

**Proposition 4.4.** *Let $E$ be an unconditional sequence space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Then $E(X_n)^* \equiv E^*(X_n^*)$.*

Proof. Each $(x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$ clearly defines an element of $E(X_n)^*$, so we only have to show the converse identification.

Let $z^* \in E(X_n)^*$, let $J_n\colon X_n \longrightarrow E(X_n)$ be the canonical inclusion of $X_n$ into $E(X_n)$ for each $n \in \mathbb{N}$ and let $x_n^* = J_n^*(z^*) \in X_n^*$ for each $n \in \mathbb{N}$; we will prove that $z^* = (x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$.

To prove that $(x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$, choose $x_n \in X_n$ such that $\|x_n\| = 1$ and $\langle x_n^*, x_n\rangle \geq \frac{1}{2}\|x_n^*\|$ for each $n \in \mathbb{N}$, and take any non-negative $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in E$. By the definition of $E(X_n)$, we have that $(\alpha_n x_n)_{n\in\mathbb{N}} \in E(X_n)$, so

$$\begin{aligned}
\sum_{n=1}^{\infty} \|x_n^*\|\alpha_n &\leq \sum_{n=1}^{\infty} 2\langle x_n^*, x_n\rangle \alpha_n = 2\sum_{n=1}^{\infty}\langle x_n^*, \alpha_n x_n\rangle \\
&= 2\sum_{n=1}^{\infty}\langle J_n^*(z^*), \alpha_n x_n\rangle = 2\sum_{n=1}^{\infty}\langle z^*, J_n(\alpha_n x_n)\rangle \\
&= 2\langle z^*, (\alpha_n x_n)_{n\in\mathbb{N}}\rangle \leq 2\|z^*\|\|(\alpha_n x_n)_{n\in\mathbb{N}}\| = 2\|z^*\|\,\|\alpha\|.
\end{aligned}$$

This proves that $(\|x_n^*\|) \in E^*$ and, as a consequence, $(x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$.

Now, given $i \in \mathbb{N}$ and $x_i \in X_i$, we have $\langle (x_n^*)_{n\in\mathbb{N}}, J_i(x_i)\rangle = \langle x_i^*, x_i\rangle = \langle z^*, J_i(x_i)\rangle$, so $(x_n^*)_{n\in\mathbb{N}}$ and $z^*$ coincide over the finitely non-null sequences of $E(X_n)$ and therefore $z^* = (x_n^*)_{n\in\mathbb{N}}$. $\square$

As we said, we will prove that the sum of superprojective spaces is also superprojective, if the sum is done in a superprojective way, which translates to the requirement that the space $E$ governing the sum must be superprojective itself. This excludes $\ell_1$ and, more generally, imposes that any unconditional basis in $E$ be shrinking, for the same reasons that $\ell_1$ is not superprojective, or precisely because of this.

**Proposition 4.5.** *Let $X$ be a superprojective Banach space and let $(x_n)_{n\in\mathbb{N}}$ be an unconditional basis of $X$. Then $(x_n)_{n\in\mathbb{N}}$ is shrinking.*

Proof. If $(x_n)_{n\in\mathbb{N}}$ is unconditional but not shrinking, then $X$ contains a (complemented) copy of $\ell_1$ [3, Theorem 3.3.1] and cannot be superprojective by Proposition 3.3. $\square$

**Theorem 4.6.** *Let $E$ be an unconditional sequence space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Then $E(X_n)$ is superprojective if and only if all of $E$ and $X_n$ are superprojective.*

Proof. Let $X = E(X_n)$. All of $E$ and $X_n$ are quotients of $X$; if $X$ is superprojective, then so are $E$ and each $X_n$.

Assume now that $E$ and each $X_n$ are superprojective, and define the projections $P_n\colon X \longrightarrow X$ as $P_n((x_n)_{n\in\mathbb{N}}) = (x_1, \ldots, x_n, 0, \ldots)$ for each $n \in \mathbb{N}$. We will prove that the sequence $(P_n)_{n\in\mathbb{N}}$ meets the criteria of Theorem 4.3. The fact that $(P_n)_{n\in\mathbb{N}}$ is associated with the natural Schauder decomposition of $X = E(X_n)$ is enough for condition (ii) to hold. For condition (iii), note that $P_n(X)$ is isometric to $\bigoplus_{i=1}^n X_i$, which is superprojective by Proposition 4.1. As for condition (i), $E$ is superprojective and its canonical basis $(e_n)_{n\in\mathbb{N}}$ is unconditional, therefore shrinking by Proposition 4.5, so $E^*$ is unconditional and $(P_n^*)_{n\in\mathbb{N}}$ is the sequence of projections associated with the natural Schauder decomposition of $E(X_n)^* \equiv E^*(X_n^*)$.

To prove condition (iv), let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of integers, let $T_1 = P_{n_1}$ and $T_k = P_{n_k} - P_{n_{k-1}}$ for $k > 1$, and let $x_k^* \in R(T_k^*)$

be non-null for each $k \in \mathbb{N}$, as in Theorem 4.3. Define $M = [x_k^* : k \in \mathbb{N}]_\perp$, which is infinite-codimensional. Then $x_k^* \in X^* \equiv E^*(X_n^*)$, so

$$x_k^* = (0, \ldots, 0, z_{n_{k-1}+1}^*, \ldots, z_{n_k}^*, 0, \ldots),$$

where $z_i^* \in X_i^*$. Pick a normalised $z_i \in X_i$ such that $\langle z_i^*, z_i \rangle \geq \|z_i^*\|/2$ for each $i \in \mathbb{N}$, and consider the operator $J: E \longrightarrow X$ defined as $J((\alpha_n)_{n\in\mathbb{N}}) = (\alpha_n z_n)_{n\in\mathbb{N}}$, which is an isometric embedding by the definition of $X = E(X_n)$.

We claim that $Q_M J: E \longrightarrow X/M$ is a surjection. Indeed, given $x = (x_n)_{n\in\mathbb{N}} \in X$, with each $x_n \in X_n$, let $\alpha_n = \langle z_n^*, x_n \rangle / \langle z_n^*, z_n \rangle$ if $z_n^* \neq 0$, else $\alpha_n = 0$, for each $n \in \mathbb{N}$, and define $\alpha = (\alpha_n)_{n\in\mathbb{N}}$. Then $|\alpha_n| \leq 2\|x_n\|$ for every $n \in \mathbb{N}$, so $\alpha \in E$, and $\langle x_k^*, x - J(\alpha) \rangle = \sum_{i=n_{k-1}-1}^{n_k} \langle z_i^*, x_i - \alpha_i z_i \rangle = 0$ for every $k \in \mathbb{N}$, so $x - J(\alpha) \in M$ and $Q_M(x) = Q_M J(\alpha) \in R(Q_M J)$.

Now, by the superprojectivity of $E$ and Proposition 2.1, there exists another surjection $S: X/M \longrightarrow Z$ such that $N(SQ_M J)$ is complemented in $E$; by Proposition 2.2, $N(SQ_M)$ is complemented in $X$, where $M \subseteq N(SQ_M)$ and $R(SQ_M) = Z$, which is infinite-dimensional. □

The following result will help us check for the last condition in Theorems 4.2 and 4.3.

**Lemma 4.7.** *Let $X$ be a Banach space, let $E$ be an unconditional sequence space and let $T$, $(T_k)_{k\in\mathbb{N}}$ be projections in $X$ such that*

- (i) $T_i T_j = 0$ *if* $i \neq j$;
- (ii) $T_k T = T T_k = T_k$ *for every* $k \in \mathbb{N}$;
- (iii) $R(T)$ *embeds into* $E(R(T_k))$ *via the mapping that takes* $x \in R(T)$ *to* $(T_k(x))_{k\in\mathbb{N}}$.

*Let $x_k^* \in R(T_k^*)$ be non-null for each $k \in \mathbb{N}$. Then $[x_k^* : k \in \mathbb{N}]_\perp$ is complemented in $X$.*

Proof. We will assume without loss of generality that $\|x_k^*\| = 1$ for every $k \in \mathbb{N}$. Let $Z = E(R(T_k))$ and let $U: R(T) \longrightarrow Z$ be the isomorphism into $Z$ defined as $U(x) = (T_k(x))_{k\in\mathbb{N}}$.

Note that, in fact, $(T_k(x))_{k\in\mathbb{N}} = (T_k(T(x)))_{k\in\mathbb{N}} = U(T(x)) \in Z$ for every $x \in X$, so $(\|T_k(x)\|)_{k\in\mathbb{N}} \in E$ and $\|(\|T_k(x)\|)_{k\in\mathbb{N}}\|_E = \|U(T(x))\|_Z$ for every $x \in X$. Define $Q: X \longrightarrow E$ as $Q(x) = (\langle x_k^*, x \rangle)_{k\in\mathbb{N}}$; then

$$|\langle x_k^*, x \rangle| = |\langle T_k^*(x_k^*), x \rangle| = |\langle x_k^*, T_k(x) \rangle| \leq \|T_k(x)\|$$

for every $x \in X$, so $Q$ is well defined and $\|Q\| \leq \|UT\|$. Also, $(\|T_k(x)\|)_{k\in\mathbb{N}} \in E$ implies that $T_k x \underset{k}{\longrightarrow} 0$ for every $x \in X$, so there exists a constant $C$ such that $\|T_k\| \leq C$ for every $k \in \mathbb{N}$.

Take now $x_k \in X$ such that $\langle x_k^*, x_k\rangle = 1$ and $\|x_k\| \leq 2$ for each $k \in \mathbb{N}$, so that $\langle x_i^*, T_j x_j\rangle = \langle T_j^* x_i^*, x_j\rangle = \delta_{ij}$ for every $i$, $j \in \mathbb{N}$, and define $J\colon E \longrightarrow X$ as $J((\alpha_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \alpha_n T_n(x_n)$. Then $U(J((\alpha_n)_{n\in\mathbb{N}})) = (\alpha_k T_k(x_k))_{k\in\mathbb{N}}$, as seen by considering the action of $UJ$ over the finitely non-null sequences of $E$, where $1 \leq \|T_k(x_k)\| \leq C$ for every $k \in \mathbb{N}$, so $UJ\colon E \longrightarrow Z$ is an isomorphism, and so must be $J$. Finally,

$$QJ((\alpha_n)_{n\in\mathbb{N}}) = \left(\left\langle x_k^*, \sum_{n=1}^{\infty} \alpha_n T_n(x_n)\right\rangle\right)_{k\in\mathbb{N}} = (\alpha_k)_{k\in\mathbb{N}},$$

so $QJ = I_E$ and $JQ$ is a projection in $X$ with kernel $[x_k^* : k \in \mathbb{N}]_\perp$. $\square$

**Theorem 4.8.** *Let $X$ and $Y$ be $c_0$ or $\ell_p$ for $1 < p < \infty$. Then $X \hat{\otimes}_\varepsilon Y$ is superprojective.*

Proof. Let $R_n\colon X \longrightarrow X$ be the projection given by $R_n((\alpha_k)_{k\in\mathbb{N}}) = (\alpha_1, \ldots, \alpha_n, 0, \ldots)$ for each $n \in \mathbb{N}$, and similarly for $Y$. (We are abusing the notation here for the sake of simplicity in that $R_n$ is really a different operator on each of $X$ and $Y$ unless they are the same space.) Define the projections

$$\begin{aligned} P_n &= R_n \otimes R_n \\ Q_n &= I_{X \hat{\otimes}_\varepsilon Y} - (I_X - R_n) \otimes (I_Y - R_n) \\ &= R_n \otimes R_n + (I_X - R_n) \otimes R_n + R_n \otimes (I_Y - R_n) \end{aligned}$$

We will prove that the sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ meet the criteria of Theorem 4.2.

Conditions (ii) and (iii) are readily satisfied, because they clearly hold for the elementary tensors $e_i \otimes e_j$. For condition (i), both $X^*$ and $Y^*$ are $\ell_q$ spaces for some $1 \leq q < \infty$, so $R_n^*(x^*) \underset{n}{\longrightarrow} x^*$ for every $x^* \in X^*$, and similarly for $Y^*$, so $P_n^*(z^*) = (R_n^* \otimes R_n^*)(z^*) \underset{n}{\longrightarrow} z^*$ for every $z^* \in (X \hat{\otimes}_\varepsilon Y)^* = X^* \hat{\otimes}_\pi Y^*$ [16, Theorem 5.33], again because it holds for the elementary tensors. For condition (iv), note that the range of $Q_n$ is the direct sum of the ranges of $R_n \otimes R_n$, $(I_X - R_n) \otimes R_n$ and $R_n \otimes (I_Y - R_n)$, where the first one is finite-dimensional and the other two are the sum of finitely many copies of $N(R_n)$ in $X$ and $Y$, respectively, which are finite-codimensional subspaces of $X$ and $Y$, respectively, hence superprojective.

To prove condition (v), let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of integers and let $T_1 = P_{n_1}$ and $T_k = (I - Q_{n_{k-1}})P_{n_k}$ for $k > 1$, as in Theorem 4.2. Note that, for $k > 1$, $T_k$ is the projection $T_k = (R_{n_k} - R_{n_{k-1}}) \otimes (R_{n_k} - R_{n_{k-1}})$, so $T_i T_j = 0$ if $i \neq j$. Using Tong's result on diagonal submatrices [18] [13, Proposition 1.c.8], the operator $T = \sum_{k=1}^{\infty} T_k$ is a norm-one projection in $X \hat{\otimes}_\varepsilon Y$, with $T_k T = T T_k = T_k$ for every $k \in \mathbb{N}$, and $R(T)$ embeds into $c_0(R(T_k))$ or $\ell_s(R(T_k))$ for suitable $1 < s < \infty$ in the natural way [14, Lemma 3.6], so Lemma 4.7 ensures that $[x_k^* : k \in \mathbb{N}]_\perp$ is complemented in $X \hat{\otimes}_\varepsilon Y$ for any choice of non-null elements $x_k^* \in R(T_k^*)$. $\square$

Theorem 4.8 can actually be extended to injective tensor products of finitely many copies of $c_0$ and $\ell_p$ $(1 < p < \infty)$ inductively in the obvious way with only minor modifications.

Lastly, we will show that $C(K, X)$ is superprojective whenever so is $X$ at least if $K$ is an interval of ordinals, which includes the case where $K$ is scattered and metrisable [17, Section 8].

**Theorem 4.9.** *Let $X$ be a superprojective Banach space and let $\lambda$ be an ordinal. Then $C_0([0,\lambda], X)$ and $C([0,\lambda], X)$ are superprojective.*

Proof. The proof will proceed by induction in $\lambda$. Assume that $C_0([0,\mu], X)$ and $C([0,\mu], X)$ are indeed superprojective for all $\mu < \lambda$; we will first prove that $C_0([0,\lambda], X)$ is superprojective too. If $\lambda$ is not a limit ordinal, then $\lambda = \mu + 1$ for some $\mu$ and $C_0([0,\lambda], X) \equiv C([0,\mu], X)$, which is superprojective by the induction hypothesis.

Otherwise, if $\lambda$ is a limit ordinal, define the projections

$$P_\mu : C_0([0,\lambda], X) \longrightarrow C_0([0,\lambda], X)$$

as $P_\mu(f) = f\chi_{[0,\mu]}$ for each $\mu < \lambda$. We will prove that the family $(P_\mu)_{\mu<\lambda}$ meets the criteria of Theorem 4.3. Condition (ii) is immediate to check. For condition (iii), $P_\mu(C_0([0,\lambda], X))$ is isometric to $C([0,\mu], X)$, which is superprojective by the induction hypothesis.

For condition (i), we have $C_0([0,\lambda])^* = \ell_1([0,\lambda))$ [7, Theorem 14.24] and $C_0([0,\lambda], X)^* = (C_0([0,\lambda]) \hat{\otimes}_\varepsilon X)^* = C_0([0,\lambda])^* \hat{\otimes}_\pi X^*$ [16, Theorem 5.33], so $C_0([0,\lambda], X)^* = \ell_1([0,\lambda)) \hat{\otimes}_\pi X^* = \ell_1([0,\lambda), X^*)$ and $P_\mu^*(z) = z\chi_{[0,\mu]} \underset{\mu}{\longrightarrow} z$ for every $z \in \ell_1([0,\lambda), X^*)$.

As for condition (iv), let $(\lambda_k)_{k\in\mathbb{N}}$ be an unbounded strictly increasing sequence of elements in $[0,\lambda)$, should it exist, and let $T_1 = P_{\lambda_1}$ and $T_k = P_{\lambda_k} - P_{\lambda_{k-1}}$ for $k > 1$, as in Theorem 4.3. Then $T_k$ is the projection given by $T_k(f) = f\chi_{[\lambda_{k-1}+1,\lambda_k]}$ for $k > 1$, so $T_iT_j = 0$ if $i \neq j$. Since $(\lambda_k)_{k\in\mathbb{N}}$ is unbounded in $[0,\lambda)$, its supremum must be $\lambda$ itself, so $C_0([0,\lambda],X) = c_0(R(T_k)) = c_0(C([\lambda_{k-1}+1,\lambda_k],X))$ and Lemma 4.7, with $T = I$, ensures that $[x_k^* : k \in \mathbb{N}]_\perp$ is complemented in $C_0([0,\lambda],X)$ for any choice of non-null elements $x_k^* \in R(T_k^*)$.

Finally, $C([0,\lambda],X) = C_0([0,\lambda],X) \oplus X$, which is superprojective by Proposition 4.1. □

## 5. Some open problems

In order to better understand the duality relations between subprojective and superprojective Banach spaces we need to answer the following questions:

(1) Does $X$ being superprojective imply that $X^*$ is subprojective?

(2) Does $X^*$ being superprojective imply that $X$ is subprojective?

The following two problems seem to be related:

(3) Is $L_p \hat{\otimes}_\pi L_q$ subprojective for $2 \leq p,q < \infty$?

(4) Is $L_p \hat{\otimes}_\varepsilon L_q$ superprojective for $1 < p,q \leq 2$?

We saw in Section 3 that $L_p \hat{\otimes}_\varepsilon L_q$ is subprojective for $2 \leq p,q < \infty$, and $L_p \hat{\otimes}_\pi L_q$ is not superprojective for $1 < p,q \leq 2$.

Suppose that $K$ is a scattered compact.

(5) Does $X$ being subprojective imply that $C(K,X)$ or $C(K) \hat{\otimes}_\pi X$ is subprojective?

(6) Does $X$ being superprojective imply that $C(K,X)$ or $C(K) \hat{\otimes}_\pi X$ is superprojective?

Note that $C(K,X) \equiv C(K) \hat{\otimes}_\varepsilon X$. There are some partial positive answers when $K$ is an interval of ordinals, namely [14, Section 4.1] and [8] for (5) and Theorem 4.9 for (6).

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